

A Markovian Growth-Collapse Model

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Abstract

We consider growth-collapse processes (GCPs) that grow linearly between random partial collapse times at which they jump down according to some distribution depending on their current level. The jump occurrences are governed by a state-dependent rate function $r(x)$. We deal with the stationary distribution of such a GCP X_t , $t \geq 0$, and the distributions of the hitting times $T_a = \inf\{t \geq 0 \mid X_t = a\}$, $a > 0$. After presenting the general theory of these GCPs, several important special cases are studied. We also take a short look at the Markov-modulated case. In particular, we present a method to compute the distribution of $\min[T_a, \sigma]$ in this case (where σ is the time of the first jump) and apply it to determine the long-run average cost of running a certain Markov-modulated disaster-ridden system.

Keywords: Growth-collapse process; piecewise deterministic Markov process; stationary distribution; hitting time; uniform cut-off; duality; Markov modulation.

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1 Introduction

Growth-collapse processes (GCPs) are real-valued stochastic processes which grow (possibly in a random fashion) between random collapse times at which they jump down according to some distribution depending on their current level. This pattern of behavior can be encountered in a large variety of physical phenomena, for example the build-up of friction, earthquakes, avalanches and neuron firing; cf. Bak's [7] paradigm of "systems of self-organized criticality". Also in population growth models it seems reasonable to assume that the growth rate and the extent of occasional 'disasters' (e.g. epidemics) depend on the current population size. In the realm of operations research, GCPs occur in insurance mathematics and related fields and in models of production/inventory systems (Asmussen [5, 6], Rolski et al. [22]) and queueing (see e.g. Bekker et al. [8]).

Eliazar and Klafter [16] consider a GCP $X = (X_t)_{t \geq 0}$ composed of three random sources: (a) a steady random inflow with stationary independent positive increments; (b) crash times $\tau_1 < \tau_2 < \dots$ which form a renewal process; (c) crash proportions U_1, U_2, \dots which are i.i.d. random variables on $(0, 1)$. The three sources are assumed to be stochastically independent. At the n th crash time τ_n , the process jumps down by the amount $U_n X_{\tau_n-}$ so that the new system level at τ_n is $X_{\tau_n} = (1 - U_n) X_{\tau_n-}$. Eliazar and Klafter [16] compute several system characteristics (means, variances, Laplace transforms, probability tails) and focus on crash proportions governed by power-law distributions.

In the spirit of [16] we study in this paper a class of $[0, \infty)$ -valued, piecewise deterministic Markov processes (PDMPs) $X = (X_t)_{t \geq 0}$ characterized by the following features:

- (i) X increases linearly at rate 1 between jumps and is right-continuous.
- (ii) Given $X_t = x$, the probability of a jump in $(t, t + \eta)$ is equal to $r(x)\eta + o(\eta)$ as $\eta \rightarrow 0$, where $r : (0, \infty) \rightarrow [0, \infty)$ is a continuous function.
- (iii) If a jump occurs at time t , and $X_{t-} = x$, then the distribution of X_t is μ_x , which is a probability measure on $[0, x)$.

If $X_0 = 0$, the function $r(x)$ in (ii) is just the failure rate of the first jump

time $\sigma = \inf\{t \geq 0 \mid X_t \neq X_{t-}\}$, so that

$$\mathbb{P}(\sigma > t) = \exp \left\{ - \int_0^t r(x) dx \right\}, \quad t \geq 0.$$

It is reasonable to assume that $\mathbb{P}(\sigma < \infty) = 1$, which is equivalent to $\int_0^\infty r(x) dx = \infty$. In most examples the probability measure μ_x has a density $p(x, y)$ on $(0, x)$ and possibly an atom at zero. A particularly interesting case which we will study is that of ‘uniform’ crash proportions, i.e., $p(x, y) = x^{-1}1_{(0,x)}(y)$. Another example is the age process $(A_t)_{t \geq 0}$ of a standard renewal process which can be described as follows: It starts at $A_0 = 0$, increases at rate 1 between jumps, every jump takes the process back to 0, and a jump occurs in $(t, t + \eta)$ with probability $r(A_t)\eta + o(\eta)$ as $\eta \rightarrow 0$, where $r(\cdot)$ is the failure rate of the underlying distribution function.

As an alternative to the growth condition (i), we also consider the case that X increases linearly between jumps at a rate $c(J_t)$, where $(J_t)_{t \geq 0}$ is a modulating irreducible Markov chain having state space $\{1, \dots, n\}$ and the rates $c(j)$ satisfy $c(1) > c(2) > \dots > c(n) \geq 0$.

In a variety of examples we will in particular deal with the stationary distribution of X and the distribution of the hitting time $T_a = \inf\{t \geq 0 \mid X_t = a\}$. In Section 2 we present the general theory of these distributions. In the subsequent sections these results are applied to important special cases. In Section 3 we give explicit formulas in the case when the jump rate is proportional to the achieved level, i.e. $r(x) = \lambda x$, and the jump sizes are uniformly distributed ($\mu_x([0, y]) = \min[y/x, 1]$). Moreover we prove that $a^{-2} \log T_a \rightarrow \lambda/2$ in probability as $a \rightarrow \infty$. In Section 4 we consider uniform cut-offs with Poisson jump arrivals. It is shown that the stationary distribution of X is Erlang(2, λ) and the LT of T_a can be given in terms of degenerate hypergeometric functions. Furthermore, we derive explicit formulas for the first two moments of the GCP at fixed time t and just after the n th jump. In Section 5 we consider GCPs for which the time periods between the jumps are i.i.d. random variables with an arbitrary distribution. Based on a duality argument it is proved that the level of the GCP after its n th jump has the same distribution as the level of a certain shot noise process which has been studied before in [8]. Using this connection the LT of the stationary density of the GCP can be derived. Finally, in Section 5 we take a short look at the Markov-modulated case, as described above. In particular, we present a method to compute the distribution of $\min[T_a, \sigma]$ in this case and apply it to determine the long-run average cost of running a certain Markov-modulated

disaster-ridden system.

We close this section with a brief discussion of some related GCP models. The sample paths of GCPs of the above type look like those of risk processes. However, in classical risk theory the surplus process of an insurance portfolio increases linearly between claims (leading to downward jumps), and the claim sizes and claim intervals are independent. We are not aware of risk studies in which the claim size depends on the size of the portfolio at the claim time. However, several authors did recently look at more general models where the independence assumption is relaxed. See Asmussen [1] for a survey of the subject. In [1] the distribution of the claim interval depends on the previous claim size (see [11] for a queueing model with a similar dependence structure). In [2] a more general, semi-Markovian, dependence structure is introduced in the risk model. The claim sizes now depend on some underlying Markov chain – but not on the size of the portfolio at the claim time.

Motivated by various applications in communication systems, there have recently been several studies about fluid systems that alternate between “on” and “off”. During off-periods, the buffer content increases in some state-dependent way, and during on-periods it decreases with a state-dependent rate (unless it is at zero). In [9] the off-periods are exponentially distributed. In [10, 23] the rate at which the system switches from on to off (and back) depends on the actual buffer content. It should be noted that if the on-period is compressed to zero, and the state-dependent decrease rate goes to infinity in an appropriate way, one could obtain a process in which the size of the downward jump depends on the buffer content.

Altman et al. [4] study GCPs in relation to the TCP (Transmission Control Protocol) of the Internet. Variants of TCP lead to AWP (Adaptive Window Protocols) in which the window size alternately grows for a certain period and decreases instantaneously according to some function of the present window size. For example, in AIMD (Additive Increase Multiplicative Decrease) the window size grows linearly, and at jump epochs it decreases proportional to its present size. For a quite general AWP, Altman et al. [4] obtain stability conditions and derive the steady-state window size distribution in analytic form. See [3, 12, 19] for studies of the AIMD protocol. As shown in Section 5, there is a one-to-one correspondence between a particular GCP and the well-known shot noise process.

2 GCPs as PDMPs

PDMPs in general were analyzed by Davis [14, 15]; see also the presentation in Rolski et al. [22]. We will specialize this theory to the growth-collapse processes defined by (i)-(iii) above. This yields the following key results.

Proposition 1 *The infinitesimal generator of X is given by*

$$(\mathcal{G}f)(x) = f'(x) + r(x) \int_{[0,x)} (f(y) - f(x)) \mu_x(dy). \quad (2.1)$$

Its domain $\mathcal{D}(\mathcal{G})$ contains all functions $f : \mathbb{R}_+ \rightarrow \mathbb{R}$ which are locally bounded and absolutely continuous.

Proof. This follows from Theorem 11.2.2 in Rolski et al. [22]. ■

Proposition 2 *The hitting times T_a satisfy*

$$\mathbb{E}_x(T_a) \leq \mathbb{E}_0(T_a) < \infty \quad \text{for all } a > x. \quad (2.2)$$

Proof. The first inequality is obvious. Regarding the integrability of T_a under \mathbb{P}_0 , observe that $T_a > t$ implies that there occurs at least one jump in each of the time intervals $(0, a], (a, 2a], \dots, ([t/a] - 1)a, [t/a]a$, so that

$$\begin{aligned} & \mathbb{P}_0(T_a > t) \\ & \leq \left(1 - \exp \left\{ - \int_0^a r(s) ds \right\} \right) \left(1 - \exp \left\{ - \min_{0 \leq x \leq a} \int_x^{x+a} r(s) ds \right\} \right)^{[t/a]-1} \\ & \leq (1 - r_a)^{[t/a]}, \end{aligned} \quad (2.3)$$

where

$$r_a = \exp \left\{ - \min_{0 \leq x \leq a} \int_x^{x+a} r(s) ds \right\}.$$

It follows that $r_a \in (0, 1)$ and

$$\begin{aligned} \mathbb{E}_0(T_a) &= \int_0^\infty \mathbb{P}_0(T_a > t) dt \leq \int_0^\infty (1 - r_a)^{t/a} dt \\ &= a / |\log(1 - r_a)|. \end{aligned}$$

■

Proposition 3 Define $\rho(x) = r(x) \int_0^x \mu_x([0, u)) du$. If for some $\varepsilon > 0$ and some $a > 0$

$$\rho(x) > 1 + \varepsilon \quad \text{for all } x \geq a, \quad (2.4)$$

then

$$\mathbb{E}_x(T_a) < \infty \quad \text{for all } x \geq a. \quad (2.5)$$

Proof. Let $i(x) = x$ be the identity function and note that by Dynkin's formula the process

$$X_{t \wedge T_a} - \int_0^{t \wedge T_a} (\mathcal{G}i)(X_s) ds \quad (2.6)$$

is a \mathbb{P}_x -martingale for every $x \geq a$. By Proposition 1, we have $(\mathcal{G}i)(x) = 1 - \rho(x)$, so that (2.6) and the definition of T_a yield

$$x = \mathbb{E}_x(X_0) = \mathbb{E}_x \left(X_{t \wedge T_a} - \int_0^{t \wedge T_a} (1 - \rho(X_s)) ds \right) \geq a + \varepsilon \mathbb{E}_x(t \wedge T_a) \quad \text{for every } x \geq a. \quad (2.7)$$

Letting $t \rightarrow \infty$ leads to

$$\mathbb{E}_x(T_a) \leq (x - a)/\varepsilon. \quad (2.8)$$

■

Proposition 4 If Q is a stationary distribution of X , then Q has a density $q(x)$ satisfying the integral equation

$$q(x) = \int_x^\infty r(t) \mu_t([0, x)) q(t) dt. \quad (2.9)$$

Proof. It follows from Davis [14] that if Q is a stationary distribution then

$$\int (\mathcal{G}f_s)(x) dQ(x) = 0 \quad \text{for all } s > 0,$$

where $f_s(x) = e^{-sx}$. Inserting (2.1) and integrating by parts yields

$$\int_0^\infty \left(f'_s(x) - r(x) \int_0^x f'_s(u) \mu_x([0, u)) du \right) dQ(x) = 0,$$

which is tantamount to

$$\int_0^\infty e^{-sx} dQ(x) = \int_0^\infty e^{-su} \left(\int_{u-}^\infty r(x) \mu_x([0, u)) dQ(x) \right) du$$

for all $s > 0$. It follows that Q has a density q satisfying (2.9). \blacksquare

Proposition 5 *If*

$$\liminf_{x \rightarrow \infty} \rho(x) > 1, \tag{2.10}$$

then X has exactly one stationary distribution Q and $X_t \rightarrow Q$ weakly as $t \rightarrow \infty$.

Proof. Since the expected recurrence times for any $a > 0$ are finite by Propositions 2 and 3, this follows from the ergodic theorem for regenerative processes (Asmussen [2, p. 170]). \blacksquare

Now let us turn to the hitting time T_a . To determine its Laplace transform (LT) $\mathbb{E}_x(e^{-\alpha T_a})$, we use in the sequel the method presented in Kella and Stadje [18], which requires, for any $\alpha > 0$, to find a solution f_α of the equation

$$\alpha f(x) = (\mathcal{G}f)(x) \tag{2.11}$$

which is positive and bounded on $[0, a]$. Then, by [18], the LT is simply given by

$$E_x(e^{-\alpha T_a}) = f_\alpha(x)/f_\alpha(a), \quad 0 \leq x < a. \tag{2.12}$$

Example 1. Let $\mu_x(dy)$ have a probability density of the form

$$p(x, y) = a(x)b(y)1_{(0, x)}(y). \tag{2.13}$$

This covers in particular the case $p(x, y) = x^{-1}1_{(0, x)}(y)$ in which at every crash a uniformly distributed piece of X_{t-} is cut off.

Setting $B(x) = \int_0^x b(y) dy$ we have $a(x) = 1/B(x)$ and $\mu_t([0, x]) = B(x)/B(t)$. By Proposition 5, X has a stationary distribution if

$$\liminf_{x \rightarrow \infty} \frac{r(x)}{B(x)} \int_0^x B(u) du > 1.$$

By Proposition 4 we get for the stationary density the integral equation

$$q(x) = B(x) \int_x^\infty \frac{r(t)}{B(t)} q(t) dt,$$

which is easily solved:

$$q(x) = CB(x) \exp \left\{ - \int_0^x r(u) du \right\} \quad (2.14)$$

where the normalizing constant C is given by

$$C = \left(\int_0^\infty B(x) \exp \left\{ - \int_0^x r(u) du \right\} dx \right)^{-1}.$$

To find the LT of T_a we have to solve the integral equation

$$\alpha f(x) = f'(x) - \frac{r(x)}{B(x)} \int_0^x f'(u) B(u) du. \quad (2.15)$$

Assuming that $r(x)$ is differentiable, we obtain from (2.15) the second-order differential equation

$$f''(x) - [\alpha + r(x) + g(x)]f'(x) + \alpha g(x)f(x) = 0. \quad (2.16)$$

Below we will find the suitable solution in some special cases.

Example 2. Let $r(x) \equiv \lambda$ be constant. The sufficient condition (2.10) for the existence and uniqueness of a stationary distribution becomes

$$\liminf_{x \rightarrow \infty} \mathbb{E}(J_x) > \lambda^{-1},$$

where J_x is the size of a generic jump starting from level x . Formula (2.9) for the stationary density transforms into

$$q(x) = \lambda \int_x^\infty \mu_t([0, x]) q(t) dt.$$

This integral equation can be solved in terms of a Neumann series.

Regarding the LT of T_a , equation (2.1) becomes

$$(\alpha + \lambda)f(x) = f'(x) + \lambda \int_{[0,x)} f(y)\mu_x(dy).$$

If μ_x has a density $p(x, y)$ on $(0, x)$ and an atom of mass $p_0(x)$ at zero, we get

$$(\alpha + \lambda)f(x) = f'(x) + \lambda \int_{[0,x)} f(y)p(x, y) dy + p_0(x)f(0).$$

Example 3. In the standard renewal age process we have $\mu_x = \varepsilon_0$, the point mass at 0. The well-known stationary density $q(x) = \mathbb{P}_0(\sigma > x)/\mathbb{E}_0(\sigma)$ can be easily derived from (2.9). Eq. (2.11) takes the form

$$\alpha f(x) = f'(x) + r(x)(f(0) - f(x)), \quad (2.17)$$

a first-order linear differential equation which can be easily solved explicitly. Let $R(x) = \int_0^x r(u) du$. The function $f_\alpha(x) = e^{\alpha x + R(x)} [1 - \int_0^x r(u) e^{-\alpha u - R(u)} du]$ is a positive solution of (2.17) so that the LT of T_a is given by

$$\mathbb{E}[e^{-\alpha T_a}] = \frac{e^{\alpha(x-a) + R(x) - R(a)} \left[1 - \int_0^x r(u) e^{-\alpha u - R(u)} du \right]}{1 - \int_0^a r(u) e^{-\alpha u - R(u)} du}.$$

3 Uniform cut-offs and proportional jump intensity

A very nice special case of Example 1 in Section 2 is the following:

(a) $p(x, y) = x^{-1}1_{(0,x)}(y)$;

(b) $r(x) = \lambda x$.

Thus the jump rate at time t is proportional to X_{t-} and at a jump time a uniformly distributed piece of X_{t-} is cut off. In this case the generator is of

the form

$$(\mathcal{G}f)(x) = f'(x) + \lambda \int_{[0,x)} (f(y) - f(x)) dy.$$

Propositions 4 and 5 immediately yield

Theorem 1 *The stationary density of X and the LT of T_a are given by*

$$q(x) = \lambda x e^{-\lambda x^2/2} \tag{3.1}$$

$$E_x(e^{-\alpha T_a}) = \left(1 + \alpha \int_0^x e^{\alpha y + (\lambda y^2/2)} dy\right) / \left(1 + \alpha \int_0^a e^{\alpha y + (\lambda y^2/2)} dy\right),$$

$$0 \leq x < a. \tag{3.2}$$

Proof. By (2.9), the stationary density of X satisfies

$$q(x) = \int_x^\infty \lambda t \frac{x}{t} q(t) dt = \lambda x \int_x^\infty q(t) dt, \tag{3.3}$$

and the right-hand side of (3.1) is the only density solving (3.3).

Eq. (2.11) becomes

$$\alpha f(x) = f'(x) + \lambda \int_0^x f(y) dy - \lambda x f(x). \tag{3.4}$$

Taking the derivative in (3.4) yields

$$\alpha f'(x) = f''(x) - \lambda x f'(x). \tag{3.5}$$

It follows that f is of the form

$$f(x) = C + D \int_0^x e^{\alpha y + (\lambda y^2/2)} dy \tag{3.6}$$

To find $\mathbb{E}_x(e^{-\alpha T_a})$ we may assume that $f(0) = 1$ (in view of (2.11)), i.e. $C = 1$. It follows from (3.5) that $\alpha f(0) = f'(0)$ so that $f'(0) = \alpha$. Hence, $D = 1$ so that (3.6) is equal to the numerator of (3.2) and provides a solution $f_\alpha(x)$ of (2.11) which is positive and bounded on $[0, a]$. Now (3.2) follows from (2.11). ■

The following theorem describes the asymptotic behavior of T_a as $a \rightarrow \infty$.

Theorem 2

$$\frac{\log T_a}{a^2} \rightarrow \frac{\lambda}{2} \quad \text{in probability as } a \rightarrow \infty. \quad (3.7)$$

Proof. Let $M_t = \max_{0 \leq s \leq t} X_s$. We will show that

$$(\log t)^{-\frac{1}{2}} M_t \xrightarrow{D} (2/\lambda)^{\frac{1}{2}}, \text{ as } t \rightarrow \infty. \quad (3.8)$$

Relation (3.8) implies (3.7). Indeed, let $g(t) = [(2/\lambda) \log t]^{1/2}$. Then, by (3.8),

$$\lim_{t \rightarrow \infty} \mathbb{P}_0(M_t \geq (1 + \delta)g(t)) = 0 \quad (3.9)$$

$$\lim_{t \rightarrow \infty} \mathbb{P}_0(M_t \geq (1 - \delta)g(t)) = 1 \quad (3.10)$$

for every $\delta > 0$. As $T_a \leq t$ if and only if $M_t \geq a$, (3.9) and (3.10) are tantamount to

$$\lim_{t \rightarrow \infty} \mathbb{P}_0(T_{(1+\delta)g(t)} \leq t) = 0 \quad (3.11)$$

$$\lim_{t \rightarrow \infty} \mathbb{P}_0(T_{(1-\delta)g(t)} \leq t) = 1. \quad (3.12)$$

Setting first $a = (1 - \delta)g(t)$ and then $a = (1 + \delta)g(t)$ we obtain from (3.11) and (3.12)

$$\lim_{a \rightarrow \infty} \mathbb{P}_0(T_a \leq \exp \left\{ (1 + \varepsilon)(\lambda/2)a^2 \right\}) = 0 \quad (3.13)$$

$$\lim_{a \rightarrow \infty} \mathbb{P}_0(T_a \leq \exp \left\{ (1 - \varepsilon)(\lambda/2)a^2 \right\}) = 1 \quad (3.14)$$

for all $\varepsilon > 0$. Relations (3.13) and (3.14) immediately yield (3.7).

Thus it is sufficient to prove (3.8). We will now actually show that

$$\lim_{t \rightarrow \infty} \mathbb{E}_0 \left(\left[((2/\lambda) \log t)^{-\frac{1}{2}} M_t \right]^n \right) = 1 \text{ for all } n \in \mathbb{N}. \quad (3.15)$$

By (3.15), all moments of $[(2/\lambda) \log t]^{-\frac{1}{2}} M_t$ converge to 1. This implies (3.8).

We know from (3.2) that

$$\mathbb{E}_0(e^{-\alpha T_a}) = \left(1 + \alpha \int_0^a e^{\alpha y + (\lambda y^2/2)} dy \right)^{-1}.$$

Let ν be the measure on $[0, \infty)$ defined by $\nu([0, t]) = \mathbb{E}_0(M_t^n)$. As $\{T_a \leq t\} = \{M_t \geq a\}$, we have

$$\begin{aligned}
\int_0^\infty e^{-\alpha t} d\nu(t) &= \int_0^\infty e^{-\alpha t} \frac{d}{dt} \mathbb{E}_0(M_t^n) dt \\
&= \int_0^\infty e^{-\alpha t} \left(\frac{d}{dt} \int_0^\infty n a^{n-1} \mathbb{P}_0(M_t \geq a) da \right) dt \\
&= \int_0^\infty \int_0^\infty n a^{n-1} e^{-\alpha t} \mathbb{P}_0(T_a \in dt) da \\
&= n \int_0^\infty a^{n-1} \mathbb{E}_0(e^{-\alpha T_a}) da \\
&= n \int_0^\infty a^{n-1} \left(1 + \alpha \int_0^a e^{\alpha y + (\lambda y^2/2)} dy \right)^{-1} da. \tag{3.16}
\end{aligned}$$

We now prove that

$$\lim_{\alpha \downarrow 0} ((2/\lambda)|\log \alpha|)^{-n/2} \int_0^\infty e^{-\alpha t} d\nu(t) = 1 \quad \text{for all } n \in \mathbb{N}. \tag{3.17}$$

As $|\log \alpha|$ is slowly varying at 0, the standard Tauberian theorem yields

$$\lim_{t \rightarrow \infty} ((2/\lambda)|\log(1/t)|)^{-n/2} \nu([0, t]) = 1 \quad \text{for all } n \in \mathbb{N},$$

which is what we want to show.

Of course, we have to consider the right-hand integral in (3.16) as $\alpha \rightarrow 0$. Its integrand tends to the nonintegrable a^{n-1} , which is why we have to introduce an appropriate normalizing factor.

Denote the right-hand side of (3.16) by $H_n(\alpha)$. We have to estimate $H_n(\alpha)$ from below and from above. For any $b > 0$ we have

$$1 + \alpha \int_0^a e^{\alpha y + (\lambda y^2/2)} dy \leq 1 + \alpha b e^{\alpha b + (\lambda b^2/2)} \quad \text{for all } a \in [0, b].$$

Hence,

$$\begin{aligned}
H_n(\alpha) &\geq n \int_0^b a^{n-1} \left(1 + \int_0^a e^{\alpha y + (\lambda y^2)/2} dy \right)^{-1} da \\
&\geq b^n / (1 + \alpha b e^{\alpha b + (\lambda b^2/2)}) \quad \text{for all } b > 0. \tag{3.18}
\end{aligned}$$

Let $\delta \in (0, 1)$ and set $b = b(\delta, \alpha) = [(1 - \delta)(2/\lambda)|\log \alpha|]^{1/2}$ in (3.18). It follows that

$$H_n(\alpha) b(\delta, \alpha)^{-n} \geq [1 + \alpha^\delta b(\delta, \alpha) e^{\alpha b(\delta, \alpha)}]^{-1}. \tag{3.19}$$

Since $\lim_{\alpha \downarrow 0} \alpha^\delta b(\delta, \alpha) = 0$, (3.19) yields

$$\liminf_{\alpha \downarrow 0} (1 - \delta)^{-n/2} ((2/\lambda) |\log \alpha|)^{-n/2} H_n(\alpha) \geq 1$$

for all $\delta \in (0, 1)$ and all $n \in \mathbb{N}$. Thus,

$$\liminf_{\alpha \downarrow 0} ((2/\lambda) |\log \alpha|)^{-n/2} H_n(\alpha) \geq 1. \quad (3.20)$$

To find an upper bound, we use

$$\begin{aligned} \int_0^a e^{\alpha y + (\lambda y^2)/2} dy &\geq \int_0^a e^{\lambda y^2/2} dy \geq \int_{a-1}^a e^{\lambda y^2/2} dy \\ &\geq e^{\lambda(a-1)^2/2} \quad \text{for all } a \geq 1. \end{aligned}$$

Hence, for any $b > 0$,

$$\begin{aligned} H_n(\alpha) &\leq \int_0^{b+1} a^{n-1} da + \alpha^{-1} n \int_{b+1}^\infty a^{n-1} e^{-\lambda(a-1)^2/2} da \\ &\leq (b+1)^n + \alpha^{-1} n \int_{b+1}^\infty \left(\frac{b+1}{b} (a-1) \right)^{n-1} e^{-\lambda(a-1)^2/2} da \\ &= (b+1)^n + \alpha^{-1} n (1+b^{-1})^{n-1} \lambda^{-(n-1)/2} \int_{\lambda^{1/2} b}^\infty x^{n-1} e^{-x^2/2} dx. \quad (3.21) \end{aligned}$$

The integral on the right-hand side of (3.21) is bounded as follows: For every $N \in \mathbb{N}$ there is a constant $C_N > 0$ such that

$$\int_t^\infty x^N e^{-x^2/2} dx \leq C_N t^{N-1} e^{-t^2/2} \quad \text{for all } t > 0. \quad (3.22)$$

By (3.21) and (3.22),

$$\begin{aligned} H_n(\alpha) &\leq (b+1)^n + \alpha^{-1} n (1+b^{-1})^{n-1} \lambda^{-(n-1)/2} C_{n-1} (\lambda^{1/2} b)^{n-2} e^{-\lambda b^2/2} \\ &\quad \text{for all } b > 0. \end{aligned} \quad (3.23)$$

Now set $b = b(\alpha) = [(2/\lambda) |\log \alpha|]^{1/2}$. From (3.23) we conclude that

$$\begin{aligned} b(\alpha)^{-n} H_n(\alpha) &\leq (1 + b(\alpha)^{-1})^n \\ &\quad + n C_{n-1} (1 + b(\alpha)^{-1})^{n-1} \lambda^{-1/2} \alpha^{-1} b(\alpha)^{-2} e^{-\lambda b(\alpha)^2/2} \\ &= (1 + b(\alpha)^{-1})^{n-1} \left[1 + b(\alpha)^{-1} + \frac{1}{2} n C_{n-1} \lambda^{1/2} |\log \alpha|^{-2} \right] \\ &\rightarrow 1, \quad \text{as } \alpha \searrow 0. \end{aligned}$$

Thus, $\limsup_{\alpha \searrow 0} b(\alpha)^{-n} H_n(\alpha) \leq 1$, which together with (3.20) yields

$$\lim_{\alpha \searrow 0} b(\alpha)^{-n} H_n(\alpha) = 1$$

for every $n \in \mathbb{N}$, and this is equivalent to (3.17). ■

4 Poisson jump times with uniform cut-offs

We now consider the case

(a) $p(x, y) = x^{-1}1_{(0,x)}(y)$

(b) $r(x) \equiv \lambda > 0$.

Thus, jumps arrive at Poisson times with intensity λ and the cut-off mechanism is the same as in Section 3.

Theorem 3 *The stationary distribution of X is Erlang(2, λ), i.e., has density*

$$q(x) = \lambda^2 x e^{-\lambda x}. \quad (4.1)$$

The LT of T_a is given by $\mathbb{E}_x(e^{-\alpha T_a}) = f_\alpha(x)/f_\alpha(a)$, $0 \leq x < a$, where $f_\alpha(x)$ is the unique solution of the differential equation

$$x f''(x) + (1 - (\lambda + \alpha)x)f'(x) - \alpha f(x) = 0, \quad x \geq 0 \quad (4.2)$$

subject to

$$f(0) = 1, \quad f'(0) = \alpha. \quad (4.3)$$

Proof. By (2.9), the stationary density of X satisfies

$$q(x) = \int_x^\infty \lambda \frac{x}{t} q(t) dt = \lambda x \int_x^\infty \frac{q(t)}{t} dt, \quad (4.4)$$

and the right-hand side of (4.1) is the only density solving (4.4).

Eq. (2.11) becomes

$$\alpha f(x) = f'(x) + \frac{\lambda}{x} \int_0^x f(y) dy - \lambda f(x). \quad (4.5)$$

Multiplying by x and taking the derivative in (4.5) yields (4.2). The initial condition $f(0) = 1$ can be fixed arbitrarily, and then the condition $f'(0) = \alpha$ follows by letting x tend to zero in (4.5). ■

Eq. (4.2) is a variant of the degenerate hypergeometric differential equation. Its general solution is given by

$$f(x) = e^{(\lambda+\alpha)x} \left[C_1 \Phi \left(\frac{\lambda}{2(\lambda+\alpha)}, 1; -2(\lambda+\alpha)x \right) + C_2 \Psi \left(\frac{\lambda}{2(\lambda+\alpha)}, 1; -2(\lambda+\alpha)x \right) \right],$$

where

$$\Phi(a, b; x) = 1 + \sum_{k=1}^{\infty} \frac{(a)_k x^k}{(b)_k k!}$$

is Kummer's series (here $(a)_k = a(a+1)\dots(a+k-1)$, $(a)_0 = 1$) and

$$\begin{aligned} \Psi(a, 1; x) = \frac{1}{\Gamma(a-1)} & \left\{ \Phi(a, 2; x) \log x \right. \\ & \left. + \sum_{r=0}^{\infty} [\psi(a+r) - \psi(1+r) - \psi(2+r)] \frac{(a)_r x^r}{r!(r+1)!} \right\} + \frac{1}{\Gamma(a)}, \end{aligned}$$

where $\psi(z) = \Gamma'(z)/\Gamma(z)$ is the logarithmic derivative of the gamma function (see [21], eqs. 2.1.2.103 and 2.1.2.65). In our case we can also use the formula

$$\Phi(a, 1; x) = \frac{1}{\Gamma(a)\Gamma(1-a)} \int_0^1 e^{tx} t^{a-1} (1-t)^{-a} dt, \quad 0 < a < 1.$$

The constants C_1 and C_2 are uniquely determined by the boundary conditions (4.3), which become transcendental equations involving the function Φ and Ψ .

Next we derive the expected value and the variance of X_t . Let $X_0 = 0$. We can write X_t in the form

$$X_t = t - \sum_{n=0}^{N(t)} \tau_n W_n^{(N(t))}, \quad (4.6)$$

where

(i) $\{N(t) \mid t \geq 0\}$ is a homogeneous Poisson counting process with rate λ , $\tau_0 \equiv 0$ and $0 < \tau_1 < \tau_2 < \dots$ denote the jump times of $\{N(t) \mid t \geq 0\}$;

(ii) U_1, U_2, \dots are i.i.d. random variables which are independent of $\{N(t) \mid t \geq 0\}$ and have the uniform distribution on $(0, 1)$, $W_0^m \equiv 1$ and

$$W_n^{(m)} = I(n=m)(1-U_m) + I(n < m)(1-U_n) \prod_{j=n+1}^m U_j, \quad n \geq 1. \quad (4.7)$$

Let $Y_t = t - X_t$. Clearly, Y_t is the sum of the sizes of all jumps in $[0, t]$.

Theorem 4 For every $t \in [0, \infty)$,

$$\mathbb{E}[X_t] = \frac{2}{\lambda}(1 - e^{-\lambda t/2}). \quad (4.8)$$

Proof. We have

$$\mathbb{E}[X_t] = t - \mathbb{E}[Y_t] = t - \mathbb{E}[\mathbb{E}[Y_t \mid N(t) = n]].$$

Obviously, $\mathbb{E}[Y_t \mid N(t) = 0] = 0$. For $n \geq 1$, it is well-known that

$$\tau_i \mid N(t) = n \sim t\text{Beta}(i, n + 1 - i).$$

Accordingly, due to the independence of τ_i and $W_i^{(n)}$, we obtain for all $n \geq 1$

$$\begin{aligned} \mathbb{E}[Y_t \mid N(t) = n] &= \mathbb{E}\left[\sum_{i=1}^n \tau_i W_i^{(n)} \mid N(t) = n\right] \\ &= t \sum_{i=1}^n \frac{i}{n+1} \left(\frac{1}{2}\right)^{n+1-i} \\ &= t \left(1 - \frac{2}{n+1} \left(1 - \left(\frac{1}{2}\right)^{n+1}\right)\right). \end{aligned} \quad (4.9)$$

Setting $p(n; \lambda t) = e^{-\lambda t} \frac{(\lambda t)^n}{n!}$, $n = 0, 1, \dots$, we find that

$$\begin{aligned} \mathbb{E}[X_t] &= t - \mathbb{E}[Y_t \mid N(t)] \\ &= 2t \mathbb{E}\left[\frac{1}{N(t) + 1} \left(1 - \left(\frac{1}{2}\right)^{N(t)+1}\right)\right] \\ &= 2t \sum_{n=0}^{\infty} p(n; \lambda t) \frac{1}{n+1} \left(1 - \left(\frac{1}{2}\right)^{n+1}\right) \\ &= \frac{2}{\lambda} \sum_{n=1}^{\infty} p(n; \lambda t) \left(1 - \left(\frac{1}{2}\right)^n\right). \end{aligned} \quad (4.10)$$

Eq. (4.8) is easily obtained from (4.10). ■

Theorem 5 For fixed t , the variance of X_t is

$$V\{X_t\} = \frac{1}{\lambda^2}(2 - 4e^{-\lambda t} - 16e^{-\lambda t/2} + 18e^{-2\lambda t/3}). \quad (4.11)$$

Proof. We have

$$\begin{aligned}
V\{X_t\} &= V\{Y_t\} = V\left\{\sum_{n=0}^{N(t)} \tau_n W_n^{(N(t))}\right\} \\
&= \mathbb{E}\left[V\left\{\sum_{n=0}^{N(t)} \tau_n W_n^{(N(t))} \mid N(t)\right\}\right] \\
&\quad + V\left\{\mathbb{E}\left[\sum_{n=0}^{N(t)} \tau_n W_n^{(N(t))} \mid N(t)\right]\right\}.
\end{aligned} \tag{4.12}$$

Since $V\{\mathbb{E}\{Y_t \mid N(t)\}\} = V\{\mathbb{E}\{X_t \mid N(t)\}\}$ we obtain from (4.9) that the second term on the right-hand side of (4.12) is

$$\begin{aligned}
V\{E\{X_t \mid N(t)\}\} &= 4t^2 V\left\{\frac{1}{N(t)+1} \left(1 - \left(\frac{1}{2}\right)^{N(t)+1}\right)\right\} \\
&= 4t^2 \sum_{n=0}^{\infty} p(n; \lambda t) \left[\frac{1}{(n+1)^2} \left(1 - \left(\frac{1}{2}\right)^{n+1}\right)^2\right] \\
&\quad - \frac{4}{\lambda^2} (1 - e^{-\lambda t/2})^2.
\end{aligned} \tag{4.13}$$

As for the first term on the right-hand side of (4.12), we notice that

$$V\left\{\sum_{n=0}^{N(t)} \tau_n W_n^{(N(t))} \mid N(t) = 0\right\} = 0,$$

and for $n \geq 1$

$$\begin{aligned}
V\left\{\sum_{i=1}^n \tau_i W_i^{(n)} \mid N(t) = n\right\} &= V\left\{\mathbb{E}\left[\sum_{i=1}^n \tau_i W_i^{(n)} \mid N(T) = n, \mathbf{W}^{(n)}\right] \mid N(t) = n\right\} \\
&\quad + \mathbb{E}\left[V\left\{\sum_{i=1}^n \tau_i W_i^{(n)} \mid N(t) = n, \mathbf{W}^{(n)}\right\} \mid N(t) = n\right].
\end{aligned} \tag{4.14}$$

Furthermore, due to the independence of $\mathbf{W}^{(n)}$ and $N(t)$,

$$\begin{aligned}
V \left\{ \mathbb{E} \left[\sum_{i=1}^n \tau_i W_i^{(n)} \mid N(t) = n, \mathbf{W}^{(n)} \right] \mid N(t) = n \right\} &= V \left\{ \frac{t}{n+1} \sum_{i=1}^n W_i^{(n)} \right\} \\
&= \frac{t^2}{(n+1)^2} \left[\sum_{i=1}^n (n+1-i)^2 \left(\left(\frac{1}{3} \right)^i - \left(\frac{1}{4} \right)^i \right) \right. \\
&\quad \left. + 2 \sum_{i=1}^{n-1} i \sum_{j=i+1}^n j \left(\frac{1}{2} \right)^{j-i} \left(\frac{1}{6} \left(\frac{1}{3} \right)^{n-j} - \frac{1}{4} \left(\frac{1}{4} \right)^{n-j} \right) \right] \\
&= \frac{t^2}{2(n+1)^2} \left(1 - 4 \left(\frac{1}{2} \right)^n + 5 \left(\frac{1}{3} \right)^n - 2 \left(\frac{1}{4} \right)^n \right). \tag{4.15}
\end{aligned}$$

To obtain the second term on the right-hand side of (4.14) we start with

$$\begin{aligned}
V \left(\sum_{i=1}^n \tau_i W_i^{(n)} \mid N(t) = n, \mathbf{W}^{(n)} \right) &= \sum_{i=1}^n (W_i^{(n)})^2 V \{ \tau_i \mid N(t) = n \} \\
&\quad + 2 \sum_{1 \leq i < j \leq n} W_i^{(n)} W_j^{(n)} \text{COV}(\tau_i, \tau_j \mid N_t = n). \tag{4.16}
\end{aligned}$$

Moreover,

$$V \{ \tau_i \mid N(t) = n \} = t^2 \frac{i(n+1-i)}{(n+1)^2(n+2)}, \tag{4.17}$$

$i = 1, \dots, n$, and for $i < j$,

$$\text{COV}(\tau_i, \tau_j \mid N(t) = n) = t^2 \frac{i(n+1-j)}{(n+1)^2(n+2)}. \tag{4.18}$$

Substituting (4.17)-(4.18) in (4.16) and taking expectations we obtain

$$\begin{aligned}
\mathbb{E} \left[V \left\{ \sum_{i=1}^n \tau_i W_i^{(n)} \mid N(t) = n, \mathbf{W}^{(n)} \right\} \mid N(t) = n \right] \\
= \frac{t^2}{(n+1)^2(n+2)} \left[\frac{n}{2} \left(3 - \left(\frac{1}{3} \right)^n \right) - 3 \left(1 + \left(\frac{1}{3} \right)^n \right) + 6 \left(\frac{1}{2} \right)^n \right]. \tag{4.19}
\end{aligned}$$

Summing (4.15) and (4.19) we get for $n \geq 1$

$$V\left\{\sum_{i=1}^n \tau_i W_i^{(n)} \mid N(t) = n\right\} = \frac{t^2}{(n+1)^2(n+2)} \left[2n \left(1 + \left(\frac{1}{3}\right)^n\right) - n \left(2 \left(\frac{1}{2}\right)^n + \left(\frac{1}{4}\right)^n\right) - 2 \left(1 + \left(\frac{1}{4}\right)^n\right) + 2 \left(\left(\frac{1}{3}\right)^n + \left(\frac{1}{2}\right)^n\right) \right]. \quad (4.20)$$

Finally, (4.13) and (4.20) yield

$$\begin{aligned} V\{X_t\} &= t^2 \sum_{n=1}^{\infty} p(n; \lambda t) \left[\frac{2(n-1)}{(n+1)^2(n+2)} \left(1 - \left(\frac{1}{2}\right)^n\right) + \frac{2}{(n+1)(n+2)} \left(\frac{1}{3}\right)^n - \frac{1}{(n+1)^2} \left(\frac{1}{4}\right)^n \right] \\ &\quad + 4t^2 \sum_{n=0}^{\infty} p(n; \lambda t) \frac{1}{(n+1)^2} \left(1 - \left(\frac{1}{2}\right)^n + \frac{1}{4} \left(\frac{1}{4}\right)^n\right) - \frac{4}{\lambda^2} (1 - e^{-\lambda t/2})^2 \\ &= 6t^2 \sum_{n=1}^{\infty} p(n; \lambda t) \frac{1}{(n+1)(n+2)} - 6t^2 \sum_{n=1}^{\infty} p(n; \lambda t) \frac{(1/2)^n}{(n+1)(n+2)} \\ &\quad + 2t^2 \sum_{n=1}^{\infty} p(n; \lambda t) \frac{(1/3)^n}{(n+1)(n+2)} + t^2 e^{-\lambda t} - \frac{4}{\lambda^2} (1 - e^{-\lambda t/2})^2. \end{aligned} \quad (4.21)$$

Careful simplification of (4.21) yields (4.11). ■

Corollary

$$\lim_{t \rightarrow \infty} \mathbb{E}[X_t] = \frac{2}{\lambda}, \quad \lim_{t \rightarrow \infty} V\{X_t\} = \frac{2}{\lambda^2}.$$

The distribution function of Y_t , for fixed t , is

$$H(y; t) = \begin{cases} e^{-\lambda t}, & y = 0 \\ e^{-\lambda t} + \sum_{n=1}^{\infty} p(n; \lambda t) H_n(y; t), & 0 < y \leq t \\ 1, & y > t \end{cases} \quad (4.22)$$

where, for $n \geq 1$,

$$H_n(y; t) = \mathbb{P} \left(\sum_{i=1}^n \tau_i W_i^{(n)} \leq y \mid N(t) = n \right). \quad (4.23)$$

Denote by $h(y; t)$ the density of $H(y; t)$. Another important function is $\mathbb{P}(Y_t \leq y \mid Y_s = x)$, $0 \leq s < t$. To compute it, we start from

$$\begin{aligned} & \mathbb{P}(Y_t \leq y \mid Y(s) = x, N(s) = m, N(t) = n, \tau_m) \\ &= \mathbb{P} \left(xW_1^{(n-m)} + \tau_m \sum_{i=1}^{n-m} W_i^{(n-m)} + \sum_{i=1}^{n-m} W_i^{(n-m)} U_{i:n-m}(t-s) \leq y \right), \\ & \qquad \qquad \qquad m \leq n, \quad s < t \end{aligned} \quad (4.24)$$

where $U_{i:n-m}(t-s)$ is the i th order statistic of $n-m$ i.i.d. random variables distributed uniformly on $(0, t-s)$. As before, $W_i^{(n-m)} = (1 - U_i) \prod_{j=i+1}^{n-m} U_j$, $i = 1, \dots, n-m-1$, and $W_{n-m}^{(n-m)} = 1 - U_{n-m}$, where U_1, \dots, U_{n-m} are i.i.d., uniform on $(0, 1)$ and independent of $U_{i:n-m}(t-s)$, $i = 1, \dots, n-m$.

The conditional distribution of τ_m , given $N(s) = m$, is like $s \cdot \text{Beta}(m, 1)$. Also, τ_m is independent of $\{W_i^{(n-m)} \mid i = 1, \dots, n-m\}$ and of $\{U_{i:n-m}(t-s) \mid i = 1, \dots, n-m\}$. Hence, by (4.24),

$$\begin{aligned} \mathbb{P}(Y_t \leq y \mid Y_s = x) &= \sum_{m=0}^{\infty} p(m; \lambda s) \sum_{l=0}^{\infty} p(l; \lambda(t-s)) \\ & \times \frac{1}{sB(m, 1)} \int_0^s z^{m-1} \mathbb{P} \left(xW_1^{(l)} + z \sum_{i=1}^l W_i^{(l)} \right. \\ & \quad \left. + \sum_{i=1}^l W_i^{(l)} U_{i:l}(t-s) \leq y \right) dz. \end{aligned} \quad (4.25)$$

The probability in the integrand on the right side of (4.25) can essentially be written as Lebesgue measure of a $(2l)$ -dimensional set.

Another quantity of interest is the embedded Markov chain $X(n) = X_{\tau_n}$, i.e., the sequence of levels just after jumps.

Theorem 6 *The stationary distribution of $X(n)$ is $\exp(\lambda)$.*

Proof. By PASTA, the levels $X_{\tau_n^-}$ just prior to jumps have the same stationary distribution as the continuous-time process $\{X_t \mid t \geq 0\}$, which is Erlang(2, λ) by Theorem 3. Therefore the stationary distribution of $X(n)$ is equal to that of VU , where V is Erlang(2, λ)-distributed, U is uniform on $(0, 1)$ and U and V are independent. The stationary density of $X(n)$ is thus given by

$$p(x) = \int_x^\infty \lambda^2 v e^{-\lambda v} \frac{1}{v} dv = \lambda e^{-\lambda x}, \quad x > 0.$$

■

We now compute the expected value and the variance of $X(n)$. Notice that for every $n \geq 1$

$$X(n) = U_n X(n-1) + (\tau_n - \tau_{n-1}) U_n,$$

where $X(0) = X_0 = 0$. Furthermore, $\tau_n - \tau_{n-1} \sim \exp(\lambda)$, $n \geq 1$. Hence we can write

$$X(n) = \sum_{i=1}^n R_i \tilde{W}_i^{(n)}, \quad n \geq 1$$

where R_1, R_2, \dots are i.i.d. $\exp(\lambda)$,

$$\tilde{W}_i^{(n)} = \prod_{j=i}^n U_j, \quad i = 1, \dots, n$$

and R_1, R_2, \dots are independent of $\{\tilde{W}_i^{(n)} \mid i = 1, \dots, n, n \geq 1\}$.

Theorem 7 For each $n \geq 1$,

$$\mathbb{E}[X(n)] = \frac{1}{\lambda} \left(1 - \left(\frac{1}{2} \right)^n \right), \quad (4.26)$$

and

$$V\{X(n)\} = \frac{1}{\lambda^2} \left(1 - 2 \left(\frac{1}{2} \right)^n + 2 \left(\frac{1}{3} \right)^n - \left(\frac{1}{4} \right)^n \right). \quad (4.27)$$

Proof. (i) Since R_i is independent of $W_i^{(n)}$,

$$\begin{aligned}\mathbb{E}[X(n)] &= \mathbb{E}\left[\sum_{i=1}^n R_i \tilde{W}_i^{(n)}\right] = \sum_{i=1}^n \mathbb{E}[R_i] \mathbb{E}[\tilde{W}_i^{(n)}] \\ &= \frac{1}{\lambda} \sum_{i=1}^n \left(\frac{1}{2}\right)^{n+1-i} = \frac{1}{\lambda} \left(1 - \left(\frac{1}{2}\right)^n\right), \quad n \geq 1.\end{aligned}$$

(ii) Let $\tilde{\mathbf{W}}^{(n)} = (\tilde{W}_1^{(n)}, \dots, \tilde{W}_n^{(n)})$. We compute the variance in two parts, according to

$$V\left(\sum_{i=1}^n R_i \tilde{W}_i^{(n)}\right) = \mathbb{E}\left[V\left\{\sum_{i=1}^n R_i \tilde{W}_i^{(n)} \mid \tilde{\mathbf{W}}^{(n)}\right\}\right] + V\left\{\mathbb{E}\left[\sum_{i=1}^n R_i \tilde{W}_i^{(n)} \mid \tilde{\mathbf{W}}^{(n)}\right]\right\}. \quad (4.28)$$

Due to independence,

$$V\left\{\sum_{i=1}^n R_i \tilde{W}_i^{(n)} \mid \tilde{\mathbf{W}}^{(n)}\right\} = \frac{1}{\lambda^2} \sum_{i=1}^n \left(\prod_{j=i}^n U_j^2\right).$$

Hence,

$$\mathbb{E}\left[V\left\{\sum_{i=1}^n R_i \tilde{W}_i^{(n)} \mid \tilde{\mathbf{W}}^{(n)}\right\}\right] = \frac{1}{\lambda^2} \sum_{i=1}^n \left(\frac{1}{3}\right)^{n+1-i} = \frac{1}{2\lambda^2} \left(1 - \left(\frac{1}{3}\right)^n\right). \quad (4.29)$$

For the second term of (4.28), we start with

$$\mathbb{E}\left[\sum_{i=1}^n R_i \tilde{W}_i^{(n)} \mid \tilde{\mathbf{W}}^{(n)}\right] = \frac{1}{\lambda} \sum_{i=1}^n \tilde{W}_i^{(n)}. \quad (4.30)$$

It follows that

$$\begin{aligned}V\left\{\mathbb{E}\left[\sum_{i=1}^n R_i \tilde{W}_i^{(n)} \mid \tilde{\mathbf{W}}^{(n)}\right]\right\} &= \frac{1}{\lambda^2} V\left\{\sum_{i=1}^n \tilde{W}_i^{(n)}\right\} \\ &= \frac{1}{\lambda^2} \left[\sum_{i=1}^n V\{\tilde{W}_i^{(n)}\} + 2 \sum_{1 \leq i < j \leq n} \text{COV}(\tilde{W}_i^{(n)}, \tilde{W}_j^{(n)})\right].\end{aligned} \quad (4.31)$$

In addition,

$$V\{\tilde{W}_i^{(n)}\} = \left(\frac{1}{3}\right)^{n+1-i} - \left(\frac{1}{4}\right)^{n+1-i}, \quad i = 1, \dots, n.$$

Similarly, for $i < j$,

$$\text{COV}(\tilde{W}_i^{(n)}, \tilde{W}_j^{(n)}) = \left(\frac{1}{2}\right)^{j-i} \left[\left(\frac{1}{3}\right)^{n+1-j} - \left(\frac{1}{4}\right)^{n+1-j} \right].$$

Substituting in (4.31) and summing shows that

$$\frac{1}{\lambda^2} V \left\{ \sum_{i=1}^n \tilde{W}_i^{(n)} \right\} = \frac{1}{\lambda^2} \left[\frac{1}{2} - 2 \left(\frac{1}{2}\right)^n + \frac{5}{2} \left(\frac{1}{3}\right)^n - \left(\frac{1}{4}\right)^n \right]. \quad (4.32)$$

Finally, from (4.29), (4.31) and (4.32) we obtain (4.27). ■

Corollary *The asymptotic mean and variance of $X(n)$ are*

$$\lim_{n \rightarrow \infty} \mathbb{E}[X(n)] = \frac{1}{\lambda}, \quad \lim_{n \rightarrow \infty} V\{X(n)\} = \frac{1}{\lambda^2}.$$

Finally, we develop recursive formulae for the distributions of $X(n)$, $n = 1, 2, \dots$. We start with the transition function

$$K(y; x) = \mathbb{P}(X(n) \leq y \mid X(n-1) = x)$$

and its density $k(y; x)$.

Theorem 8 *For each $n \geq 1$,*

$$K(y; x) = 1 - \exp(-\lambda(y-x)^+) + \lambda e^{\lambda x} y \int_{\max(x,y)}^{\infty} \frac{1}{u} e^{-\lambda u} du, \quad (4.33)$$

and

$$k(y; x) = \lambda e^{\lambda x} \int_{\max(x,y)}^{\infty} \frac{1}{u} e^{-\lambda u} du. \quad (4.34)$$

Proof. We have

$$K(y; x) = \mathbb{P}(X(n) \leq y \mid X(n-1) = x) = \mathbb{P}(U(x+R) \leq y),$$

where $U \sim U(0, 1)$, $R \sim \exp(\lambda)$ and U is independent of R . Notice that

$$\mathbb{P}\left(U \leq \frac{y}{x+R} \mid R\right) = I\{x+R \leq y\} + I\{x+R > y\} \frac{y}{x+R}.$$

This implies (4.33). Formula (4.34) is obtained by differentiating (4.33) with respect to y . \blacksquare

Let $f_n(x)$ denote the density function of $X(n)$. One can immediately prove that

$$f_1(y) = \lambda \int_y^\infty \frac{1}{u} e^{-\lambda u} du, \quad 0 < y < \infty.$$

Theorem 9 For each $n \geq 2$,

$$f_n(y) = f_1(y) \int_0^y e^{\lambda x} f_{n-1}(x) dx + \int_y^\infty e^{\lambda x} f_1(x) f_{n-1}(x) dx. \quad (4.35)$$

Proof. For each $n \geq 2$,

$$f_n(y) = \int_0^\infty k(y; x) f_{n-1}(x) dx. \quad (4.36)$$

Substituting (4.34) in (4.36) we obtain (4.35). \blacksquare

5 The generalized uniform cut-off process and its relation to the shot noise process

In this section we consider the following GCP $(X_t)_{t \geq 0}$. As before, X_t increases linearly, at rate 1, between downward jumps. This time the intervals B_1, B_2, \dots between the downward jumps are i.i.d. random variables with a *general* distribution with LST $\beta(\cdot)$. The downward jump Z_n after B_n depends on S_n , where S_n is the level of the X_t -process just before the n th jump. We generalize the uniform cut-off procedure of the previous sections in the following way. The remainder $W_n = S_n - Z_n$, after the jump, is given by $W_n = U^a(0, S_n)$, where $U^a(0, b)$ denotes a random variable with density at^{a-1}/b^a on $(0, b)$. (Of course, $a = 1$ yields the uniform cut-off procedure.)

We want to analyze W_n , the state of X_t immediately after the n th jump. Since $S_n = W_n + B_n$, we have

$$W_{n+1} = U^a(0, W_n + B_n), \quad n = 1, 2, \dots \quad (5.1)$$

One can show that the steady-state distribution of the W_n -process exists for all traffic parameters; see the Remark below.

It follows from (5.1) that the steady-state variable W_e of the sequence W_n satisfies

$$\begin{aligned} \mathbb{E}[e^{-\alpha W_e}] &= \int_0^\infty e^{-\alpha x} \int_x^\infty \frac{ax^{a-1}}{y^a} \mathbb{P}(W + B \in dy) dx \\ &= \int_0^\infty \frac{1}{y^a} \mathbb{P}(W + B \in dy) \int_0^y ax^{a-1} e^{-\alpha x} dx. \end{aligned} \quad (5.2)$$

Differentiate both sides of (5.2) with respect to α and use partial integration in the last integral to get

$$\frac{d}{d\alpha} \mathbb{E}[e^{-\alpha W_e}] = -a \frac{1 - \beta(\alpha)}{\alpha} \mathbb{E}[e^{-\alpha W_e}]. \quad (5.3)$$

The solution of this differential equation is readily verified to be given by

$$\mathbb{E}[e^{-\alpha W_e}] = \exp\left\{-a \int_0^\alpha \frac{1 - \beta(u)}{u} du\right\}. \quad (5.4)$$

Let us now point out a relation between the above growth-collapse process and the so-called shot noise process, which has been extensively studied in the literature on queueing models with workload-dependent service speed (see, e.g., [8, 13, 17]). First transform X_t into a ‘dual’ workload process following a procedure in [20]. This is done in two steps:

- (1) Construct a ‘mountain’ process by changing the negative jumps into negative slopes with rate -1 ;
- (2) construct a workload process by changing the positive slopes into positive jumps of size B_1, B_2, \dots

The resulting process has paths which are linearly decreasing between positive jumps; its workload just before the n th jump (of size B_n) is W_n .

Next consider the following shot noise process: jumps upward, of sizes B_1, B_2, \dots , occur after independent, $\exp(\lambda)$ -distributed time intervals. Between

jumps, the process decreases at rate rx if the process level equals x (where $r > 0$ is some constant). One can now interpret

$$R(y, x) = \int_y^x \frac{1}{rw} dw \quad (5.5)$$

as the time it takes for this shot noise process to decrease from level x to level y , when no jumps occur.

Finally consider the following transformation:

$$Y_n = -\frac{1}{r} \log\left(\frac{W_n}{S_n}\right). \quad (5.6)$$

A simple calculation shows that Y_n is $\exp(ra)$ -distributed. Taking $a = \lambda/r$ results in Y_n being $\exp(\lambda)$ -distributed. But from (5.5) and (5.6) it is also clear that Y_n as defined above denotes the time to decrease from S_n to W_n in the dual workload process, as well as an interarrival time in the shot noise process. It is thus seen that the level of the shot noise process just before the n th upward jump has exactly the same distribution as W_n in the dual process and the original GCP.

The shot noise process just described has been analyzed in [8]. On p. 546 of [8] it was shown that it has a steady-state density $v(\cdot)$ whose Laplace transform is given in (5.4). Because of the PASTA property, this is also the LT of the density of the shot noise process just before jump epochs, and the above construction confirms that this is also the LT of W_e . See p. 546/547 of [8] for special cases (like $\lambda = r$ and B_1 being exponentially distributed, resulting in an exponential workload density).

Remark. The shot noise process not only never reaches 0; it is also known to be stable for all offered traffic loads. The relation to the special process derived above implies that the same properties hold for the latter process.

6 The Markov-modulated case

Now let us look at the *Markov-modulated case*, as described in the Introduction. The underlying Markov process is two-dimensional: $Z_t = (X_t, J_t)$, and

the generator is

$$\begin{aligned}
(\mathcal{G}f)(x, i) &= c(i)f'(x, i) + \sum_{j \neq i} q_{ij}f(x, j) - (q_i + r(x))f(x, i) \\
&\quad + r(x) \int_{[0, x)} f(y, i) \mu_x(dy), \quad i = 1, \dots, n \quad (6.1)
\end{aligned}$$

or in matrix form

$$(\mathcal{G}f)(x) = Cf'(x) + (Q - r(x)E)f(x) + r(x)D_f(x),$$

where

- (a) $f(x) = (f(x, 1), \dots, f(x, n))^t$;
- (b) $Q = (q_{ij})_{i, j \in \{1, \dots, n\}}$ is the generator of the Markov chain J_t and $q_i = -q_{ii}$;
- (c) C and $D_f(x)$ are diagonal matrices with diagonal entries $c(i)$ and $\int_{[0, x)} f(y, i) \mu_x(dy)$, respectively;
- (d) E is the $n \times n$ identity matrix.

One can now derive the stationary distribution of Z_t and the LT $E_x(e^{-\alpha T_a})$ in terms of integral and differential equations.

Instead of developing this generalization we finally consider the following problem: what is the distribution of $T_a \wedge \sigma$? To see that this is a relevant question, let us for example interpret σ as the time a disaster occurs in some “system”, say some technical item. Assume that the system has to run indefinitely; at any disaster it has to be replaced by a new identical one but it can also be replaced preventively when its age reaches a certain threshold $a > 0$ which has to be specified by the controller. Thus, the first replacement takes place at time $T_a \wedge \sigma$. Suppose that after replacement the modulating chain is restarted at some fixed state i_0 . If a replacement of a still functioning system costs $\$C_1$ and a replacement upon disaster costs $\$C_2$ (where $C_1 < C_2$), then the long-run average cost of running the system when using the policy T_a is given by

$$C(a) = \frac{C_1 \mathbb{P}_{0, i_0}(T_a \wedge \sigma = T_a) + C_2(1 - \mathbb{P}_{0, i_0}(T_a \wedge \sigma = T_a))}{\mathbb{E}_{0, i_0}(T_a \wedge \sigma)}.$$

Hence, $\mathbb{P}_{x, i}(T_a \wedge \sigma = T_a)$ and $\mathbb{E}_{x, i}(T_a \wedge \sigma)$ are important quantities; once they are known as functions of a one can try to minimize $C(a)$.

We deal with this problem as follows. Suppose the process is killed at time σ by entering a coffin state ∂ . By Dynkin's formula, we have

$$f(x, i) = \mathbb{E}_{(x, i)} \left(\int_0^T (\mathcal{G}f)(Z_t) dt \right) - \mathbb{E}_{(x, i)} (f(Z_T)) \quad (6.2)$$

for f bounded and in the domain of \mathcal{G} and T any integrable stopping time. Now apply (6.2) to $T = T_a \wedge \sigma$ in two cases:

(i) $f = f_1$ such that $(\mathcal{G}f_1)(x, i) = 0$, $x \in (0, a)$, and $f_1(\partial, i) = 0$, $f_1(a, i) = 1$.

(ii) $f = f_2$ such that $(\mathcal{G}f_2)(x, i) = -1$, $x \in (0, a)$, and $f_2(\partial) = 0$, $f_2(a, i) = 0$.

A moment's reflection shows that if f_1, f_2 have these properties, then

$$f_1(x, i) = \mathbb{P}_{(x, i)}(T_a \wedge \sigma = T_a) \quad (6.3)$$

$$f_2(x, i) = \mathbb{E}_{(x, i)}(T_a \wedge \sigma). \quad (6.4)$$

Hence, we have to solve

$$Cf'(x) + (Q - r(x)E)f(x) = (0, \dots, 0)^t \quad (6.5)$$

subject to

$$f(a, \cdot) = (1, \dots, 1)^t \quad (6.6)$$

and

$$Cf'(x) + (Q - r(x)E)f(x) = (-1, \dots, -1)^t \quad (6.7)$$

subject to

$$f(a, \cdot) = (0, \dots, 0)^t. \quad (6.8)$$

Let us finally show that in the case $n = 2$ (two modulating states) these systems of linear differential equations can be solved in special cases.

Example. Let $n = 2$ and set $q_{12} = \mu_1$, $q_{21} = \mu_2$ and $f(x, i) = h_i(x)$, $i = 1, 2$. (6.5)-(6.6) and (6.7)-(6.8) become, for $x \in (0, a]$,

$$\begin{aligned} c(1)h_1'(x) - (\mu_1 + \lambda(x))h_1(x) + \mu_1h_2(x) &= 0 \\ c(2)h_1'(x) - (\mu_2 + \lambda(x))h_2(x) + \mu_2h_1(x) &= 0 \end{aligned} \quad (6.9)$$

subject to

$$h_1(a) = h_2(a) = 1 \quad (6.10)$$

and

$$\begin{aligned} c(1)h_1'(x) - (\mu_1 + \lambda(x))h_1(x) + \mu_1 h_2(x) &= -1 \\ c(2)h_1'(x) - (\mu_2 + \lambda(x))h_2(x) + \mu_2 h_1(x) &= -1 \end{aligned} \quad (6.11)$$

subject to

$$h_1(a) = h_2(a) = 0, \quad (6.12)$$

respectively. (6.9)-(6.10) can be transformed into the two second-order linear differential equations

$$\begin{aligned} c(1)c(2)h_i''(x) - [(c(1) + c(2)\lambda(x) + \mu_1 c(2) + \mu_2 c(1))]h_i'(x) \\ + [\lambda(x)^2 + (\mu_1 + \mu_2)\lambda(x)]h_i(x) = 0, \quad x \in (0, a), \quad i = 1, 2 \end{aligned} \quad (6.13)$$

with the boundary conditions

$$h_i(a) = 1, \quad h_i'(a) = \lambda(a), \quad i = 1, 2. \quad (6.14)$$

Similarly, (6.11)-(6.12) lead to

$$\begin{aligned} c(1)c(2)h_i''(x) - [(c(1) + c(2)\lambda(x) + \mu_1 c(2) + \mu_2 c(1))]h_i'(x) \\ + [\lambda(x)^2 + (\mu_1 + \mu_2)\lambda(x)]h_i(x) - \mu_1 - \mu_2 - \lambda(x) = 0, \quad x \in (0, a), \quad i = 1, 2 \end{aligned} \quad (6.15)$$

subject to

$$h_i(a) = 0, \quad h_i'(a) = -1, \quad i = 1, 2. \quad (6.16)$$

Consider now the proportional jump intensity $\lambda(x) = \lambda x$, which we have assumed in Section 3. In this case (6.13) takes the form

$$h_i''(x) + [d_1 x + d_2]h_i'(x) + [d_3 x^2 + d_4 x]h_i(x) = 0, \quad i = 1, 2, \quad x \in (0, a] \quad (6.17)$$

where

$$\begin{aligned} d_1 &= -\left(c(1)c(2)\right)^{-1} [c(1) + c(2)\lambda], \quad d_2 = -\left(c(1)c(2)\right)^{-1} [\mu_1 c(2) + \mu_2 c(1)], \\ d_3 &= \left(c(1)c(2)\right)^{-1} \lambda^2, \quad d_4 = \left(c(1)c(2)\right)^{-1} (\mu_1 + \mu_2)\lambda. \end{aligned}$$

The general solution of (6.17) can be given as a linear combination of the degenerate hypergeometric functions Φ and Ψ , which we have already used in Section 3 (see [21], eq. 2.1.2.103 on p. 143), at certain arguments, and

then the boundary conditions (6.14) uniquely determine the coefficients. We will not write down the lengthy exact formulas. Let $\Phi_0(x)$ and $\Psi_0(x)$ be two linearly independent solutions of (6.17). The corresponding general solution of the nonhomogeneous equation (6.15) is given by

$$C_1\Phi_0(x) + C_2\Psi_0(x) + (c(1)c(2))^{-1} \left[\Psi_0(x) \int_0^x \frac{\Phi_0(u)(\mu_1 + \mu_2 + \lambda(u))}{W(u)} du - \Phi_0(x) \int_0^x \frac{\Psi_0(u)(\mu_1 + \mu_2 + \lambda(u))}{W(u)} du \right],$$

where $W(u) = \Phi_0(u)\Psi_0'(u) - \Psi_0(u)\Phi_0'(u)$, and the constants C_1 and C_2 can be determined from (6.16).

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