Local asymptotics for the cycle maximum of a heavy-tailed random walk

Denis Denisov, Vsevolod Shneer

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Abstract

Let $\xi, \xi_1, \xi_2, \ldots$ be a sequence of independent and identically distributed random variables, $S_n = \xi_1 + \cdots + \xi_n$ and $M_n = \max_{k \le n} S_k$. Let $\tau = \min\{n \ge 1 : S_n \le 0\}$. We assume that ξ has a heavy-tailed distribution and negative finite mean $\mathbf{E}\xi < 0$. We find asymptotics for $\mathbf{P}(M_\tau \in (x, x+T])$ as $x \to \infty$ for a fixed positive constant $T \le \infty$.

1 Introduction and main result

Let $\xi, \xi_1, \xi_2, \ldots$ be a sequence of independent random variables with a common distribution F and mean $-\infty < -m < 0$. Consider the random walk

$$S_0 = 0, \quad S_n = \xi_1 + \cdots + \xi_n.$$

Let

$$\tau = \min\{n \ge 1 : S_n \le 0\}, \quad M_\tau = \max_{0 \le i \le \tau} S_i$$

be the first ladder epoch and the cycle maximum of the random walk respectively. Note that in this case, $\mathbf{E}\tau < \infty$ and $M_\tau < \infty$ a.s. In this work we study local asymptotics for the cycle maximum

$$\mathbf{P}(M_\tau \in (x, x+T]), \quad x \to \infty,$$

where T is a fixed positive constant. We consider the (right) heavy-tailed case, that is when $\mathbf{E}e^{\lambda \xi_1} = \infty$ for all $\lambda > 0$.

The global asymptotics for $\mathbf{P}(M_\tau > x)$ (and some related problems) are studied by various authors. In [13] these asymptotics are obtained for regularly varying distributions. In [1] (see also corrections in the proof in [2, Theorem X.9.4]) these asymptotics are found for a more general class $\mathcal{S}^*$ (see Definition 1 below). Namely, it is proved that if F belongs to $\mathcal{S}^*$ then

$$\mathbf{P}(M_\tau > x) \sim \mathbf{E}\tau \overline{F}(x) \tag{1}$$

(here and throughout $a(x) \sim b(x)$ means $\lim_{x\to\infty} \frac{a(x)}{b(x)} = 1$). A short proof of (1) may be found in [8]. Foss and Zachary [12] show that the converse is true: if F is long-tailed and (1) holds then $F \in \mathcal{S}^*$. They also prove that (1) holds even if instead of τ we take any stopping time with finite mean. In [11] this result is generalized to the case of infinite mean stopping times.

In order to state our results we require some definitions.

Definition 1. A distribution function F on $\mathbb{R}$ belongs to the class $\mathcal{S}^*$ (see Klüppelberg[14]) if and only if $\overline{F}(x) > 0$ for all x and

$$\int_0^x \overline{F}(x-y)\overline{F}(y)dy \sim 2m^+\overline{F}(x),$$

where $m^+ = \int_0^\infty \overline{F}(y)dy < \infty$.

Further, it is known that if a distribution function F belongs to the class $\mathcal{S}^*$ then it is subexponential (see [14]). In general, the converse assertion does not hold, i.e. a subexponential distribution with finite mean may not belong to $\mathcal{S}^*$, see [9] for a counterexample.

Fix $0 < T \leq \infty$ and write $\Delta = (0, T]$,

$$x + \Delta = (x, x+T], \quad x \in \mathbb{R}.$$

Let

$$F(x+\Delta) = \mathbf{P}(\xi \in x+\Delta) = \mathbf{P}(\xi \in (x, x+T]).$$

Definition 2. We say that a distribution F on $\mathbb{R}$ belongs to the class $\mathcal{L}_\Delta$ if and only if $F(x+\Delta) > 0$ for all sufficiently large x and

$$\frac{F(x+t+\Delta)}{F(x+\Delta)} \to 1 \quad \text{as} \quad x \to \infty, \tag{2}$$

for all $t \in [0, 1]$.

Remark 1. The class $\mathcal{L}_\Delta$ is introduced in [3]. Note that Definition 2 implies local uniform convergence (uniform convergence on each compact t—set in $(0, \infty)$) in (2). Indeed, it follows from Definition 2 that (2) holds for all $t \geq 0$. Put $f(x) = F(\log x + \Delta)$, then (2) is equivalent to $f(tx)/f(x) \to 1$ as $x \to \infty$. This means that function f is slowly varying (see [5] for definition and properties). Uniform convergence in (2) follows now from the Uniform Convergence Theorem for slowly varying functions (see, e.g., [5, Theorem 1.2.1]). Moreover, it follows from the uniform convergence on any compact set that one can choose a function $h(x) \to \infty$ such that (2) holds uniformly in $|t| \leq h(x)$.

Definition 3. Let F be a distribution on $\mathbb{R}_+$ with unbounded support. We say that F is Δ-subexponential and write $F \in \mathcal{S}_\Delta$ if $F \in \mathcal{L}_\Delta$ and

$$(F * F)(x+\Delta) \sim 2F(x+\Delta) \quad \text{as} \quad x \to \infty.$$

If $T = \infty$ we simply say that F is subexponential.

The notion of Δ-subexponential distributions has been introduced in [3]. The case $T = \infty$ corresponds to ordinary subexponential distributions introduced by Chistyakov [7]. In [3] it is shown that the basic properties of subexponential distributions carry over virtually without changes to the case of Δ-subexponential distributions.

In this paper we introduce a new class of distributions.

Definition 4. We say that a distribution F belongs to the class $\mathcal{S}^*_\Delta$ if $F \in \mathcal{L}_\Delta$, $m^+ < \infty$ and

$$\int_0^{x/2} F(x - y + \Delta)\overline{F}(y)dy \sim m^+ F(x + \Delta) \quad \text{as} \quad x \to \infty.$$

This class is a natural extension of the class $\mathcal{S}^*$. It is not difficult to see that $\mathcal{S}^* = \mathcal{S}^*_{(0,\infty)}$ since $S^* \subset \mathcal{L}_{(0,\infty)}$ (see [14]). We will also show that if F belongs to $\mathcal{S}^*_\Delta$ for some Δ then it belongs to $\mathcal{S}_\Delta$.

Now we are in position to state our main result.

Theorem 1. *Let $F \in \mathcal{S}^*_\Delta$ and $\left(\overline{F}(x)\right)^2 = o(F(x + \Delta))$. Then*

$$\boldsymbol{P}(M_\tau \in x + \Delta) \sim \mathbf{E}\tau F(x + \Delta). \tag{3}$$

The proof of the result is given in Section 3. It will be shown in Remark 6 that the condition $\left(\overline{F}(x)\right)^2 = o(F(x + \Delta))$ is essential for the relation (3) to hold. In other words, Remark 6 shows that asymptotics of $\mathbf{P}(M_\tau \in x + \Delta)$ may be different from (3) if we assume only that $F \in \mathcal{S}^*_\Delta$.

The paper is organized as follows. In Section 2, in the form of five lemmas, we present some properties of the new class $\mathcal{S}^*_\Delta$. We show that the main properties of the class $\mathcal{S}^*$ remain valid for the case of arbitrary positive T. We also give sufficient conditions for a distribution to belong to class $\mathcal{S}^*_\Delta$. Using these sufficient conditions we show that standard examples of subexponential distributions are contained in the class $\mathcal{S}^*_\Delta$. The proof of our main result is given in Section 3. Proofs of five lemmas formulated in Section 2 are collected in the Appendix.

2 Basic properties of the class $\mathcal{S}^*_\Delta$

First, we give some conditions for distributions to belong to the class $\mathcal{S}^*_\Delta$. These conditions show that standard examples of distributions from the class $\mathcal{S}^*$ are contained in the class $\mathcal{S}^*_\Delta$.

Lemma 1. *Let a distribution F belong to the class $\mathcal{L}_\Delta$ for some finite $T > 0$. Assume that there exist $c > 0$ and $x_0 < \infty$ such that $F(x + t + \Delta) \geq cF(x + \Delta)$ for any $t \in (0, x]$ and $x > x_0$. Assume also that $m^+ < \infty$. Then $F \in \mathcal{S}^*_\Delta$.*

Remark 2. In [3], it is shown that if a distribution F satisfies the conditions of Lemma 1 then $F \in \mathcal{S}_\Delta$. It is clear that for such distributions $\overline{F}(2x) \geq c\overline{F}(x)$, and it is shown in ([14], Theorem 3.2) that distributions with this property belong to the class $\mathcal{S}^*$.

The Pareto distribution (with the tail $\overline{F}(x) = x^{-\alpha}$, $\alpha > 1$, $x \geq 1$) satisfies the conditions of Lemma 1 for any $T > 0$. The same is true for any distribution F such that $\mathbf{P}(\xi \in x + \Delta)$ is regularly varying at infinity, i.e., for $F(x + \Delta) \sim x^{-\alpha} l(x)$, where $l(x)$ is slowly varying at infinity.

Let $Q_\Delta(x) = -\ln F(x + \Delta)$ for any finite T and $Q(x) = -\ln \overline{F}(x)$. Following, with obvious changes, the construction presented in [14] (see also [16]), it is easy to check that for any distribution $F \in \mathcal{L}_\Delta$ we can always find a distribution G such that $G \in \mathcal{L}_\Delta$, $F(x + \Delta) \sim G(x + \Delta)$ as $x \to \infty$ and $R_\Delta(x) = -\ln G(x + \Delta)$ is differentiable. In view of Lemma 4 we may give sufficient conditions for $F \in \mathcal{S}^*_\Delta$ assuming the existence of derivative $Q'_\Delta(x)$.

Lemma 2. *Assume that* $r = \limsup_{x\to\infty} \frac{xQ'_\Delta(x)}{Q(x)} < 1$, *the function* $Q(x)/x$ *is eventually non-increasing and* $\overline{F}^{1-r-\varepsilon}(x)$ *is integrable for some* $\varepsilon > 0$. *Then* $F \in \mathcal{S}^*_\Delta$.

Remark 3. Lemma 2 is a generalization of Theorem 2.8 (c) of [15] to the case of arbitrary positive T. Note that in the case $T = \infty$ the conditions of both propositions coincide, since in this case the fact that $Q(x)/x$ is a non-increasing function follows from assumption $r < 1$.

Direct computations show that any Weibull distribution (i.e., distribution with the tail $\overline{F}(x) = e^{-x^\gamma}$) satisfies the conditions of Lemma 2 for any $T > 0$ if $0 < \gamma < 1$. One can also show that so-called semi-exponential distributions (i.e., distributions with the tails $\overline{F}(x) = e^{-x^\gamma l(x)}$, where $0 \leq \gamma < 1$ and $l(x)$ is a slowly varying function such that $l'(x) = o(l(x)/x)$ as $x \to \infty$, see, for example, [6]) satisfy the conditions of Lemma 2 for any $T > 0$.

It is known (see [3]) that $\mathcal{S}_\Delta \subset S$ for any positive T. The Lemma below shows that an inclusion $\mathcal{S}^*_\Delta \subset \mathcal{S}^*$ also holds.

Lemma 3. *If* $F \in \mathcal{S}^*_\Delta$ *for some finite interval* $\Delta = (0, T]$, *then* $F \in \mathcal{S}^*$.

The following Lemma is a generalization of Theorem 2.1 (b) of [14] to the case of arbitrary positive T.

Lemma 4. *Let* $F, G \in \mathcal{L}_\Delta$ *and assume that there exist* $M_1, M_2 \in (0, \infty)$ *such that* $M_1 \leq G(x+\Delta)/F(x+\Delta) \leq M_2$ *for all sufficiently large* x. *Then* $F \in \mathcal{S}^*_\Delta \Leftrightarrow G \in \mathcal{S}^*_\Delta$.

Let H be a non-negative measure on $\mathbb{R}_+$ such that

$$\int_0^\infty \overline{F}(t) H(dt) < \infty.$$

In this case we can define the distribution G_H on $\mathbb{R}_+$ with the tail

$$\overline{G}_H(x) := \min\left(1, \int_0^\infty \overline{F}(x+t)H(dt)\right).$$

The following Lemma is a generalization of Lemma 9 of [9].

Lemma 5. *Let $F \in \mathcal{S}^*_\Delta$ and assume that $\sup_t H((t,t+1]) \le b < \infty$. Then $G_H \in \mathcal{S}_\Delta$.*

Remark 4. Here are some examples of such measures H:
(i) if $H(B) = \mathbf{I}(0 \in B)$, then $G_H = F$;
(ii) if $H(dt) = dt$ is the Lebesgue measure on $\mathbb{R}_+$, then G_H is the integrated tail distribution.

3 Proof of Theorem 1

Put $M = \sup_{n\ge 0} S_n$ and let $\pi(B) = \mathbf{P}(M \in B)$.
Let $\eta = \min\{n \ge 1 : S_n > 0\} \le \infty$ be the first (strict) ascending ladder epoch and put

$$p = \mathbf{P}\{\eta = \infty\} = \mathbf{P}(M = 0).$$

Let $\{\psi_n\}_{n\ge 1}$ be a sequence of independent random variables with common distribution

$$\mathbf{P}(\psi_1 \in B) \equiv G_+(B) = \mathbf{P}(S_\eta \in B | \eta < \infty). \tag{4}$$

Let ν be a random variable, independent of the above sequence, such that $\mathbf{P}(\nu = n) = p(1-p)^n, n = 0, 1, 2, \ldots$ Then (see [10, Chapter XII] or [2, Chapter VIII])

$$M \stackrel{d}{=} \psi_1 + \ldots + \psi_\nu. \tag{5}$$

Let $\chi = S_\tau$ be the first non-positive sum and

$$G_-(B) = \mathbf{P}(-\chi \in B). \tag{6}$$

Our proof of Theorem 1 is based on the following sequence of lemmas.

Lemma 6. *Let $M < \infty$ a.s. Then*

$$\begin{aligned}\mathbf{P}(M_\tau \in (x, x+T]) \;&\sim\; \int_0^\infty \boldsymbol{P}(\chi \in -dz)(\pi(x, x+z] - \pi(x+T, x+T+z]) \\ &+ \int_0^\infty \boldsymbol{P}(\chi \in -dz, M_\tau > x)\pi(x+z, x+z+T)\end{aligned} \tag{7}$$

Lemma 7. *Let the assumptions of Theorem 1 hold. Then, for any positive integer n,*

$$\int_0^\infty G_-(dz)\Big(G_+^{*n}((x,x+z])-G_+^{*n}((x+T,x+T+z])\Big) \sim \frac{n}{1-p}F(x+\Delta) \quad as \quad x\to\infty.$$

Lemma 8. *(exponential bound) Let the assumptions of Theorem 1 hold. Then for any $\varepsilon>0$ there exist numbers $K<\infty$ and $x_0>0$ such that for all n and $x>x_0$,*

$$\left|\int_0^\infty G_-(dz)\Big(G_+^{*n}((x,x+z])-G_+^{*n}((x+T,x+T+z])\Big)\right| \le K(1+\varepsilon)^n F(x+\Delta).$$

Lemma 6 is an extension of Lemma 1 from [8] to the case of arbitrary $T>0$. Indeed, in the case $T=\infty$ the second term (7) is negligible and one obtains Lemma 1 from [8].

Proofs of lemmas 6–8 are given in Section 4. We now present the proof of Theorem 1.

First, we will analyse the second term (7) in Lemma 6. We have,

$$(8)\quad \int_0^\infty \mathbf{P}(\chi\in -dz, M_\tau>x)\mathbf{P}(\widetilde{M}\in x+z+\Delta)$$
$$= \frac{T}{|\mathbf{E}\xi|}(1+o(1))\int_0^\infty \mathbf{P}(\chi\in -dz, M_\tau>x)\overline{F}(x+z)$$
$$\le \frac{T}{|\mathbf{E}\xi|}(1+o(1))\overline{F}(x)\int_0^\infty \mathbf{P}(\chi\in -dz, M_\tau>x) \le \frac{T}{|\mathbf{E}\xi|}(\mathbf{E}\tau)(1+o(1))(\overline{F}(x))^2.$$

Here we used the facts that $\mathbf{P}(\widetilde{M}\in x+\Delta)\sim \dfrac{T}{|\mathbf{E}\xi|}\overline{F}(x)$ and $\mathbf{P}(M_\tau>x)\sim \mathbf{E}\tau\overline{F}(x)$ as $x\to\infty$ if $F\in\mathcal{S}^*$ (see [4]). Inclusion $\mathcal{S}^*_\Delta\subset\mathcal{S}^*$ proved in Lemma 3 implies that $F\in\mathcal{S}^*$ under the assumptions of Theorem 1.

In view of our assumption $(\overline{F}(x))^2 = o(F(x+\Delta))$ it remains to prove that

$$(9)\qquad \int_0^\infty \mathbf{P}(\chi\in -dz)(\pi(x,x+z]-\pi(x+T,x+T+z]) \sim \mathbf{E}\tau F(x+\Delta)$$

as $x\to\infty$. One of the ways to show this equivalence is to find the asymptotic behaviour of

$$(10)\qquad \pi(x,x+z]-\pi(x+T,x+T+z]$$

when z and T are fixed and x goes to infinity. This approach works well when $T=\infty$. But in the case $T<\infty$, we were not able to find asymptotics (10) without imposing some additional assumptions (for example, it is possible to find these asymptotics when ξ^- has a finite third moment). Therefore, we use another approach and prove (9) directly, using lemmas 7 and 8. However, this approach uses almost the same type of arguments.

It follows from (5) that (9) can be represented in the following form

$$\int_0^\infty \mathbf{P}(\chi \in -dz)(\pi(x, x+z] - \pi(x+T, x+T+z])$$
$$= \sum_{n=0}^\infty p(1-p)^n \int_0^\infty G_-(dz)\Big(G_+^{*n}((x, x+z]) - G_+^{*n}((x+T, x+T+z])\Big).$$

Then it follows from Lemmas 7 and 8 and the dominated convergence theorem that

$$\int_0^\infty \mathbf{P}(\chi \in -dz)(\pi(x, x+z] - \pi(x+T, x+T+z])$$
$$\sim \sum_{n=0}^\infty p(1-p)^n \frac{n}{1-p} F(x+\Delta) = \mathbf{E}\tau F(x+\Delta).$$

Theorem 1 is proved.

4 Proofs of lemmas 6–8

Proof of Lemma 6. By the total probability formula

$$\mathbf{P}(M > x) = \mathbf{P}(M_\tau > x) + \mathbf{P}(M_\tau \le x, \widetilde{M} > x - \chi), \tag{11}$$

where $\widetilde{M} = \sup_{n\ge 0}(S_{\tau+n} - \chi)$. Clearly,

$$\widetilde{M} \overset{D}{=} M, \quad \widetilde{M} \text{ and } (\chi, M_\tau) \text{ are independent.}$$

The second term in the RHS of (11) is

$$\mathbf{P}(M_\tau \le x, \widetilde{M} > x - \chi) = \mathbf{P}(\widetilde{M} > x - \chi) - \mathbf{P}(M_\tau > x, \widetilde{M} > x - \chi).$$

Then,

$$\mathbf{P}(M_\tau > x) = \mathbf{P}(M > x) - \mathbf{P}(\widetilde{M} > x - \chi) + \mathbf{P}(M_\tau > x, \widetilde{M} > x - \chi)$$
$$= \int_0^\infty \mathbf{P}(\chi \in -dz)\pi(x, x+z] + \mathbf{P}(M_\tau > x, \widetilde{M} > x - \chi)$$

and it implies that

$$\mathbf{P}(M_\tau \in x+\Delta) = \int_0^\infty \mathbf{P}(\chi \in -dz)(\pi(x, x+z] - \pi(x+T, x+T+z])$$
$$+ \mathbf{P}(M_\tau > x, \widetilde{M} > x - \chi) - \mathbf{P}(M_\tau > x+T, \widetilde{M} > x+T-\chi).$$

We have

$$\mathbf{P}(M_\tau > x, \widetilde{M} > x - \chi) - \mathbf{P}(M_\tau > x+T, \widetilde{M} > x+T-\chi)$$
$$= \mathbf{P}(M_\tau > x, \widetilde{M} > x - \chi) - \mathbf{P}(M_\tau > x, \widetilde{M} > x+T-\chi)$$
$$+ \mathbf{P}(M_\tau > x, \widetilde{M} > x+T-\chi) - \mathbf{P}(M_\tau > x+T, \widetilde{M} > x+T-\chi),$$

where the latter line is estimated as follows

$$
\begin{aligned}
&0 \le \mathbf{P}(M_\tau > x, \widetilde{M} > x + T - \chi) - \mathbf{P}(M_\tau > x + T, \widetilde{M} > x + T - \chi) \\
&= \mathbf{P}(M_\tau \in x+\Delta, \widetilde{M} > x+T-\chi) \le \mathbf{P}(M_\tau \in x+\Delta, \widetilde{M} > x+T) = o(\mathbf{P}(M_\tau \in x+\Delta)).
\end{aligned}
$$

Finally,

$$
\begin{aligned}
&\mathbf{P}(M_\tau > x, \widetilde{M} > x - \chi) - \mathbf{P}(M_\tau > x, \widetilde{M} > x + T - \chi) \\
&\qquad = \int_0^\infty \mathbf{P}(\chi \in -dz, M_\tau > x)\mathbf{P}(\widetilde{M} \in x + z + \Delta).
\end{aligned}
$$

□

In the proofs of Lemmas 7 and 8 we will need the following technical result.

Lemma 9. *Let F belong to $\mathcal{S}^*_\Delta$ and $\big(\overline{F}(x)\big)^2 = o(F(x+\Delta)), x \to \infty$. Then $\big(\overline{F}(x/2)\big)^2 = o(F(x+\Delta)), x \to \infty$.*

Proof of Lemma 9. Choose a function $h(x) \uparrow \infty$ such that $h(x) \le x/2$ and (2) holds uniformly in $|t| \le h(x)$. For this function we have

$$
\int_0^{h(x)} F(x - y + \Delta)\overline{F}(y)dy \sim m^+ F(x+\Delta) \quad \text{as} \quad x \to \infty, \tag{12}
$$

and, since $F \in \mathcal{S}^*_\Delta$,

$$
\int_{h(x)}^{x/2} F(x - y + \Delta)\overline{F}(y)dy = o(F(x+\Delta)) \quad \text{as} \quad x \to \infty. \tag{13}
$$

Consider the LHS of the last relation:

$$
\begin{aligned}
\int_{h(x)}^{x/2} F(x - y + \Delta)\overline{F}(y)dy \ &\ge\ \overline{F}(x/2) \int_{h(x)}^{x/2} F(x - y + \Delta)dy \\
&=\ \overline{F}(x/2)\Big(\int_{x/2}^{x/2+T} \overline{F}(y)dy - \int_{x-h(x)}^{x-h(x)+T} \overline{F}(y)dy\Big) \\
&\ge\ T\overline{F}(x/2)\Big(\overline{F}(x/2+T) - \overline{F}(x-h(x))\Big) \\
&\ge\ T\overline{F}(x-h(x))\Big(\overline{F}(x/2+T) - \overline{F}(x-h(x))\Big).
\end{aligned}
$$

It follows from the latter bounds and (13) that

$$
\overline{F}(x/2)\Big(\overline{F}(x/2+T) - \overline{F}(x-h(x))\Big) = o(F(x+\Delta)) \quad \text{as} \quad x \to \infty \tag{14}
$$

and

$$(15)\qquad \overline{F}(x-h(x))\Big(\overline{F}(x/2+T)-\overline{F}(x-h(x))\Big) = o(F(x+\Delta)) \quad \text{as} \quad x\to\infty.$$

Summing (14) and (15) we obtain that

$$\overline{F}(x/2)\overline{F}(x/2+T)-\overline{F}(x-h(x))F(x/2,x/2+T]-\overline{F}(x-h(x))^2 = o(F(x+\Delta)).$$

First, it follows from (12) and $\overline{F}(x)^2 = o(F(x+\Delta))$ that

$$\overline{F}(x-h(x))^2 = o(F(x+\Delta)).$$

Second, it follows from (12) and $F\in\mathcal{S}^*_\Delta$ that

$$\overline{F}(x-h(x))F(x/2,x/2+T] = o(F(x+\Delta)).$$

Therefore, $\big(\overline{F}(x/2)\big)^2 = o(F(x+\Delta))$.

□

Proof of Lemma 7. The starting point of our analysis is the well-known Wiener-Hopf identity (see, e.g. [10, Chapter XII, (3.11)]):

$$(16)\qquad \overline{F}(x) = (1-p)\int_0^\infty G_-(dz)G_+((x,x+z]).$$

Recall that $G_-(dz) = \mathbf{P}(-\chi\in dz)$.
In the case $n=1$, (16) yields

$$\int_0^\infty G_-(dz)\Big(G_+((x,x+z])-G_+((x+T,x+T+z])\Big) = \frac{1}{1-p}F(x+\Delta).$$

Denote $V_n = \sum_{k=1}^n \psi_k$. We will use induction arguments. Assume that the assertion of the Lemma is valid for n and let us prove it for $n+1$. For any $z>0$, let $\Delta_z = (0,z]$. By the total probability formula,

$$\begin{aligned}
&\int_0^\infty G_-(dz)\Big(G_+^{*(n+1)}((x,x+z])-G_+^{*(n+1)}((x+T,x+T+z])\Big)\\
&= \int_0^\infty G_-(dz)\Big(\mathbf{P}(V_n\le x/2, V_{n+1}\in x+\Delta_z)-\mathbf{P}(V_n\le x/2, V_{n+1}\in x+T+\Delta_z)\Big)\\
&+\int_0^\infty G_-(dz)\Big(\mathbf{P}(\psi_{n+1}\le x/2, V_{n+1}\in x+\Delta_z)-\mathbf{P}(\psi_{n+1}\le x/2, V_{n+1}\in x+T+\Delta_z)\Big)\\
&+\int_0^\infty G_-(dz)\Big(\mathbf{P}(V_n> x/2, \psi_{n+1}>x/2, V_{n+1}\in x+\Delta_z)\\
&-\mathbf{P}(V_n> x/2, \psi_{n+1}>x/2, V_{n+1}\in x+T+\Delta_z)\Big)\\
(17)\qquad &\equiv I_1(x)+I_2(x)+I_3(x).
\end{aligned}$$

First,

$$I_1(x) = \int_0^{x/2} G_+^{*n}(dy) \int_0^\infty G_-(dz) \Big(G_+((x-y, x-y+z]) - G_+((x+T-y, x+T-y+z]) \Big)$$
$$= \frac{1}{1-p} \int_0^{x/2} G_+^{*n}(dy) F(x-y+\Delta).$$

Since $F \in \mathcal{L}_\Delta$ we can choose a function $h(x) \uparrow \infty$ such that $h(x) \le x/2$ and (2) holds uniformly for $|t| \le h(x)$. For this function we have

$$\int_0^{h(x)} G_+^{*n}(dy) F(x-y+\Delta) \sim F(x+\Delta) \quad \text{as} \quad x \to \infty, \tag{18}$$

and, by Proposition 2 from the appendix,

$$\int_{h(x)}^{x/2} G_+^{*n}(dy) F(x-y+\Delta) \le \sum_{k=[h(x)]-1}^{[x/2]+1} G_+^{*n}(k, k+1] \sup_{y \in [0,1]} F(x-k-y+\Delta) \tag{19}$$
$$\sim n \frac{1-p}{pm} T \sum_{k=[h(x)]-1}^{[x/2]+1} \overline{F}(k) F(x-k+\Delta) = o(F(x+\Delta))$$

since $F \in \mathcal{S}_\Delta^*$.

Second, by the induction hypothesis,

$$I_2(x) = \int_0^{x/2} G_+(dy) \int_0^\infty G_-(dz) \Big(G_+^{*n}((x-y, x-y+z]) - G_+^{*n}((x+T-y, x+T-y+z]) \Big)$$
$$\sim \frac{n}{1-p} \int_0^{x/2} G_+(dy) F(x-y+\Delta).$$

Following the arguments used to prove (18) and (19) we conclude that

$$n \int_0^{x/2} G_+(dy) F(x-y+\Delta) \sim n F(x+\Delta) \quad \text{as} \quad x \to \infty. \tag{20}$$

Finally, in view of (18) – (20), it remains to prove that $I_3(x) = o(F(x+\Delta))$. We have,

$$\begin{aligned} I_3(x) &= \int_0^\infty G_-(dz) \Big(\mathbf{P}(V_n > x/2, \psi_{n+1} > x/2, V_{n+1} \in x + \Delta_z) \\ &\quad - \mathbf{P}(V_n > x/2, \psi_{n+1} > x/2, V_{n+1} \in x + T + \Delta_z) \Big) \\ &= \int_0^\infty G_-(dz) \Big(\mathbf{P}(V_n > x/2, \psi_{n+1} > x/2, V_{n+1} \in x + \Delta) \\ &\quad - \mathbf{P}(V_n > x/2, \psi_{n+1} > x/2, V_{n+1} \in x + z + \Delta) \Big) \\ &\equiv I_{31}(x) - I_{32}(x). \end{aligned}$$

Here, we have used the fact that for any measurable event B,

$$\begin{aligned}&\mathbf{P}(B, V_n \in x + \Delta_z) - \mathbf{P}(B, V_n \in x + T + \Delta_z])\\ &= \mathbf{P}(B, V_n > x) - \mathbf{P}(B, V_n > x + z) - \mathbf{P}(B, V_n > x + T) + \mathbf{P}(B, V_n > x + T + z)\\ &\qquad\qquad\qquad\qquad = \mathbf{P}(B, V_n \in x + \Delta) - \mathbf{P}(B, V_n \in x + z + \Delta).\end{aligned}$$

The following estimate is valid:

$$\begin{aligned} I_{31}(x) &\leq \mathbf{P}(V_n \in x/2 + \Delta)\mathbf{P}(\psi_{n+1} \in x/2 + \Delta)\\ &= G_+^{*n}(x/2 + \Delta)G_+(x/2 + \Delta). \end{aligned} \tag{21}$$

By Proposition 2 from the appendix,

$$G_+^{*n}(x/2 + \Delta)G_+(x/2 + \Delta) \sim n\frac{1-p}{pm}T\overline{F}(x/2)\frac{1-p}{pm}T\overline{F}(x/2) = O(\overline{F}(x/2)^2).$$

Therefore, by Lemma 9, $I_{31}(x) = o(F(x + \Delta))$. Similarly, using Proposition 2 from the appendix, we obtain

$$\begin{aligned} I_{32}(x) &\leq \int_0^\infty G_-(dz)\int_{x/2}^{x/2+z+T} G_+^{*n}(dy)G_+((x+z-y, x+z-y+T])\\ &\sim \frac{1-p}{pm}T\int_0^\infty G_-(dz)\int_{x/2}^{x/2+z+T} G_+^{*n}(dy)\overline{F}(x+z-y)\\ &\leq \frac{1-p}{pm}T\overline{F}(x/2-T)\int_0^\infty G_-(dz)G_+^{*n}((x/2, x/2+z+T]). \end{aligned}$$

Then, by the induction hypothesis,

$$\int_0^\infty G_-(dz)G_+^{*n}((x/2, x/2+z]) \sim \frac{n}{1-p}\overline{F}(x/2)$$

and, by Proposition 2 from the appendix,

$$\int_0^\infty G_-(dz)G_+^{*n}((x/2+z, x/2+z+T]) \sim n\frac{1-p}{pm}T\int_0^\infty G_-(dz)\overline{F}(x/2+z) \leq n\frac{1-p}{pm}T\overline{F}(x/2).$$

As a result, we obtain that

$$I_{32}(x) = O(\overline{F}(x/2)^2) = o(F(x+\Delta))$$

due to Lemma 9.

□

Proof of Lemma 8. We will give only the proof of the upper bound. The proof of the lower bound is similar. For $x_0 \geq 0$ and $k \geq 1$, put

$$A_k(x_0) = \sup_{x > x_0} \frac{\int_0^\infty G_-(dz)\Big(G_+^{*k}((x, x+z]) - G_+^{*k}((x+T, x+T+z])\Big)}{F(x+\Delta)}.$$

Take any $\varepsilon > 0$. Pick x_0 such that for all $x > x_0$ the following holds: $F(x+\Delta) > 0$ and

$$\text{(22)} \qquad \frac{\sum_{k=[x_0]-1}^{[x/2]+1} \overline{F}(k) \sup_{y\in[0,1]} F(x-k-y+\Delta)}{F(x+\Delta)} \leq \varepsilon/2;$$

$$\text{(23)} \qquad \frac{\int_0^{x/2} G_+(dy)F(x-y+\Delta)}{F(x+\Delta)} \leq 1+\varepsilon/2;$$

$$\text{(24)} \qquad \frac{\overline{F}(x/2)^2}{F(x+\Delta)} \leq \varepsilon/2.$$

It follows from the fact that $F \in \mathcal{S}^*_\Delta$, from Lemma 9 and (20) that such x_0 always exists.

For any $k \geq 1$, put $A_k \equiv A_k(x_0)$. Then,

$$A_k \leq \max\Bigg\{ \frac{1}{\inf_{x_0\leq x\leq 2x_0} F(x+\Delta)}, \sup_{x>2x_0} \frac{\int_0^\infty G_-(dz) \left(G_+^{*k}((x,x+z]) - G_+^{*k}((x+T,x+T+z])\right)}{F(x+\Delta)} \Bigg\}.$$

Let us now estimate the second term in the maximum. As in the proof of Lemma 7, denote $V_n = \sum_{k=1}^{n} \psi_k$ and $\Delta_z = (0,z]$, where z is any positive number. We will use representation (17). First, by the same arguments as in Lemma 7, we have

$$I_1(x) = \frac{1}{1-p}\int_0^{x/2} G_+^{*n}(dy)F(x-y+\Delta) = \frac{1}{1-p}\left(\int_0^{x_0} + \int_{x_0}^{x/2}\right) G_+^{*n}(dy)F(x-y+\Delta).$$

Then,

$$\sup_{x>2x_0} \frac{\int_0^{x_0} G_+^{*n}(dy)F(x-y+\Delta)}{F(x+\Delta)} \leq \sup_{x>2x_0} \frac{\sup_{0\leq y\leq x_0} F(x-y+\Delta)}{F(x+\Delta)} \equiv R_1.$$

Constant R_1 is finite since $F \in \mathcal{L}_\Delta$. Further, it follows from (22) and Proposition 2 from the appendix that

$$\frac{\int_{x_0}^{x/2} G_+^{*n}(dy)F(x-y+\Delta)}{F(x+\Delta)} \leq \frac{\sum_{k=[x_0]-1}^{[x/2]+1} G_+^{*n}(k,k+1] \sup_{y\in[0,1]} F(x-k-y+\Delta)}{F(x+\Delta)}$$
$$\leq C_1(1+\varepsilon/2)^n \frac{\sum_{k=[x_0]-1}^{[x/2]+1} \overline{F}(k) \sup_{y\in[0,1]} F(x-k-y+\Delta)}{F(x+\Delta)} \leq C_1(1+\varepsilon/2)^n \varepsilon/2.$$

Second,

$$I_2(x) = \int_0^{x/2} G_+(dy) \int_0^\infty G_-(dz)\Big(G_+^{*n}((x-y,x-y+z]) - G_+^{*n}((x+T-y,x+T-y+z])\Big)$$
$$\leq A_n \int_0^{x/2} G_+(dy)F(x-y+\Delta) \leq A_n(1+\varepsilon/2)F(x+\Delta),$$

where the latter inequality follows from (22).

Third, we use the same representation $I_3(x) = I_{31}(x) - I_{32}(x)$ as in the proof of Lemma 7. Then, the following estimate holds

$$
\begin{aligned}
I_{31}(x) &\leq \mathbf{P}(V_n \in x/2 + \Delta)\mathbf{P}(\psi_{n+1} \in x/2 + \Delta) \\
&\leq C_1(1+\varepsilon/2)^n \mathbf{P}(\psi_{n+1} \in x/2 + \Delta)^2 \leq C_2(1+\varepsilon/2)^n F(x+\Delta). \qquad (25)
\end{aligned}
$$

Therefore, for some positive constants C and R,

$$A_{n+1} \leq (1+\varepsilon/2)A_n + C(1+\varepsilon/2)^n + R,$$

and, by recursion,

$$A_{n+1} \leq (1+\varepsilon/2)^n A_1 + Cn(1+\varepsilon/2)^n + Rn(1+\varepsilon/2)^n.$$

The latter implies the assertion of the Lemma.

□

A Appendix

A.1 Properties of $\Delta-$ subexponential distributions

In this Proposition we list properties of Δ-subexponential distributions that are used in the proof of Theorem 1. Proofs and some other properties may be found in [3].

Proposition 1. *Let $F \in \mathcal{S}_\Delta$. Then*

(i) (see [3, Corollary 2])

$$F^{*n}(x+\Delta) \sim nF(x+\Delta), \quad x \to \infty;$$

(ii) (see [3, Proposition 4]) for any $\varepsilon > 0$, there exist $x_0 = x_0(\varepsilon) > 0$ and $V(\varepsilon) > 0$ such that, for any $x > x_0$ and $n \geq 1$,

$$F^{*n}(x+\Delta) \leq V(\varepsilon)(1+\varepsilon)^n F(x+\Delta).$$

We also need the following Proposition (see[4]).

Proposition 2. *Let $\mathbf{E}\xi = -m \in (-\infty, 0)$ and $F \in \mathcal{S}^*$ be a non-lattice distribution. Then for any $T > 0$,*

$$G_+(x+\Delta) \sim \frac{1-p}{pm} T\overline{F}(x)$$

and $G_+ \in \mathcal{S}_\Delta$.

Remark 5. In the lattice case this Proposition holds as well with some obvious changes.

A.2 Proofs

We present here the proofs of the Lemmas stated in Section 2. Throughout the Appendix, a function $h(x)$ is such that $h(x) \uparrow \infty$ as $x \to \infty$, $h(x) < x/2$ for all x and (2) holds uniformly in $|t| \leq h(x)$.

Proof of Lemma 1. For the function $h(x)$ we have

$$\int_0^{h(x)} F(x-y+\Delta)\overline{F}(y)dy \sim m^+ F(x+\Delta) \quad \text{as} \quad x \to \infty$$

and

$$\int_{h(x)}^{x/2} F(x-y+\Delta)\overline{F}(y)dy \leq \frac{1}{c}F(x+\Delta) \int_{h(x)}^{x/2} \overline{F}(y)dy = o(F(x+\Delta)) \quad \text{as} \quad x \to \infty.$$

□

Proof of Lemma 2. We need to prove that

$$\int_{h(x)}^{x/2} \frac{F(x-y+\Delta)}{F(x+\Delta)}\overline{F}(y)dy \to 0 \quad \text{as} \quad x \to \infty$$

for the function $h(x)$. Consider the integrand:

$$\begin{aligned} \frac{F(x-y+\Delta)}{F(x+\Delta)}\overline{F}(y) &= \exp\{Q_\Delta(x) - Q_\Delta(x-y) - Q(y)\} \\ &\leq \exp\left\{y(r+\varepsilon)\frac{Q(x-y)}{x-y} - Q(y)\right\} \end{aligned}$$

for sufficiently large x. Since the function $\dfrac{Q(x)}{x}$ is eventually non-increasing and $y \leq x/2$,

$$\exp\left\{y(r+\varepsilon)\frac{Q(x-y)}{x-y} - Q(y)\right\} \leq \exp\{-(1-r-\varepsilon)Q(y)\} = \overline{F}^{1-r-\varepsilon}(y),$$

and the result follows.

□

Proof of Lemma 3. Indeed, note that $F \in \mathcal{S}^*$ if and only if $\int_0^{x/2} \overline{F}(x-y)\overline{F}(y)dy \sim m^+\overline{F}(x)$ as $x \to \infty$. First,

$$\int_0^{x/2} \overline{F}(x-y)\overline{F}(y)dy \geq \overline{F}(x) \int_0^{x/2} \overline{F}(y)dy = m^+(1+o(1))\overline{F}(x).$$

Second, if $F \in \mathcal{S}^*_\Delta$ then

$$\begin{aligned}\int\limits_0^{x/2} \overline{F}(x-y)\overline{F}(y)dy &= \sum_{n=0}^{\infty} \int_0^{x/2} F(x-y+nT+\Delta)\overline{F}(y)dy \\ &\le \sum_{n=0}^{\infty} \int_0^{(x+nT)/2} F(x-y+nT+\Delta)\overline{F}(y)dy \\ &= (1+o(1))\sum_{n=0}^{\infty} m^+ F(x+nT+\Delta) = m^+(1+o(1))\overline{F}(x).\end{aligned}$$

□

Proof of Lemma 4. Indeed, from the assumptions of Lemma 4 we conclude that $M_1 \le \overline{G}(x)/\overline{F}(x) \le M_2$ for all sufficiently large x. Suppose $F \in \mathcal{S}^*_\Delta$. Then G has finite expectation. Since $F, G \in \mathcal{L}_\Delta$, there exists a function $h(x) \to \infty$ such that (2) and the same relation for G hold uniformly in $|t| \le h(x)$. For this function $h(x)$ we have

$$\int\limits_0^{h(x)} G(x-y+\Delta)\overline{G}(y)dy \sim G(x+\Delta)\int_0^\infty \overline{G}(y)dy \quad \text{as} \quad x \to \infty$$

and

$$\int\limits_{h(x)}^{x/2} \frac{G(x-y+\Delta)}{G(x+\Delta)}\overline{G}(y)dy \le \frac{M_2^2}{M_1}\int\limits_{h(x)}^{x/2} \frac{F(x-y+\Delta)}{F(x+\Delta)}\overline{F}(y)dy \to 0 \quad \text{as} \quad x \to \infty.$$

□

Proof of Lemma 5.

Note that for all sufficiently large x

$$G_H(x+\Delta) = \int\limits_0^\infty F(x+t+\Delta)H(dt).$$

Thus $G_H \in \mathcal{L}_\Delta$ since $F \in \mathcal{L}_\Delta$.

Consider two independent random variables η_1 and η_2 both having distribution G_H. We need to show that $\mathbf{P}(\eta_1+\eta_2 \in x+\Delta) \sim 2\mathbf{P}(\eta_1 \in x+\Delta)$. Consider the following equality:

$$\begin{aligned}(26)\quad \mathbf{P}(\eta_1+\eta_2 \in x+\Delta) &= 2\mathbf{P}(\eta_1 \le x/2, \eta_1+\eta_2 \in x+\Delta) \\ &+ \mathbf{P}(\eta_1 > x/2, \eta_2 > x/2, \eta_1+\eta_2 \in x+\Delta) \equiv 2I_1(x) + I_2(x).\end{aligned}$$

Choose a function $h(x)$ such that $G_H(x+t+\Delta) \sim G_H(x+\Delta)$ holds uniformly in $|t| \le h(x)$ and note that

$$(27)\qquad \int\limits_0^{h(x)} dG_H(y)G_H(x-y+\Delta) \sim G_H(x+\Delta) \quad \text{as} \quad x \to \infty.$$

The mean value of F is finite. Thus, $\overline{F}(t)H((0,t]) = o(1/t)O(t) \to 0$ as $t \to \infty$ and integration by parts yields, for x large enough,

$$\overline{G}_H(x) = \int_x^\infty H((0,t-x])F(dt).$$

Hence,

$$G_H((x,x+1]) \;=\; \int_x^\infty H((t-x-1,t-x])\,F(dt).$$

Therefore, $\int\limits_{h(x)}^{x/2} dG_H(y)G_H(x-y+\Delta) \to 0$ as $x \to \infty$ if $\int\limits_{h(x)}^{x/2} G_H(x-y+\Delta)\overline{F}(y)dy \to 0$ as $x \to \infty$ (we used the last equality and our assumption that $\sup\limits_t H((t,t+1]) \leq b < \infty$). Fix $\varepsilon > 0$. Since $F \in \mathcal{S}^*_\Delta$, there exists x_0 such that, for all $x \geq x_0$,

$$\int_{h(x)}^{x/2} F(x-y+\Delta)\overline{F}(y)dy \leq \varepsilon F(x+\Delta).$$

Then, for $x \geq x_0$,

$$\begin{aligned}
\int_{h(x)}^{x/2} G_H(x-y+\Delta)\overline{F}(y)dy &= \int_{h(x)}^{x/2}\left(\int_0^\infty F(x+t-y+\Delta)H(dt)\right)\overline{F}(y)dy \\
&\leq \int_0^\infty \left(\int_{h(x)}^{\frac{x+t}{2}} F(x+t-y+\Delta)\overline{F}(y)dy\right)H(dt) \\
&\leq \varepsilon \int_0^\infty F(x+t+\Delta)H(dt) = \varepsilon G_H(x+\Delta).
\end{aligned}$$

Using (27), we conclude that $I_1(x) \sim G_H(x+\Delta)$ as $x \to \infty$. Consider now

$$\begin{aligned}
I_2(x) &\leq \int_{x/2}^{x/2+T} G_H(dy)G_H(x-y+\Delta) \\
&= \int_0^{x/2+T} G_H(dy)G_H(x-y+\Delta) - \int_0^{x/2} G_H(dy)G_H(x-y+\Delta).
\end{aligned}$$

Note that

$$\begin{aligned}
\int_0^{x/2+T} G_H(dy)G_H(x-y+\Delta) &\sim \int_0^{x/2+T} G_H(dy)G_H(x-y+2T+\Delta) \\
&\sim G_H(x+2T+\Delta) \sim G_H(x+\Delta)
\end{aligned}$$

as $x \to \infty$, and

$$\int_0^{x/2} G_H(dy)G_H(x-y+\Delta) \sim G_H(x+\Delta)$$

as $x \to \infty$. Therefore, $I_2(x) = o(G_H(x+\Delta))$ and the result follows from (26).

□

Remark 6. We will now show that the condition $\left(\overline{F}(x)\right)^2 = o(F(x+\Delta))$ is essential for the relation (3) to hold. Assume that there exists $0 < y < \infty$ such that $\xi_1 \geq -y$ a.s. Then $\chi \geq -y$ a.s. In this case instead of the upper bound given in (8) we can give the following lower bound:

$$\begin{aligned}
&\mathbf{P}(M_\tau > x, \widetilde{M} > x - \chi) - \mathbf{P}(M_\tau > x, \widetilde{M} > x + t - \chi) \\
&\qquad = \int_0^\infty \mathbf{P}(\chi \in -dz, M_\tau > x)\mathbf{P}(\widetilde{M} \in x + z + \Delta) \\
&\qquad = \frac{T}{|\mathbf{E}\xi|}(1 + o(1)) \int_0^{-y} \mathbf{P}(\chi \in -dz, M_\tau > x)\overline{F}(x + z) \\
&\qquad \geq \frac{T}{|\mathbf{E}\xi|}(1 + o(1))\mathbf{P}(M_\tau > x)\overline{F}(x) \geq \frac{T}{|\mathbf{E}\xi|}\mathbf{E}\tau(1 + o(1))(\overline{F}(x))^2,
\end{aligned}$$

where we used the fact that F belongs to the class $\mathcal{L}_\Delta$ and therefore is long-tailed. Since the rest of the proof of Theorem 1 remains valid in this case, the given upper estimate shows that the asymptotics of $\mathbf{P}(M_\tau \in x + \Delta)$ may be different from (3) if we assume only that $F \in \mathcal{S}^*_\Delta$.

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