The survival probability for critical spread-out oriented percolation above 4 + 1 dimensions. I. Induction

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Abstract

We consider critical spread-out oriented percolation above 4+1 dimensions. Our main result is that the extinction probability at time n (i.e., the probability for the origin to be connected to the hyperplane at time n but not to the hyperplane at time $n+1$) decays like $1/Bn^2$ as $n \to \infty$, where B is a finite positive constant. This in turn implies that the survival probability at time n (i.e., the probability that the origin is connected to the hyperplane at time n) decays like $1/Bn$ as $n \to \infty$. The latter has been shown in an earlier paper to have consequences for the geometry of large critical clusters and for the incipient infinite cluster.

The present paper is Part I in a series of two papers. In Part II, we derive a lace expansion for the survival probability, adapted so as to deal with point-to-plane connections. This lace expansion leads to a nonlinear recursion relation for the survival probability. In Part I, we use this recursion relation to deduce the asymptotics via induction.

1 Introduction and results

For oriented bond percolation on $\mathbb{Z}^d \times \mathbb{Z}_+$ with parameter p, the survival probability $\theta_n = \theta_n(p)$ at time $n \in \mathbb{Z}_+$ is the probability that there exists an $x \in \mathbb{Z}^d$ such that $(0,0)$ is connected to (x,n). In the oriented setting, it is known that there is no percolation at the critical threshold $p = p_c$ [3, 7], so $\lim_{n\to\infty} \theta_n(p_c) = 0$. In this paper, we study the manner in which $\theta_n(p_c)$ tends to zero as $n \to \infty$ when $d > 4$.

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Our main result is that for sufficiently "spread-out" oriented bond percolation, with the degree to which connections are spread out in space parameterized by $L \in \mathbb{N}$,

$$\lim_{n\to\infty} n\theta_n(p_c) = \frac{1}{B} \in (0,\infty) \quad \text{for } d>4 \text{ and for } L \text{ sufficiently large.} \tag{1.1}$$

In terms of the critical exponent ρ, defined by the conjecture that $\theta_n(p_c)$ behaves like $n^{-1/\rho}$ as $n \to \infty$, (1.1) implies that $\rho = 1$. Our proof of (1.1) makes use of a result in Part II ([12]), which consists of an extension of the lace expansion to deal with *point-to-plane connections* and which leads to a *nonlinear recursion relation* for $\theta_n(p)$. In Section 2, we use this recursion relation, together with bounds on its coefficients that are valid when $p = p_c$, $d > 4$ and L sufficiently large, to deduce (1.1) via induction.

The outline of this section is as follows. In Section 1.1, we define spread-out oriented percolation and recall a few basic facts. In Section 1.2, we formulate our main theorem, a sharp asymptotic formula for $\Delta\theta_n(p_c) = \theta_n(p_c)-\theta_{n+1}(p_c)$, which is the probability that extinction occurs at time $n+1$. In Section 1.3, we explain that (1.1) has interesting consequences for the geometry of large critical clusters and for the incipient infinite cluster, as shown in an earlier paper [11]. In Section 1.4, we indicate that our main theorem can be viewed as a perturbation of a sharp asymptotic formula for the extinction probability of a critical branching process. Finally, in Section 1.5, we list the three main ingredients that go into the proof of the main theorem, two of which are treated in Part II.

1.1 The model

The spread-out oriented bond percolation model is defined as follows. Consider the graph with vertices $\mathbb{Z}^d \times \mathbb{Z}_+$ and with directed bonds $((x,n),(y,n+1))$, for $n \in \mathbb{Z}_+ = \{0,1,2,\ldots\}$ and $x, y \in \mathbb{Z}^d$. Let D be a fixed function $D : \mathbb{Z}^d \to [0,1]$, satisfying

$$\sum_{x\in\mathbb{Z}^d} D(x) = 1. \tag{1.2}$$

Let $p \in [0, \|D\|_\infty^{-1}]$, where $\|\cdot\|_\infty$ denotes the supremum norm, so that $pD(x) \le 1$ for all $x \in \mathbb{Z}^d$. We associate to each directed bond $((x,n),(y,n+1))$ an independent random variable taking the value 1 with probability $pD(y-x)$ and the value 0 with probability $1-pD(y-x)$. We say that a bond is *occupied* when the corresponding random variable is 1 and *vacant* when it is 0. Note that p is *not* a probability. Rather, p is the average number of occupied bonds from a given vertex. The joint probability distribution of the bond variables will be denoted by $\mathbb{P}_p$, the corresponding expectation by $\mathbb{E}_p$.

The function D will be assumed to obey the properties of Assumption D in [16, Section 1.2] (whose precise form is not important for the present paper), together with [17, Equation (1.2)]. This assumption involves a parameter $L \in \mathbb{N}$, which serves to spread out the connections and which will be taken to be large. The assumption implies, in particular, that there exists a finite positive constant C such that

$$\sup_{x\in\mathbb{Z}^d} D(x) \le CL^{-d}. \tag{1.3}$$

Examples of functions D obeying the assumption are given in [16, Section 1.2]. A simple and basic example is

$$D(x) = \begin{cases} (2L+1)^{-d} & \text{if } \|x\|_\infty \le L, \\ 0 & \text{otherwise.} \end{cases} \tag{1.4}$$

In this example, the bonds are given by $((x,n),(y,n+1))$ with $\|x-y\|_\infty \le L$, and a bond is occupied with probability $p(2L+1)^{-d}$.

We say that (x,n) is *connected* to (y,m), and write $(x,n) \longrightarrow (y,m)$, if there is an oriented path from (x,n) to (y,m) consisting of occupied bonds. Note that this is only possible when $m \ge n$. By convention, (x,n) is connected to itself. We write $(x,n) \longrightarrow m$ if $m \ge n$ and there is a $y \in \mathbb{Z}^d$ such that $(x,n) \longrightarrow (y,m)$.

The event $\{(0,0) \longrightarrow \infty\}$ is the event that $\{(0,0) \longrightarrow n\}$ occurs for all n. There is a critical threshold $p_c > 0$ such that the event $\{(0,0) \longrightarrow \infty\}$ has probability zero for $p \le p_c$ and has positive probability for $p > p_c$. The parameterization we have chosen is convenient, since p_c is close to 1 for large L. In fact, it is shown in [14] that there is a finite positive constant c such that

$$p_c = 1 + cL^{-d} + \mathcal{O}(L^{-d-1}) \quad \text{as } L \to \infty \text{ for } d > 4. \tag{1.5}$$

The survival probability at time n is defined by

$$\theta_n(p) = \mathbb{P}_p((0,0) \longrightarrow n), \tag{1.6}$$

and the extinction probability at time n is defined by

$$\Delta\theta_n(p) = \theta_n(p) - \theta_{n+1}(p) = \mathbb{P}_p((0,0) \longrightarrow n, (0,0) \not\longrightarrow n+1). \tag{1.7}$$

General results of [3, 7] imply that $\lim_{n\to\infty} \theta_n(p_c) = 0$. The same conclusion was shown in [2] to follow from the triangle condition. The triangle condition was verified in [17, 18], for $d > 4$ and L sufficiently large, yielding an alternate proof that $\lim_{n\to\infty} \theta_n(p_c) = 0$ in this setting.

1.2 The main theorem

Henceforth we will assume that $p = p_c$ and suppress p from the notation.

Our main result is the following theorem. Constants implied by the $\mathcal{O}$ notation below are independent of both L and n. Although only the dimension $d = 5$ lies in the interval $4 < d < 6$, we indicate the d-dependence of our estimate in this range to display its degeneracy as $d \downarrow 4$.

Theorem 1.1. *Let $d > 4$ and $p = p_c$. There are finite positive constants $L_0 = L_0(d)$ and $B = B(d,L) = \frac{1}{2} + \mathcal{O}(L^{-d})$ such that, for $L \ge L_0$,*

$$\Delta\theta_n = \frac{1}{Bn^2}\left[1 + \mathcal{O}(n^{-1}\log n) + L^{-d}\mathcal{O}(\delta_n)\right] \qquad \text{as } n \to \infty \tag{1.8}$$

with

$$\delta_n = \begin{cases} n^{-(d-4)/2}\log n & (4 < d < 6), \\ n^{-1}\log^2 n & (d = 6), \\ n^{-1}\log n & (d > 6). \end{cases} \tag{1.9}$$

From (1.8) we obtain

$$\theta_n = \sum_{m=n}^{\infty} \Delta\theta_m = \frac{1}{Bn}\left[1 + \mathcal{O}(n^{-1}\log n) + L^{-d}\mathcal{O}(\delta_n)\right] \qquad \text{as } n \to \infty, \tag{1.10}$$

which proves (1.1).

In Section 1.5, we sketch the main ingredients in the proof of Theorem 1.1. As explained in Section 1.4, for critical branching processes a result like (1.8) without the second error term is well known. In [17], the scaling behaviour of the critical oriented percolation r-point functions ($r \geq 2$) was computed, for $d > 4$ and L sufficiently large. It was shown that all moment measures of critical oriented percolation converge to the moment measures of the canonical measure of super-Brownian motion in the scaling limit. The latter shows that the nature of large critical spread-out oriented percolation clusters is similar to that of large critical spread-out branching random walk clusters, for $d > 4$. This intuition will guide our proof of Theorem 1.1.

1.3 Consequences of the main theorem

We formulate four consequences of (1.1), which we have seen is a consequence of Theorem 1.1. For this, we first recall some results from [17]. Let

$$\tau_n(x) = \mathbb{P}((0,0) \longrightarrow (x,n)), \tag{1.11}$$
$$\tau_{n_1,n_2}(x_1,x_2) = \mathbb{P}((0,0) \longrightarrow (x_1,n_1),\ (0,0) \longrightarrow (x_2,n_2)) \tag{1.12}$$

denote the two-point and three-point functions, respectively. It follows from [17, Theorem 1.1 and Equation (2.52)] (see [10] for a review) that there are finite positive constants $A = A(d,L)$ and $V = V(d,L)$ such that for $p = p_c$, $d > 4$ and L sufficiently large,

$$\lim_{n\to\infty} \sum_{x\in\mathbb{Z}^d} \tau_n(x) = A, \tag{1.13}$$
$$\sum_{x_1,x_2\in\mathbb{Z}^d} \tau_{n_1,n_2}(x_1,x_2) = A^3 V(n_1 \wedge n_2)[1+o(1)] \qquad \text{as } n_1 \wedge n_2 \to \infty. \tag{1.14}$$

Moreover, $A = 1 + \mathcal{O}(L^{-d})$ and $V = 1 + \mathcal{O}(L^{-d})$ as $L \to \infty$.

I. Relation between constants A, V and B. In [11, Theorem 1.5] we proved that, subject to (1.1),

$$B = \frac{AV}{2}. \tag{1.15}$$

Thus we can now conclude that this formula is true. It follows that $B = \frac{1}{2} + \mathcal{O}(L^{-d})$ as $L \to \infty$. We will provide a direct proof of the latter below, based on the explicit series representation for B given in (2.2). The identity (1.15) is discussed further in Section 1.3 of Part II.

It is worth noting that for $p = p_c$, $d > 4$ and L sufficiently large, (1.13)–(1.14) imply the elementary lower bound

$$\theta_n \geq \frac{1}{AVn}[1+o(1)], \tag{1.16}$$

which gives the correct power of n and only misses the correct constant by a factor 2. To prove (1.16), we let

$$N_n = \#\{x \in \mathbb{Z}^d : (0,0) \longrightarrow (x,n)\} \tag{1.17}$$

denote the number of vertices at time n to which the origin is connected. From the Cauchy–Schwarz inequality, we obtain

$$\mathbb{E}[N_n] = \mathbb{E}\Big[N_n I[N_n > 0]\Big] \leq \Big(\mathbb{E}[N_n^2]\Big)^{1/2} \theta_n^{1/2}. \tag{1.18}$$

Since $\mathbb{E}[N_n] = \sum_{x\in\mathbb{Z}^d} \tau_n(x)$ and $\mathbb{E}[N_n^2] = \sum_{x_1,x_2\in\mathbb{Z}^d} \tau_{n,n}(x_1,x_2)$, it follows from (1.13)–(1.14) that

$$A[1+o(1)] \leq \Big(A^3 V n[1+o(1)]\Big)^{1/2} \theta_n^{1/2}, \tag{1.19}$$

which implies (1.16).

II. Incipient infinite cluster. The formula (1.1) has implications for the incipient infinite cluster (IIC). Let $\mathcal{F}$ denote the σ-algebra of events, and let $\mathcal{F}_0$ denote the algebra of cylinder events, i.e., the events that depend on the occupation status of a finite set of bonds. In [11], we constructed a measure $\mathbb{P}_\infty$, the IIC measure, as follows. We defined $\mathbb{P}_n$ by

$$\mathbb{P}_n(E) = \frac{1}{\tau_n} \sum_{x\in\mathbb{Z}^d} \mathbb{P}(E \cap \{(0,0) \longrightarrow (x,n)\}) \quad (E \in \mathcal{F}), \tag{1.20}$$

where $\tau_n = \sum_{x\in\mathbb{Z}^d} \tau_n(x)$. In [11, Theorem 1.1], we showed that for $p = p_c$, $d > 4$ and L sufficiently large, the limit

$$\mathbb{P}_\infty(E) = \lim_{n\to\infty} \mathbb{P}_n(E) \quad (E \in \mathcal{F}_0) \tag{1.21}$$

exists and extends to a measure on $\mathcal{F}$. A second and more natural construction of the IIC arises when we condition on survival up to time n, as follows. For $E \in \mathcal{F}$, let $\mathbb{Q}_n(E) = \mathbb{P}(E \mid (0,0) \longrightarrow n)$. In [11, Theorem 1.2], we showed that, subject to (1.1), for $p = p_c$, $d > 4$ and L sufficiently large, the limit

$$\mathbb{Q}_\infty(E) = \lim_{n\to\infty} \mathbb{Q}_n(E) \quad (E \in \mathcal{F}_0) \tag{1.22}$$

exists and extends to a measure on $\mathcal{F}$, with $\mathbb{Q}_\infty = \mathbb{P}_\infty$. Thus we now have the following corollary.

Corollary 1.2. *Let $d > 4$ and $p = p_c$. There is a finite positive constant $L_0 = L_0(d)$ such that, for $L \geq L_0$, the measure $\mathbb{Q}_\infty$ exists and equals $\mathbb{P}_\infty$.*

III. Size of survival set. Recall from (1.17) that N_n denotes the number of vertices at time n to which the origin is connected. The following is a consequence of [11, Theorem 1.5] and (1.1).

Corollary 1.3. *Let $d > 4$ and $p = p_c$. There is a finite positive constant $L_0 = L_0(d)$ such that, for $L \geq L_0$, $n^{-1}N_n$ converges weakly to an exponential random variable with parameter $\lambda = 2/(A^2V) = 1/(AB)$, under the measure $\mathbb{Q}_n$ as $n \to \infty$.*

We have already used the fact that $\mathbb{E}[N_n] = A[1 + o(1)]$ by (1.13). According to (1.1) and Corollary 1.3, we can understand this statement to correspond to the two statements

$$\mathbb{P}(N_n > 0) = \frac{1}{Bn}[1+o(1)], \qquad \mathbb{E}[N_n | N_n > 0] = ABn[1+o(1)]. \tag{1.23}$$

In other words, clusters rarely survive to time n, but when they do, they are large.

IV. Critical exponent for size of cluster of origin. Let $p = p_c$. Let $C(x,n) = \{(y,m) \in \mathbb{Z}^d \times \mathbb{Z}_+ : (x,n) \longrightarrow (y,m)\}$ denote the forward cluster of (x,n), of cardinality $|C(x,n)|$. Let

$$P_n = \mathbb{P}(|C(0,0)| = n), \tag{1.24}$$

and let

$$P_{\geq n} = \sum_{m=n}^{\infty} P_m = \mathbb{P}(|C(0,0)| \geq n) \tag{1.25}$$

denote the probability that the size of the cluster of the origin is at least n. For $h \geq 0$, let $M(h) = 1 - \sum_{n=1}^{\infty} P_n e^{-hn}$. The critical exponent δ is defined by the conjectured asymptotic relation $M(h) \sim \text{const} \cdot h^{1/\delta}$ as $h \downarrow 0$, or, in a stronger statement, by $P_n \sim \text{const} \cdot n^{-1-1/\delta}$, which implies that $P_{\geq n} \sim \text{const} \cdot n^{-1/\delta}$. It is known quite generally that $M(h) \geq \text{const} \cdot h^{1/2}$ for $h \geq 0$ [1]. For $d > 4$ and L sufficiently large, it is a consequence of the triangle condition that also $M(h) \leq \text{const} \cdot h^{1/2}$ [2, 18], so that $M(h) \simeq h^{1/2}$ (where "$\simeq$" denotes upper and lower bounds with possibly different constants), and thus $\delta = 2$ in this sense. It is also known that $\frac{dM}{dh} = \sum_{n=1}^{\infty} n P_n e^{-hn} \simeq h^{-1/2}$ [2, 18] for $d > 4$ and L sufficiently large. The following corollary to (1.1) gives a somewhat different statement that $\delta = 2$, and is proved directly without invoking the triangle condition.

Corollary 1.4. *Let $d > 4$ and $p = p_c$. There are finite positive constants $L_0 = L_0(d)$ and $c_i = c_i(d, L)$, $i = 1, 2$, such that, for $L \geq L_0$,*

$$\frac{c_1}{\sqrt{n}} \leq P_{\geq n} \leq \frac{c_2}{\sqrt{n}} \quad (n \geq 1). \tag{1.26}$$

The proof is given in Section 3. For ordinary (non-oriented) nearest-neighbour bond percolation in dimensions $d \gg 6$, the asymptotic formula $\mathbb{P}(|C(0)| = n) \sim \text{const} \cdot n^{-3/2}$ was proved in [9, Theorem 1.1], where $C(0)$ denotes the cluster of the origin. Our present methods are not sufficient to prove the corresponding statement for oriented percolation for $d > 4$, which would imply an asymptotic formula in place of (1.26).

1.4 Critical branching processes

Above the critical dimension 4, the connectivity functions of critical oriented percolation have been shown to have the same scaling as their analogues for critical branching random walk [17]. It is therefore natural to expect that the same will be true for the survival probability, and our analysis is based on a comparison of the recursion relation (1.37) with its counterpart for critical branching random walk, or, equivalently, the survival probability for critical branching processes.

In this section, we derive the analogue of (1.8) for critical branching processes. Consider a branching process with a critical offspring distribution $\hat{q} = (\hat{q}_m)_{m=0}^{\infty}$, i.e.,

$$\mu_{\hat{q}} = \sum_{m=0}^{\infty} m \hat{q}_m = 1. \tag{1.27}$$

Let $\sigma_{\hat{q}}^2$ denote the variance of $\hat{q}$, which we assume is positive and finite. By (1.27),

$$\sigma_{\hat{q}}^2 = \sum_{m=0}^{\infty} m(m-1)\hat{q}_m. \tag{1.28}$$

We write $\hat{\mathbb{P}}$ for the law of the critical branching process, Z_n for the number of particles alive at time n, and we let

$$\hat{\theta}_n = \hat{\mathbb{P}}(Z_n > 0) \tag{1.29}$$

denote the survival probability at time n. By conditioning on the number of offspring of the initial particle that survive to time $n+1$, and assuming for simplicity that the third moment of $\hat{q}$ is finite as well, we obtain the recursion relation

$$\hat{\theta}_{n+1} = \sum_{m=1}^{\infty} m\hat{q}_m \hat{\theta}_n (1-\hat{\theta}_n)^{m-1} + \sum_{m=2}^{\infty} \frac{m(m-1)}{2} \hat{q}_m \hat{\theta}_n^2 (1-\hat{\theta}_n)^{m-2} + \mathcal{O}(\hat{\theta}_n^3). \tag{1.30}$$

We expand the power of $1-\hat{\theta}_n$ in (1.30) to obtain

$$\begin{aligned} \hat{\theta}_{n+1} &= \sum_{m=1}^{\infty} m\hat{q}_m [\hat{\theta}_n - (m-1)\hat{\theta}_n^2] + \sum_{m=2}^{\infty} \frac{m(m-1)}{2} \hat{q}_m \hat{\theta}_n^2 + \mathcal{O}(\hat{\theta}_n^3) \\ &= \hat{\theta}_n - \frac{\sigma_{\hat{q}}^2}{2} \hat{\theta}_n^2 + \mathcal{O}(\hat{\theta}_n^3). \end{aligned} \tag{1.31}$$

Note the cancellation that results in a negative coefficient for the quadratic term in (1.31).

From (1.31) it is straightforward to deduce that

$$\hat{\theta}_n = \frac{2}{\sigma_{\hat{q}}^2 n} [1 + \mathcal{O}(n^{-1} \log n)], \tag{1.32}$$

which is the analogue of (1.10). Indeed, following [5, Section 8.5], we put $\hat{v}_n = 1/\hat{\theta}_n$ and note that (1.31) yields the recursion relation

$$\hat{v}_{n+1} = \frac{\hat{v}_n}{1 - \frac{\sigma_{\hat{q}}^2}{2} \hat{v}_n^{-1} + \mathcal{O}(\hat{v}_n^{-2})} = \hat{v}_n + \frac{\sigma_{\hat{q}}^2}{2} + \mathcal{O}(\hat{v}_n^{-1}). \tag{1.33}$$

For later reference (see under (2.28) below), we note also that the constant in the error term of (1.33) depends only on the third moment of $\hat{q}$. It is a classical result that $\hat{\theta}_n \to 0$, and hence $\hat{v}_n \to \infty$. Using the latter in the right-hand side of (1.33), we obtain

$$\hat{v}_{n+1} = \frac{\sigma_{\hat{q}}^2 n}{2} [1 + o(1)]. \tag{1.34}$$

Inserting this into (1.33), we get the recursion

$$\hat{v}_{n+1} = \hat{v}_n + \frac{\sigma_{\hat{q}}^2}{2} + \mathcal{O}(n^{-1}). \tag{1.35}$$

From this, in turn, we obtain

$$\hat{v}_n = \frac{\sigma_{\hat{q}}^2 n}{2} + \mathcal{O}(\log n), \tag{1.36}$$

which proves (1.32).

1.5 Main ingredients in the proof

There are three main ingredients in the proof of Theorem 1.1. The first two are proved in Part II, and we prove the third here in Part I.

A. The first ingredient is the derivation of a *recursion relation* for θ_n that replaces the simple recursion (1.31) for branching processes. We do this in Part II by extending the lace expansion from an expansion for the two-point function (a point-to-point expansion) to an expansion for the survival probability (a point-to-plane expansion). It turns out that this is *not* a minor modification of previous lace expansions. The result of the expansion is that the recursion relation (1.31) for branching processes is replaced by the following recursion relation for oriented percolation:

$$\theta_n = \sum_{m=0}^{n-1} \pi_m p \theta_{n-1-m} - \sum_{m_1=1}^{\lfloor n/2 \rfloor} \sum_{m_2=m_1}^{n} \phi_{m_1,m_2} \theta_{n-m_1} \theta_{n-m_2} + e_n. \tag{1.37}$$

Here, (π_m) are the coefficients appearing in the lace expansion for the *two-point function* [17, 18] (in terms of the notation $\pi_m(x)$ of [17, Section 3], we have $\pi_0 = 1$, $\pi_1 = 0$, and $\pi_m = \sum_{x \in \mathbb{Z}^d} \pi_m(x)$ for $m \geq 2$), (ϕ_{m_1,m_2}) are certain coefficients similar to those appearing in the lace expansion for the *three-point function* [17], while (e_n) are error terms. The expansion (1.37) holds rather generally, but to be useful it is necessary to have estimates for the coefficients appearing in its right-hand side.

B. The second ingredient in the proof consists of estimates on π_m, ϕ_{m_1,m_2} and e_n, which we refer to as *diagrammatic estimates.* These diagrammatic estimates are valid for $p = p_c$, $d > 4$ and L sufficiently large. Diagrammatic estimates for ϕ_{m_1,m_2} and e_n are obtained in Part II, and good diagrammatic estimates for π_m are known already from [17, Proposition 2.2]. Also, it was shown in [17, Section 2.1.2] that

$$p_c \sum_{m=0}^{\infty} \pi_m = 1. \tag{1.38}$$

We may think of (1.38) as an analogue of (1.27). The following theorem summarises the diagrammatic bounds. Here, and throughout the paper, we use the abbreviation

$$\beta = L^{-d}. \tag{1.39}$$

The dimension d enters our analysis only as a parameter and not with any geometric meaning. To emphasise this, and to facilitate the possible extension of our analysis to models other than oriented percolation, we replace the parameter d by

$$\kappa = \frac{d}{2}, \tag{1.40}$$

and we assume that $\kappa > 2$. We replace (1.9) by

$$\delta_n = \begin{cases} \frac{1}{3-\kappa} n^{-(\kappa-2)} \log n & (2 < \kappa < 3), \\ n^{-1} \log^2 n & (\kappa = 3), \\ n^{-1} \log n & (\kappa > 3), \end{cases} \tag{1.41}$$

and also define

$$\Delta_n = \begin{cases} n^{-(\kappa-2)} \log n & (2 < \kappa < 3), \\ n^{-1} \log n & (\kappa = 3), \\ n^{-1} & (\kappa > 3). \end{cases} \tag{1.42}$$

Part (i) of the following theorem is proved in [17], and parts (ii-iii) are proved in [12].

Theorem 1.5. (Diagrammatic estimates)
Fix $d > 4$, $\kappa = d/2$, and $p = p_c$. There are positive constants C_π, C_ϕ, C_e and β_0 such that for $0 < \beta \leq \beta_0$ the following hold:
(i) $\pi_0 = 1$, $\pi_1 = 0$ and

$$|\pi_m| \leq C_\pi \beta (m+1)^{-\kappa} \qquad (m \geq 2). \tag{1.43}$$

(ii) $\phi_{1,1} = \frac{1}{2} p_c^2 \sum_{x \in \mathbb{Z}^d} D(x)(1 - D(x)) = \frac{1}{2}[1 + \mathcal{O}(\beta)]$ and

$$|\phi_{m_1,m_2}| \leq C_\phi \beta (m_1+1)^{-(\kappa-1)} (m_2 - m_1 + 1)^{-(\kappa-1)} \qquad (m_2 \geq m_1 \geq 1,\ m_1 + m_2 \geq 3). \tag{1.44}$$

(iii) If $\theta_m \leq C_\theta (m+1)^{-1}$ for $0 \leq m \leq n$ and some $C_\theta \geq 1$, then

$$|e_{n+1}| \leq C_e C_\theta^3 (n+1)^{-2} \left[(n+1)^{-1} + \beta \Delta_{n+1} \right]. \tag{1.45}$$

C. The third ingredient in the proof is an *inductive analysis* of the recursion relation (1.37), using the diagrammatic estimates of Theorem 1.5 to bound the coefficients in (1.37). The inductive analysis is carried out in Section 2 and is the main content of Part I. Note that the diagrammatic estimate in (1.45) for e_{n+1}, which is the error term in (1.37) for θ_{n+1}, assumes a bound for θ_m with $0 \leq m \leq n$. *This is precisely what opens up the possibility of an inductive analysis.* The recursion relation (1.37) is a *nonlinear* equation for θ_n. Our induction hypothesis is on $v_n = 1/\theta_n$, as in Section 1.4. From the induction we will conclude that $v_n = Bn[1 + \mathcal{O}(n^{-1}\log n) + \beta\mathcal{O}(\delta_n)]$, which will in turn imply the result (1.8) for $\Delta\theta_n$.

The initialisation of the induction in ingredient C uses specific properties of oriented percolation, but the advancement of the induction uses only the recursion relation (1.37) and the estimates of Theorem 1.5, and does not otherwise use specific properties of oriented percolation. Because of this *model-independent* aspect, our analysis can potentially serve to study the extinction probability for other models as well, such as critical unoriented percolation, lattice trees, and the critical contact process. In particular, a program to apply the lace expansion to the critical spread-out contact process on $\mathbb{Z}^d$ for $d > 4$ was initiated in [19], and extended in [13, 15], via an approximation by critical spread-out oriented percolation. We expect that it is possible to combine our methods with those of [15] to prove the analogue of Theorem 1.1, and hence also the analogue of the asymptotic formula (1.10) for the survival probability, for the critical spread-out contact process in dimensions $d > 4$.

For the voter model, the survival probability is the probability that the opinion of the origin survives to time t when initially all other vertices hold the opposite opinion. Methods quite different from ours have been used to prove the analogue of (1.1) for the voter model for all dimensions $d \geq 2$ (with a logarithmic correction when $d = 2$) [4, 6]. However, these methods do not have a known extension to the critical contact process or critical oriented percolation.

2 The induction analysis

In this section we prove Theorem 1.1, subject to (1.37) and Theorem 1.5. We use the bounds on the coefficients in the recursion relation (1.37), given in Theorem 1.5, to deduce the asymptotics in (1.8) via induction on n. The induction is carried out on the quantity

$$v_n = \frac{1}{\theta_n}. \tag{2.1}$$

Since $\theta_n \to 0$ as $n \to \infty$, we know that $v_n \to \infty$ as $n \to \infty$.

The outline of this section is as follows. In Section 2.1, we formulate our induction hypothesis. In Section 2.2, we initialise the induction by comparing critical oriented percolation with branching random walk. Finally, in Section 2.3, we advance the induction.

We assume throughout this section that $\kappa > 2$, which for oriented percolation is the statement that $d > 4$. Also, we fix $p = p_c$ throughout this section.

2.1 The induction hypothesis

In the course of the induction, we will show that the constant B in Theorem 1.1 is given by

$$B = \frac{\sum_{m_1=1}^{\infty} \sum_{m_2=m_1}^{\infty} \phi_{m_1,m_2}}{1 + p_c \sum_{m=2}^{\infty} m\pi_m}. \tag{2.2}$$

By Theorem 1.5(i–ii), $B < \infty$ for $d > 4$ and L sufficiently large, with

$$B = \frac{1}{2} + \mathcal{O}(\beta) \quad \text{as} \quad \beta \downarrow 0. \tag{2.3}$$

The formula (2.2) can be guessed from the following rough calculation, in which '$\approx$' denotes an uncontrolled approximation. Let $\Phi = \sum_{m_1=1}^{\infty} \sum_{m_2=m_1}^{\infty} \phi_{m_1,m_2}$. We first approximate (1.37) by

$$\theta_n \approx \sum_{m=0}^{n-1} \pi_m p_c \theta_{n-1-m} - \Phi \theta_n^2. \tag{2.4}$$

We then replace θ_j by $(Bj)^{-1}$, use

$$\frac{1}{n-1-m} = \frac{1}{n} + \frac{m+1}{n(n-1-m)} \approx \frac{1}{n} + \frac{m+1}{n^2}, \tag{2.5}$$

and, recalling (1.38) and $\pi_1 = 0$, use $\sum_{m=0}^{n-1} \pi_m p_c \approx 1$ and $\sum_{m=0}^{n-1} (m+1)\pi_m p_c \approx 1 + \sum_{m=2}^{\infty} m\pi_m p_c$. This leads to

$$\frac{1}{Bn} \approx \frac{1}{Bn} + \frac{1}{Bn^2}\left(1 + \sum_{m=2}^{\infty} m\pi_m p_c\right) - \frac{\Phi}{(Bn)^2}, \tag{2.6}$$

and (2.2) results after we solve for B. Note the cancellation of the first-order term in (2.6), due to (1.38).

Our induction hypothesis is the following analogue of (1.35):

(IH) There are $K, \widetilde{K} > 0$ (independent of β) and $\beta_0 > 0$ such that

$$|v_j - v_{j-1} - B| \leq K(j+1)^{-1} + \widetilde{K}\beta\Delta_{j+1} \qquad (1 \leq j \leq n,\ 0 < \beta \leq \beta_0), \tag{2.7}$$

with Δ_{j+1} given by (1.42), B given by (2.2), and $v_0 = 1$.

Note that

$$\sum_{j=1}^{n} \Delta_{j+1} \leq (n+1)\delta_{n+1}, \tag{2.8}$$

with δ_n defined in (1.41). It follows from (2.7)–(2.8) that

$$\begin{aligned} |v_n - Bn| &\le v_0 + \sum_{j=1}^{n} |v_j - v_{j-1} - B| \\ &\le 1 + K\log(n+1) + \widetilde{K}\beta \sum_{j=1}^{n} \Delta_{j+1} \\ &\le 2K\log(n+1) + \widetilde{K}\beta(n+1)\delta_{n+1} \end{aligned} \tag{2.9}$$

(when K is not too small). This says that

$$v_n = Bn[1 + \mathcal{O}(n^{-1}\log n) + \beta\mathcal{O}(\delta_n)], \tag{2.10}$$

which is (1.10). Combining (2.7), the inequality $\Delta_{n+1} \le \delta_{n+1}$ and (2.10) with the relation

$$\Delta\theta_n = \theta_n - \theta_{n+1} = \frac{1}{v_n v_{n+1}}(v_{n+1} - v_n) = \frac{1}{v_n v_{n+1}}[B + (v_{n+1} - v_n - B)], \tag{2.11}$$

we get (1.8).

Thus, to prove Theorem 1.1, it suffices to initialise and advance the induction hypothesis (IH). The initialisation is via the following proposition, which is proved in Section 2.2. Proposition 2.1 shows that (IH) holds for $1 \le n \le N_0$, for a suitable choice of constants, and with β small enough (depending on N_0).

Proposition 2.1. (Initialisation of the induction)
Fix $d > 4$ and $p = p_c$. There are constants K_0 and β_1 such that for every $1 \le N_0 < \infty$ there exists a $\widetilde{K}_0 = \widetilde{K}_0(N_0) > 0$ such that (IH) holds for all $K \ge K_0$, $\widetilde{K} \ge \widetilde{K}_0$, $1 \le n \le N_0$ and $0 < \beta \le \beta_1$.

Note that to prove Proposition 2.1 it suffices to obtain (IH) with $K = K_0$ and $\widetilde{K} = \widetilde{K}_0$, by the monotonicity of (2.7) in K and $\widetilde{K}$.

It is important to initialise the induction for all $1 \le n \le N_0$, with N_0 large, for two reasons. First, for the advancement of the induction, we find it useful to start from $n = N_0$ with N_0 large, since this allows us to only keep track of leading order terms in n while being generous with constants. More importantly, the bound (1.45) on the error term e_{n+1} is not useful unless n is large compared to C_θ and C_3, and indeed unless we have the existence of the constant C_θ of Theorem 1.5(iii). We will need large n in this regard.

The following proposition, which is proved in Section 2.3, makes a choice of N_0 and advances the induction to all $n > N_0$.

Proposition 2.2. (Advancement of the induction)
Fix $d > 4$ and $p = p_c$. There are constants K, $N_0 = N_0(K)$, $\widetilde{K} = \widetilde{K}(K)$ and $\beta_2 = \beta_2(K, \widetilde{K})$ such that if (IH) holds for all $1 \le j \le n$, for some $n \ge N_0$, for all $0 < \beta \le \beta_2$ and with the constants K and $\widetilde{K}$, then (IH) holds for $n+1$ with the same constants.

Note that we are free also to require that the constants K and $\widetilde{K}$ of Proposition 2.2 obey $K \ge K_0$ and $\widetilde{K} \ge \widetilde{K}_0$, with K_0 and $\widetilde{K}_0$ given by Proposition 2.1. The various constants will be chosen in the following order:

(∗) First $K \geq K_0$ is chosen according to Propositions 2.1–2.2, next N_0 is chosen large (depending on K), next $\widetilde{K} \geq \widetilde{K}_0$ is chosen large (depending on K and N_0), and finally L_0 is chosen so large (depending on K, $\widetilde{K}$ and hence on N_0) that $\beta = L^{-d} \leq \min\{\beta_0, \beta_1, \beta_2\}$ for $L \geq L_0$, where the β_i are the constants of Theorem 1.5 and Propositions 2.1–2.2.

Together, Propositions 2.1–2.2 imply that (IH) holds for all n, with a suitable choice of constants. Thus our remaining task is to prove Propositions 2.1–2.2. The proof of Proposition 2.1 is via a comparison of oriented percolation with branching random walk, and is model-dependent. The advancement of the induction is a model-independent argument that relies only on (1.37) and Theorem 1.5, provided that $\kappa > 2$.

2.2 Initialisation of the induction

In this section, we prove Proposition 2.1 by showing that for any fixed N_0, (IH) holds for $1 \leq n \leq N_0$, provided we choose $\widetilde{K}_0$ depending on N_0. For this, we will make use of *three* related branching random walk models.

We first define a *critical branching random walk*, with law $\hat{\mathbb{P}}$ and offspring distribution $\hat{q}$, as follows. An initial particle at the origin gives birth to a particle at x with probability $D(x)$, for each $x \in \mathbb{Z}^d$, after which it dies. In the next time step, each particle at x gives birth to a particle at y with probability $D(y-x)$, for each $y \in \mathbb{Z}^d$, after which it dies, etc. Thus, the number of offspring per particle is a random variable

$$X = \sum_{x \in \mathbb{Z}^d} I_x, \tag{2.12}$$

where I_x $(x \in \mathbb{Z}^d)$ are independent Bernoulli random variables with

$$\hat{\mathbb{P}}(I_x = 1) = D(x). \tag{2.13}$$

By (1.2)–(1.3),

$$\mu_{\hat{q}} = \hat{\mathbb{E}}[X] = \sum_{x \in \mathbb{Z}^d} D(x) = 1, \tag{2.14}$$

$$\sigma^2_{\hat{q}} = \hat{\mathbb{E}}[X^2] - (\hat{\mathbb{E}}[X])^2 = \sum_{x \in \mathbb{Z}^d} D(x)(1 - D(x)) = 1 + \mathcal{O}(\beta). \tag{2.15}$$

Let Z_j denote the number of particles alive at time j, and let

$$\hat{\theta}_j = \hat{\mathbb{P}}(Z_j > 0). \tag{2.16}$$

We next define a *supercritical branching random walk*, with law $\mathbb{P}^*$ and offspring distribution q^*, by replacing $D(x)$ by $p_c D(x)$ in (2.13), i.e.,

$$\mathbb{P}^*(I_x = 1) = p_c D(x). \tag{2.17}$$

As in (2.14)–(2.15), we have (recall (1.5))

$$\mu_{q^*} = \sum_{x \in \mathbb{Z}^d} p_c D(x) = p_c = 1 + \mathcal{O}(\beta), \qquad \sigma^2_{q^*} = \sum_{x \in \mathbb{Z}^d} p_c D(x)(1 - p_c D(x)) = 1 + \mathcal{O}(\beta), \tag{2.18}$$

and, as in (2.16), we define

$$\theta_j^* = \mathbb{P}^*(Z_j > 0). \tag{2.19}$$

It is an elementary fact (see, e.g., [8, p. 172]) that

$$\mathbb{E}^*[Z_j] = \mu_{q^*}^j, \qquad \mathbb{E}^*[Z_j^2] - (\mathbb{E}^*[Z_j])^2 = \sigma_{q^*}^2 \, \frac{\mu_{q^*}^{j-1}(\mu_{q^*}^j - 1)}{\mu_{q^*} - 1}, \tag{2.20}$$

where we use that $\mu_{q^*} = p_c > 1$ (recall (1.5)).

The supercritical branching random walk in the previous paragraph is closely related to critical oriented percolation, but with the important difference that particles can coexist at the same vertex in the supercritical branching random walk, whereas in oriented percolation each vertex contains at most one particle. However, we can think of oriented percolation as corresponding to a *supercritical branching random walk with killing*, as follows. An initial particle at the origin gives birth to a particle at x with probability $p_c D(x)$, for each $x \in \mathbb{Z}^d$, after which it dies. In the next time step, each particle at x gives birth to a particle at y with probability $p_c D(y-x)$, for each $y \in \mathbb{Z}^d$, after which it dies, but *if two or more particles land on the same vertex in $\mathbb{Z}^d$ then all but one are killed.* Each particle in the resulting configuration generates its offspring and dies, but again at each vertex all but one of the particles are killed, etc. If $\mathbb{P}$ denotes the law of the *supercritical branching random walk with killing* that is thus obtained, then by definition the survival probability at time j of our oriented percolation model is given by

$$\theta_j = \mathbb{P}(Z_j > 0). \tag{2.21}$$

From this representation we immediately obtain the sandwich

$$\mathbb{P}^*(Z_j > 0, T > j) \le \theta_j \le \mathbb{P}^*(Z_j > 0), \tag{2.22}$$

where $\mathbb{P}^*$ is the law of the *supercritical branching random walk without killing* and T denotes the first time that two particles meet at the same vertex. Note that $\mathbb{P}^*(T=1) = 0$ because the initial particle at the origin puts at most one child at a vertex. We will estimate the upper and lower bounds in (2.22) using the following two lemmas.

Lemma 2.3. *The branching random walks with offspring distributions $\hat{q}$ and q^* are related by*

$$\hat{\theta}_j \le \theta_j^* \le p_c^j \hat{\theta}_j, \qquad j \in \mathbb{Z}_+. \tag{2.23}$$

Proof. Since $p_c \ge 1$ by (1.5), the lower bound in (2.23) is trivial (see (2.13) and (2.17)). The upper bound is proved as follows. For $i \in \mathbb{Z}_+$, let $I_x^{(i)}$ denote independent copies of I_x ($x \in \mathbb{Z}^d$) with law (2.17). For $\omega : 0 \to j$ an oriented path of length j connecting 0 to level j, let

$$\begin{aligned} E^{(j)}(\omega) &= \bigcap_{i=1}^{j} \{I^{(i-1)}_{\omega(i)-\omega(i-1)} = 1\}, \\ E^{(j),<}(\omega) &= \bigcap_{\omega' < \omega} [E^{(j)}(\omega')]^c, \end{aligned} \tag{2.24}$$

where $\omega' < \omega$ means that ω' is lexicographically smaller than ω, i.e., $\omega'(k) < \omega(k)$ when k is the first time at which the two paths disagree. In terms of these quantities, we have

$$\begin{aligned}
\mathbb{P}^*(Z_j > 0) &= \sum_{\omega:0\to j} \mathbb{P}^*\Big(E^{(j)}(\omega) \cap E^{(j),<}(\omega)\Big) \\
&= \sum_{\omega:0\to j} \mathbb{P}^*\Big(E^{(j),<}(\omega) \mid E^{(j)}(\omega)\Big)\mathbb{P}^*\Big(E^{(j)}(\omega)\Big) \\
&= \sum_{\omega:0\to j} \mathbb{P}^*\Big(E^{(j),<}(\omega) \mid E^{(j)}(\omega)\Big)\ p_c^j \prod_{i=1}^{j} D(\omega(i)-\omega(i-1)) \qquad (2.25)\\
&\le \sum_{\omega:0\to j} \hat{\mathbb{P}}\Big(E^{(j),<}(\omega) \mid E^{(j)}(\omega)\Big)\ p_c^j \hat{\mathbb{P}}\Big(E^{(j)}(\omega)\Big) \\
&= p_c^j \hat{\mathbb{P}}(Z_j > 0),
\end{aligned}$$

where the inequality comes from the fact that $E^{(j),<}(\omega)$ is a decreasing function of each I_x. □

Lemma 2.4. *For each $1 \le N_0 < \infty$, there is a positive constant $C(N_0)$ such that $\mathbb{P}^*(T \le j) \le C(N_0)\beta$ for all $2 \le j \le N_0$.*

Proof. Let S_j denote the set of vertices where particles live at time j. Then

$$\begin{aligned}
\mathbb{P}^*(T \le j) &= \sum_{k=2}^{j} \mathbb{P}^*(T = k) = \sum_{k=2}^{j}\sum_{A\neq\emptyset} \mathbb{P}^*(S_{k-1} = A, T = k) \\
&= \sum_{k=2}^{j}\sum_{A\neq\emptyset} \mathbb{P}^*(S_{k-1} = A, T > k-1)\ \mathbb{P}^*(T = k \mid S_{k-1} = A, T > k-1) \\
&\le \sum_{k=2}^{j}\sum_{A\neq\emptyset} \mathbb{P}^*(S_{k-1} = A) \sum_{x_1,x_2\in A, x_1\neq x_2}\ \sum_{y\in\mathbb{Z}^d} p_c D(y-x_1) p_c D(y-x_2) \qquad (2.26)\\
&\le \sum_{k=2}^{j}\sum_{A\neq\emptyset} \mathbb{P}^*(S_{k-1} = A)\ C^2 p_c^2 \beta\ |A|(|A|-1) \\
&= C^2 p_c^2 \beta \sum_{k=2}^{j} \mathbb{E}^*(Z_{k-1}(Z_{k-1}-1)),
\end{aligned}$$

where the last inequality uses (1.3). Substituting (2.18)–(2.20) into (2.26) and using the inequality $(\mu_q^l - 1)/(\mu_q - 1) = \sum_{i=0}^{l-1} \mu_q^i \le l\mu_q^{l-1}$, we arrive at $\mathbb{P}^*(T \le j) \le C\beta j^2$ for some $C > 0$. This proves the lemma with $C(N_0) = CN_0^2$. □

With the above preliminaries, we are now able to prove Proposition 2.1.

Proof of Proposition 2.1. Recall from (2.1) that $v_j = 1/\theta_j$, and let

$$\hat{v}_j = \frac{1}{\hat{\theta}_j}. \qquad (2.27)$$

By the triangle inequality,

$$|v_j - v_{j-1} - B| \le |v_j - \hat{v}_j| + |v_{j-1} - \hat{v}_{j-1}| + |B - \tfrac{\sigma_{\hat{q}}^2}{2}| + |\hat{v}_j - \hat{v}_{j-1} - \tfrac{\sigma_{\hat{q}}^2}{2}|. \qquad (2.28)$$

For the fourth term, we first observe that it is easily verified that the third moment of $\hat{q}$ is bounded by a universal constant. By (1.33)–(1.35) (and the comment below (1.33)), the fourth term is therefore at most $K_0(j+1)^{-1}$ for $j \geq 1$ and some $0 < K_0 < \infty$, where K_0 is a universal constant. Hence it is also at most $K(j+1)^{-1}$ for any $K \geq K_0$.

The third term on the right-hand side is at most $C'\beta$ for some $C' > 0$, since $\sigma_{\hat{q}}^2 = 1 + \mathcal{O}(\beta)$ and $B = \frac{1}{2} + \mathcal{O}(\beta)$. For the first and second terms, we fix $1 \leq N_0 < \infty$, let $1 \leq j \leq N_0$, and write $v_j - \hat{v}_j = (\hat{\theta}_j - \theta_j)/\theta_j\hat{\theta}_j$. It can be seen from (2.22) and Lemmas 2.3–2.4 that

$$\theta_j \geq \hat{\theta}_j - C(N_0)\beta. \tag{2.29}$$

In addition, it follows from (1.32) that, for small β, $\hat{\theta}_n$ is eventually close to a β-independent constant multiplied by n^{-1}. By the monotonicity of $\hat{\theta}_j$ in j, this implies that $\hat{\theta}_j$ is bounded below by an N_0-dependent positive constant, uniformly in $0 < \beta \leq \beta_1$ with β_1 small enough, and in $1 \leq j \leq N_0$. Therefore, $\theta_j\hat{\theta}_j \geq 1/C(N_0)$ for $1 \leq j \leq N_0$ and some $C(N_0) > 0$, and hence, once we prove that

$$|\theta_j - \hat{\theta}_j| \leq C(N_0)\beta \qquad (1 \leq j \leq N_0), \tag{2.30}$$

(2.7) follows if $\widetilde{K} \geq \widetilde{K}_0 = (C' + 2C(N_0)^2)\Delta_{N_0+1}^{-1}$. To prove (2.30), we combine (2.29) with (2.22)–(2.23) to obtain

$$\hat{\theta}_j - C(N_0)\beta \leq \theta_j \leq p_c^j\hat{\theta}_j. \tag{2.31}$$

Since $p_c = 1 + \mathcal{O}(\beta)$, (2.30) now follows. □

2.3 Advancement of the induction

In this section, we prove Proposition 2.2 by showing that the induction hypothesis (IH) can be advanced from $n = N_0$ onwards when N_0 is chosen large enough. In the proof, we assume that $\kappa > 2$, and make use of (1.37) and Theorem 1.5, but we not otherwise use specific properties of oriented percolation.

To begin, we recall the definition of B in (2.2), and define

$$B_{n+1} = \frac{\sum_{m_1=1}^{\lfloor (n+1)/2 \rfloor} \sum_{m_2=m_1}^{n+1} \phi_{m_1,m_2}}{1 + p_c \sum_{m=2}^{n+1} m\pi_m}, \qquad \partial B_{n+1} = B - B_{n+1}. \tag{2.32}$$

We also define

$$u_{n+1} = \frac{\theta_n - \theta_{n+1}}{\theta_n}, \tag{2.33}$$

and we write

$$v_{n+1} - v_n = v_n \frac{u_{n+1}}{1 - u_{n+1}}. \tag{2.34}$$

The main step in the advancement of (IH) will be to prove the following proposition.

Proposition 2.5. (Key estimate for advancement of the induction)
Let K and $\widetilde{K}$ be the constants of (IH). For every N_0 sufficiently large depending on K, there exists a $\beta_3 = \beta_3(\widetilde{K})$ such that if (IH) holds for some $n \geq N_0$ and for all $0 < \beta \leq \beta_3$, with these constants K and $\widetilde{K}$, then

$$|v_n u_{n+1} - B_{n+1}| \leq 600 B^2 C_e (n+1)^{-1} + C(K)\beta\Delta_{n+1}, \tag{2.35}$$

where the constant $C(K)$ depends only on K, and where C_e is the constant of (1.45).

Before proving Proposition 2.5, we first show how it can be used to prove Proposition 2.2. Recall that our choice of constants is taken in the order indicated in item $(*)$ in Section 2.1. The constants $C_\pi, C_\phi, C_e, C_\theta$ are the constants of Theorem 1.5. We use C to denote a generic constant whose value may change from line to line. If C depends on variables such as K or N_0, then we make this explicit by writing $C = C(K, N_0)$. Otherwise, C denotes a constant that is independent of K, $\widetilde{K}$, N_0 and β.

Note that it follows from (2.9), and hence from (IH), that

$$|v_n - Bn| \leq Bn \left[\frac{2K}{Bn} \log(n+1) + \frac{\widetilde{K}\beta}{B} \Big(1 + \frac{1}{n}\Big) \delta_{n+1} \right]. \tag{2.36}$$

Thus, if we choose $n \geq N_0(K)$ sufficiently large, and β sufficiently small depending on $N_0(K)$, then

$$\frac{1}{2} Bn \leq v_n \leq 2Bn. \tag{2.37}$$

Proof of Proposition 2.2. (Advancement of (IH)). Suppose that (2.7) holds for all $1 \leq j \leq n$. By (2.32) and (2.34),

$$v_{n+1} - v_n - B = \frac{1}{1 - u_{n+1}} \left\{ -\partial B_{n+1} + (v_n u_{n+1} - B_{n+1}) + B u_{n+1} \right\}. \tag{2.38}$$

From (1.43)–(1.44), (2.2) and (2.32), it is easily deduced that

$$|\partial B_{n+1}| \leq C(C_\pi + C_\phi)\beta(n+1)^{-(\kappa-2)}. \tag{2.39}$$

This term is of smaller order in n than the second term on the right-hand side of (2.7). The middle term of (2.38) is handled using Proposition 2.5. To control the denominator and last term on the right-hand side of (2.38), we first note that by (2.32),

$$u_{n+1} - \frac{1}{n} = \frac{1}{v_n} \left\{ -\frac{1}{n}(v_n - Bn) - \partial B_{n+1} + (v_n u_{n+1} - B_{n+1}) \right\}. \tag{2.40}$$

It follows from (1.42), (2.8)–(2.9), (2.35), (2.37) and (2.39) that

$$\left| u_{n+1} - \frac{1}{n} \right| \leq C(K, C_e)(n+1)^{-2} \log(n+1) + C(K, \widetilde{K}, C_\pi, C_\phi)\beta(n+1)^{-1}\delta_{n+1}. \tag{2.41}$$

In particular, $|u_{n+1}| \leq 2/n$, and hence $(1 - u_{n+1})^{-1} \leq 2$, if we choose $n \geq N_0(K, C_e)$ large and $\beta \leq \beta_2(K, \widetilde{K}, C_\pi, C_\phi)$ small, as indicated in $(*)$. It then follows from (2.35) and (2.38)–(2.39) that

$$|v_{n+1} - v_n - B| \leq 2\left(600B^2 C_e + 2B\right)(n+1)^{-1} + C(K, C_\pi, C_\phi)\beta\Delta_{n+1}. \tag{2.42}$$

This proves (IH) for $j = n+1$, provided we take $K \geq 2(600B^2C_e + 2B)$ and $\widetilde{K} \geq C(K, C_\pi, C_\phi)$. $\square$

Proof of Proposition 2.5. To prove (2.35), we first use the recursion relation (1.37) and the identity (1.38) to write

$$\begin{aligned} v_n u_{n+1} - B_{n+1} &= v_n^2(\theta_n - \theta_{n+1}) - B_{n+1} \\ &= v_n^2 \Big\{ \sum_{m=0}^{n} \pi_m p_c (\theta_n - \theta_{n-m}) + \theta_n \sum_{m=n+1}^{\infty} \pi_m p_c \\ &\qquad + \sum_{m_1=1}^{\lfloor (n+1)/2 \rfloor} \sum_{m_2=m_1}^{n+1} \phi_{m_1,m_2} \theta_{n+1-m_1} \theta_{n+1-m_2} - e_{n+1} \Big\} - B_{n+1}. \end{aligned} \tag{2.43}$$

The first equation in (2.32) can be rewritten as

$$B_{n+1} = -B_{n+1} \sum_{m=2}^{n+1} m\pi_m p_c + \sum_{m_1=1}^{\lfloor (n+1)/2 \rfloor} \sum_{m_2=m_1}^{n+1} \phi_{m_1,m_2}. \tag{2.44}$$

Thus we may rewrite (2.43) as

$$v_n u_{n+1} - B_{n+1} = -v_n^2 e_{n+1} + v_n \sum_{m=n+1}^{\infty} \pi_m p_c - X_n + Y_n, \tag{2.45}$$

with

$$X_n = \sum_{m=0}^{n} \pi_m p_c \left[\frac{v_n}{v_{n-m}} (v_n - v_{n-m}) - B_{n+1} m \right], \tag{2.46}$$

$$Y_n = \sum_{m_1=1}^{\lfloor (n+1)/2 \rfloor} \sum_{m_2=m_1}^{n+1} \phi_{m_1,m_2} \left[\frac{v_n^2}{v_{n+1-m_1} v_{n+1-m_2}} - 1 \right]. \tag{2.47}$$

Note that the terms with $m = 0, 1$ and $m_1 = m_2 = 1$ vanish (recall that $\pi_1 = 0$). Thus, by (1.43)–(1.44), X_n and Y_n are both of order β.

For the first term on the right-hand side of (2.45), we note that (2.37) supplies the hypothesis of Theorem 1.5(iii) with $C_\theta = 5 > 2/B$ (by (2.3)). It then follows from (1.45) that

$$|v_n^2 e_{n+1}| \leq 4B^2 C_e C_\theta^3 [(n+1)^{-1} + \beta \Delta_{n+1}]. \tag{2.48}$$

The right-hand side of (2.48) has precisely the form of the right-hand side of (2.35). Similarly, the second term on the right-hand side of (2.45) can be estimated with the help of (1.43) as

$$v_n \Big| \sum_{m=n+1}^{\infty} \pi_m p_c \Big| \leq 2Bnp_c \sum_{m=n+1}^{\infty} |\pi_m| \leq CC_\pi \beta (n+1)^{-(\kappa-2)}, \tag{2.49}$$

which is of smaller order in n than the last term in (2.48) (recall (1.42)).

To estimate $|X_n|$, we first rewrite the expression in square brackets in (2.46) as

$$\frac{v_n}{v_{n-m}} (v_n - v_{n-m}) - B_{n+1} m$$
$$= \frac{v_n}{v_{n-m}} [v_n - v_{n-m} - Bm] \left(1 + \frac{Bm}{v_n}\right) + \frac{v_n}{v_{n-m}} \frac{B^2 m^2}{v_n} + \partial B_{n+1} m. \tag{2.50}$$

Hence, using the lower bound $v_n \geq \frac{1}{2} Bn$ of (2.37), we have

$$|X_n| \leq I + II + III \tag{2.51}$$

with

$$I = 3p_c \sum_{m=2}^{n} |\pi_m| \; \frac{v_n}{v_{n-m}} \; |v_n - v_{n-m} - Bm|,$$
$$II = \frac{2B}{n} p_c \sum_{m=2}^{n} m^2 |\pi_m| \; \frac{v_n}{v_{n-m}}, \tag{2.52}$$
$$III = |\partial B_{n+1}| p_c \sum_{m=2}^{n} m |\pi_m|.$$

The easiest to estimate is III, for which we use (1.43) and (2.39) to obtain

$$III \leq C(C_\pi + C_\phi)\beta^2(n+1)^{-(\kappa-2)}, \tag{2.53}$$

which is of smaller order, both in n and in β, than the right-hand side of (2.35).

To estimate I and II, we first consider the factor v_n/v_{n-m}. Since $v_{n-m} \geq 1$, given any $\ell_0 \geq 1$, it follows from (2.37) that

$$\frac{v_n}{v_{n-m}} \leq v_n \leq 2Bn \leq \frac{2B\ell_0 n}{n+1-m} \qquad (n+1-m \leq \ell_0). \tag{2.54}$$

On the other hand, if we choose $\ell_0 \geq N_0(K)$, then it follows from (2.37) that $v_{n-m} \geq B(n-m)/2$ when $n+1-m > \ell_0$. Therefore,

$$\frac{v_n}{v_{n-m}} \leq \frac{C(K)n}{n+1-m} \qquad (2 \leq m \leq n). \tag{2.55}$$

It follows from (IH) that

$$|v_n - v_{n-m} - Bm| \leq \sum_{j=n+1-m}^{n} |v_j - v_{j-1} - B| \leq m\left[\frac{K}{n+2-m} + C\widetilde{K}\beta\Delta_{n+2-m}\right], \tag{2.56}$$

where we use that Δ_{j+1} is decreasing in j for j large. From (1.43) and (2.55)–(2.56) we obtain

$$\begin{aligned} I &\leq 3p_c C_\pi \beta \sum_{m=2}^{n} (m+1)^{-\kappa} \frac{C(K)nm}{n+1-m}\left[\frac{K}{n+2-m} + C\widetilde{K}\beta\Delta_{n+2-m}\right] \\ &\leq CC(K)C_\pi[K\beta + \widetilde{K}\beta^2]\Delta_{n+1}, \end{aligned} \tag{2.57}$$

where we use (1.42) and the convolution bounds (2.65)–(2.66) stated in Lemma 2.6 below (with $a = \kappa - 1$, $b = 2$, $c = 0$). Similarly, we obtain

$$II \leq C(K)C_\pi\beta\Delta_{n+1}. \tag{2.58}$$

Note that I carries an extra factor β compared to the second term on the right-hand side of (2.35), while II does not.

To estimate $|Y_n|$, we make the decomposition

$$\frac{v_n^2}{v_{n+1-m_1}v_{n+1-m_2}} - 1 = \frac{v_n - v_{n+1-m_1}}{v_{n+1-m_1}} + \frac{v_n - v_{n+1-m_2}}{v_{n+1-m_2}} + \frac{v_n - v_{n+1-m_1}}{v_{n+1-m_1}}\,\frac{v_n - v_{n+1-m_2}}{v_{n+1-m_2}}. \tag{2.59}$$

We use (2.37) and (2.55)–(2.56) to estimate

$$\begin{aligned} 0 \leq \frac{v_n - v_{n+1-m}}{v_{n+1-m}} &= \frac{1}{v_n}\,\frac{v_n}{v_{n+1-m}}(v_n - v_{n+1-m}) \\ &\leq \frac{2}{Bn}\,\frac{C(K)n}{n+2-m}(m-1)\left[B + \frac{K}{n+3-m} + C\widetilde{K}\beta\Delta_{n+3-m}\right] \\ &\leq \frac{m-1}{n+2-m}C(K)\big[1 + \widetilde{K}\beta\big]. \end{aligned} \tag{2.60}$$

Note that when $m_1 \leq m_2 \leq n+1$ we have

$$\frac{m_1 - 1}{n + 2 - m_1} \leq \frac{m_2 - 1}{n + 2 - m_2}, \tag{2.61}$$

and when $m_1 \leq \lfloor (n+1)/2 \rfloor$ we have

$$\frac{m_1 - 1}{n + 2 - m_1} \leq C. \tag{2.62}$$

Combining (1.44), (2.47) and (2.59)–(2.62), we find that

$$|Y_n| \leq C(K)\Big[1 + \widetilde{K}\beta\Big] C_\phi \beta \sum_{m_1=1}^{\lfloor (n+1)/2 \rfloor} \sum_{m_2=m_1}^{n+1} (m_1+1)^{-(\kappa-1)} (m_2 - m_1 + 1)^{-(\kappa-1)} \frac{m_2 - 1}{n + 2 - m_2}. \tag{2.63}$$

With the bound (2.67) stated in Lemma 2.6 below, this implies that

$$|Y_n| \leq C(K)\Big[1 + \widetilde{K}\beta\Big] C_\phi \beta \Delta_{n+1}. \tag{2.64}$$

This is of the same order as the estimate for I in (2.57).

Finally, recalling (2.45) and collecting the estimates in (2.48)–(2.49), (2.51), (2.53), (2.57)–(2.58) and (2.63), we see that we have proved the claim in (2.35), provided we take β_3 sufficiently small depending on $\widetilde{K}$. □

The following elementary lemma was used in the proof of Proposition 2.5.

Lemma 2.6. *(i) For $a, b > 1$ and $c \geq 0$,*

$$\sum_{m=2}^{n} (m+1)^{-a} (n+2-m)^{-b} [\log (n+2-m)]^c \leq C(n+1)^{-a \wedge b} [\log (n+1)]^c. \tag{2.65}$$

(ii) For $\kappa > 2$,

$$\sum_{m=2}^{n} (m+1)^{-(\kappa-1)} (n+1-m)^{-1} \Delta_{n+2-m} \leq \frac{C}{n+1} \Delta_{n+1}. \tag{2.66}$$

(iii) For $\kappa > 2$,

$$\sum_{m_1=1}^{\lfloor (n+1)/2 \rfloor} \sum_{m_2=m_1}^{n+1} (m_1+1)^{-(\kappa-1)} (m_2 - m_1 + 1)^{-(\kappa-1)} \frac{m_2 - 1}{n + 2 - m_2} \leq C \Delta_{n+1}. \tag{2.67}$$

Proof. (i) The inequality (2.65) is obtained by estimating the logarithmic factor on the left side by the logarithmic factor on the right-hand side, and then considering separately the cases $m \leq n/2$ (for which $(n+2-m)^{-b} \leq C(n+1)^{-b}$) and $m > n/2$ (for which $(m+1)^{-a} \leq C(n+1)^{-a}$).
(ii) The inequality (2.66) follows from (2.65) and the definition of Δ_n in (1.42).
(iii) For the inequality (2.67), the contribution to the sum due to $m_2 \geq \lfloor (n+1)/2 \rfloor$ is at most

$$\begin{aligned} C \sum_{m_1=1}^{\lfloor (n+1)/2 \rfloor} (m_1+1)^{-(\kappa-1)} (\lfloor (n+1)/2 \rfloor - m_1 + 1)^{-(\kappa-1)} \sum_{m_2=\lfloor (n+1)/2 \rfloor}^{n+1} \frac{n}{n+2-m_2} \\ \leq C(n+1)^{-(\kappa-2)} \log(n+1) \leq C\Delta_{n+1}, \end{aligned} \tag{2.68}$$

where we use (2.65) to perform the sum over m_1. The contribution due to $m_2 < \lfloor (n+1)/2 \rfloor$ is at most

$$C \sum_{m_1=1}^{\lfloor (n+1)/2 \rfloor} \sum_{m_2=m_1}^{\lfloor (n+1)/2 \rfloor} (m_1+1)^{-(\kappa-1)}(m_2-m_1+1)^{-(\kappa-1)} \frac{m_2-1}{n+1}. \tag{2.69}$$

Since $m_2 - 1 \leq (m_1+1) + (m_2-m_1+1)$, this is at most

$$\begin{aligned} &\frac{C}{n+1} \sum_{m_1=1}^{\lfloor (n+1)/2 \rfloor} (m_1+1)^{-(\kappa-2)} \sum_{m_2=m_1}^{\lfloor (n+1)/2 \rfloor} (m_2-m_1+1)^{-(\kappa-1)} \\ &\quad + \frac{C}{n+1} \sum_{m_1=1}^{\lfloor (n+1)/2 \rfloor} (m_1+1)^{-(\kappa-1)} \sum_{m_2=m_1}^{\lfloor (n+1)/2 \rfloor} (m_2-m_1+1)^{-(\kappa-2)}. \end{aligned} \tag{2.70}$$

In the first term, the sum over m_2 is bounded by C, and the sum over m_1 together with the factor $(n+1)^{-1}$ is at most $C\Delta_{n+1}$. The second term is similar. □

3 Critical exponent for size of cluster of origin

Proof of Corollary 1.4. For the upper bound, we write

$$P_{\geq n} \leq \mathbb{P}((0,0) \longrightarrow \sqrt{n}) + \mathbb{P}(|C(0,0)| \geq n,\ (0,0) \not\longrightarrow \sqrt{n}). \tag{3.1}$$

By (1.1), the first term on the right-hand side decays like $(B\sqrt{n})^{-1}[1+o(1)]$. By the Markov inequality and (1.13), the second term can be bounded by

$$\begin{aligned} \mathbb{P}(|C(0,0)| \geq n,\ (0,0) \not\longrightarrow \sqrt{n}) &\leq n^{-1}\mathbb{E}(|C(0,0)| I[(0,0) \not\longrightarrow \sqrt{n}]) \\ &\leq n^{-1} \sum_{m=0}^{\sqrt{n}} \sum_{x \in \mathbb{Z}^d} \tau_m(x) = \frac{A}{\sqrt{n}}[1+o(1)]. \end{aligned} \tag{3.2}$$

This proves the upper bound.

For the lower bound, given $C_1 > 0$, we define

$$X_n = \#\{(x,m) \in C(0,0) :\ C_1\sqrt{n} \leq m \leq 2C_1\sqrt{n}\}. \tag{3.3}$$

Then, for $C_2 \geq 1$,

$$\begin{aligned} P_{\geq n} &\geq \mathbb{P}\Big(n \leq X_n \leq C_2 n\Big) \geq (C_2 n)^{-1}\mathbb{E}\Big(X_n I[n \leq X_n \leq C_2 n]\Big) \\ &= (C_2 n)^{-1}\Big[\mathbb{E}(X_n) - \mathbb{E}\Big(X_n I[1 \leq X_n < n]\Big) - \mathbb{E}\Big(X_n I[X_n > C_2 n]\Big)\Big]. \end{aligned} \tag{3.4}$$

The first term on the right-hand side is

$$\mathbb{E}(X_n) = \sum_{m=C_1\sqrt{n}}^{2C_1\sqrt{n}} \sum_{x \in \mathbb{Z}^d} \tau_m(x) = AC_1\sqrt{n}[1+o(1)]. \tag{3.5}$$

The second term can be bounded by

$$\mathbb{E}\Big(X_n I[1 \leq X_n < n]\Big) \leq n\mathbb{P}(X_n \geq 1) = n\theta_{C_1\sqrt{n}} = \frac{\sqrt{n}}{BC_1}[1+o(1)]. \tag{3.6}$$

The third term can be bounded, using (1.14), by

$$\mathbb{E}\Big(X_n I[X_n > C_2 n]\Big) \leq (C_2 n)^{-1}\mathbb{E}(X_n^2) = (C_2 n)^{-1} \sum_{m_1=C_1\sqrt{n}}^{2C_1\sqrt{n}} \sum_{m_2=C_1\sqrt{n}}^{2C_1\sqrt{n}} \sum_{x_1,x_2\in\mathbb{Z}^d} \tau^{(3)}_{m_1,m_2}(x_1,x_2)$$
$$\leq (C_2 n)^{-1}(C_1\sqrt{n})^2 A^3 V(2C_1\sqrt{n})[1+o(1)] = \frac{2A^3VC_1^3}{C_2}\sqrt{n}[1+o(1)]. \tag{3.7}$$

Combining (3.4)–(3.7), we arrive at

$$P_{\geq n} \geq \frac{1}{C_2}\Big(C_1 A - \frac{1}{BC_1} - \frac{2A^3VC_1^3}{C_2}\Big)\frac{1}{\sqrt{n}}[1+o(1)]. \tag{3.8}$$

The desired lower bound now follows if we first choose C_1 large and then choose C_2 large compared to C_1. □

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References

[1] M. Aizenman and D.J. Barsky. Sharpness of the phase transition in percolation models. *Commun. Math. Phys.*, **108**:489–526, (1987).

[2] D.J. Barsky and M. Aizenman. Percolation critical exponents under the triangle condition. *Ann. Probab.*, **19**:1520–1536, (1991).

[3] C. Bezuidenhout and G. Grimmett. The critical contact process dies out. *Ann. Probab.*, **18**:1462–1482, (1990).

[4] M. Bramson and D. Griffeath. Asymptotics for interacting particle systems on $\mathbb{Z}^d$. *Z. Wahrsch. Verw. Gebiete*, **53**:183–196, (1980).

[5] N.G. de Bruijn. *Asymptotic Methods in Analysis.* North Holland, Amsterdam, 3rd edition, (1981).

[6] J.T. Cox and E.A. Perkins. An application of the voter model–super-Brownian motion invariance principle. *Ann. Inst. H. Poincaré Probab. Statist.*, **40**:25–32, (2004).

[7] G. Grimmett and P. Hiemer. Directed percolation and random walk. In V. Sidoravicius, editor, *In and Out of Equilibrium*, pages 273–297. Birkhäuser, Boston, (2002).

[8] G.R. Grimmett and D.R. Stirzaker. *Probability and Random Processes.* Oxford University Press, Oxford, 3rd edition, (2001).

[9] T. Hara and G. Slade. The scaling limit of the incipient infinite cluster in high-dimensional percolation. II. Integrated super-Brownian excursion. *J. Math. Phys.*, **41**:1244–1293, (2000).

[10] R. van der Hofstad. Spread-out oriented percolation and related models above the upper critical dimension: induction and superprocesses. *Ensaios Matemáticos*, **9**:91–181, (2005).

[11] R. van der Hofstad, F. den Hollander, and G. Slade. Construction of the incipient infinite cluster for spread-out oriented percolation above 4 + 1 dimensions. *Commun. Math. Phys.*, **231**:435–461, (2002).

[12] R. van der Hofstad, F. den Hollander, and G. Slade. The survival probability for critical spread-out oriented percolation above 4 + 1 dimensions. II. Expansion. Preprint, (2005).

[13] R. van der Hofstad and A. Sakai. Gaussian scaling for the critical spread-out contact process above the upper critical dimension. *Electron. J. Probab.*, **9**:710–769, (2004).

[14] R. van der Hofstad and A. Sakai. Critical points for spread-out contact processes and oriented percolation. *Probab. Th. Rel. Fields*, **132**:438–470, (2005).

[15] R. van der Hofstad and A. Sakai. Convergence of the critical finite-range contact process to super-Brownian motion above the upper critical dimension. In preparation.

[16] R. van der Hofstad and G. Slade. A generalised inductive approach to the lace expansion. *Probab. Th. Rel. Fields*, **122**:389–430, (2002).

[17] R. van der Hofstad and G. Slade. Convergence of critical oriented percolation to super-Brownian motion above 4+1 dimensions. *Ann. Inst. H. Poincaré Probab. Statist.*, **39**:415–485, (2003).

[18] B.G. Nguyen and W-S. Yang. Triangle condition for oriented percolation in high dimensions. *Ann. Probab.*, **21**:1809–1844, (1993).

[19] A. Sakai. Mean-field critical behavior for the contact process. *J. Stat. Phys.*, **104**:111–143, (2001).