Lace expansion for the Ising model

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October 26, 2005

Abstract: The lace expansion has been a powerful tool to investigate mean-field behavior for various stochastic-geometrical models, such as self-avoiding walk and percolation, above their respective upper-critical dimension. In this paper, we prove for the first time the lace expansion for the Ising model, which is independent of the property of the spin-spin coupling. In the ferromagnetic case, we provide key propositions to prove that, without requiring the reflection positivity of the spin-spin coupling, the two-point function obeys a Gaussian infrared bound for the nearest-neighbor model with $d \gg 4$ and for the spread-out model with $d > 4$ and $L \gg 1$, as well as that the critical two-point function exhibits a Gaussian asymptotics for the spread-out model with $d > 4$ and $L \gg 1$. As a result, these models exhibit the ferromagnetic mean-field behavior.

Keywords: Ising model; random-current representation; lace expansion; ferromagnetic mean-field behavior.

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1 Introduction

The Ising model is a classical statistical mechanical model that was first introduced in [21] as a model of magnets. We consider the d-dimensional integer lattice $\mathbb{Z}^d$, and a spin variable $\varphi_x = \pm 1$ is assigned to each site $x \in \mathbb{Z}^d$. The energy of the system is formally given by $H(\varphi) = -\sum_{\{x,y\}\subset\mathbb{Z}^d} J_{x,y}\varphi_x\varphi_y$, where $\varphi = \{\varphi_x\}_{x\in\mathbb{Z}^d}$ is a spin configuration and each $J_{x,y} \in \mathbb{R}$ is a given spin-spin coupling. If the model is ferromagnetic (i.e., $J_{x,y} \geq 0$), then the energy becomes lower as more spins align. In addition, if $d \geq 2$ and the spin-spin coupling is translation-invariant and summable (i.e., $\sum_{x\in\mathbb{Z}^d} J_{o,x} < \infty$), then there is a critical inverse temperature $\beta_c \in (0,\infty)$ such that the susceptibility $\chi(\beta)$ is finite if and only if $\beta < \beta_c$ and diverges as $\beta \uparrow \beta_c$ (e.g., [1]). The susceptibility $\chi(\beta)$ is the sum of the two-point function $\langle\varphi_o\varphi_x\rangle_\beta$, where $\langle f\rangle_\beta$ denotes the thermal average of a function $f = f(\varphi)$ at the inverse temperature β.

We are interested in the critical phenomena around $\beta = \beta_c$. For example, it is expected that there is a critical exponent $\gamma = \gamma(d)$ such that $\chi(\beta) \approx (\beta_c - \beta)^{-\gamma}$ as $\beta \uparrow \beta_c$ (in some appropriate sense). Other observables, such as the spontaneous magnetization, are also believed to exhibit power-law behavior characterized by their respective critical exponents that depend only on d and are insensitive to the precise definition of $J_{o,x} \geq 0$, as long as its range is finite (universality).

For such ferromagnetic models, it was proved in [1, 4] that, if $\sum_{x\in\mathbb{Z}^d}\langle\varphi_o\varphi_x\rangle_\beta^2$ is bounded uniformly in $\beta < \beta_c$, then the aforementioned critical exponents take on their respective d-independent mean-field values: e.g., $\gamma = 1$. This sufficient condition for the ferromagnetic mean-field behavior has been verified above four dimensions for a class of models that satisfy a special property, called the reflection positivity (e.g., [6, 8, 9]). However, more general finite-range models do not always satisfy this property, and therefore their mean-field behavior has not been completely established yet, even in high dimensions. If we believe in universality, we expect that these finite-range models also exhibit the same mean-field behavior, as soon as $d > 4$. (On the other hand, because of the hyperscaling inequalities in, e.g., [28], the mean-field exponents for finite-range models are incompatible with $d < 4$.)

In this paper, we prove the lace expansion for the Ising model. The lace expansion has been applied to various stochastic-geometrical models, such as self-avoiding walk and percolation, to prove their mean-field behavior above the upper-critical dimension (e.g., [18]). The expansion obtained below for the two-point function gives rise to an identity that is similar to the recursion equation for the random-walk Green's function, and is valid independently of the property of the spin-spin coupling: $J_{x,y}$ is not required to be nonnegative, translation-invariant or $\mathbb{Z}^d$-symmetric.

For the ferromagnetic case, we obtain bounds on the expansion coefficients in terms of two-point functions, and prove that, assuming translation-invariance and $\mathbb{Z}^d$-symmetry of the spin-spin coupling, the two-point function obeys a Gaussian infrared bound for the nearest-neighbor model with $d \gg 4$ and for the spread-out model (defined below in Section 2) with $d > 4$ and the range of interaction $L \gg 1$, and that $\langle\varphi_o\varphi_x\rangle_{\beta_c}$ exhibits a Gaussian asymptotics for the spread-out model with $d > 4$ and $L \gg 1$. As a result, the aforementioned sufficient condition for the ferromagnetic mean-field behavior holds.

We emphasize that our approach using the lace expansion does not require the reflection positivity of the spin-spin coupling, so that it can be used to prove the same results for, e.g., the next-nearest-neighbor model with $d \gg 4$.

In the next section, we define the model and state the main results.

2 Model and the main results

Let Λ be a finite subset of $\mathbb{Z}^d$ containing the origin $o \in \mathbb{Z}^d$, such as a d-dimensional hypercube centered at the origin. The Hamiltonian represents the energy of the system, and is defined by

$$H_\Lambda(\varphi) = -\sum_{\{x,y\}\subset\Lambda} J_{x,y}\varphi_x\varphi_y \qquad (\varphi = \{\varphi_x\}_{x\in\Lambda} \in \{\pm1\}^\Lambda), \tag{2.1}$$

where $J_{x,y} \in \mathbb{R}$ for $x, y \in \mathbb{Z}^d$ is the spin-spin coupling. The partition function $Z_{\beta;\Lambda}$ is defined to be the expectation of the Boltzmann factor $e^{-\beta H_\Lambda(\varphi)}$ with respect to the product of the single-spin measures $d\mu_\Lambda(\varphi) = \prod_{x\in\Lambda}(\frac{1}{2}\mathbb{1}_{\{\varphi_x=+1\}} + \frac{1}{2}\mathbb{1}_{\{\varphi_x=-1\}})$, i.e.,

$$Z_{\beta;\Lambda} = \int d\mu_\Lambda(\varphi)\ e^{-\beta H_\Lambda(\varphi)} = 2^{-|\Lambda|}\sum_{\varphi\in\{\pm1\}^\Lambda} e^{-\beta H_\Lambda(\varphi)}. \tag{2.2}$$

We define the thermal average of a function $f = f(\varphi)$ by

$$\langle f\rangle_{\beta;\Lambda} = \frac{1}{Z_{\beta;\Lambda}}\int d\mu_\Lambda(\varphi)\ f(\varphi)\, e^{-\beta H_\Lambda(\varphi)} = \frac{2^{-|\Lambda|}}{Z_{\beta;\Lambda}}\sum_{\varphi\in\{\pm1\}^\Lambda} f(\varphi)\, e^{-\beta H_\Lambda(\varphi)}. \tag{2.3}$$

In particular, the two-point function is defined by

$$\langle\varphi_o\varphi_x\rangle_{\beta;\Lambda} = \frac{2^{-|\Lambda|}}{Z_{\beta;\Lambda}}\sum_{\varphi\in\{\pm1\}^\Lambda} \varphi_o\varphi_x\, e^{-\beta H_\Lambda(\varphi)} \qquad (x\in\Lambda). \tag{2.4}$$

In this paper, we prove the following identity for the two-point function, in which we use

$$\tau_{x,y} = \tanh(\beta J_{x,y}). \tag{2.5}$$

Theorem 2.1 (Lace expansion). *For any $\Lambda \subset \mathbb{Z}^d$ and $j \geq 0$, there exist $\pi^{(i)}_{\beta;\Lambda}(x)$ and $R^{(j+1)}_{\beta;\Lambda}(x)$ for $x \in \Lambda$ and $i = 0, \dots, j$ such that, by defining*

$$\Pi^{(j)}_{\beta;\Lambda}(x) = \sum_{i=0}^{j}(-1)^i\,\pi^{(i)}_{\beta;\Lambda}(x), \tag{2.6}$$

we have

$$\langle\varphi_o\varphi_x\rangle_{\beta;\Lambda} = \Pi^{(j)}_{\beta;\Lambda}(x) + \sum_{u,v\in\Lambda}\Pi^{(j)}_{\beta;\Lambda}(u)\ \tau_{u,v}\ \langle\varphi_v\varphi_x\rangle_{\beta;\Lambda} + (-1)^{j+1}R^{(j+1)}_{\beta;\Lambda}(x). \tag{2.7}$$

Moreover, if the spin-spin coupling is nonnegative, then we have the bounds

$$\pi^{(i)}_{\beta;\Lambda}(x) \geq 0, \qquad\qquad 0 \leq R^{(j+1)}_{\beta;\Lambda}(x) \leq \sum_{u,v\in\Lambda}\pi^{(j)}_{\beta;\Lambda}(u)\ \tau_{u,v}\ \langle\varphi_v\varphi_x\rangle_{\beta;\Lambda}. \tag{2.8}$$

We defer to Section 3.2.3 giving the exact expressions of $\pi_{\beta;\Lambda}^{(i)}(x)$ and $R_{\beta;\Lambda}^{(j+1)}(x)$, since we need a certain representation to describe these functions. We introduce this representation in Section 3.1 and complete the proof of Theorem 2.1 in Section 3.2.

Whether the above expansion is useful or not depends very much on the existence of nice bounds on the expansion coefficients and the remainder. This is indeed the case for the ferromagnetic models whose spin-spin coupling is translation-invariant and $\mathbb{Z}^d$-symmetric, as explained below. Let

$$\tau = \sum_{x\in\mathbb{Z}^d} \tau_{o,x}, \qquad D(x) = \frac{\tau_{o,x}}{\tau}, \qquad \sigma^2 = \sum_{x\in\mathbb{Z}^d} |x|^2 D(x), \tag{2.9}$$

where $|\cdot|$ is the Euclidean norm, and let

$$G_\beta(x) = \lim_{\Lambda\uparrow\mathbb{Z}^d} \langle\varphi_o\varphi_x\rangle_{\beta;\Lambda}, \tag{2.10}$$

which exists as a nondecreasing limit, due to the second Griffiths inequality (e.g., [8, 9]). For functions f, g on $\mathbb{Z}^d$, we write $(f*g)(x) = \sum_y f(y)\, g(x-y)$ and $f^{*(i+1)}(x) = (f^{*i}*f)(x)$.

Proposition 2.2. *Let $J_{u,v}$ be nonnegative, translation-invariant and $\mathbb{Z}^d$-symmetric, and define*

$$F_1(\beta) = \tau - 1, \qquad F_2(\beta) = \sup_x (D * G_\beta^{*2})(x), \qquad F_3(\beta) = \sup_{x\neq o} \bigg(\frac{|x|^2}{\sigma^2} \vee 1\bigg) G_\beta(x). \tag{2.11}$$

Suppose

$$F(\beta) \equiv \max_{i=1,2,3} F_i(\beta) \le \theta. \tag{2.12}$$

Then, there is a θ-independent constant $C<\infty$ such that, for any $\Lambda\subset\mathbb{Z}^d$,

$$\sum_x \pi_{\beta;\Lambda}^{(i)}(x) \le \begin{cases} 1+(C\theta)^2 & (i=0), \\ (C\theta)^i & (i\ge 1), \end{cases} \qquad \sum_x |x|^2 \pi_{\beta;\Lambda}^{(i)}(x) \le (C\theta)^{i\vee 2}\sigma^2. \tag{2.13}$$

This is a consequence of Proposition 4.1 below, which provides upper bounds on $\pi_{\beta;\Lambda}^{(i)}(x)$ in terms of two-point functions (diagrammatic bounds). For $\pi_{\beta;\Lambda}^{(0)}(x)$, for example, Proposition 4.1 reads

$$\pi_{\beta;\Lambda}^{(0)}(x) \le \langle\varphi_o\varphi_x\rangle_{\beta;\Lambda}^3 \le G_\beta(x)^3, \tag{2.14}$$

where the last inequality is due to the second Griffiths inequality. These diagrammatic bounds replace the results of the BK inequality for percolation [5]. For example, the zeroth expansion coefficient for percolation is bounded, by using the BK inequality, as [16]

$$\pi_p^{(0)}(x) \le G_p(x)^2, \tag{2.15}$$

where p is the bond-occupation parameter and the percolation two-point function $G_p(x)$ is the probability of o and x being connected by a sequence of occupied bonds. See [18, 26] for the diagrammatic bounds on the expansion coefficients for self-avoiding walk, lattice trees and lattice animals.

We now briefly explain a consequence of Theorem 2.1 and Proposition 2.2. Suppose that $\beta < \beta_c$ (i.e., $\chi(\beta) = \sum_x G_\beta(x) < \infty$) and θ in Proposition 2.2 is sufficiently small. Then, by (2.8) and (2.13), $\sum_x R^{(j+1)}_{\beta;\Lambda}(x)$ decays as $j \uparrow \infty$. By (2.10) and dominated convergence, the Fourier transform of $\langle \varphi_o \varphi_x \rangle_{\beta;\Lambda}$ converges to $\hat{G}_\beta(k) \equiv \sum_x G_\beta(x)\, e^{ik\cdot x}$, independently of the choice of a sequence $\Lambda_1 \subset \Lambda_2 \subset \cdots \uparrow \mathbb{Z}^d$. Since $\Pi_{\beta;\Lambda}(x) \equiv \sum_{i=0}^\infty (-1)^i \pi^{(i)}_{\beta;\Lambda}(x)$ is absolutely summable, there is a subsequence $\Lambda'_1 \subset \Lambda'_2 \subset \cdots \uparrow \mathbb{Z}^d$ such that the limit $\Pi_\beta(x) \equiv \lim_{n\uparrow\infty} \Pi_{\beta;\Lambda'_n}(x)$ exists for all $x \in \mathbb{Z}^d$ (provided that $\Pi_{\beta;\Lambda'_n}(x) \equiv 0$ for all $x \notin \Lambda'_n$, for every n) and satisfies $\sum_x |\Pi_\beta(x)| \le 1 + O(\theta)$ and $\sum_x |x|^2 |\Pi_\beta(x)| \le O(\theta^2)\sigma^2$. Then, by dominated convergence, we have $\hat{\Pi}_\beta(k) = \sum_x \Pi_\beta(x)\, e^{ik\cdot x}$, and hence

$$\hat{G}_\beta(k) = \hat{\Pi}_\beta(k) + \hat{\Pi}_\beta(k)\, \tau \hat{D}(k)\, \hat{G}_\beta(k). \tag{2.16}$$

Rearranging this identity and using $\hat{G}_\beta(0) = \chi(\beta)$, the symmetry of the model and then (2.13), we obtain

$$\begin{aligned} |\hat{G}_\beta(k)| &= \left| \frac{\hat{\Pi}_\beta(k)\, \hat{\Pi}_\beta(0)^{-1}}{\chi(\beta)^{-1} + \tau(1 - \hat{D}(k)) + (\hat{\Pi}_\beta(0) - \hat{\Pi}_\beta(k))\tau \hat{D}(k)\, \hat{\Pi}_\beta(0)^{-1}} \right| \\ &\le \frac{1 + O(\theta)}{\tau(1 - \hat{D}(k) - O(\theta^2)\sigma^2 d^{-1}|k|^2)}. \end{aligned} \tag{2.17}$$

For the nearest-neighbor model (i.e., $J_{o,x} = \mathbb{1}_{\{|x|=1\}}$), $1 - \hat{D}(k) \ge 2\pi^{-2} d^{-1} |k|^2$, and thus

$$|\hat{G}_\beta(k)| \le \frac{1 + O(\theta)}{\tau(1 - \hat{D}(k))}. \tag{2.18}$$

Note that we have obtained this Gaussian infrared bound under the assumption that (2.12) holds. Now, we use (2.18) to verify this assumption. In fact, following the calculations in the previous lace-expansion works (e.g., [22]), we obtain that $F(\beta)$ is bounded by $c(d-4)^{-1}$ for $d > 4$, where $c < \infty$ is independent of d and θ. This implies that, if θ in (2.12) is initially chosen as, say, $2c(d-4)^{-1}$, and if d is sufficiently large, then the stronger version of (2.12) with $\theta = c(d-4)^{-1}$ holds, i.e., $F(\beta) \notin \frac{c}{d-4}(1,2]$ if $d \gg 4$. Let β_0 satisfy $\tau(\beta_0) \equiv \sum_x \tanh(\beta_0 J_{o,x}) = 1$. It is not so hard to show that $F(\beta_0)$ is indeed bounded by $c(d-4)^{-1}$ using random-walk estimates (see the footnote around (4.4) below), and that $F(\beta)$ is continuous in $\beta \in [\beta_0, \beta_c)$. Therefore, $F(\beta) \le c(d-4)^{-1}$ and (2.18) hold for all $\beta \in [\beta_0, \beta_c)$ if $d \gg 4$. (In particular, $1 \le \tau(\beta_c) = \lim_{\beta\uparrow\beta_c} \tau(\beta) \le 1 + c(d-4)^{-1}$.) As a result, $\sum_x G_\beta(x)^2$ is bounded uniformly in $\beta < \beta_c$, and hence the critical exponents take on their respective mean-field values [1, 2, 3, 4][1].

Another example is the following spread-out interaction (often called the Kac potential):

$$J_{o,x} = L^{-d} \rho(L^{-1}x) \qquad (1 \le L < \infty), \tag{2.19}$$

[1] Since there is a unique translation-invariant measure in the high-temperature phase, our $G_\beta(x)$ coincides with the infinite-volume limit of the two-point function under the periodic-boundary condition, which was used in [1, 2, 3, 4] to prove differential inequalities for $\chi(\beta)$ and other observables. These differential inequalities are the foundation of the proof of the ferromagnetic mean-field behavior.

where[2] $\rho : [-1,1]^d \setminus \{o\} \mapsto [0,\infty)$ is a bounded probability distribution, which is symmetric under rotations by $\pi/2$ and reflections in coordinate hyperplanes, and is piecewise continuous so that the Riemann sum $L^{-d}\sum_{x\in\mathbb{Z}^d}\rho(L^{-1}x)$ approximates $\int_{\mathbb{R}^d} d^dx\,\rho(x) \equiv 1$. The parameter L is the range of the spin-spin coupling, and will be taken to be large in the analysis. The simplest example would be

$$J_{o,x} = \frac{\mathbb{1}_{\{0<|x|\le L\}}}{\sum_{z\in\mathbb{Z}^d}\mathbb{1}_{\{0<|z|\le L\}}} = O(L^{-d})\,\mathbb{1}_{\{0<|L^{-1}x|\le 1\}}. \tag{2.20}$$

For this model with $L \gg 1$, $1-\hat{D}(k)$ is bounded from below by $\sigma^2|k|^2 \wedge 1$ multiplied by a d-dependent positive constant [19]. Following the same strategy as explained above for the nearest-neighbor model, we obtain (2.18) with $\theta = O(L^{-d})$, uniformly in $\beta < \beta_{\mathrm{c}}$, if $d > 4$ and $L \gg 1$, and thus prove the ferromagnetic mean-field behavior.

Here, we summarize the above results.

Theorem 2.3 (**Gaussian infrared bound and the mean-field behavior**). *For the nearest-neighbor model ($J_{o,x} = \mathbb{1}_{\{|x|=1\}}$) with $d \gg 4$ and the spread-out model (2.19) with $d > 4$ and $L \gg 1$, the infrared bound (2.18), with $\theta = (d-4)^{-1}$ and $\theta = L^{-d}$ respectively, holds uniformly in $\beta < \beta_{\mathrm{c}}$, and hence the susceptibility exponent γ and several other critical exponents exist and take on their mean-field values. In addition, $1 \le \tau(\beta_{\mathrm{c}}) \le 1 + O(\theta)$.*

We emphasize that, to arrive at the above conclusion, the reflection positivity of the spin-spin coupling has not been required. The class of reflection-positive models includes the nearest-neighbor model, a "variant" next-nearest-neighbor model, Yukawa potentials, power-law decaying interactions, and their combinations [6]. For the reflection-positive models, it has been proved [11] (see also [9]) that, for $d > 2$,

$$0 \le \hat{G}_\beta(k) \le \frac{\text{const.}}{\beta|k|^2} \qquad \text{uniformly in } \beta < \beta_{\mathrm{c}}, \tag{2.21}$$

and hence the susceptibility exponent and several other critical exponents take on their respective mean-field values for $d > 4$. However, since this class of models is rather restricted, and in some cases the Gaussian infrared bound (2.21) is not expected to be sharp, it has been longed to have different approaches than using the reflection positivity. Our approach using the lace expansion is one of them.

Furthermore, it has been known for the nearest-neighbor model [27] that the two-point function also obeys the following x-space bound:

$$G_\beta(x) \le \frac{\text{const.}}{\beta\|x\|^{d-2}} \qquad \text{uniformly in } \beta < \beta_{\mathrm{c}}, \tag{2.22}$$

where $\|\cdot\| = |\cdot| \vee 1$. There has been no similar result for the spread-out model. We can improve this situation by using the lace expansion (2.7) and the following proposition:

Proposition 2.4. *Let $J_{u,v}$ be the spread-out interaction defined in (2.19). Suppose $\frac{1}{2}d < q < d$ and*

$$\tau \le 2, \qquad\qquad G_\beta(x) \le \delta_{o,x} + \theta\|x\|^{-q}. \tag{2.23}$$

[2]For the Gaussian infrared bound, the finite-support condition on ρ can be replaced by the existence of the $(2+\epsilon)$-moment for some $\epsilon > 0$, but not for the x-space asymptotics (2.26) below.

Then, there is a $C = C(d,q) < \infty$ such that, for any $\Lambda \subset \mathbb{Z}^d$ and sufficiently small θ, with θL^{d-q} being bounded away from zero (which requires L to be large),

$$\pi^{(i)}_{\beta;\Lambda}(x) \le \begin{cases} \delta_{o,x} + (C\theta)^3 \|\!|x|\!\|^{-3q} & (i=0), \\ C\theta\delta_{o,x} + (C\theta)^3 \|\!|x|\!\|^{-3q} & (i=1), \\ (C\theta)^i \|\!|x|\!\|^{-3q} & (i \ge 2). \end{cases} \tag{2.24}$$

Following the analysis of the lace expansion in [14], we can indeed prove that, if $\beta < \beta_c$, $d > 4$ and $L \gg 1$, then (2.23) with $q = d-2$ and $\theta = O(L^{-2+\epsilon})$, where $\epsilon > 0$ is an arbitrarily small number, holds [25], and thus

$$|\Pi_{\beta;\Lambda}(x) - \delta_{o,x}| \le C\theta\delta_{o,x} + O(\theta^2)\|\!|x|\!\|^{3(2-d)} \qquad ({}^\forall\Lambda \subset \mathbb{Z}^d). \tag{2.25}$$

Since $3(d-2) = d+2+2(d-4) > d+2$ if $d > 4$, we can say that $\Pi_{\beta;\Lambda}(x)$ is close to $\delta_{o,x}$ up to the second moment. As a result, with the help of the continuity in $\beta \le \beta_c$ of $G_\beta(x)$, we can prove the following x-space asymptotics at $\beta = \beta_c$ [25]:

Theorem 2.5 (Asymptotic behavior for the spread-out model). *Fix $\kappa = 2(d-4) \wedge 2 > 0$ and $\epsilon > 0$, and let $a_d = \frac{d}{2}\pi^{-d/2}\Gamma(\frac{d}{2}-1)$. Then, there exist $L_0 = L_0(d,\epsilon)$ and $A = A(d,L,\epsilon) = 1 + O(L^{-6+\epsilon})$ such that, for $L \ge L_0$,*

$$G_{\beta_c}(x) = \frac{A}{\tau(\beta_c)} \frac{a_d\sigma^{-2}}{\|\!|x|\!\|^{d-2}} \Big(1 + O(L^\kappa \|\!|x|\!\|^{-\kappa+\epsilon}) + O(L^2\|\!|x|\!\|^{-2+\epsilon})\Big), \tag{2.26}$$

where $A-1$ and constants in the error terms in (2.26) depend on ϵ.

We note that the factor $a_d\sigma^{-2}\|\!|x|\!\|^{-(d-2)}$ in (2.26) is exactly equal to the leading asymptotics of the random-walk Green's function [14]. Therefore, (2.26) reads that the anomalous dimension η takes on the mean-field value $\eta = 0$. For the nearest-neighbor model, we may obtain the same asymptotics for $G_{\beta_c}(x)$ (with different A and error estimates) by using the method in [13].

In the next section, we prove the lace expansion (2.7). In Section 4, we prove the diagrammatic bounds on the expansion coefficients, mentioned below Proposition 2.2. The proof of Propositions 2.2 and 2.4 using these diagrammatic bounds is based on a common philosophy, and is not so difficult as soon as we understand the composition of the diagrams in terms of two-point functions. For simplicity, we will only prove Proposition 2.4 in detail in Section 4.

3 Lace expansion

The lace expansion was first invented by Brydges and Spencer [7] to investigate weakly self-avoiding walk for $d > 4$. Later, it was developed for various stochastic-geometrical models, such as strictly self-avoiding walk for $d > 4$ (e.g., [17]), lattice trees and lattice animals for $d > 8$ (e.g., [15]), unoriented percolation for $d > 6$ (e.g., [16]), oriented percolation for $d > 4$ (e.g., [23]) and the contact process for $d > 4$ (e.g., [24]). See [26] for an extensive list of references. This is the first lace-expansion paper that deals with the Ising model.

There might be several ways to obtain the lace expansion for $\langle\varphi_o\varphi_x\rangle_{\beta;\Lambda}$ via, e.g., the high-temperature expansion, the random-walk representation (e.g., [9]) or the FK random-cluster representation (e.g., [10]). In this paper, we use the random-current representation (Section 3.1), which applies to the models in the Griffiths-Simon class (e.g., [1, 4]). This representation is similar in philosophy to the high-temperature expansion, but it turned out to be much stronger in investigating the critical phenomena [1, 2, 3, 4]. The main advantage in this representation is the source-switching lemma (Lemma 3.3 below in Section 3.2.2) by which we have an identity for $\langle\varphi_o\varphi_x\rangle_{\beta;\Lambda} - \langle\varphi_o\varphi_x\rangle_{\beta;\mathcal{A}}$ with $\mathcal{A} \subset \Lambda$. We will repeatedly use this identity to complete the lace expansion for $\langle\varphi_o\varphi_x\rangle_{\beta;\Lambda}$ in Section 3.2.3.

In the rest of this paper, we omit the subscript β and write, e.g., $\langle\varphi_o\varphi_x\rangle_\Lambda = \langle\varphi_o\varphi_x\rangle_{\beta;\Lambda}$.

3.1 Random-current representation

In this section, we describe the random-current representation and introduce some notation that will be essential in the derivation of the lace expansion.

First, we consider the partition function. We call a pair of sites $b = \{u, v\}$ with $J_b > 0$ a *bond*. For $\mathcal{A} \subset \Lambda$, we denote by $\mathbb{B}_\mathcal{A}$ the set of bonds whose both endvertices are in $\mathcal{A}$. By expanding the Boltzmann factor in (2.2), the partition function $Z_\mathcal{A}$ on $\mathcal{A}$ (i.e., $J_b = 0$ for all $b \in \mathbb{B}_\Lambda \setminus \mathbb{B}_\mathcal{A}$) can be written as

$$\begin{aligned} Z_\mathcal{A} &= 2^{-|\mathcal{A}|} \sum_{\varphi\in\{\pm1\}^\mathcal{A}} \prod_{\{u,v\}\in\mathbb{B}_\mathcal{A}} \Bigg(\sum_{n_{u,v}\in\mathbb{Z}_+} \frac{(\beta J_{u,v})^{n_{u,v}}}{n_{u,v}!} \varphi_u^{n_{u,v}} \varphi_v^{n_{u,v}} \Bigg) \\ &= \sum_{\mathbf{n}\in\mathbb{Z}_+^{\mathbb{B}_\mathcal{A}}} \Bigg(\prod_{b\in\mathbb{B}_\mathcal{A}} \frac{(\beta J_b)^{n_b}}{n_b!} \Bigg) \prod_{v\in\mathcal{A}} \Bigg(\frac{1}{2} \sum_{\varphi_v=\pm1} \varphi_v^{\sum_{b\ni v} n_b} \Bigg), \end{aligned} \tag{3.1}$$

where $\mathbf{n} = \{n_b\}_{b\in\mathbb{B}_\mathcal{A}}$ is called a *current configuration.* Note that the single-spin average in the second parentheses in the last line is 1 if $\sum_{b\ni x} n_b$ is an even integer, and 0 otherwise. Denoting by $\partial\mathbf{n}$ the set of *sources* $x \in \Lambda$ at which $\sum_{b\ni x} n_b$ is an *odd* integer, and defining

$$w_\mathcal{A}(\mathbf{n}) = \prod_{b\in\mathbb{B}_\mathcal{A}} \frac{(\beta J_b)^{n_b}}{n_b!} \qquad (\mathbf{n} \in \mathbb{Z}_+^{\mathbb{B}_\mathcal{A}}), \tag{3.2}$$

we obtain

$$Z_\mathcal{A} = \sum_{\mathbf{n}\in\mathbb{Z}_+^{\mathbb{B}_\mathcal{A}}} w_\mathcal{A}(\mathbf{n}) \prod_{x\in\mathcal{A}} \mathbb{1}_{\{\sum_{b\ni x} n_b \text{ even}\}} = \sum_{\partial\mathbf{n}=\varnothing} w_\mathcal{A}(\mathbf{n}). \tag{3.3}$$

To achieve the above representation, we have assumed that $J_b = 0$ for $b \in \mathbb{B}_\Lambda \setminus \mathbb{B}_\mathcal{A}$. Instead, we can think of $Z_\mathcal{A}$ as the sum of $w_\Lambda(\mathbf{n})$ over $\mathbf{n} \in \mathbb{Z}_+^{\mathbb{B}_\Lambda}$ satisfying $\mathbf{n}|_{\mathcal{A}^c} \equiv 0$, where $\mathbf{n}|_{\mathcal{A}^c}$ is the projection of $\mathbf{n}$ over the bonds incident on $\mathcal{A}^c \equiv \Lambda \setminus \mathcal{A}$, i.e.,

$$\mathbf{n}|_{\mathcal{A}^c} = \{n_b : b \in \mathbb{B}_\Lambda \setminus \mathbb{B}_\mathcal{A}\}. \tag{3.4}$$

By this observation, (3.3) can be written as

$$Z_\mathcal{A} = \sum_{\substack{\partial\mathbf{n}=\varnothing \\ \mathbf{n}|_{\mathcal{A}^c}\equiv0}} w_\Lambda(\mathbf{n}). \tag{3.5}$$

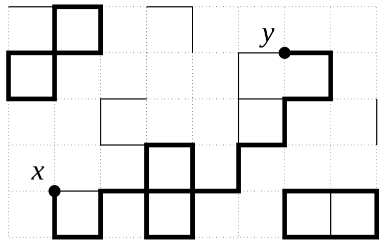

Figure 1: A current configuration with sources at x and y. The thick-solid line segments stand for bonds with odd currents, while the thin-solid line segments stand for bonds with positive even currents, which cannot be seen in the high-temperature expansion.

Following the same calculation, we can rewrite $Z_{\mathcal{A}}\langle\varphi_x\varphi_y\rangle_{\mathcal{A}}$ for $x, y \in \mathcal{A}$ as

$$\begin{aligned} Z_{\mathcal{A}}\langle\varphi_x\varphi_y\rangle_{\mathcal{A}} &= \sum_{\mathbf{n}\in\mathbb{Z}_+^{\mathbb{B}_{\mathcal{A}}}} \Bigg(\prod_{b\in\mathbb{B}_{\mathcal{A}}} \frac{(\beta J_b)^{n_b}}{n_b!}\Bigg) \prod_{v\in\mathcal{A}} \Bigg(\frac{1}{2}\sum_{\varphi_v=\pm1} \varphi_v^{\mathbb{1}_{\{v\in x\triangle y\}}+\sum_{b\ni v} n_b}\Bigg) \\ &= \sum_{\partial\mathbf{n}=x\triangle y} w_{\mathcal{A}}(\mathbf{n}) = \sum_{\substack{\partial\mathbf{n}=x\triangle y \\ \mathbf{n}|_{\mathcal{A}^c}\equiv 0}} w_\Lambda(\mathbf{n}), \end{aligned} \tag{3.6}$$

where $x \triangle y$ is an abbreviation for $\{x\} \triangle \{y\}$. If x or y is in $\mathcal{A}^c$, then we define both sides of (3.6) to be zero. This is consistent with the above representation when $x \neq y$, since, for example, if $x \in \mathcal{A}^c$, then the leftmost expression of (3.6) is a multiple of $\frac{1}{2}\sum_{\varphi_x=\pm1}\varphi_x = 0$, while the last expression in (3.6) is also zero because there is no way of connecting x and y on a current configuration $\mathbf{n}$ with $\mathbf{n}|_{\mathcal{A}^c} \equiv 0$.

The key observation in the representation (3.6) is that the right-hand side is nonzero only when x and y are connected by a chain of bonds with *odd* currents (see Figure 1). We will exploit this peculiar underlying percolation picture to derive the lace expansion for the two-point function.

3.2 Derivation of the lace expansion

In this subsection, we derive the lace expansion for $\langle\varphi_o\varphi_x\rangle_\Lambda$ using the random-current representation. In Section 3.2.1, we introduce some definition and perform the first stage of the expansion, namely (2.7) for $j = 0$, simply by inclusion-exclusion. In Section 3.2.2, we perform the second stage of the expansion, where the source-switching lemma plays a significant role. Finally, in Section 3.2.3, we complete the proof of Theorem 2.1.

3.2.1 The first stage of the expansion

As mentioned in the previous section, the underlying picture in the random-current representation is quite similar to percolation. We exploit this similarity to obtain the lace expansion.

First, we introduce some notions and notation.

Definition 3.1. (i) Given a current configuration $\mathbf{n} \in \mathbb{Z}_+^{\mathbb{B}_\Lambda}$ and a site set $\mathcal{A} \subset \Lambda$, we say that x is $\mathbf{n}$*-connected to* y *in* $\mathcal{A}$, and write $x \underset{\mathbf{n}}{\longleftrightarrow} y$ *in* $\mathcal{A}$, if either $x = y \in \mathcal{A}$ or there is a path from x to y consisting of bonds $b \in \mathbb{B}_\mathcal{A}$ with $n_b > 0$. If $\mathbf{n} \in \mathbb{Z}_+^{\mathbb{B}_\mathcal{A}}$, we omit "in $\mathcal{A}$" and simply write $x \underset{\mathbf{n}}{\longleftrightarrow} y$. We also define

$$\{x \overset{\mathcal{A}}{\underset{\mathbf{n}}{\longleftrightarrow}} y\} = \{x \underset{\mathbf{n}}{\longleftrightarrow} y\} \setminus \{x \underset{\mathbf{n}}{\longleftrightarrow} y \text{ in } \mathcal{A}^{\mathrm{c}}\}, \tag{3.7}$$

and say that x is $\mathbf{n}$-connected to y *through* $\mathcal{A}$.

(ii) For an event E (i.e., a set of current configurations), we define $\{E \text{ off } b\}$ to be the set of current configurations whose restriction to the bonds other than b are in E. Let $\mathcal{C}_\mathbf{n}^b(x) = \{y \in \Lambda : x \underset{\mathbf{n}}{\longleftrightarrow} y \text{ off } b\}$.

(iii) For a *directed* bond $b = (u, v)$, we write $\underline{b} = u$ and $\overline{b} = v$. We say that a directed bond b is *pivotal* for $x \underset{\mathbf{n}}{\longleftrightarrow} y$ from x, if $\{x \underset{\mathbf{n}}{\longleftrightarrow} \underline{b} \text{ off } b\} \cap \{\overline{b} \underset{\mathbf{n}}{\longleftrightarrow} y \notin \mathcal{C}_\mathbf{n}^b(x)\}$ occurs. If $\{x \underset{\mathbf{n}}{\longleftrightarrow} y\}$ occurs with no pivotal bonds, we say that x is $\mathbf{n}$*-doubly connected to* y, and write $x \underset{\mathbf{n}}{\Longleftrightarrow} y$.

We begin with the first stage of the lace expansion. First, by using the above percolation language, the two-point function can be written as

$$\langle \varphi_o \varphi_x \rangle_\Lambda = \sum_{\partial\mathbf{n}=o\triangle x} \frac{w_\Lambda(\mathbf{n})}{Z_\Lambda} = \sum_{\partial\mathbf{n}=o\triangle x} \frac{w_\Lambda(\mathbf{n})}{Z_\Lambda} \mathbb{1}_{\{o \underset{\mathbf{n}}{\longleftrightarrow} x\}}. \tag{3.8}$$

We decompose the indicator on the right-hand side into two parts depending on whether there is or is not a pivotal bond for $o \underset{\mathbf{n}}{\longleftrightarrow} x$ from o; if there is, we take the *first* bond among them. Then, we have

$$\mathbb{1}_{\{o \underset{\mathbf{n}}{\longleftrightarrow} x\}} = \mathbb{1}_{\{o \underset{\mathbf{n}}{\Longleftrightarrow} x\}} + \sum_{b\in\mathbb{B}_\Lambda} \mathbb{1}_{\{o \underset{\mathbf{n}}{\Longleftrightarrow} \underline{b} \text{ off } b\}} \mathbb{1}_{\{n_b>0\}} \mathbb{1}_{\{\overline{b} \underset{\mathbf{n}}{\longleftrightarrow} x \notin \mathcal{C}_\mathbf{n}^b(o)\}}. \tag{3.9}$$

Let

$$\pi_\Lambda^{(0)}(x) = \sum_{\partial\mathbf{n}=o\triangle x} \frac{w_\Lambda(\mathbf{n})}{Z_\Lambda} \mathbb{1}_{\{o \underset{\mathbf{n}}{\Longleftrightarrow} x\}}. \tag{3.10}$$

Substituting (3.9) into (3.8), we obtain (see Figure 2)

$$\langle \varphi_o \varphi_x \rangle_\Lambda = \pi_\Lambda^{(0)}(x) + \sum_{b\in\mathbb{B}_\Lambda} \sum_{\partial\mathbf{n}=o\triangle x} \frac{w_\Lambda(\mathbf{n})}{Z_\Lambda} \mathbb{1}_{\{o \underset{\mathbf{n}}{\Longleftrightarrow} \underline{b} \text{ off } b\}} \mathbb{1}_{\{n_b>0\}} \mathbb{1}_{\{\overline{b} \underset{\mathbf{n}}{\longleftrightarrow} x \notin \mathcal{C}_\mathbf{n}^b(o)\}}. \tag{3.11}$$

Next, we consider the sum over $\mathbf{n}$ in (3.11). Since b is pivotal for $o \underset{\mathbf{n}}{\longleftrightarrow} x$ from o ($\neq x$, due to the last indicator), n_b is an *odd* integer. We alternate the parity of n_b, with changing the source constraint into $o \triangle b \triangle x \equiv \{o\} \triangle \{\underline{b}, \overline{b}\} \triangle \{x\}$ and multiplying

$$\frac{\sum_{n \text{ odd}} (\beta J_b)^n/n!}{\sum_{n \text{ even}} (\beta J_b)^n/n!} = \tanh(\beta J_b) \equiv \tau_b. \tag{3.12}$$

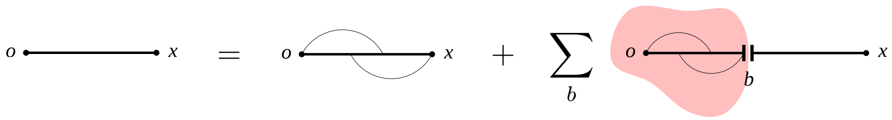

Figure 2: A schematic representation of (3.11). The thick lines are connections consisting of bonds with odd currents, while the thin lines are connections made of bonds with positive (not necessarily odd) currents. The shaded region stands for $\mathcal{C}^b_{\mathbf{n}}(o)$.

Then, the sum over $\mathbf{n}$ in (3.11) equals

$$\sum_{\partial\mathbf{n}=o\triangle b\triangle x} \frac{w_\Lambda(\mathbf{n})}{Z_\Lambda} \mathbb{1}_{\{o \underset{\mathbf{n}}{\Longleftrightarrow} \underline{b} \text{ off } b\}} \tau_b \mathbb{1}_{\{n_b \text{ even}\}} \mathbb{1}_{\{\overline{b} \underset{\mathbf{n}}{\longleftrightarrow} x \notin \mathcal{C}^b_{\mathbf{n}}(o)\}}. \tag{3.13}$$

We note that there are no positive currents on the boundary bonds, except for b, of $\mathcal{C}^b_{\mathbf{n}}(o)$. Let $\bar{\mathcal{A}} \subset \Lambda$ be the set of sites at which there is at least one bond that is incident on $\mathcal{A}$, so that $\mathbb{B}_{\bar{\mathcal{A}}} = \mathbb{B}_\Lambda \setminus \mathbb{B}_{\mathcal{A}^{\rm c}}$. Conditioning on $\mathcal{C}^b_{\mathbf{n}}(o) = \mathcal{A}$ (with denoting $\mathbf{k} = \mathbf{n}|_{\mathcal{A}}$ and $\mathbf{m} = \mathbf{n} - \mathbf{k}$) and then summing over $\mathcal{A} \subset \Lambda$, we can write (3.13) as

$$\begin{aligned}
&\sum_{\mathcal{A}\subset\Lambda} \sum_{\substack{\partial\mathbf{k}=o\triangle\underline{b}\\ \partial\mathbf{m}=\overline{b}\triangle x}} \frac{Z_{\mathcal{A}^{\rm c}}\, w_{\bar{\mathcal{A}}}(\mathbf{k})}{Z_\Lambda} \frac{w_{\mathcal{A}^{\rm c}}(\mathbf{m})}{Z_{\mathcal{A}^{\rm c}}} \mathbb{1}_{\{o \underset{\mathbf{k}}{\Longleftrightarrow} \underline{b} \text{ off } b\}\cap\{\mathcal{C}^b_{\mathbf{k}}(o)=\mathcal{A}\}} \tau_b \mathbb{1}_{\{k_b \text{ even}\}} \mathbb{1}_{\{\overline{b} \underset{\mathbf{m}}{\longleftrightarrow} x \text{ in } \mathcal{A}^{\rm c}\}}\\
&= \sum_{\mathcal{A}\subset\Lambda} \sum_{\partial\mathbf{n}=o\triangle\underline{b}} \frac{w_\Lambda(\mathbf{n})}{Z_\Lambda} \mathbb{1}_{\{o \underset{\mathbf{n}}{\Longleftrightarrow} \underline{b} \text{ off } b\}\cap\{\mathcal{C}^b_{\mathbf{n}}(o)=\mathcal{A}\}} \tau_b \mathbb{1}_{\{n_b \text{ even}\}} \sum_{\partial\mathbf{m}=\overline{b}\triangle x} \frac{w_{\mathcal{A}^{\rm c}}(\mathbf{m})}{Z_{\mathcal{A}^{\rm c}}} \mathbb{1}_{\{\overline{b} \underset{\mathbf{m}}{\longleftrightarrow} x\}}\\
&= \sum_{\mathcal{A}\subset\Lambda} \sum_{\partial\mathbf{n}=o\triangle\underline{b}} \frac{w_\Lambda(\mathbf{n})}{Z_\Lambda} \mathbb{1}_{\{o \underset{\mathbf{n}}{\Longleftrightarrow} \underline{b} \text{ off } b\}\cap\{\mathcal{C}^b_{\mathbf{n}}(o)=\mathcal{A}\}} \tau_b \mathbb{1}_{\{n_b \text{ even}\}} \langle \varphi_{\overline{b}}\varphi_x \rangle_{\mathcal{A}^{\rm c}}\\
&= \sum_{\partial\mathbf{n}=o\triangle\underline{b}} \frac{w_\Lambda(\mathbf{n})}{Z_\Lambda} \mathbb{1}_{\{o \underset{\mathbf{n}}{\Longleftrightarrow} \underline{b} \text{ off } b\}} \tau_b \mathbb{1}_{\{n_b \text{ even}\}} \langle \varphi_{\overline{b}}\varphi_x \rangle_{\mathcal{C}^b_{\mathbf{n}}(o)^{\rm c}},
\end{aligned} \tag{3.14}$$

where we have omitted "in $\mathcal{A}^{\rm c}$" in the second line, due to the abbreviation rule in Definition 3.1(i). Since $\langle \varphi_{\overline{b}}\varphi_x \rangle_{\mathcal{C}^b_{\mathbf{n}}(o)^{\rm c}}$ is zero on the event $\{o \underset{\mathbf{n}}{\Longleftrightarrow} \underline{b}\} \setminus \{o \underset{\mathbf{n}}{\Longleftrightarrow} \underline{b} \text{ off } b\} \subset \{\overline{b} \in \mathcal{C}^b_{\mathbf{n}}(o)\}$, we can omit "off b" in the last line of (3.14). Moreover, with the help of the source constraint $\partial\mathbf{n} = o \triangle \underline{b}$, we can also omit $\mathbb{1}_{\{n_b \text{ even}\}}$. (If n_b is odd, then again $\overline{b}$ is required to be in $\mathcal{C}^b_{\mathbf{n}}(o)$, since $\overline{b}$ is not a source.) Therefore, (3.14) equals

$$\sum_{\partial\mathbf{n}=o\triangle\underline{b}} \frac{w_\Lambda(\mathbf{n})}{Z_\Lambda} \mathbb{1}_{\{o \underset{\mathbf{n}}{\Longleftrightarrow} \underline{b}\}} \tau_b \langle \varphi_{\overline{b}}\varphi_x \rangle_{\mathcal{C}^b_{\mathbf{n}}(o)^{\rm c}}. \tag{3.15}$$

By (3.11) and (3.15), we arrive at

$$\langle \varphi_o\varphi_x \rangle_\Lambda = \pi^{(0)}_\Lambda(x) + \sum_{b\in\mathbb{B}_\Lambda} \pi^{(0)}_\Lambda(\underline{b})\, \tau_b \langle \varphi_{\overline{b}}\varphi_x \rangle_\Lambda - R^{(1)}_\Lambda(x), \tag{3.16}$$

where

$$R_\Lambda^{(1)}(x) = \sum_{b\in\mathbb{B}_\Lambda} \sum_{\partial\mathbf{n}=o\triangle\underline{b}} \frac{w_\Lambda(\mathbf{n})}{Z_\Lambda} \mathbb{1}_{\{o \underset{\mathbf{n}}{\Longleftrightarrow} \underline{b}\}} \tau_b \Big(\langle\varphi_{\bar{b}}\varphi_x\rangle_\Lambda - \langle\varphi_{\bar{b}}\varphi_x\rangle_{\mathcal{C}^b_{\mathbf{n}}(o)^{\mathrm{c}}} \Big). \tag{3.17}$$

This completes the proof of (2.7) for $j = 0$, with $\pi_\Lambda^{(0)}(x)$ defined in (3.10) and $R_\Lambda^{(1)}(x)$ defined in (3.17).

3.2.2 The second stage of the expansion

To expand $R_\Lambda^{(1)}(x)$ further, we investigate the difference $\langle\varphi_{\bar{b}}\varphi_x\rangle_\Lambda - \langle\varphi_{\bar{b}}\varphi_x\rangle_{\mathcal{C}^b_{\mathbf{n}}(o)^{\mathrm{c}}}$ in (3.17). First, we prove the following key proposition[3]:

Proposition 3.2. *For $v, x \in \Lambda$ and $\mathcal{A} \subset \Lambda$, we have*

$$\langle\varphi_v\varphi_x\rangle_\Lambda - \langle\varphi_v\varphi_x\rangle_{\mathcal{A}^{\mathrm{c}}} = \sum_{\substack{\partial\mathbf{m}=\varnothing \\ \partial\mathbf{n}=v\triangle x}} \frac{w_{\mathcal{A}^{\mathrm{c}}}(\mathbf{m})}{Z_{\mathcal{A}^{\mathrm{c}}}} \frac{w_\Lambda(\mathbf{n})}{Z_\Lambda} \mathbb{1}_{\{v \underset{\mathbf{m}+\mathbf{n}}{\overset{\mathcal{A}}{\longleftrightarrow}} x\}}. \tag{3.18}$$

Proof. Since both sides of (3.18) are equal to $\mathbb{1}_{\{x\in\mathcal{A}\}}$ when $v = x$ (see below (3.6)), it suffices to prove (3.18) when $v \neq x$.

First, we let

$$\tilde{Z} = Z_\Lambda - Z_{\mathcal{A}^{\mathrm{c}}} = \sum_{\partial\mathbf{n}=\varnothing} w_\Lambda(\mathbf{n}) - \sum_{\substack{\partial\mathbf{n}=\varnothing \\ \mathbf{n}|_{\mathcal{A}}\equiv 0}} w_\Lambda(\mathbf{n}) = \sum_{\substack{\partial\mathbf{n}=\varnothing \\ \mathbf{n}|_{\mathcal{A}}\not\equiv 0}} w_\Lambda(\mathbf{n}), \tag{3.19}$$

where we have used the representation (3.5). Similarly, we let $W_{\mathcal{A}^{\mathrm{c}}} = Z_{\mathcal{A}^{\mathrm{c}}}\langle\varphi_v\varphi_x\rangle_{\mathcal{A}^{\mathrm{c}}}$ and

$$\tilde{W} = Z_\Lambda \langle\varphi_v\varphi_x\rangle_\Lambda - W_{\mathcal{A}^{\mathrm{c}}} = \sum_{\partial\mathbf{n}=\{v,x\}} w_\Lambda(\mathbf{n}) - \sum_{\substack{\partial\mathbf{n}=\{v,x\} \\ \mathbf{n}|_{\mathcal{A}}\equiv 0}} w_\Lambda(\mathbf{n}) = \sum_{\substack{\partial\mathbf{n}=\{v,x\} \\ \mathbf{n}|_{\mathcal{A}}\not\equiv 0}} w_\Lambda(\mathbf{n}), \tag{3.20}$$

where we have used (3.6). Then, we obtain

$$\langle\varphi_v\varphi_x\rangle_\Lambda - \langle\varphi_v\varphi_x\rangle_{\mathcal{A}^{\mathrm{c}}} = \frac{W_{\mathcal{A}^{\mathrm{c}}} + \tilde{W}}{Z_{\mathcal{A}^{\mathrm{c}}} + \tilde{Z}} - \frac{W_{\mathcal{A}^{\mathrm{c}}}}{Z_{\mathcal{A}^{\mathrm{c}}}} = \frac{Z_{\mathcal{A}^{\mathrm{c}}}\tilde{W} - W_{\mathcal{A}^{\mathrm{c}}}\tilde{Z}}{Z_{\mathcal{A}^{\mathrm{c}}}Z_\Lambda}, \tag{3.21}$$

where the numerator is

$$Z_{\mathcal{A}^{\mathrm{c}}}\tilde{W} - W_{\mathcal{A}^{\mathrm{c}}}\tilde{Z} = \sum_{\substack{\partial\mathbf{m}=\varnothing,\ \mathbf{m}|_{\mathcal{A}}\equiv 0 \\ \partial\mathbf{n}=\{v,x\},\ \mathbf{n}|_{\mathcal{A}}\not\equiv 0}} w_\Lambda(\mathbf{m})\, w_\Lambda(\mathbf{n}) - \sum_{\substack{\partial\mathbf{m}=\{v,x\},\ \mathbf{m}|_{\mathcal{A}}\equiv 0 \\ \partial\mathbf{n}=\varnothing,\ \mathbf{n}|_{\mathcal{A}}\not\equiv 0}} w_\Lambda(\mathbf{m})\, w_\Lambda(\mathbf{n}). \tag{3.22}$$

We note that the only difference between these two terms is the alternation of the source constraints.

[3] The differential inequalities mentioned in the footnote at the end of the paragraph below (2.18) can be derived, under the free-boundary condition as well, by simply using Proposition 3.2, instead of using the random-walk representation introduced in [2, 3, 4].

Next, we consider the second term of (3.22), whose explicit form is

$$\sum_{\substack{\partial\mathbf{m}=\{v,x\},\ \mathbf{m}|_{\mathcal{A}}\equiv 0\\ \partial\mathbf{n}=\varnothing,\ \mathbf{n}|_{\mathcal{A}}\not\equiv 0}} \Bigg(\prod_{b\in\mathbb{B}_\Lambda\setminus\mathbb{B}_{\mathcal{A}^c}} \frac{(\beta J_b)^{n_b}}{n_b!}\Bigg)\Bigg(\prod_{b\in\mathbb{B}_{\mathcal{A}^c}} \frac{(\beta J_b)^{m_b+n_b}}{m_b!\,n_b!}\Bigg) = \sum_{\substack{\partial\mathbf{N}=\{v,x\}\\ \mathbf{N}|_{\mathcal{A}}\not\equiv 0}} w_\Lambda(\mathbf{N}) \sum_{\substack{\partial\mathbf{m}=\{v,x\}\\ \mathbf{m}|_{\mathcal{A}}\equiv 0}} \prod_{b\in\mathbb{B}_{\mathcal{A}^c}} \binom{N_b}{m_b}. \tag{3.23}$$

The following is a variant of the source-switching lemma [1, 12] and allows us to change the source constraints in (3.23).

Lemma 3.3 (**Source-switching lemma**).

$$\sum_{\substack{\partial\mathbf{m}=\{v,x\}\\ \mathbf{m}|_{\mathcal{A}}\equiv 0}} \prod_{b\in\mathbb{B}_{\mathcal{A}^c}} \binom{N_b}{m_b} = \mathbb{1}_{\{v \underset{\mathbf{N}}{\longleftrightarrow} x \text{ in } \mathcal{A}^c\}} \sum_{\substack{\partial\mathbf{m}=\varnothing\\ \mathbf{m}|_{\mathcal{A}}\equiv 0}} \prod_{b\in\mathbb{B}_{\mathcal{A}^c}} \binom{N_b}{m_b}. \tag{3.24}$$

We refer the readers to [1, Lemma 3.1] for more general cases in which the number of sources in $\mathbf{m}$ is more than two. Lemma 3.3 will be explained after completing the proof of Proposition 3.2.

We continue with the proof of Proposition 3.2. Substituting (3.24) into (3.23), we obtain

$$\sum_{\substack{\partial\mathbf{N}=\{v,x\}\\ \mathbf{N}|_{\mathcal{A}}\not\equiv 0}} w_\Lambda(\mathbf{N})\, \mathbb{1}_{\{v \underset{\mathbf{N}}{\longleftrightarrow} x \text{ in } \mathcal{A}^c\}} \sum_{\substack{\partial\mathbf{m}=\varnothing\\ \mathbf{m}|_{\mathcal{A}}\equiv 0}} \prod_{b\in\mathbb{B}_{\mathcal{A}^c}} \binom{N_b}{m_b} = \sum_{\substack{\partial\mathbf{m}=\varnothing,\ \mathbf{m}|_{\mathcal{A}}\equiv 0\\ \partial\mathbf{n}=\{v,x\},\ \mathbf{n}|_{\mathcal{A}}\not\equiv 0}} w_\Lambda(\mathbf{m})\, w_\Lambda(\mathbf{n})\, \mathbb{1}_{\{v \underset{\mathbf{m}+\mathbf{n}}{\longleftrightarrow} x \text{ in } \mathcal{A}^c\}}. \tag{3.25}$$

Note that the source constraint in the right-hand side is identical to that in the first term of (3.22), under which $\mathbb{1}_{\{v \underset{\mathbf{m}+\mathbf{n}}{\longleftrightarrow} x\}}$ is always 1. Using (3.7), we can rewrite (3.21) as

$$\langle \varphi_v\varphi_x\rangle_\Lambda - \langle \varphi_v\varphi_x\rangle_{\mathcal{A}^c} = \sum_{\substack{\partial\mathbf{m}=\varnothing,\ \mathbf{m}|_{\mathcal{A}}\equiv 0\\ \partial\mathbf{n}=\{v,x\},\ \mathbf{n}|_{\mathcal{A}}\not\equiv 0}} \frac{w_\Lambda(\mathbf{m})}{Z_{\mathcal{A}^c}} \frac{w_\Lambda(\mathbf{n})}{Z_\Lambda}\, \mathbb{1}_{\{v \underset{\mathbf{m}+\mathbf{n}}{\overset{\mathcal{A}}{\longleftrightarrow}} x\}}. \tag{3.26}$$

We can omit "$\mathbf{n}|_{\mathcal{A}} \not\equiv 0$" because $\mathbb{1}_{\{v \underset{\mathbf{m}+\mathbf{n}}{\overset{\mathcal{A}}{\longleftrightarrow}} x\}} = 0$ when $\mathbf{m}|_{\mathcal{A}} \equiv \mathbf{n}|_{\mathcal{A}} \equiv 0$ and $v \neq x$. Finally, by using (3.3) and (3.5) to replace $w_\Lambda(\mathbf{m})$ in (3.26) with $w_{\mathcal{A}^c}(\mathbf{m})$ and omit $\mathbf{m}|_{\mathcal{A}} \equiv 0$, we arrive at (3.18). This completes the proof of Proposition 3.2. ☐

Sketch of the proof of Lemma 3.3. We briefly explain the meaning of the identity (3.24) and the idea of its proof. Given $\mathbf{N} = \{N_b\}_{b\in\mathbb{B}_\Lambda}$, we denote by $\mathbb{G}_{\mathbf{N}}$ the graph consisting of N_b *labeled* edges between $\underline{b}$ and $\overline{b}$ for every $b \in \mathbb{B}_\Lambda$ (see Figure 3). For a subgraph $\mathbb{S} \subset \mathbb{G}_{\mathbf{N}}$, we denote by $\partial\mathbb{S}$ the set of vertices at which the total number of incident edges in $\mathbb{S}$ is *odd*, and by $\mathbb{S}|_{\mathcal{A}}$ the subgraph consisting of all edges in $\mathbb{S}$ that are incident on $\mathcal{A}$. Then, the left-hand side of (3.24) is equivalent to the cardinality $|\mathfrak{S}|$ of

$$\mathfrak{S} = \{\mathbb{S} \subset \mathbb{G}_{\mathbf{N}} : \partial\mathbb{S} = \{v, x\},\ \mathbb{S}|_{\mathcal{A}} = \varnothing\}, \tag{3.27}$$

and the sum in the right-hand side of (3.24) is the cardinality $|\mathfrak{S}'|$ of

$$\mathfrak{S}' = \{\mathbb{S} \subset \mathbb{G}_{\mathbf{N}} : \partial\mathbb{S} = \varnothing,\ \mathbb{S}|_{\mathcal{A}} = \varnothing\}. \tag{3.28}$$

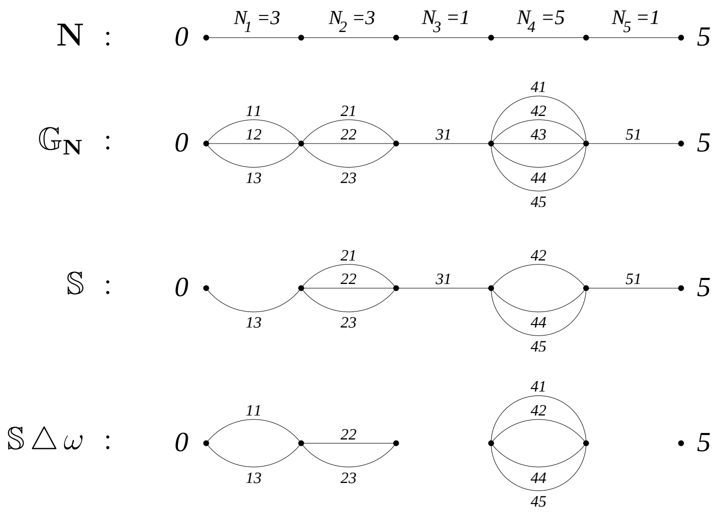

Figure 3: $\mathbf{N} = \{N_b\}_{b=1}^5 = (3,3,1,5,1)$ is an example of a current configuration on $[0,5]\cap\mathbb{Z}_+$ satisfying $\partial\mathbf{N} = \{0,5\}$, and $\mathbb{G}_\mathbf{N}$ is the corresponding labeled graph consisting of edges $e = b\ell_b$, where $\ell_b \in \{1,\dots,N_b\}$. The third and fourth pictures show the relation between a subgraph $\mathbb{S}$ with $\partial\mathbb{S} = \{0,5\}$ and its image $\mathbb{S}\,\triangle\,\omega$ in (3.29), where ω is a path of edges $(11,21,31,41,51)$.

We note that $|\mathfrak{S}|$ is zero when there are no paths on $\mathbb{G}_\mathbf{N}$ between v and x consisting of edges whose both endvertices are in $\mathcal{A}^c$, while $|\mathfrak{S}'|$ may not be zero. The identity (3.24) reads that $|\mathfrak{S}|$ equals $|\mathfrak{S}'|$ if we compensate this discrepancy.

Suppose that there is a path ω from v to x consisting of edges in $\mathbb{G}_\mathbf{N}$ whose both endvertices are in $\mathcal{A}^c$. Then, the map

$$\mathbb{S}\in\mathfrak{S} \;\mapsto\; \mathbb{S}\,\triangle\,\omega \in \mathfrak{S}' \tag{3.29}$$

is a bijection [1, 12], and therefore $|\mathfrak{S}| = |\mathfrak{S}'|$. This implies (3.24). $\square$

We now start with the second stage of the expansion by using Proposition 3.2 and applying inclusion-exclusion as in the first stage of the expansion in Section 3.2.1. First, we decompose the indicator in (3.18) into two parts depending on whether there is or is not a pivotal bond b for $v \xleftrightarrow[\mathbf{m}+\mathbf{n}]{} x$ from v such that $v \xleftrightarrow[\mathbf{m}+\mathbf{n}]{\mathcal{A}} \underline{b}$. Let

$$E_\mathbf{n}(v,x;\mathcal{A}) = \{v \xleftrightarrow[\mathbf{n}]{\mathcal{A}} x\} \cap \{\nexists \text{ pivotal bond } b \text{ for } v \xleftrightarrow[\mathbf{n}]{} x \text{ from } v \text{ such that } v \xleftrightarrow[\mathbf{n}]{\mathcal{A}} \underline{b}\}. \tag{3.30}$$

On the event $\{v \xleftrightarrow[\mathbf{m}+\mathbf{n}]{\mathcal{A}} x\} \setminus E_{\mathbf{m}+\mathbf{n}}(v,x;\mathcal{A})$, we take the *first* pivotal bond b for $v \xleftrightarrow[\mathbf{m}+\mathbf{n}]{} x$ from v satisfying $v \xleftrightarrow[\mathbf{m}+\mathbf{n}]{\mathcal{A}} \underline{b}$. Similarly to (3.9), we have

$$\mathbb{1}_{\{v \xleftrightarrow[\mathbf{m}+\mathbf{n}]{\mathcal{A}} x\}} = \mathbb{1}_{E_{\mathbf{m}+\mathbf{n}}(v,x;\mathcal{A})} + \sum_{b\in\mathbb{B}_\Lambda} \mathbb{1}_{\{E_{\mathbf{m}+\mathbf{n}}(v,\underline{b};\mathcal{A}) \text{ off } b\}}\, \mathbb{1}_{\{m_b+n_b>0\}}\, \mathbb{1}_{\{\overline{b} \xleftrightarrow[\mathbf{m}+\mathbf{n}]{} x \notin \mathcal{C}^b_{\mathbf{m}+\mathbf{n}}(v)\}}. \tag{3.31}$$

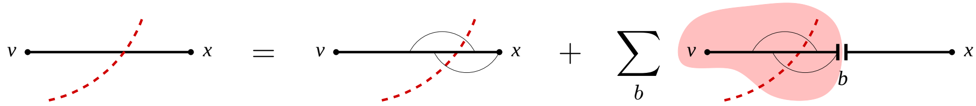

Figure 4: A schematic representation of (3.33). The dashed lines stand for $\mathcal{A}$, the thick-solid lines stand for connections consisting of bonds b' such that $m_{b'}+n_{b'}$ is odd, and the thin-solid lines are connections made of bonds b'' such that $m_{b''}+n_{b''}$ is positive (not necessarily odd). The shaded region stands for $\mathcal{C}^b_{\mathbf{m}+\mathbf{n}}(v)$.

Let

$$\Theta_{v,x;\mathcal{A}}[X]=\sum_{\substack{\partial\mathbf{m}=\varnothing\\ \partial\mathbf{n}=v\triangle x}}\frac{w_{\mathcal{A}^{\rm c}}(\mathbf{m})}{Z_{\mathcal{A}^{\rm c}}}\frac{w_\Lambda(\mathbf{n})}{Z_\Lambda}\mathbb{1}_{E_{\mathbf{m}+\mathbf{n}}(v,x;\mathcal{A})}X(\mathbf{m}+\mathbf{n}),\qquad \Theta_{v,x;\mathcal{A}}=\Theta_{v,x;\mathcal{A}}[1]. \tag{3.32}$$

Then, by substituting (3.31) into (3.18), we obtain (see Figure 4)

$$\begin{aligned}&\langle\varphi_v\varphi_x\rangle_\Lambda-\langle\varphi_v\varphi_x\rangle_{\mathcal{A}^{\rm c}} \\ &=\Theta_{v,x;\mathcal{A}}+\sum_{b\in\mathbb{B}_\Lambda}\sum_{\substack{\partial\mathbf{m}=\varnothing\\ \partial\mathbf{n}=v\triangle x}}\frac{w_{\mathcal{A}^{\rm c}}(\mathbf{m})}{Z_{\mathcal{A}^{\rm c}}}\frac{w_\Lambda(\mathbf{n})}{Z_\Lambda}\mathbb{1}_{\{E_{\mathbf{m}+\mathbf{n}}(v,\underline{b};\mathcal{A})\text{ off }b\}}\mathbb{1}_{\{m_b\text{ even},\,n_b\text{ odd}\}}\mathbb{1}_{\{\overline{b}\underset{\mathbf{m}+\mathbf{n}}{\longleftrightarrow}x\notin\mathcal{C}^b_{\mathbf{m}+\mathbf{n}}(v)\}},\end{aligned}\tag{3.33}$$

where we have replaced "$m_b+n_b>0$" by "m_b even, n_b odd" that is the only possible combination which is consistent with the source constraints and the conditions in the indicators. As in (3.13), we alternate the parity of n_b, with changing the source constraint into $\partial\mathbf{n}=v\triangle b\triangle x$ and multiplying τ_b. Then, as in (3.14), by conditioning on $\mathcal{C}^b_{\mathbf{m}+\mathbf{n}}(v)=\mathcal{B}$, multiplying $(Z_{\mathcal{B}^{\rm c}}/Z_{\mathcal{B}^{\rm c}})^2\equiv1$ and summing over $\mathcal{B}\subset\Lambda$, we can rewrite the sum over $\mathbf{m},\mathbf{n}$ in (3.33) by

$$\begin{aligned}&\sum_{\mathcal{B}\subset\Lambda}\sum_{\substack{\partial\mathbf{m}=\varnothing\\ \partial\mathbf{n}=v\triangle b\triangle x}}\frac{w_{\mathcal{A}^{\rm c}}(\mathbf{m})}{Z_{\mathcal{A}^{\rm c}}}\frac{w_\Lambda(\mathbf{n})}{Z_\Lambda}\mathbb{1}_{\{E_{\mathbf{m}+\mathbf{n}}(v,\underline{b};\mathcal{A})\text{ off }b\}\cap\{\mathcal{C}^b_{\mathbf{m}+\mathbf{n}}(v)=\mathcal{B}\}}\tau_b\mathbb{1}_{\{m_b,n_b\text{ even}\}}\mathbb{1}_{\{\overline{b}\underset{\mathbf{m}+\mathbf{n}}{\longleftrightarrow}x\text{ in }\mathcal{B}^{\rm c}\}} \\ &=\sum_{\mathcal{B}\subset\Lambda}\sum_{\substack{\partial\mathbf{m}=\varnothing\\ \partial\mathbf{n}=v\triangle\underline{b}}}\frac{w_{\mathcal{A}^{\rm c}}(\mathbf{m})}{Z_{\mathcal{A}^{\rm c}}}\frac{w_\Lambda(\mathbf{n})}{Z_\Lambda}\mathbb{1}_{\{E_{\mathbf{m}+\mathbf{n}}(v,\underline{b};\mathcal{A})\text{ off }b\}\cap\{\mathcal{C}^b_{\mathbf{m}+\mathbf{n}}(v)=\mathcal{B}\}}\tau_b\mathbb{1}_{\{m_b,n_b\text{ even}\}}\langle\varphi_{\overline{b}}\varphi_x\rangle_{\mathcal{B}^{\rm c}} \\ &=\sum_{\substack{\partial\mathbf{m}=\varnothing\\ \partial\mathbf{n}=v\triangle\underline{b}}}\frac{w_{\mathcal{A}^{\rm c}}(\mathbf{m})}{Z_{\mathcal{A}^{\rm c}}}\frac{w_\Lambda(\mathbf{n})}{Z_\Lambda}\mathbb{1}_{E_{\mathbf{m}+\mathbf{n}}(v,\underline{b};\mathcal{A})}\tau_b\langle\varphi_{\overline{b}}\varphi_x\rangle_{\mathcal{C}^b_{\mathbf{m}+\mathbf{n}}(v)^{\rm c}},\end{aligned}\tag{3.34}$$

where we have omitted "off b" and $\mathbb{1}_{\{m_b,n_b\text{ even}\}}$ in the last line using the source constraints on $\mathbf{m},\mathbf{n}$ and the fact that $\langle\varphi_{\overline{b}}\varphi_x\rangle_{\mathcal{C}^b_{\mathbf{m}+\mathbf{n}}(v)^{\rm c}}$ is zero whenever $\overline{b}\in\mathcal{C}^b_{\mathbf{m}+\mathbf{n}}(v)$ (cf., the derivation of (3.15) from (3.14)). By (3.32)–(3.34), we finally arrive at

$$\begin{aligned}\langle\varphi_v\varphi_x\rangle_\Lambda-\langle\varphi_v\varphi_x\rangle_{\mathcal{A}^{\rm c}}=\Theta_{v,x;\mathcal{A}}&+\sum_{b\in\mathbb{B}_\Lambda}\Theta_{v,\underline{b};\mathcal{A}}\tau_b\langle\varphi_{\overline{b}}\varphi_x\rangle_\Lambda \\ &-\sum_{b\in\mathbb{B}_\Lambda}\tau_b\Theta_{v,\underline{b};\mathcal{A}}\Big[\langle\varphi_{\overline{b}}\varphi_x\rangle_\Lambda-\langle\varphi_{\overline{b}}\varphi_x\rangle_{\mathcal{C}^b_{\mathbf{m}+\mathbf{n}}(v)^{\rm c}}\Big],\end{aligned}\tag{3.35}$$

where $\mathcal{C}^b_{\mathbf{m}+\mathbf{n}}(v)$ is the variable of the operation $\Theta_{v,\underline{b};\mathcal{A}}[\,\cdots]$. This completes the second stage of the expansion.

3.2.3 Completion of the lace expansion

For notational convenience, we let $w_\varnothing(\mathbf{m})/Z_\varnothing = \mathbb{1}_{\{\mathbf{m}\equiv 0\}}$. Then, since $E_{\mathbf{n}}(o,x;\Lambda) = \{o \underset{\mathbf{n}}{\Longleftrightarrow} x\}$ (cf., (3.30)), we can write

$$\pi^{(0)}_\Lambda(x) = \Theta_{o,x;\Lambda}. \tag{3.36}$$

Repeated application of (3.35) to (3.16)–(3.17) results in (2.6)–(2.7) in Theorem 2.1 with, for $j \ge 1$,

$$\pi^{(j)}_\Lambda(x) = \sum_{b_1,\dots,b_j} \Theta^{(0)}_{o,\underline{b}_1;\Lambda}\Big[\tau_{b_1}\Theta^{(1)}_{\overline{b}_1,\underline{b}_2;\tilde{\mathcal{C}}_0}\Big[\cdots\tau_{b_{j-1}}\Theta^{(j-1)}_{\overline{b}_{j-1},\underline{b}_j;\tilde{\mathcal{C}}_{j-2}}\Big[\tau_{b_j}\Theta^{(j)}_{\overline{b}_j,x;\tilde{\mathcal{C}}_{j-1}}\Big]\cdots\Big]\Big], \tag{3.37}$$

$$R^{(j)}_\Lambda(x) = \sum_{b_1,\dots,b_j} \Theta^{(0)}_{o,\underline{b}_1;\Lambda}\Big[\tau_{b_1}\Theta^{(1)}_{\overline{b}_1,\underline{b}_2;\tilde{\mathcal{C}}_0}\Big[\cdots\tau_{b_{j-1}}\Theta^{(j-1)}_{\overline{b}_{j-1},\underline{b}_j;\tilde{\mathcal{C}}_{j-2}}\Big[\tau_{b_j}\Big(\langle\varphi_{\overline{b}_j}\varphi_x\rangle_\Lambda - \langle\varphi_{\overline{b}_j}\varphi_x\rangle_{\tilde{\mathcal{C}}^{\mathrm{c}}_{j-1}}\Big)\Big]\cdots\Big]\Big], \tag{3.38}$$

where the operation $\Theta^{(i)}$ determines the variable $\tilde{\mathcal{C}}_i = \mathcal{C}^{b_{i+1}}_{\mathbf{m}_i+\mathbf{n}_i}(\overline{b}_i)$ (provided that $\overline{b}_0 = o$).

If every J_b is nonnegative, then, by definition, τ_b and $w_{\mathcal{A}}(\mathbf{n})$ for any $\mathcal{A} \subset \Lambda$ and $\mathbf{n} \in \mathbb{Z}_+^{\mathbb{B}_{\mathcal{A}}}$ are nonnegative. This proves the first inequality in (2.8) and the nonnegativity of $R^{(j+1)}_\Lambda(x)$. To prove the upper bound on $R^{(j+1)}_\Lambda(x)$, we simply ignore $\langle\varphi_{\overline{b}_j}\varphi_x\rangle_{\tilde{\mathcal{C}}^{\mathrm{c}}_{j-1}}$ in (3.38) and replace j by $j+1$, where $b_{j+1} = \{u,v\}$. This completes the proof of Theorem 2.1. □

4 Bounds on the expansion coefficients

From now on, we assume that the spin-spin coupling is nonnegative. Then, by (2.8), we only need to control the expansion coefficients (3.36)–(3.37). In this section, we prove diagrammatic bounds on the expansion coefficients, and then apply these bounds to prove Proposition 2.4.

Before going into details, we compare the expansion coefficients (3.36)–(3.37) for the Ising model with those for percolation; the j^{th}-expansion coefficient for percolation is (cf., [16])

$$\pi^{(j)}_p(x) = \begin{cases} \mathbb{E}^{(0)}_p\big[\mathbb{1}_{\{o \underset{\mathbf{n}_0}{\Longleftrightarrow} x\}}\big] \equiv \mathbb{P}_p(o \Longleftrightarrow x) & (j=0), \\ \displaystyle\sum_{b_1,\dots,b_j} \mathbb{E}^{(0)}_p\Big[\mathbb{1}_{\{o \underset{\mathbf{n}_0}{\Longleftrightarrow} \underline{b}_1\}} p_{b_1}\mathbb{E}^{(1)}_p\Big[\mathbb{1}_{E_{\mathbf{n}_1}(\overline{b}_1,\underline{b}_2;\tilde{\mathcal{C}}_0)}\cdots p_{b_j}\mathbb{E}^{(j)}_p\Big[\mathbb{1}_{E_{\mathbf{n}_j}(\overline{b}_j,x;\tilde{\mathcal{C}}_{j-1})}\Big]\cdots\Big]\Big] & (j \ge 1), \end{cases} \tag{4.1}$$

where each $\mathbb{E}^{(i)}_p$ denotes the expectation with respect to the product of the Bernoulli measures $\prod_b(p_b\mathbb{1}_{\{\mathbf{n}_i(b)=1\}} + (1-p_b)\mathbb{1}_{\{\mathbf{n}_i(b)=0\}})$, with the bond-occupation parameter $p = \sum_{x\in\mathbb{Z}^d} p_{o,x}$ (see (2.15)). Since we exploited the underlying percolation picture to derive (2.7) for the Ising model, it is not so surprising that the expansion coefficients for both models are quite similar; in particular, the events involved in (3.36)–(3.37) are identical to those in (4.1). However, they are indeed different. The major differences between these two models are the following:

(a) Each current configuration must satisfy not only the conditions in the indicators, but also its source constraint that is absent in percolation.

(b) An operation Θ is not an expectation, since the source constraints in the numerator and denominator in the definition (3.32) of Θ are different.

(c) In each $\Theta^{(i)}$ for $i \geq 1$, the sum $\mathbf{m}_i + \mathbf{n}_i$ of two current configurations is coupled with $\mathbf{m}_{i-1}+\mathbf{n}_{i-1}$ via the cluster $\tilde{\mathcal{C}}_{i-1}$ determined by $\mathbf{m}_{i-1}+\mathbf{n}_{i-1}$. (By contrast, in each $\mathbb{E}_p^{(i)}$ in (4.1), a single percolation configuration $\mathbf{n}_i$ is coupled with $\mathbf{n}_{i-1}$ via $\tilde{\mathcal{C}}_{i-1} = \mathcal{C}^{b_i}_{\mathbf{n}_{i-1}}(\overline{b}_{i-1})$.) In addition, $\mathbf{m}_i$ is nonzero only on bonds in $\mathbb{B}_{\tilde{\mathcal{C}}^{\rm c}_{i-1}}$, while the current configuration $\mathbf{n}_i$ has no such restriction.

Take $\pi_\Lambda^{(0)}(x)$ for example, which is

$$\pi_\Lambda^{(0)}(x) = \frac{\sum_{\partial\mathbf{n}=o\triangle x} w_\Lambda(\mathbf{n})\, \mathbb{1}_{\{o \underset{\mathbf{n}}{\Longleftrightarrow} x\}}}{\sum_{\partial\mathbf{n}=\varnothing} w_\Lambda(\mathbf{n})}. \tag{4.2}$$

Due to the indicator function, every current configuration $\mathbf{n} \in \mathbb{Z}_+^{\mathbb{B}_\Lambda}$ that gives nonzero contribution to the numerator has at least *two bond-disjoint* paths ζ_1, ζ_2 from o to x such that $n_b > 0$ for all $b \in \zeta_1 \dot\cup \zeta_2$. Also, due to the source constraint, there should be at least one path ζ from o to x such that n_b is odd for all $b \in \zeta$. Suppose, for example, that $\zeta = \zeta_1$ and that n_b for $b \in \zeta_2$ are all positive-even. Since a positive-even integer can split into two odd integers, on the labeled graph $\mathbb{G}_\mathbf{n}$ with $\partial\mathbb{G}_\mathbf{n} = o \triangle x$ (recall the notation introduced above (3.27)) there are at least *three edge-disjoint* paths from o to x. This observation leads us to expect that $\pi_\Lambda^{(0)}(x)$ is bounded by $\langle\varphi_o\varphi_x\rangle_\Lambda^3$, as in (2.14), for the ferromagnetic Ising model.

To state bounds on the expansion coefficients, we first introduce diagrammatic functions consisting of two-point functions. Let

$$\tilde{G}_\Lambda(y,x) = \sum_{b:\underline{b}=y} \tau_b \langle\varphi_{\overline{b}}\varphi_x\rangle_\Lambda = \sum_{b:\overline{b}=x} \langle\varphi_y\varphi_{\underline{b}}\rangle_\Lambda \tau_b. \tag{4.3}$$

We note that $\langle\varphi_y\varphi_x\rangle_\Lambda \leq \tilde{G}_\Lambda(y,x)$ for $y \neq x$, since[4]

$$\langle\varphi_y\varphi_x\rangle_\Lambda \leq \delta_{y,x} + \sum_{b:\underline{b}=y} \sum_{\substack{\partial\mathbf{n}=y\triangle x \\ n_b \text{ odd}}} \frac{w_\Lambda(\mathbf{n})}{Z_\Lambda} = \delta_{y,x} + \sum_{b:\underline{b}=y} \tau_b \sum_{\substack{\partial\mathbf{n}=\overline{b}\triangle x \\ n_b \text{ even}}} \frac{w_\Lambda(\mathbf{n})}{Z_\Lambda} \leq \delta_{y,x} + \tilde{G}_\Lambda(y,x). \tag{4.4}$$

Using this notation, we let

$$\psi_\Lambda(y,x) = \sum_{j=0}^\infty \big(\tilde{G}_\Lambda^2\big)^{*j}(y,x) \equiv \delta_{y,x} + \sum_{j=1}^\infty \sum_{\substack{u_0,\dots,u_j \\ u_0=y,\, u_j=x}} \prod_{l=1}^j \tilde{G}_\Lambda(u_{l-1},u_l)^2, \tag{4.5}$$

and define (see the first line in Figure 5)

$$P_\Lambda^{(1)}(v_1,v_1') = \big(\psi_\Lambda(v_1,v_1') - \delta_{v_1,v_1'}\big)\, \langle\varphi_{v_1}\varphi_{v_1'}\rangle_\Lambda, \tag{4.6}$$

[4]Repeated application of (4.4) results in the random-walk bound: $\langle\varphi_o\varphi_x\rangle_\Lambda \leq \sum_{n=0}^\infty \tau^n D^{*n}(x)$ for $\Lambda \subset \mathbb{Z}^d$, $\tau \leq 1$ and $d > 2$.

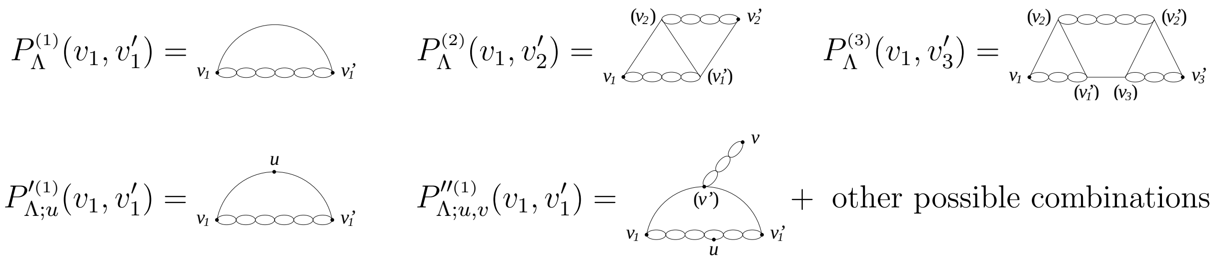

Figure 5: Schematic representations of $P^{(j)}_\Lambda(v_1, v_j')$ for $j = 1, 2, 3$, as well as those of $P'^{(1)}_{\Lambda;u}(v_1, v_1')$ and $P''^{(1)}_{\Lambda;u,v}(v_1, v_1')$, where the labels in the parentheses stand for the vertices that are summed over, the sequence of bubbles from v_i and v_i' represents $\psi_\Lambda(v_i, v_i') - \delta_{v_i,v_i'}$, and the sequence of bubbles from v' to v in $P''^{(1)}_{\Lambda;u,v}(v_1, v_1')$ is $\psi_\Lambda(v', v)$.

and, for $j \geq 2$,

$$\begin{aligned} P^{(j)}_\Lambda(v_1, v_j') = \sum_{\substack{v_2,\dots,v_j \\ v_1',\dots,v_{j-1}'}} & \big(\psi_\Lambda(v_1, v_1') - \delta_{v_1,v_1'}\big) \langle \varphi_{v_1}\varphi_{v_2}\rangle_\Lambda \langle \varphi_{v_2}\varphi_{v_1'}\rangle_\Lambda \\ & \times \left(\prod_{i=2}^{j-1} \big(\psi_\Lambda(v_i, v_i') - \delta_{v_i,v_i'}\big) \langle \varphi_{v_{i-1}'}\varphi_{v_{i+1}}\rangle_\Lambda \langle \varphi_{v_{i+1}}\varphi_{v_i'}\rangle_\Lambda \right) \\ & \times \big(\psi_\Lambda(v_j, v_j') - \delta_{v_j,v_j'}\big) \langle \varphi_{v_{j-1}'}\varphi_{v_j'}\rangle_\Lambda, \end{aligned} \tag{4.7}$$

where, by convention, the empty product for $j = 2$ is 1. Then, we define $P'^{(j)}_{\Lambda;u}(v_1, v_j')$ by replacing one of the $2j-1$ two-point functions explicitly consisting of $P^{(j)}_\Lambda(v_1, v_j')$ in (4.6)–(4.7), say, $\langle\varphi_z\varphi_{z'}\rangle_\Lambda$ (e.g., $(z, z') = (v_1, v_1')$ for $j = 1$, and either $(z, z') = (v_1, v_2)$, $(z, z') = (v_2, v_1')$ or $(z, z') = (v_1', v_2)$ for $j = 2$, and so on) with $\langle\varphi_z\varphi_u\rangle_\Lambda\langle\varphi_u\varphi_{z'}\rangle_\Lambda$, and then summing over all $2j-1$ choices of this replacement (see the second line in Figure 5). Similarly, we define $P''^{(j)}_{\Lambda;u,v}(v_1, v_j')$ by replacing two distinct two-point functions consisting of $P^{(j)}_\Lambda(v_1, v_j')$, one of which is among the aforementioned $2j-1$ two-point functions and the other is among those of which $\psi_\Lambda(v_i, v_i') - \delta_{v_i,v_i'}$ for $i = 1, \dots, j$ are composed, say, $\prod_{i=1,2}\langle\varphi_{z_i}\varphi_{z_i'}\rangle_\Lambda$, with $\langle\varphi_{z_1}\varphi_u\rangle_\Lambda\langle\varphi_u\varphi_{z_1'}\rangle_\Lambda\sum_{v'}\langle\varphi_{z_2}\varphi_{v'}\rangle_\Lambda\langle\varphi_{v'}\varphi_{z_2'}\rangle_\Lambda\psi_\Lambda(v', v)$, and then summing over all possible combinations of these two distinct two-point functions (see the second line in Figure 5 again). Moreover, we let

$$P'^{(0)}_{\Lambda;u}(y, x) = \langle\varphi_y\varphi_x\rangle_\Lambda^2 \langle\varphi_y\varphi_u\rangle_\Lambda \langle\varphi_u\varphi_x\rangle_\Lambda, \tag{4.8}$$

$$P''^{(0)}_{\Lambda;u,v}(y, x) = \langle\varphi_y\varphi_x\rangle_\Lambda \langle\varphi_y\varphi_u\rangle_\Lambda \langle\varphi_u\varphi_x\rangle_\Lambda \sum_{v'} \langle\varphi_y\varphi_{v'}\rangle_\Lambda \langle\varphi_{v'}\varphi_x\rangle_\Lambda \psi_\Lambda(v', v), \tag{4.9}$$

and define

$$P'_{\Lambda;u}(y, x) = \sum_{j\geq 0} P'^{(j)}_{\Lambda;u}(y, x), \qquad P''_{\Lambda;u,v}(y, x) = \sum_{j\geq 0} P''^{(j)}_{\Lambda;u,v}(y, x), \tag{4.10}$$

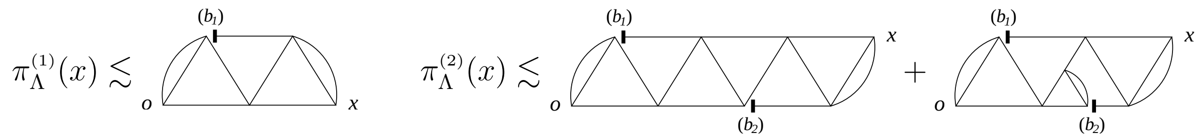

Figure 6: Leading diagrammatic bounds on $\pi_\Lambda^{(1)}(x)$ and $\pi_\Lambda^{(2)}(x)$, where, in particular, the line segments that terminate with b_i for $i = 1, 2$ represent $\delta + \tilde{G}_\Lambda$ (cf., (4.11)–(4.12)). The labels in the parentheses stand for the bonds that are summed over.

where, as shown below in Section 4.3, $P'^{(0)}_{\Lambda;u}(y,x)$ and $P''^{(0)}_{\Lambda;u,v}(y,x)$ are the leading contributions to $P'_{\Lambda;u}(y,x)$ and $P''_{\Lambda;u,v}(y,x)$, respectively. Finally, we define

$$Q'_{\Lambda;u}(y,x) = \sum_z \big(\delta_{y,z} + \tilde{G}_\Lambda(y,z)\big)\, P'_{\Lambda;u}(z,x), \tag{4.11}$$

$$Q''_{\Lambda;u,v}(y,x) = \sum_z \big(\delta_{y,z} + \tilde{G}_\Lambda(y,z)\big)\, P''_{\Lambda;u,v}(z,x) + \sum_{v',z} \langle \varphi_y \varphi_{v'} \rangle_\Lambda\, \tilde{G}_\Lambda(v',z)\, P'_{\Lambda;u}(z,x)\, \psi_\Lambda(v',v). \tag{4.12}$$

Now, we can state diagrammatic bounds on the expansion coefficients as follows (see Figure 6):

Proposition 4.1 (**Diagrammatic bounds**). *For the ferromagnetic Ising model, we have*

$$\pi_\Lambda^{(j)}(x) \le \begin{cases} P'^{(0)}_{\Lambda;o}(o,x) \equiv \langle \varphi_o \varphi_x \rangle_\Lambda^3 & (j=0), \\ \displaystyle\sum_{\substack{b_1,\dots,b_j \\ v_1,\dots,v_j}} P'^{(0)}_{\Lambda;v_1}(o,\underline{b}_1) \left(\prod_{i=1}^{j-1} \tau_{b_i} Q''_{\Lambda;v_i,v_{i+1}}(\overline{b}_i, \underline{b}_{i+1}) \right) \tau_{b_j} Q'_{\Lambda;v_j}(\overline{b}_j, x) & (j \ge 1), \end{cases} \tag{4.13}$$

where, by convention, the empty product for $j = 1$ is 1.

We prove (4.13) for $j = 0$ in Section 4.1 and (4.13) for $j \ge 1$ in Section 4.2. Then, in Section 4.3, we use Proposition 4.1 to prove Proposition 2.4.

4.1 Bound on $\pi_\Lambda^{(0)}(x)$

The key ingredient of the proof of (4.13) is Lemma 4.2 below, which is an extension of the GHS idea used in the proof of Lemma 3.3. In this subsection, we demonstrate how this extension works to prove the bound on $\pi_\Lambda^{(0)}(x)$ and the inequality

$$\sum_{\partial \mathbf{n} = o \triangle x} \frac{w_\Lambda(\mathbf{n})}{Z_\Lambda} \mathbb{1}_{\{o \underset{\mathbf{n}}{\Longleftrightarrow} x\} \cap \{o \underset{\mathbf{n}}{\longleftrightarrow} y\}} \le P'^{(0)}_{\Lambda;y}(o,x), \tag{4.14}$$

which will be used in Section 4.2 to obtain the bounds on $\pi_\Lambda^{(j)}(x)$ for $j \ge 1$.

Proof of (4.13) for $j=0$. Since the inequality is trivial if $x=o$, we restrict our attention to the case of $x \neq o$.

We note that, for each current configuration $\mathbf{n}$ that satisfies $\partial\mathbf{n}=\{o,x\}$ and $\mathbb{1}_{\{o \underset{\mathbf{n}}{\Longleftrightarrow} x\}}=1$, there are at least *three edge-disjoint* paths on $\mathbb{G}_{\mathbf{n}}$ between o and x. (See the first term on the right-hand side in Figure 2. For example, if the thick line in that picture, referred to as λ_1 and decomposed as $\lambda_{11} \dot\cup \lambda_{12} \dot\cup \lambda_{13}$ from o to x, consists of bonds b with $n_b=1$, and if the thin lines, referred to as λ_2 and λ_3 that terminate at o and x, respectively, consist of bonds b' with $n_{b'}=2$, then the decomposition into three edge-disjoint paths is $\{\lambda_2 \dot\cup \lambda_{13}, \lambda'_2 \dot\cup \lambda_{12} \dot\cup \lambda_3, \lambda_{11} \dot\cup \lambda'_3\}$, where λ'_i is the duplication of λ_i.) Multiplying $\pi_\Lambda^{(0)}(x)$ by *two* dummies $(Z_\Lambda/Z_\Lambda)^2$ ($\equiv 1$), we obtain

$$\begin{aligned}\pi_\Lambda^{(0)}(x) &= \sum_{\substack{\partial\mathbf{n}=\{o,x\}\\ \partial\mathbf{m}'=\partial\mathbf{m}''=\varnothing}} \frac{w_\Lambda(\mathbf{n})}{Z_\Lambda}\frac{w_\Lambda(\mathbf{m}')}{Z_\Lambda}\frac{w_\Lambda(\mathbf{m}'')}{Z_\Lambda}\mathbb{1}_{\{o \underset{\mathbf{n}}{\Longleftrightarrow} x\}}\\ &= \sum_{\partial\mathbf{N}=\{o,x\}}\frac{w_\Lambda(\mathbf{N})}{Z_\Lambda^3}\sum_{\substack{\partial\mathbf{n}=\{o,x\}\\ \partial\mathbf{m}'=\partial\mathbf{m}''=\varnothing\\ \mathbf{N}\equiv\mathbf{n}+\mathbf{m}'+\mathbf{m}''}}\mathbb{1}_{\{o \underset{\mathbf{n}}{\Longleftrightarrow} x\}}\prod_{b\in\mathbb{B}_\Lambda}\frac{N_b!}{n_b!\,m'_b!\,m''_b!},\end{aligned} \tag{4.15}$$

where the sum over $\mathbf{n},\mathbf{m}',\mathbf{m}''$ in the second line equals the cardinality of the following set of partitions:

$$\left\{(\mathbb{S}_0,\mathbb{S}_1,\mathbb{S}_2):\mathbb{G}_{\mathbf{N}}=\dot{\bigcup_{i=0,1,2}}\mathbb{S}_i,\ \partial\mathbb{S}=\{o,x\},\ \partial\mathbb{S}_1=\partial\mathbb{S}_2=\varnothing,\ o\Longleftrightarrow x\text{ in }\mathbb{S}_0\right\}, \tag{4.16}$$

where "$o \Longleftrightarrow x$ in $\mathbb{S}_0$" means that there are at least two *bond*-disjoint paths in $\mathbb{S}_0$. We prove below that the cardinality of (4.16) is bounded from above by

$$\left|\left\{(\mathbb{S}_0,\mathbb{S}_1,\mathbb{S}_2):\mathbb{G}_{\mathbf{N}}=\dot{\bigcup_{i=0,1,2}}\mathbb{S}_i,\ \partial\mathbb{S}_0=\partial\mathbb{S}_1=\partial\mathbb{S}_2=\{o,x\}\right\}\right| = \sum_{\substack{\partial\mathbf{n}=\{o,x\}\\ \partial\mathbf{m}'=\partial\mathbf{m}''=\{o,x\}\\ \mathbf{N}\equiv\mathbf{n}+\mathbf{m}'+\mathbf{m}''}}\prod_{b\in\mathbb{B}_\Lambda}\frac{N_b!}{n_b!\,m'_b!\,m''_b!}. \tag{4.17}$$

This implies (4.13) for $j=0$, since

$$\sum_{\partial\mathbf{N}=\{o,x\}}\frac{w_\Lambda(\mathbf{N})}{Z_\Lambda^3}\sum_{\substack{\partial\mathbf{n}=\partial\mathbf{m}'=\partial\mathbf{m}''=\{o,x\}\\ \mathbf{N}\equiv\mathbf{n}+\mathbf{m}'+\mathbf{m}''}}\prod_{b\in\mathbb{B}_\Lambda}\frac{N_b!}{n_b!\,m'_b!\,m''_b!}=\langle\varphi_o\varphi_x\rangle_\Lambda^3. \tag{4.18}$$

It remains to prove that the cardinality of (4.16) is bounded from above by (4.17). For this, we use the following lemma, in which we denote the set of paths on $\mathbb{G}_{\mathbf{N}}$ from z to z' by $\Omega^{\mathbf{N}}_{z\to z'}$ and write $\omega\cap\omega'=\varnothing$ to mean that ω and ω' are *edge*-disjoint (not necessarily *bond*-disjoint).

Lemma 4.2. *Given $\mathbf{N}\in\mathbb{Z}_+^{\mathbb{B}_\Lambda}$, $k\ge 1$, $\mathcal{V}\subset\Lambda$ and $z_i\neq z'_i\in\Lambda$ for $i=1,\dots,k$, we let*

$$\mathfrak{S}=\left\{(\mathbb{S}_0,\mathbb{S}_1,\dots,\mathbb{S}_k):\begin{array}{c}\mathbb{G}_{\mathbf{N}}=\dot\bigcup_{i=0}^k\mathbb{S}_i,\ \partial\mathbb{S}_0=\mathcal{V},\ \partial\mathbb{S}_i=\varnothing\ {}^\forall i=1,\dots,k\\ {}^\exists\omega_i\in\Omega^{\mathbf{N}}_{z_i\to z'_i}\ {}^\forall i=1,\dots,k\ \text{ s.t. }\omega_i\cap\omega_j=\varnothing\ {}^\forall i\neq j\\ \omega_i\subset\mathbb{S}_0\dot\cup\mathbb{S}_i\ {}^\forall i=1,\dots,k\end{array}\right\}, \tag{4.19}$$

and let $\mathfrak{S}'$ be the right-hand side of (4.19) with "$\partial\mathbb{S}_0 = \mathcal{V}$, $\partial\mathbb{S}_i = \varnothing$ $^\forall i = 1, \dots, k$" being replaced by "$\partial\mathbb{S}_0 = \mathcal{V} \triangle \{z_1, z_1'\} \triangle \cdots \triangle \{z_k, z_k'\}$, $\partial\mathbb{S}_i = \{z_i, z_i'\}$ $^\forall i = 1, \dots, k$". Then, $|\mathfrak{S}| = |\mathfrak{S}'|$.

We prove this lemma at the end of this subsection.

Now, we use Lemma 4.2 with $k = 2$ and $\mathcal{V} = \{z_1, z_1'\} = \{z_2, z_2'\} = \{o, x\}$. Note that (4.16) is a subset of $\mathfrak{S}$, since $\mathfrak{S}$ includes the partitions $(\mathbb{S}_0, \mathbb{S}_1, \mathbb{S}_2)$ in which there does not exist two *bond*-disjoint paths on $\mathbb{S}_0$. In addition, $\mathfrak{S}'$ is trivially a subset of the set in the left-hand side of (4.17). Therefore, the cardinality of (4.16) is bounded from above by (4.17). This completes the proof of (4.13) for $j = 0$. □

Here, we summarize the basic steps that we have followed to bound $\pi_\Lambda^{(0)}(x)$ and which we generalize to prove (4.14) below and the bounds on $\pi_\Lambda^{(j)}(x)$ for $j \geq 1$ in Section 4.2.2.

(i) Count the number, say, $k+1$, of *edge-disjoint* paths on $\mathbb{G}_{\mathbf{n}}$ that satisfies the source constraint (as well as other additional conditions, if there are) of the considered function $f(x)$, such as $\pi_\Lambda^{(0)}(x) = \frac{1}{Z_\Lambda}\sum_{\partial\mathbf{n}=\{o,x\}} w_\Lambda(\mathbf{n})\,\mathbb{1}_{\{o \underset{\mathbf{n}}{\Longleftrightarrow} x\}}$ with $k = 2$.

(ii) Multiply $f(x)$ by $(\frac{Z_\Lambda}{Z_\Lambda})^k = \prod_{i=1}^k(\frac{1}{Z_\Lambda}\sum_{\partial\mathbf{m}^{(i)}=\varnothing} w_\Lambda(\mathbf{m}^{(i)}))$ $(\equiv 1)$, and then overlap the k dummies $\mathbf{m}^{(1)}, \dots, \mathbf{m}^{(k)}$ over the original current configuration $\mathbf{n}$. Choose any k paths among the $(k+1)$-edge-disjoint paths on $\mathbb{G}_{\mathbf{n}+\sum_{i=1}^k \mathbf{m}^{(i)}}$, and denote them by $\omega_1, \dots, \omega_k$.

(iii) Use Lemma 4.2 to exchange the occupation status of edges on ω_i between $\mathbb{G}_{\mathbf{n}}$ and $\mathbb{G}_{\mathbf{m}^{(i)}}$ for every $i = 1, \dots, k$. The current configurations after the mapping, denoted by $\tilde{\mathbf{n}}, \tilde{\mathbf{m}}^{(1)}, \dots, \tilde{\mathbf{m}}^{(k)}$, satisfy $\partial\tilde{\mathbf{n}} = \partial\mathbf{n} \vartriangle \partial\omega_1 \vartriangle \cdots \vartriangle \partial\omega_k$ and $\partial\tilde{\mathbf{m}}^{(i)} = \partial\omega_i$ for $i = 1, \dots, k$.

Proof of (4.14). When $y = o$ or x, (4.14) is reduced to the inequality for $\pi_\Lambda^{(0)}(x)$. Also, the case of $o = x \neq y$ is trivial, since

$$\sum_{\partial\mathbf{n}=\varnothing} \frac{w_\Lambda(\mathbf{n})}{Z_\Lambda}\,\mathbb{1}_{\{o \underset{\mathbf{n}}{\longleftrightarrow} y\}} \leq \sum_{\partial\mathbf{n}=\partial\mathbf{m}=\varnothing} \frac{w_\Lambda(\mathbf{n})}{Z_\Lambda}\,\frac{w_\Lambda(\mathbf{m})}{Z_\Lambda}\,\mathbb{1}_{\{o \underset{\mathbf{n}+\mathbf{m}}{\longleftrightarrow} y\}} = \langle\varphi_o\varphi_y\rangle_\Lambda^2, \tag{4.20}$$

due to Lemma 3.3. Therefore, we can assume $o \neq x \neq y \neq o$.

We follow the three steps described above.

(i) Because of the source constraint $\partial\mathbf{n} = \{o, x\}$ and the events in the indicator function, there are at least 4 $(= k+1)$ edge-disjoint paths on $\mathbb{G}_{\mathbf{n}}$, one of which is from o to y, another is from y to x, and the remaining two are from o to x. (It is not so hard to realize that there is an *edge*-disjoint cycle, $o \to y \to x \to o$, due to the fact that y is not a source, but o and x are. Since a cycle does not have a source, the existence of another edge-disjoint connection from o to x is assured by the source constraint $\partial\mathbf{n} = \{o, x\}$.)

(ii) Then, by multiplying $(\frac{Z_\Lambda}{Z_\Lambda})^3$, the inequality (4.14) is equivalent to

$$\begin{aligned}
&\sum_{\partial\mathbf{N}=\{o,x\}} \frac{w_\Lambda(\mathbf{N})}{Z_\Lambda^4} \sum_{\substack{\partial\mathbf{n}=\{o,x\} \\ \partial\mathbf{m}^{(i)}=\varnothing\ ^\forall i=1,2,3 \\ \mathbf{N}=\mathbf{n}+\sum_{i=1}^3 \mathbf{m}^{(i)}}} \mathbb{1}_{\{o \underset{\mathbf{n}}{\Longleftrightarrow} x\} \cap \{o \underset{\mathbf{n}}{\longleftrightarrow} y\}} \prod_{b\in\mathbb{B}_\Lambda} \frac{N_b!}{n_b!\,m_b^{(1)}!\,m_b^{(2)}!\,m_b^{(3)}!} \\
&\leq \sum_{\partial\mathbf{N}=\{o,x\}} \frac{w_\Lambda(\mathbf{N})}{Z_\Lambda^4} \sum_{\substack{\partial\mathbf{n}=\partial\mathbf{m}^{(3)}=\{o,x\} \\ \partial\mathbf{m}^{(1)}=\{o,y\},\ \partial\mathbf{m}^{(2)}=\{y,x\} \\ \mathbf{N}=\mathbf{n}+\sum_{i=1}^3 \mathbf{m}^{(i)}}} \prod_{b\in\mathbb{B}_\Lambda} \frac{N_b!}{n_b!\,m_b^{(1)}!\,m_b^{(2)}!\,m_b^{(3)}!}.
\end{aligned} \tag{4.21}$$

Therefore, it suffices to prove that the second sum on the left-hand side is less than or equal to that on the right-hand side.

(iii) We note that the second sum on the left-hand side of (4.21) equals the cardinality of

$$\left\{(\mathbb{S}_0,\mathbb{S}_1,\mathbb{S}_2,\mathbb{S}_3) : \begin{array}{c}\mathbb{G}_{\mathbf{N}}=\dot\bigcup_{i=0}^3\mathbb{S}_i,\ \partial\mathbb{S}_0=\{o,x\},\ \partial\mathbb{S}_1=\partial\mathbb{S}_2=\partial\mathbb{S}_3=\varnothing\\ o\Longleftrightarrow x \text{ in } \mathbb{S}_0,\ o\longleftrightarrow y \text{ in } \mathbb{S}_0\end{array}\right\}, \tag{4.22}$$

and the second sum on the right-hand side of (4.21) equals the cardinality of

$$\Big\{(\mathbb{S}_0,\mathbb{S}_1,\mathbb{S}_2,\mathbb{S}_3) : \mathbb{G}_{\mathbf{N}}=\dot\bigcup_{i=0}^3\mathbb{S}_i,\ \partial\mathbb{S}_0=\{o,x\},\ \partial\mathbb{S}_1=\{o,y\},\ \partial\mathbb{S}_2=\{y,x\},\ \partial\mathbb{S}_3=\{o,x\}\Big\}. \tag{4.23}$$

Now, we use Lemma 4.2 with $k=3$ and $\mathcal{V}=\{z_3,z_3'\}=\{o,x\}$, $\{z_1,z_1'\}=\{o,y\}$ and $\{z_2,z_2'\}=\{y,x\}$. Since (4.22) is a subset of $\mathfrak{S}$ for this setting, while $\mathfrak{S}'$ is a subset of (4.23), we obtain (4.21). This completes the proof of (4.14). □

Proof of Lemma 4.2. For every $\Omega^{\mathbf{N}}_{z_i\to z_i'}$, we introduce an arbitrarily fixed order. For $\omega,\omega'\in\Omega^{\mathbf{N}}_{z_i\to z_i'}$, we write $\omega\prec\omega'$ if ω is earlier than ω' in this order. For $\omega_1\in\Omega^{\mathbf{N}}_{z_1\to z_1'}$, we denote by $\Omega^{\mathbf{N};\omega_1}_{z_2\to z_2'}$ the set of paths $\omega\in\Omega^{\mathbf{N}}_{z_2\to z_2'}$ such that $\omega\cap\omega_1=\varnothing$ and that $\zeta\not\subset\omega$ for any $\zeta\in\Omega^{\mathbf{N}}_{z_1\to z_1'}$ with $\zeta\prec\omega_1$. Moreover, for $1<l<k$, we define $\Omega^{\mathbf{N};\vec\omega_l}_{z_{l+1}\to z_{l+1}'}$, where $\vec\omega_l=(\omega_1,\dots,\omega_l)$ with $\omega_1\in\Omega^{\mathbf{N}}_{z_1\to z_1'}$, $\omega_2\in\Omega^{\mathbf{N};\omega_1}_{z_2\to z_2'},\dots,\omega_l\in\Omega^{\mathbf{N};\vec\omega_{l-1}}_{z_l\to z_l'}$, to be the set of paths $\zeta\in\Omega^{\mathbf{N}}_{z_{l+1}\to z_{l+1}'}$ such that $\zeta\cap\dot\bigcup_{i=1}^l\omega_i=\varnothing$ and that $\xi\not\subset\zeta$ for any $\xi\in\Omega^{\mathbf{N};\vec\omega_{i-1}}_{z_i\to z_i'}$ satisfying $\xi\prec\omega_i$, for every $i=1,\dots,l$, where we have denoted $\Omega^{\mathbf{N};\vec\omega_0}_{z_1\to z_1'}=\Omega^{\mathbf{N}}_{z_1\to z_1'}$.

Using the above notation, we decompose $\mathfrak{S}^{(\prime)}$ disjointly as follows. Given $\omega_1\in\Omega^{\mathbf{N}}_{z_1\to z_1'},\dots,\omega_k\in\Omega^{\mathbf{N};\vec\omega_{k-1}}_{z_k\to z_k'}$, we denote by $\mathfrak{S}^{(\prime)}_{\vec\omega_k}$ the set of partitions $(\mathbb{S}_0,\mathbb{S}_1,\dots,\mathbb{S}_k)\in\mathfrak{S}^{(\prime)}$ such that ω_i is the earliest element of $\Omega^{\mathbf{N};\vec\omega_{i-1}}_{z_i\to z_i'}$ contained in $\mathbb{S}_0\,\dot\cup\,\mathbb{S}_i$, for every $i=1,\dots,k$. Then, $\mathfrak{S}^{(\prime)}$ can be decomposed as

$$\mathfrak{S}^{(\prime)}=\dot\bigcup_{\omega_1\in\Omega^{\mathbf{N}}_{z_1\to z_1'}}\ \dot\bigcup_{\omega_2\in\Omega^{\mathbf{N};\omega_1}_{z_2\to z_2'}}\cdots\dot\bigcup_{\omega_k\in\Omega^{\mathbf{N};\vec\omega_{k-1}}_{z_k\to z_k'}}\mathfrak{S}^{(\prime)}_{\vec\omega_k}. \tag{4.24}$$

The proof of Lemma 4.2 will be completed if we can find a bijection from $\mathfrak{S}_{\vec\omega_k}$ to $\mathfrak{S}'_{\vec\omega_k}$ for every $\vec\omega_k$. For $(\mathbb{S}_0,\dots,\mathbb{S}_k)\in\mathfrak{S}_{\vec\omega_k}$, we define

$$F_{\vec\omega_k}(\mathbb{S}_0,\dots,\mathbb{S}_k)\equiv\big(F^{(0)}_{\vec\omega_k}(\mathbb{S}_0),\dots,F^{(k)}_{\vec\omega_k}(\mathbb{S}_k)\big)=\Big(\mathbb{S}_0\triangle\dot\bigcup_{i=1}^k\omega_i,\ \mathbb{S}_1\triangle\omega_1,\dots,\mathbb{S}_k\triangle\omega_k\Big), \tag{4.25}$$

where $\partial F^{(0)}_{\vec\omega_k}(\mathbb{S}_0)=\mathcal{V}\triangle\{z_1,z_1'\}\triangle\cdots\triangle\{z_k,z_k'\}$ and $\partial F^{(i)}_{\vec\omega_k}(\mathbb{S}_i)=\{z_i,z_i'\}$ for $i=1,\dots,k$. Note that, by definition, we have $F_{\vec\omega_k}(F_{\vec\omega_k}(\mathbb{S}_0,\dots,\mathbb{S}_k))=(\mathbb{S}_0,\dots,\mathbb{S}_k)$. Also, by simple arithmetic using $\omega_i\cap\omega_j=\mathbb{S}_i\cap\mathbb{S}_j=\varnothing$ and $\omega_j\subset\mathbb{S}_0\,\dot\cup\,\mathbb{S}_j$ for $1\le j\le k$ and $i\ne j$, we have

$$F^{(i)}_{\vec\omega_k}(\mathbb{S}_i)\cap F^{(j)}_{\vec\omega_k}(\mathbb{S}_j)=\varnothing,\qquad F^{(0)}_{\vec\omega_k}(\mathbb{S}_0)\,\dot\cup\,F^{(j)}_{\vec\omega_k}(\mathbb{S}_j)=\Big(\mathbb{S}_0\triangle\dot\bigcup_{i\ne j}\omega_i\Big)\,\dot\cup\,\mathbb{S}_j. \tag{4.26}$$

Since $\omega_j \subset \mathbb{S}_0 \dot\cup \mathbb{S}_j$ and $\omega_j \cap \dot\bigcup_{i\neq j} \omega_i = \varnothing$, we have $\omega_j \subset F^{(0)}_{\vec\omega_k}(\mathbb{S}_0) \dot\cup F^{(j)}_{\vec\omega_k}(\mathbb{S}_j)$. In addition, since $\Omega^{\mathbf{N};\vec\omega_{j-1}}_{z_j\to z'_j}$ is a set of paths that do not use any edge in $\dot\bigcup_{i<j} \omega_i$, its earliest element contained in $(\mathbb{S}_0 \triangle \dot\bigcup_{i<j} \omega_i) \dot\cup \mathbb{S}_j$ is ω_j. Furthermore, since each $\Omega^{\mathbf{N};\vec\omega_{i-1}}_{z_i\to z'_i}$ for $i > j$ is a set of paths that do not fully contain ω_j or any earlier element of $\Omega^{\mathbf{N};\vec\omega_{j-1}}_{z_j\to z'_j}$ as a subset, ω_j is the earliest element of

$$\bigg(\Big(\mathbb{S}_0 \triangle \dot\bigcup_{i<j} \omega_i\Big) \dot\cup \mathbb{S}_j\bigg) \triangle \Big(\dot\bigcup_{i>j} \omega_i\Big) \equiv \Big(\mathbb{S}_0 \triangle \dot\bigcup_{i\neq j} \omega_i\Big) \dot\cup \mathbb{S}_j. \tag{4.27}$$

Therefore, $F_{\vec\omega_k}$ is a bijection from $\mathfrak{S}_{\vec\omega_k}$ to $\mathfrak{S}'_{\vec\omega_k}$. This completes the proof of Lemma 4.2. □

4.2 Bounds on $\pi^{(j)}_\Lambda(x)$ for $j \geq 1$

In this subsection, we prove (4.13) for $j \geq 1$ using the following two lemmas, in which we use

$$E'_{\mathbf{n}}(z,x;\mathcal{A}) = \{z \xleftrightarrow[\mathbf{n}]{\mathcal{A}} x\} \cap \{z \underset{\mathbf{n}}{\Longleftrightarrow} x\}, \qquad E''_{\mathbf{n}}(z,x,v;\mathcal{A}) = E'_{\mathbf{n}}(z,x;\mathcal{A}) \cap \{z \underset{\mathbf{n}}{\longleftrightarrow} v\}, \tag{4.28}$$

$$\Theta'_{z,x;\mathcal{A}} = \sum_{\substack{\partial\mathbf{m}=\varnothing\\ \partial\mathbf{n}=z\triangle x}} \frac{w_{\mathcal{A}^{\rm c}}(\mathbf{m})}{Z_{\mathcal{A}^{\rm c}}} \frac{w_\Lambda(\mathbf{n})}{Z_\Lambda} \mathbb{1}_{E'_{\mathbf{m}+\mathbf{n}}(z,x;\mathcal{A})}, \qquad \Theta''_{z,x,v;\mathcal{A}} = \sum_{\substack{\partial\mathbf{m}=\varnothing\\ \partial\mathbf{n}=z\triangle x}} \frac{w_{\mathcal{A}^{\rm c}}(\mathbf{m})}{Z_{\mathcal{A}^{\rm c}}} \frac{w_\Lambda(\mathbf{n})}{Z_\Lambda} \mathbb{1}_{E''_{\mathbf{m}+\mathbf{n}}(z,x,v;\mathcal{A})}. \tag{4.29}$$

Lemma 4.3. *For the ferromagnetic Ising model, we have*

$$\Theta_{y,x;\mathcal{A}} \leq \sum_z \big(\delta_{y,z} + \tilde G_\Lambda(y,z)\big)\, \Theta'_{z,x;\mathcal{A}}, \tag{4.30}$$

$$\Theta_{y,x;\mathcal{A}}\big[\mathbb{1}_{\{y\longleftrightarrow v\}}\big] \leq \sum_z \big(\delta_{y,z} + \tilde G_\Lambda(y,z)\big)\, \Theta''_{z,x,v;\mathcal{A}} + \sum_{v',z} \langle\varphi_y\varphi_{v'}\rangle_\Lambda\, \tilde G_\Lambda(v',z)\, \Theta'_{z,x;\mathcal{A}}\, \psi_\Lambda(v',v). \tag{4.31}$$

Lemma 4.4. *For the ferromagnetic Ising model, we have*

$$\Theta'_{y,x;\mathcal{A}} \leq \sum_{u\in\mathcal{A}} P'_{\Lambda;u}(y,x), \qquad \Theta''_{y,x,v;\mathcal{A}} \leq \sum_{u\in\mathcal{A}} P''_{\Lambda;u,v}(y,x). \tag{4.32}$$

Proof of (4.13) for $j \geq 1$ assuming Lemmas 4.3–4.4. Recalling (3.37) and using (4.30), (4.32) and (4.11), we obtain

$$\begin{aligned}\Theta^{(j-1)}_{\bar b_{j-1},\underline{b}_j;\tilde{\mathcal{C}}_{j-2}}\big[\tau_{b_j}\Theta^{(j)}_{\bar b_j,x;\tilde{\mathcal{C}}_{j-1}}\big] &\leq \sum_z \big(\delta_{\bar b_j,z} + \tilde G_\Lambda(\bar b_j,z)\big)\, \Theta^{(j-1)}_{\bar b_{j-1},\underline{b}_j;\tilde{\mathcal{C}}_{j-2}}\Bigg[\tau_{b_j} \sum_{v_j\in\tilde{\mathcal{C}}_{j-1}} P'_{\Lambda;v_j}(z,x)\Bigg]\\ &\leq \sum_{v_j} \Theta^{(j-1)}_{\bar b_{j-1},\underline{b}_j;\tilde{\mathcal{C}}_{j-2}}\Big[\mathbb{1}_{\{\bar b_{j-1}\longleftrightarrow v_j\}}\Big]\, \tau_{b_j} Q'_{\Lambda;v_j}(\bar b_j,x).\end{aligned} \tag{4.33}$$

If $j = 1$, we use (4.14). Otherwise, we use (4.11)–(4.12) and (4.31)–(4.32) $j-1$ times and at last use (4.14). This completes the proof of (4.13) for $j \geq 1$. □

We prove Lemma 4.3 in Section 4.2.1, and Lemma 4.4 in Section 4.2.2.

4.2.1 Proof of Lemma 4.3

Proof of (4.30). Recalling (3.30) and (4.28), we have

$$
\begin{aligned}
E_{\mathbf{n}}(y,x;\mathcal{A}) &= E'_{\mathbf{n}}(y,x;\mathcal{A}) \\
&\qquad \dot\cup \dot\bigcup_{b\in\mathbb{B}_\Lambda} \Big\{\{E'_{\mathbf{n}}(\overline{b},x;\mathcal{A}) \text{ off } b\} \cap \{n_b>0\} \cap \big\{y \underset{\mathbf{n}}{\longleftrightarrow} \underline{b} \text{ in } \mathcal{A}^{\rm c},\ \underline{b}\notin \mathcal{C}^b_{\mathbf{n}}(x)\big\}\Big\}.
\end{aligned}
\tag{4.34}
$$

Therefore, we obtain

$$
\begin{aligned}
\Theta_{y,x;\mathcal{A}} &= \Theta'_{y,x;\mathcal{A}} \\
&\quad + \sum_{b\in\mathbb{B}_\Lambda} \sum_{\substack{\partial\mathbf{m}=\varnothing\\ \partial\mathbf{n}=y\triangle x}} \frac{w_{\mathcal{A}^{\rm c}}(\mathbf{m})}{Z_{\mathcal{A}^{\rm c}}} \frac{w_\Lambda(\mathbf{n})}{Z_\Lambda} \mathbb{1}_{\{E'_{\mathbf{m}+\mathbf{n}}(\overline{b},x;\mathcal{A}) \text{ off } b\}} \mathbb{1}_{\{m_b+n_b>0\}} \mathbb{1}_{\{y \underset{\mathbf{m}+\mathbf{n}}{\longleftrightarrow} \underline{b} \text{ in } \mathcal{A}^{\rm c},\ \underline{b}\notin\mathcal{C}^b_{\mathbf{m}+\mathbf{n}}(x)\}}.
\end{aligned}
\tag{4.35}
$$

It remains to bound the second line of (4.35), which is nonzero only if m_b is even and n_b is odd, due to the source constraints and the conditions in the indicators. By alternating the parity of n_b with changing the source constraint into $\partial\mathbf{n} = y \triangle b \triangle x$ and multiplying τ_b as in (3.33), and then conditioning on $\mathcal{C}^b_{\mathbf{m}+\mathbf{n}}(x)$ as in (3.34), the second line of (4.35) can be written as

$$
\begin{aligned}
&\sum_{b\in\mathbb{B}_\Lambda} \sum_{\substack{\partial\mathbf{m}=\varnothing\\ \partial\mathbf{n}=\overline{b}\triangle x}} \frac{w_{\mathcal{A}^{\rm c}}(\mathbf{m})}{Z_{\mathcal{A}^{\rm c}}} \frac{w_\Lambda(\mathbf{n})}{Z_\Lambda} \mathbb{1}_{\{E'_{\mathbf{m}+\mathbf{n}}(\overline{b},x;\mathcal{A}) \text{ off } b\}} \tau_b \mathbb{1}_{\{m_b,n_b \text{ even}\}} \\
&\qquad \times \sum_{\substack{\partial\mathbf{h}=\varnothing\\ \partial\mathbf{k}=y\triangle\underline{b}}} \frac{w_{\mathcal{A}^{\rm c}\cap\mathcal{C}^b_{\mathbf{m}+\mathbf{n}}(x)^{\rm c}}(\mathbf{h})}{Z_{\mathcal{A}^{\rm c}\cap\mathcal{C}^b_{\mathbf{m}+\mathbf{n}}(x)^{\rm c}}} \frac{w_{\mathcal{C}^b_{\mathbf{m}+\mathbf{n}}(x)^{\rm c}}(\mathbf{k})}{Z_{\mathcal{C}^b_{\mathbf{m}+\mathbf{n}}(x)^{\rm c}}} \mathbb{1}_{\{y \underset{\mathbf{h}+\mathbf{k}}{\longleftrightarrow} \underline{b} \text{ in } \mathcal{A}^{\rm c}\cap\mathcal{C}^b_{\mathbf{m}+\mathbf{n}}(x)^{\rm c}\}} \\
&= \sum_{b\in\mathbb{B}_\Lambda} \sum_{\substack{\partial\mathbf{m}=\varnothing\\ \partial\mathbf{n}=\overline{e}\triangle x}} \frac{w_{\mathcal{A}^{\rm c}}(\mathbf{m})}{Z_{\mathcal{A}^{\rm c}}} \frac{w_\Lambda(\mathbf{n})}{Z_\Lambda} \mathbb{1}_{\{E'_{\mathbf{m}+\mathbf{n}}(\overline{b},x;\mathcal{A}) \text{ off } b\}} \tau_b \mathbb{1}_{\{m_b,n_b \text{ even}\}} \langle \varphi_y\varphi_{\underline{b}}\rangle_{\mathcal{A}^{\rm c}\cap\mathcal{C}^b_{\mathbf{m}+\mathbf{n}}(x)^{\rm c}},
\end{aligned}
\tag{4.36}
$$

where we have used Lemma 3.3 for $y \neq \underline{b}$ to obtain the last line. Since $\langle \varphi_y\varphi_{\underline{b}}\rangle_{\mathcal{A}^{\rm c}\cap\mathcal{C}^b_{\mathbf{m}+\mathbf{n}}(x)^{\rm c}}$ is zero on the event $E'_{\mathbf{m}+\mathbf{n}}(\overline{b},x;\mathcal{A}) \setminus \{E'_{\mathbf{m}+\mathbf{n}}(\overline{b},x;\mathcal{A}) \text{ off } b\} \subset \{\underline{b}\in\mathcal{C}^b_{\mathbf{m}+\mathbf{n}}(x)\}$, or on the event that m_b or n_b is odd (cf., the argument below (3.14) or below (3.34)), we can omit "off b" and $\mathbb{1}_{\{m_b,n_b \text{ even}\}}$ to obtain that (4.36) is

$$
\sum_{b\in\mathbb{B}_\Lambda} \sum_{\substack{\partial\mathbf{m}=\varnothing\\ \partial\mathbf{n}=\overline{b}\triangle x}} \frac{w_{\mathcal{A}^{\rm c}}(\mathbf{m})}{Z_{\mathcal{A}^{\rm c}}} \frac{w_\Lambda(\mathbf{n})}{Z_\Lambda} \mathbb{1}_{E'_{\mathbf{m}+\mathbf{n}}(\overline{b},x;\mathcal{A})} \tau_b \langle \varphi_y\varphi_{\underline{b}}\rangle_{\mathcal{A}^{\rm c}\cap\mathcal{C}^b_{\mathbf{m}+\mathbf{n}}(x)^{\rm c}} \le \sum_{b\in\mathbb{B}_\Lambda} \langle \varphi_y\varphi_{\underline{b}}\rangle_\Lambda \tau_b\, \Theta'_{\overline{b},x;\mathcal{A}},
\tag{4.37}
$$

due to the second Griffiths inequality. This completes the proof of (4.30). □

Proof of (4.31). Recall (4.34). Since b in (4.34) is the last pivotal bond for $y \underset{\mathbf{n}}{\longleftrightarrow} x$ from y, we have

$$
\begin{aligned}
&E_{\mathbf{n}}(y,x;\mathcal{A}) \cap \{y \underset{\mathbf{n}}{\longleftrightarrow} v\} \\
&= E''_{\mathbf{n}}(y,x,v;\mathcal{A}) \,\dot\cup \dot\bigcup_{b\in\mathbb{B}_\Lambda} \Big\{\{E''_{\mathbf{n}}(\overline{b},x,v;\mathcal{A}) \text{ off } b\} \cap \{n_b>0\} \cap \big\{y \underset{\mathbf{n}}{\longleftrightarrow} \underline{b} \text{ in } \mathcal{A}^{\rm c},\ \underline{b}\notin\mathcal{C}^b_{\mathbf{n}}(x)\big\}\Big\} \\
&\qquad \dot\cup \dot\bigcup_{b\in\mathbb{B}_\Lambda} \Big\{\{E'_{\mathbf{n}}(\overline{b},x;\mathcal{A}) \text{ off } b\} \cap \{n_b>0\} \cap \big\{y \underset{\mathbf{n}}{\longleftrightarrow} \underline{b} \text{ in } \mathcal{A}^{\rm c},\ y \underset{\mathbf{n}}{\longleftrightarrow} v,\ \underline{b}\notin\mathcal{C}^b_{\mathbf{n}}(x)\big\}\Big\},
\end{aligned}
\tag{4.38}
$$

where $v \in \mathcal{C}^b_{\mathbf{n}}(x)$ in the event subject to the first big union, and $v \in \mathcal{C}^b_{\mathbf{n}}(y)$ in the event subject to the second big union. By the same computation between (4.35) and (4.37), the contribution from the second line of (4.38) is bounded by

$$\Theta''_{y,x,v;\mathcal{A}} + \sum_{b\in\mathbb{B}_\Lambda} \sum_{\substack{\partial\mathbf{m}=\varnothing \\ \partial\mathbf{n}=\bar{b}\triangle x}} \frac{w_{\mathcal{A}^c}(\mathbf{m})}{Z_{\mathcal{A}^c}} \frac{w_\Lambda(\mathbf{n})}{Z_\Lambda} \mathbb{1}_{E''_{\mathbf{m}+\mathbf{n}}(\bar{b},x,v;\mathcal{A})} \tau_b \left\langle \varphi_y \varphi_{\underline{b}} \right\rangle_{\mathcal{A}^c \cap \mathcal{C}^b_{\mathbf{m}+\mathbf{n}}(x)^c}$$
$$\leq \sum_z \big(\delta_{y,z} + \tilde{G}_\Lambda(y,z)\big)\, \Theta''_{z,x,v;\mathcal{A}}. \tag{4.39}$$

Similarly, the contribution from the third line of (4.38) is bounded by

$$\sum_{b\in\mathbb{B}_\Lambda} \sum_{\substack{\partial\mathbf{m}=\varnothing \\ \partial\mathbf{n}=\bar{b}\triangle x}} \frac{w_{\mathcal{A}^c}(\mathbf{m})}{Z_{\mathcal{A}^c}} \frac{w_\Lambda(\mathbf{n})}{Z_\Lambda} \mathbb{1}_{\{E'_{\mathbf{m}+\mathbf{n}}(\bar{b},x;\mathcal{A}) \text{ off } b\}} \tau_b \mathbb{1}_{\{m_b, n_b \text{ even}\}}$$
$$\times \sum_{\substack{\partial\mathbf{h}=\varnothing \\ \partial\mathbf{k}=y\triangle\underline{b}}} \frac{w_{\mathcal{A}^c \cap \mathcal{C}^b_{\mathbf{m}+\mathbf{n}}(x)^c}(\mathbf{h})}{Z_{\mathcal{A}^c \cap \mathcal{C}^b_{\mathbf{m}+\mathbf{n}}(x)^c}} \frac{w_{\mathcal{C}^b_{\mathbf{m}+\mathbf{n}}(x)^c}(\mathbf{k})}{Z_{\mathcal{C}^b_{\mathbf{m}+\mathbf{n}}(x)^c}} \mathbb{1}_{\{y \underset{\mathbf{h}+\mathbf{k}}{\longleftrightarrow} \underline{b} \text{ in } \mathcal{A}^c\cap\mathcal{C}^b_{\mathbf{m}+\mathbf{n}}(x)^c,\ y \underset{\mathbf{h}+\mathbf{k}}{\longleftrightarrow} v \text{ (in } \mathcal{C}^b_{\mathbf{m}+\mathbf{n}}(x)^c)\}}$$
$$\leq \sum_{b\in\mathbb{B}_\Lambda} \sum_{\substack{\partial\mathbf{m}=\varnothing \\ \partial\mathbf{n}=\bar{b}\triangle x}} \frac{w_{\mathcal{A}^c}(\mathbf{m})}{Z_{\mathcal{A}^c}} \frac{w_\Lambda(\mathbf{n})}{Z_\Lambda} \mathbb{1}_{E'_{\mathbf{m}+\mathbf{n}}(\bar{b},x;\mathcal{A})} \tau_b \Psi_{y,\underline{b},v;\mathcal{A},\mathcal{C}^b_{\mathbf{m}+\mathbf{n}}(x)}, \tag{4.40}$$

where we have omitted "off b" and $\mathbb{1}_{\{m_b,n_b \text{ even}\}}$ using $0 \leq \Psi_{y,\underline{b},v;\mathcal{A},\mathcal{C}^b_{\mathbf{m}+\mathbf{n}}(x)} \leq \langle \varphi_y \varphi_{\underline{b}} \rangle_{\mathcal{C}^b_{\mathbf{m}+\mathbf{n}}(x)^c}$ and the fact that $\langle \varphi_y \varphi_{\underline{b}} \rangle_{\mathcal{C}^b_{\mathbf{m}+\mathbf{n}}(x)^c}$ is zero when $\underline{b} \in \mathcal{C}^b_{\mathbf{m}+\mathbf{n}}(x)$, where

$$\Psi_{y,z,v;\mathcal{A},\mathcal{B}} = \sum_{\substack{\partial\mathbf{h}=\varnothing \\ \partial\mathbf{k}=y\triangle z}} \frac{w_{\mathcal{A}^c\cap\mathcal{B}^c}(\mathbf{h})}{Z_{\mathcal{A}^c\cap\mathcal{B}^c}} \frac{w_{\mathcal{B}^c}(\mathbf{k})}{Z_{\mathcal{B}^c}} \mathbb{1}_{\{y \underset{\mathbf{h}+\mathbf{k}}{\longleftrightarrow} v\}}. \tag{4.41}$$

To complete the proof of (4.31), it thus suffices to show

$$\Psi_{y,z,v;\mathcal{A},\mathcal{B}} \leq \sum_{v'} \langle \varphi_y \varphi_{v'} \rangle_\Lambda \langle \varphi_{v'} \varphi_z \rangle_\Lambda \, \psi_\Lambda(v',v). \tag{4.42}$$

We note that, by Lemma 3.3, $\Psi_{y,z,v;\mathcal{B},\mathcal{B}} = \langle \varphi_y \varphi_v \rangle_{\mathcal{B}^c} \langle \varphi_v \varphi_z \rangle_{\mathcal{B}^c}$. However, to deal with a general $\mathcal{A} \subset \Lambda$, we use

$$\{y \underset{\mathbf{h}+\mathbf{k}}{\longleftrightarrow} v\} = \{y \underset{\mathbf{k}}{\longleftrightarrow} v\} \,\dot\cup\, \big\{\{y \underset{\mathbf{h}+\mathbf{k}}{\longleftrightarrow} v\} \setminus \{y \underset{\mathbf{k}}{\longleftrightarrow} v\}\big\}, \tag{4.43}$$

and consider the two events in the right-hand side separately.

The contribution to (4.41) from $\{y \underset{\mathbf{k}}{\longleftrightarrow} v\}$ is bounded, similarly to (4.20), by

$$\sum_{\partial\mathbf{k}=y\triangle z} \frac{w_{\mathcal{B}^c}(\mathbf{k})}{Z_{\mathcal{B}^c}} \mathbb{1}_{\{y \underset{\mathbf{k}}{\longleftrightarrow} v\}} \leq \sum_{\substack{\partial\mathbf{k}=y\triangle z \\ \partial\mathbf{k}'=\varnothing}} \frac{w_{\mathcal{B}^c}(\mathbf{k})}{Z_{\mathcal{B}^c}} \frac{w_{\mathcal{B}^c}(\mathbf{k}')}{Z_{\mathcal{B}^c}} \mathbb{1}_{\{y \underset{\mathbf{k}+\mathbf{k}'}{\longleftrightarrow} v\}} = \langle \varphi_y \varphi_v \rangle_{\mathcal{B}^c} \langle \varphi_v \varphi_z \rangle_{\mathcal{B}^c}$$
$$\leq \langle \varphi_y \varphi_v \rangle_\Lambda \langle \varphi_v \varphi_z \rangle_\Lambda, \tag{4.44}$$

due to the second Griffiths inequality.

Next, we consider the contribution from $\{y \underset{\mathbf{h}+\mathbf{k}}{\longleftrightarrow} v\} \setminus \{y \underset{\mathbf{k}}{\longleftrightarrow} v\}$ in (4.43). We denote by $\mathcal{C}_{\mathbf{k}}(y)$ the set of $\mathbf{k}$-connected sites from y. Since y is $(\mathbf{h}+\mathbf{k})$-connected, but not $\mathbf{k}$-connected, to v, there is a *nonzero* alternating chain of mutually-disjoint $\mathbf{h}$-connected clusters and mutually-disjoint $\mathbf{k}$-connected clusters, from some $u_0 \in \mathcal{C}_{\mathbf{k}}(y)$ to v. Therefore, we have

$$\mathbb{1}_{\{y \underset{\mathbf{h}+\mathbf{k}}{\longleftrightarrow} v\} \setminus \{y \underset{\mathbf{k}}{\longleftrightarrow} v\}} \leq \sum_{j=1}^{\infty} \sum_{\substack{u_0,\dots,u_j \\ u_l \neq u_{l'} \, \forall l \neq l' \\ u_j = v}} \mathbb{1}_{\{y \underset{\mathbf{k}}{\longleftrightarrow} u_0\}} \bigg(\prod_{l \geq 0} \mathbb{1}_{\{u_{2l} \underset{\mathbf{h}}{\longleftrightarrow} u_{2l+1}\}} \bigg) \bigg(\prod_{l \geq 1} \mathbb{1}_{\{u_{2l-1} \underset{\mathbf{k}}{\longleftrightarrow} u_{2l}\}} \bigg)$$
$$\times \bigg(\prod_{\substack{l,l' \geq 0 \\ l \neq l'}} \mathbb{1}_{\{\mathcal{C}_{\mathbf{h}}(u_{2l}) \cap \mathcal{C}_{\mathbf{h}}(u_{2l'}) = \varnothing\}} \, \mathbb{1}_{\{\mathcal{C}_{\mathbf{k}}(u_{2l}) \cap \mathcal{C}_{\mathbf{k}}(u_{2l'}) = \varnothing\}} \bigg). \tag{4.45}$$

Because of this bound, we can now treat the sums over $\mathbf{h}$ and $\mathbf{k}$ in (4.41) separately.

Fix $j \geq 1$ and a sequence of *distinct* sites $u_0, \dots, u_j$ $(= v)$, and consider the contribution to the sum over $\mathbf{k}$ in (4.41) from the relevant indicators in the right-hand side of (4.45), which is

$$\sum_{\partial \mathbf{k} = y \triangle z} \frac{w_{\mathcal{B}^{\mathrm{c}}}(\mathbf{k})}{Z_{\mathcal{B}^{\mathrm{c}}}} \mathbb{1}_{\{y \underset{\mathbf{k}}{\longleftrightarrow} u_0\}} \bigg(\prod_{l \geq 1} \mathbb{1}_{\{u_{2l-1} \underset{\mathbf{k}}{\longleftrightarrow} u_{2l}\}} \bigg) \bigg(\prod_{\substack{l,l' \geq 0 \\ l \neq l'}} \mathbb{1}_{\{\mathcal{C}_{\mathbf{k}}(u_{2l}) \cap \mathcal{C}_{\mathbf{k}}(u_{2l'}) = \varnothing\}} \bigg). \tag{4.46}$$

Conditioning over $\mathcal{U}_{\mathbf{k};1} \equiv \dot{\bigcup}_{l \geq 1} \mathcal{C}_{\mathbf{k}}(u_{2l})$ and using (4.44), we obtain

$$\sum_{\partial \mathbf{k} = \varnothing} \frac{w_{\mathcal{B}^{\mathrm{c}}}(\mathbf{k})}{Z_{\mathcal{B}^{\mathrm{c}}}} \bigg(\prod_{l \geq 1} \mathbb{1}_{\{u_{2l-1} \underset{\mathbf{k}}{\longleftrightarrow} u_{2l}\}} \bigg) \bigg(\prod_{\substack{l,l' \geq 1 \\ l \neq l'}} \mathbb{1}_{\{\mathcal{C}_{\mathbf{k}}(u_{2l}) \cap \mathcal{C}_{\mathbf{k}}(u_{2l'}) = \varnothing\}} \bigg) \sum_{\partial \mathbf{k}' = y \triangle z} \frac{w_{\mathcal{B}^{\mathrm{c}} \cap \mathcal{U}_{\mathbf{k};1}^{\mathrm{c}}}(\mathbf{k}')}{Z_{\mathcal{B}^{\mathrm{c}} \cap \mathcal{U}_{\mathbf{k};1}^{\mathrm{c}}}} \mathbb{1}_{\{y \underset{\mathbf{k}'}{\longleftrightarrow} u_0\}}$$
$$\leq \langle \varphi_y \varphi_{u_0} \rangle_\Lambda \, \langle \varphi_{u_0} \varphi_z \rangle_\Lambda \sum_{\partial \mathbf{k} = \varnothing} \frac{w_{\mathcal{B}^{\mathrm{c}}}(\mathbf{k})}{Z_{\mathcal{B}^{\mathrm{c}}}} \bigg(\prod_{l \geq 1} \mathbb{1}_{\{u_{2l-1} \underset{\mathbf{k}}{\longleftrightarrow} u_{2l}\}} \bigg) \bigg(\prod_{\substack{l,l' \geq 1 \\ l \neq l'}} \mathbb{1}_{\{\mathcal{C}_{\mathbf{k}}(u_{2l}) \cap \mathcal{C}_{\mathbf{k}}(u_{2l'}) = \varnothing\}} \bigg). \tag{4.47}$$

Then, by conditioning on $\mathcal{U}_{\mathbf{k};2} \equiv \dot{\bigcup}_{l \geq 2} \mathcal{C}_{\mathbf{k}}(u_{2l})$, following the same computation as above and using (4.4), the sum in (4.47) is bounded by

$$\sum_{\partial \mathbf{k} = \varnothing} \frac{w_{\mathcal{B}^{\mathrm{c}}}(\mathbf{k})}{Z_{\mathcal{B}^{\mathrm{c}}}} \bigg(\prod_{l \geq 2} \mathbb{1}_{\{u_{2l-1} \underset{\mathbf{k}}{\longleftrightarrow} u_{2l}\}} \bigg) \bigg(\prod_{\substack{l,l' \geq 2 \\ l \neq l'}} \mathbb{1}_{\{\mathcal{C}_{\mathbf{k}}(u_{2l}) \cap \mathcal{C}_{\mathbf{k}}(u_{2l'}) = \varnothing\}} \bigg) \sum_{\partial \mathbf{k}' = \varnothing} \frac{w_{\mathcal{B}^{\mathrm{c}} \cap \mathcal{U}_{\mathbf{k};2}^{\mathrm{c}}}(\mathbf{k}')}{Z_{\mathcal{B}^{\mathrm{c}} \cap \mathcal{U}_{\mathbf{k};2}^{\mathrm{c}}}} \mathbb{1}_{\{u_1 \underset{\mathbf{k}'}{\longleftrightarrow} u_2\}}$$
$$\leq \tilde{G}_\Lambda(u_1, u_2)^2 \sum_{\partial \mathbf{k} = \varnothing} \frac{w_{\mathcal{B}^{\mathrm{c}}}(\mathbf{k})}{Z_{\mathcal{B}^{\mathrm{c}}}} \bigg(\prod_{l \geq 2} \mathbb{1}_{\{u_{2l-1} \underset{\mathbf{k}}{\longleftrightarrow} u_{2l}\}} \bigg) \bigg(\prod_{\substack{l,l' \geq 2 \\ l \neq l'}} \mathbb{1}_{\{\mathcal{C}_{\mathbf{k}}(u_{2l}) \cap \mathcal{C}_{\mathbf{k}}(u_{2l'}) = \varnothing\}} \bigg). \tag{4.48}$$

We repeat this computation until all the indicators for $\mathbf{k}$ are used up. Also, we apply the same argument to the indicators for $\mathbf{h}$. Summarizing these bounds with (4.44) and replacing u_0 in (4.45)–(4.47) by v', we obtain (4.42). This completes the proof of (4.31). ☐

4.2.2 Proof of Lemma 4.4

We note that the common factor $\mathbb{1}_{\{y \underset{\mathbf{m}+\mathbf{n}}{\Longleftrightarrow} x\}}$ in $\Theta'_{y,x;\mathcal{A}}$ and $\Theta''_{y,x,v;\mathcal{A}}$ can be divided as

$$\mathbb{1}_{\{y \underset{\mathbf{m}+\mathbf{n}}{\Longleftrightarrow} x\}} = \mathbb{1}_{\{y \underset{\mathbf{n}}{\Longleftrightarrow} x\}} + \mathbb{1}_{\{y \underset{\mathbf{m}+\mathbf{n}}{\Longleftrightarrow} x\} \setminus \{y \underset{\mathbf{n}}{\Longleftrightarrow} x\}}. \tag{4.49}$$

We first estimate the contribution from $\mathbb{1}_{\{y \underset{\mathbf{n}}{\Longleftrightarrow} x\}}$ to $\Theta'_{y,x;\mathcal{A}}$ and $\Theta''_{y,x,v;\mathcal{A}}$, and then estimate the contribution from $\mathbb{1}_{\{y \underset{\mathbf{m}+\mathbf{n}}{\Longleftrightarrow} x\} \setminus \{y \underset{\mathbf{n}}{\Longleftrightarrow} x\}}$ to $\Theta'_{y,x;\mathcal{A}}$ and $\Theta''_{y,x,v;\mathcal{A}}$.

Contribution to $\Theta'_{y,x;\mathcal{A}}$ from $\mathbb{1}_{\{y \underset{\mathbf{n}}{\Longleftrightarrow} x\}}$. For a set of events $E_1, \dots, E_N$, let $E_1 \circ \cdots \circ E_N$ be the event that $E_1, \dots, E_N$ occur *bond*-disjointly. Then, we have

$$\mathbb{1}_{\{y \overset{\mathcal{A}}{\underset{\mathbf{m}+\mathbf{n}}{\longleftrightarrow}} x\} \cap \{y \underset{\mathbf{n}}{\Longleftrightarrow} x\}} \le \mathbb{1}_{\{y \overset{\mathcal{A}}{\underset{\mathbf{n}}{\longleftrightarrow}} x\} \cap \{y \underset{\mathbf{n}}{\Longleftrightarrow} x\}} \le \sum_{u \in \mathcal{A}} \mathbb{1}_{\{y \underset{\mathbf{n}}{\longleftrightarrow} u\} \circ \{u \underset{\mathbf{n}}{\longleftrightarrow} x\} \circ \{y \underset{\mathbf{n}}{\longleftrightarrow} x\}}, \tag{4.50}$$

where the right-hand side does not depend on $\mathbf{m}$. Therefore, the contribution to $\Theta'_{y,x;\mathcal{A}}$ is bounded by

$$\sum_{u \in \mathcal{A}} \sum_{\partial \mathbf{n} = y \triangle x} \frac{w_\Lambda(\mathbf{n})}{Z_\Lambda} \mathbb{1}_{\{y \underset{\mathbf{n}}{\longleftrightarrow} u\} \circ \{u \underset{\mathbf{n}}{\longleftrightarrow} x\} \circ \{y \underset{\mathbf{n}}{\longleftrightarrow} x\}} \le \sum_{u \in \mathcal{A}} P'^{(0)}_{\Lambda;u}(y,x), \tag{4.51}$$

where we have applied the same argument used in the proof of (4.14). □

Contribution to $\Theta''_{y,x,v;\mathcal{A}}$ from $\mathbb{1}_{\{y \underset{\mathbf{n}}{\Longleftrightarrow} x\}}$ of (4.49). First, by using (4.43), we have

$$\mathbb{1}_{\{y \overset{\mathcal{A}}{\underset{\mathbf{m}+\mathbf{n}}{\longleftrightarrow}} x\} \cap \{y \underset{\mathbf{n}}{\Longleftrightarrow} x\} \cap \{y \underset{\mathbf{m}+\mathbf{n}}{\longleftrightarrow} v\}} \le \mathbb{1}_{\{y \overset{\mathcal{A}}{\underset{\mathbf{n}}{\longleftrightarrow}} x\} \cap \{y \underset{\mathbf{n}}{\Longleftrightarrow} x\}} \Big(\mathbb{1}_{\{y \underset{\mathbf{n}}{\longleftrightarrow} v\}} + \mathbb{1}_{\{y \underset{\mathbf{m}+\mathbf{n}}{\longleftrightarrow} v\} \setminus \{y \underset{\mathbf{n}}{\longleftrightarrow} v\}} \Big). \tag{4.52}$$

We investigate the contributions from the two indicators in the parentheses separately.

We begin with the contribution from $\mathbb{1}_{\{y \underset{\mathbf{n}}{\longleftrightarrow} v\}}$, which is independent of $\mathbf{m}$. Since

$$\{y \underset{\mathbf{n}}{\Longleftrightarrow} x\} \cap \{y \underset{\mathbf{n}}{\longleftrightarrow} v\} \subset \{y \underset{\mathbf{n}}{\longleftrightarrow} x\} \circ \{y \underset{\mathbf{n}}{\longleftrightarrow} x,\ y \underset{\mathbf{n}}{\longleftrightarrow} v\}, \tag{4.53}$$

$$\{y \overset{\mathcal{A}}{\underset{\mathbf{n}}{\longleftrightarrow}} x\} \subset \bigcup_{u \in \mathcal{A}} \{y \underset{\mathbf{n}}{\longleftrightarrow} u\} \circ \{u \underset{\mathbf{n}}{\longleftrightarrow} x\}, \tag{4.54}$$

the contribution to $\Theta''_{y,x,v;\mathcal{A}}$ is bounded by

$$\sum_{u \in \mathcal{A}} \sum_{\partial \mathbf{n} = y \triangle x} \frac{w_\Lambda(\mathbf{n})}{Z_\Lambda} \mathbb{1}_{\{y \underset{\mathbf{n}}{\longleftrightarrow} u\} \circ \{u \underset{\mathbf{n}}{\longleftrightarrow} x\} \circ \{y \underset{\mathbf{n}}{\longleftrightarrow} x,\ y \underset{\mathbf{n}}{\longleftrightarrow} v\}}. \tag{4.55}$$

We follow Steps (i)–(iii) described in Section 4.1. Without loss of generality, we can assume that all four sites y, u, x and v are different; otherwise, the argument below can be simplified. Similarly to the argument below (4.20), since y and x are sources, but u and v are not, there is an edge-disjoint cycle $y \to u \to x \to v \to y$, with an extra edge-disjoint path from y to x.

Therefore, we have in total at least 5 $(= 4+1)$ edge-disjoint paths. By multiplying $(\frac{Z_\Lambda}{Z_\Lambda})^4$, (4.55) equals

$$\sum_{u\in\mathcal{A}}\sum_{\partial\mathbf{N}=y\Delta x}\frac{w_\Lambda(\mathbf{N})}{Z_\Lambda^5}\sum_{\substack{\partial\mathbf{n}=y\Delta x\\ \partial\mathbf{m}^{(i)}=\varnothing\ ^\forall i=1,\dots,4\\ \mathbf{N}=\mathbf{n}+\sum_{i=1}^4\mathbf{m}^{(i)}}}\mathbb{1}_{\{y\underset{\mathbf{n}}{\longleftrightarrow}u\}\circ\{u\underset{\mathbf{n}}{\longleftrightarrow}x\}\circ\{y\underset{\mathbf{n}}{\longleftrightarrow}x,\ y\underset{\mathbf{n}}{\longleftrightarrow}v\}}\prod_{b\in\mathbb{B}_\Lambda}\frac{N_b!}{n_b!\prod_{i=1}^4 m_b^{(i)}!},\tag{4.56}$$

where the sum over $\mathbf{n},\mathbf{m}^{(1)},\dots,\mathbf{m}^{(4)}$ is bounded by the cardinality of $\mathfrak{S}$ in Lemma 4.2 with $k=4$, $\mathcal{V}=\{y,x\}$, $\{z_1,z_1'\}=\{y,u\}$, $\{z_2,z_2'\}=\{u,x\}$, $\{z_3,z_3'\}=\{y,v\}$ and $\{z_4,z_4'\}=\{v,x\}$. Bounding $|\mathfrak{S}'|$ for this setting, we obtain that (4.56) is bounded by

$$\begin{aligned}&\sum_{u\in\mathcal{A}}\sum_{\partial\mathbf{N}=y\Delta x}\frac{w_\Lambda(\mathbf{N})}{Z_\Lambda^5}\sum_{\substack{\partial\mathbf{n}=y\Delta x\\ \partial\mathbf{m}^{(1)}=y\Delta u,\ \partial\mathbf{m}^{(2)}=u\Delta x\\ \partial\mathbf{m}^{(3)}=y\Delta v,\ \partial\mathbf{m}^{(4)}=v\Delta x\\ \mathbf{N}=\mathbf{n}+\sum_{i=1}^4\mathbf{m}^{(i)}}}\prod_{b\in\mathbb{B}_\Lambda}\frac{N_b!}{n_b!\prod_{i=1}^4 m_b^{(i)}!}\\ &\le\sum_{u\in\mathcal{A}}\langle\varphi_y\varphi_x\rangle_\Lambda\langle\varphi_y\varphi_u\rangle_\Lambda\langle\varphi_u\varphi_x\rangle_\Lambda\langle\varphi_y\varphi_v\rangle_\Lambda\langle\varphi_v\varphi_x\rangle_\Lambda.\end{aligned}\tag{4.57}$$

Next, we investigate the contribution from $\mathbb{1}_{\{y\underset{\mathbf{m}+\mathbf{n}}{\longleftrightarrow}v\}\setminus\{y\underset{\mathbf{n}}{\longleftrightarrow}v\}}$ of (4.52). On the event $\{y\underset{\mathbf{n}}{\Longleftrightarrow}x\}\cap\{\{y\underset{\mathbf{m}+\mathbf{n}}{\longleftrightarrow}v\}\setminus\{y\underset{\mathbf{n}}{\longleftrightarrow}v\}\}$, there exists a v_0 such that $\{y\underset{\mathbf{n}}{\longleftrightarrow}x\}\circ\{y\underset{\mathbf{n}}{\longleftrightarrow}x,\ y\underset{\mathbf{n}}{\longleftrightarrow}v_0\}$ occurs and that v_0 and v are connected via a nonzero alternating chain of mutually-disjoint $\mathbf{m}$-connected clusters and mutually-disjoint $\mathbf{n}$-connected clusters. Therefore, by using (4.45) and $\{y\overset{\mathcal{A}}{\underset{\mathbf{n}}{\longleftrightarrow}}x\}\subset\bigcup_{u\in\mathcal{A}}\{y\underset{\mathbf{n}}{\longleftrightarrow}u\}\circ\{u\underset{\mathbf{n}}{\longleftrightarrow}x\}$, we obtain (cf., (4.55))

$$\begin{aligned}&\mathbb{1}_{\{y\overset{\mathcal{A}}{\underset{\mathbf{n}}{\longleftrightarrow}}x\}\cap\{y\underset{\mathbf{n}}{\Longleftrightarrow}x\}\cap\{\{y\underset{\mathbf{m}+\mathbf{n}}{\longleftrightarrow}v\}\setminus\{y\underset{\mathbf{n}}{\longleftrightarrow}v\}\}}\\ &\le\sum_{u\in\mathcal{A}}\sum_{j\ge1}\sum_{\substack{v_0,\dots,v_j\\ v_l\ne v_{l'}\ ^\forall l\ne l'\\ v_j=v}}\mathbb{1}_{\{y\underset{\mathbf{n}}{\longleftrightarrow}u\}\circ\{u\underset{\mathbf{n}}{\longleftrightarrow}x\}\circ\{y\underset{\mathbf{n}}{\longleftrightarrow}x,\ y\underset{\mathbf{n}}{\longleftrightarrow}v_0\}}\bigg(\prod_{l\ge0}\mathbb{1}_{\{v_{2l}\underset{\mathbf{m}}{\longleftrightarrow}v_{2l+1}\}}\bigg)\\ &\quad\times\bigg(\prod_{l\ge1}\mathbb{1}_{\{v_{2l-1}\underset{\mathbf{n}}{\longleftrightarrow}v_{2l}\}}\bigg)\bigg(\prod_{\substack{l,l'\ge0\\ l\ne l'}}\mathbb{1}_{\{\mathcal{C}_\mathbf{m}(v_{2l})\cap\mathcal{C}_\mathbf{m}(v_{2l'})=\varnothing\}}\,\mathbb{1}_{\{\mathcal{C}_\mathbf{n}(v_{2l})\cap\mathcal{C}_\mathbf{n}(v_{2l'})=\varnothing\}}\bigg).\end{aligned}\tag{4.58}$$

For the products of indicators, we repeatedly use the "conditioning-over-clusters" argument, as in (4.46)–(4.48). As a result, because the first indicator in the right-hand side of (4.58) does not depend on $\mathbf{m}$, we can apply (4.55)–(4.57) to obtain

$$\begin{aligned}&\sum_{\substack{\partial\mathbf{m}=\varnothing\\ \partial\mathbf{n}=y\Delta x}}\frac{w_{\mathcal{A}^c}(\mathbf{m})}{Z_{\mathcal{A}^c}}\frac{w_\Lambda(\mathbf{n})}{Z_\Lambda}\mathbb{1}_{\{y\overset{\mathcal{A}}{\underset{\mathbf{n}}{\longleftrightarrow}}x\}\cap\{y\underset{\mathbf{n}}{\Longleftrightarrow}x\}\cap\{\{y\underset{\mathbf{m}+\mathbf{n}}{\longleftrightarrow}v\}\setminus\{y\underset{\mathbf{n}}{\longleftrightarrow}v\}\}}\\ &\le\sum_{v_0}\big(\psi_\Lambda(v_0,v)-\delta_{v_0,v}\big)\sum_{u\in\mathcal{A}}\sum_{\partial\mathbf{n}=y\Delta x}\frac{w_\Lambda(\mathbf{n})}{Z_\Lambda}\mathbb{1}_{\{y\underset{\mathbf{n}}{\longleftrightarrow}u\}\circ\{u\underset{\mathbf{n}}{\longleftrightarrow}x\}\circ\{y\underset{\mathbf{n}}{\longleftrightarrow}x,\ y\underset{\mathbf{n}}{\longleftrightarrow}v_0\}}\\ &\le\sum_{u\in\mathcal{A},\ v_0}\big(\psi_\Lambda(v_0,v)-\delta_{v_0,v}\big)\langle\varphi_y\varphi_x\rangle_\Lambda\langle\varphi_y\varphi_u\rangle_\Lambda\langle\varphi_u\varphi_x\rangle_\Lambda\langle\varphi_y\varphi_{v_0}\rangle_\Lambda\langle\varphi_{v_0}\varphi_x\rangle_\Lambda.\end{aligned}\tag{4.59}$$

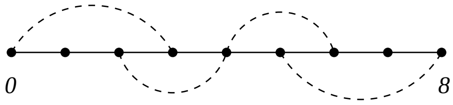

Figure 7: An element in $\mathcal{L}^{(4)}_{[0,8]}$, which consists of $s_1t_1 = \{0,3\}$, $s_2t_2 = \{2,4\}$, $s_3t_3 = \{4,6\}$ and $s_4t_4 = \{5,8\}$.

Summarizing (4.52), (4.57) and (4.59), we arrive at

$$\sum_{\substack{\partial\mathbf{m}=\varnothing\\ \partial\mathbf{n}=y\triangle x}} \frac{w_{\mathcal{A}^{\rm c}}(\mathbf{m})}{Z_{\mathcal{A}^{\rm c}}}\,\frac{w_\Lambda(\mathbf{n})}{Z_\Lambda}\,\mathbb{1}_{\{y \underset{\mathbf{m}+\mathbf{n}}{\overset{\mathcal{A}}{\longleftrightarrow}} x\}\cap\{y \underset{\mathbf{n}}{\Longleftrightarrow} x\}\cap\{y \underset{\mathbf{m}+\mathbf{n}}{\longleftrightarrow} v\}} \le \sum_{u\in\mathcal{A}} P''^{(0)}_{\Lambda;u,v}(y,x). \tag{4.60}$$

This complets the bound on the contribution to $\Theta''_{y,x,v;\mathcal{A}}$ from $\mathbb{1}_{\{y \underset{\mathbf{n}}{\Longleftrightarrow} x\}}$ of (4.49). $\square$

Contribution to $\Theta'_{y,x;\mathcal{A}}$ *from* $\mathbb{1}_{\{y \underset{\mathbf{m}+\mathbf{n}}{\Longleftrightarrow} x\}\setminus\{y \underset{\mathbf{n}}{\Longleftrightarrow} x\}}$ *of (4.49).* When $\mathbb{1}_{\{\partial\mathbf{n}=y\triangle x\}\setminus\{y \underset{\mathbf{n}}{\Longleftrightarrow} x\}} = 1$, then y is $\mathbf{n}$-connected, but not $\mathbf{n}$-doubly connected, to x, and therefore there exists at least one pivotal bond for $y \underset{\mathbf{n}}{\longleftrightarrow} x$. Given an ordered set $\vec{b}_T = (b_1,\dots,b_T)$, we define

$$H_{\mathbf{n};\vec{b}_T}(y,x) = \{y \underset{\mathbf{n}}{\Longleftrightarrow} \underline{b}_1\} \cap \bigcap_{i=1}^{T} \Big\{\{\overline{b}_i \underset{\mathbf{n}}{\Longleftrightarrow} \underline{b}_{i+1}\} \cap \{n_{b_i} > 0,\ b_i \text{ is pivotal for } y \underset{\mathbf{n}}{\longleftrightarrow} x\}\Big\}, \tag{4.61}$$

where, by convention, $\underline{b}_{T+1} = x$. Then, the contribution to $\Theta'_{y,x;\mathcal{A}}$ can be written as

$$\sum_{T\ge1}\sum_{\vec{b}_T}\sum_{\substack{\partial\mathbf{m}=\varnothing\\ \partial\mathbf{n}=y\triangle x}} \frac{w_{\mathcal{A}^{\rm c}}(\mathbf{m})}{Z_{\mathcal{A}^{\rm c}}}\,\frac{w_\Lambda(\mathbf{n})}{Z_\Lambda}\,\mathbb{1}_{\{y \underset{\mathbf{m}+\mathbf{n}}{\overset{\mathcal{A}}{\longleftrightarrow}} x\}\cap H_{\mathbf{n};\vec{b}_T}(y,x)\cap\{\{y \underset{\mathbf{m}+\mathbf{n}}{\Longleftrightarrow} x\}\setminus\{y \underset{\mathbf{n}}{\Longleftrightarrow} x\}\}}. \tag{4.62}$$

On the event $H_{\mathbf{n};\vec{b}_T}(y,x)$, we denote the $\mathbf{n}$-double connections between the pivotal bonds $b_1,\dots,b_T$ by

$$\mathcal{D}_{\mathbf{n};i} = \begin{cases} \mathcal{C}^{b_1}_{\mathbf{n}}(y) & (i=0),\\ \mathcal{C}^{b_{i+1}}_{\mathbf{n}}(y)\setminus\mathcal{C}^{b_i}_{\mathbf{n}}(y) & (i=1,\dots,T-1),\\ \mathcal{C}_{\mathbf{n}}(y)\setminus\mathcal{C}^{b_T}_{\mathbf{n}}(y) & (i=T). \end{cases} \tag{4.63}$$

We can think of $\mathcal{C}_{\mathbf{n}}(y)$ as the interval $[0,T]$, where each integer $i\in[0,T]$ corresponds to $\mathcal{D}_{\mathbf{n};i}$ and the unit interval $(i-1,i)\subset[0,T]$ corresponds to the pivotal bond b_i. Since y is $(\mathbf{m}+\mathbf{n})$-doubly connected to x, for every b_i there must be an $(\mathbf{m}+\mathbf{n})$-bypath (i.e., an $(\mathbf{m}+\mathbf{n})$-connection that does not go through b_i) from some $z\in\mathcal{D}_{\mathbf{n};s}$ with $s\in\{0,\dots,i-1\}$ to some $z'\in\mathcal{D}_{\mathbf{n};t}$ with $t\in\{i,\dots,T\}$. Let $\mathcal{L}^{(1)}_{[0,T]} = \{\{0T\}\}$, $\mathcal{L}^{(2)}_{[0,T]} = \{\{0t_1, s_2T\} : 0 < s_2 \le t_1 < T\}$ and, generally for $j\le T$ (see Figure 7),

$$\mathcal{L}^{(j)}_{[0,T]} = \Big\{\{s_it_i\}_{i=1}^{j} : 0 = s_1 < s_2 \le t_1 < s_3 \le \cdots \le t_{j-2} < s_j \le t_{j-1} < t_j = T\Big\}. \tag{4.64}$$

By this definition, we have $\bigcup_{st\in\Gamma}[s,t]=[0,T]$ for any $\Gamma\in\mathcal{L}^{(j)}_{[0,T]}$, for every $j\in\{1,\dots,T\}$. Using this notation and $\mathbb{1}_{\{y\underset{\mathbf{m}+\mathbf{n}}{\overset{\mathcal{A}}{\longleftrightarrow}}x\}\cap\{\partial\mathbf{n}=y\vartriangle x\}}\le\mathbb{1}_{\{y\underset{\mathbf{n}}{\overset{\mathcal{A}}{\longleftrightarrow}}x\}\cap\{\partial\mathbf{n}=y\vartriangle x\}}$ and conditioning over $\mathcal{C}_{\mathbf{n}}(y)$, we can bound (4.62) by

$$\sum_{T\ge1}\sum_{\vec b_T}\sum_{\partial\mathbf{n}=y\vartriangle x}\frac{w_\Lambda(\mathbf{n})}{Z_\Lambda}\,\mathbb{1}_{\{y\underset{\mathbf{n}}{\overset{\mathcal{A}}{\longleftrightarrow}}x\}\cap H_{\mathbf{n};\vec b_T}(y,x)}\sum_{j=1}^{T}\sum_{\mathcal{L}^{(j)}_{[0,T]}=\{s_it_i\}_{i=1}^j}\sum_{\substack{z_1,\dots,z_j\\ z'_1,\dots,z'_j}}\bigg(\prod_{i=1}^j\mathbb{1}_{\{z_i\in\mathcal{D}_{\mathbf{n};s_i},\ z'_i\in\mathcal{D}_{\mathbf{n};t_i}\}}\bigg)$$
$$\times\sum_{\partial\mathbf{m}=\partial\mathbf{k}=\varnothing}\frac{w_{\mathcal{A}^{\rm c}}(\mathbf{m})}{Z_{\mathcal{A}^{\rm c}}}\,\frac{w_{\mathcal{C}_{\mathbf{n}}(y)^{\rm c}}(\mathbf{k})}{Z_{\mathcal{C}_{\mathbf{n}}(y)^{\rm c}}}\bigg(\prod_{i=1}^j\mathbb{1}_{\{z_i\underset{\mathbf{m}+\mathbf{k}}{\longleftrightarrow}z'_i\}}\bigg)\bigg(\prod_{i\ne l}\mathbb{1}_{\{\mathcal{C}_{\mathbf{m}+\mathbf{k}}(z_i)\cap\mathcal{C}_{\mathbf{m}+\mathbf{k}}(z_l)=\varnothing\}}\bigg),\tag{4.65}$$

where the first line determines $\mathcal{C}_{\mathbf{n}}(y)$ that contains vertices z_i,z'_i for $i=1,\dots,j$ in a specific manner, and the second line determines the bypathes $\mathcal{C}_{\mathbf{m}+\mathbf{k}}(z_i)$, for $i=1,\dots,j$, joining z_i and z'_i.

First, we estimate the second line of (4.65). Since $\mathcal{C}_{\mathbf{m}+\mathbf{k}}(z_i)$ for $i=1,\dots,j$ are mutually-disjoint, we can treat each cluster separately by using the "conditioning-over-clusters" argument. By abbreviating $\mathcal{C}_{\mathbf{n}}(y)$ to $\mathcal{C}$ and conditioning over $\mathcal{V}_{\mathbf{m}+\mathbf{k}}\equiv\dot{\bigcup}_{i\ge2}\mathcal{C}_{\mathbf{m}+\mathbf{k}}(z_i)$, the second line of (4.65) equals

$$\sum_{\partial\mathbf{m}=\partial\mathbf{k}=\varnothing}\frac{w_{\mathcal{A}^{\rm c}}(\mathbf{m})}{Z_{\mathcal{A}^{\rm c}}}\,\frac{w_{\mathcal{C}^{\rm c}}(\mathbf{k})}{Z_{\mathcal{C}^{\rm c}}}\bigg(\prod_{i=2}^j\mathbb{1}_{\{z_i\underset{\mathbf{m}+\mathbf{k}}{\longleftrightarrow}z'_i\}}\bigg)\bigg(\prod_{\substack{i,l\ge2\\ i\ne l}}\mathbb{1}_{\{\mathcal{C}_{\mathbf{m}+\mathbf{k}}(z_i)\cap\mathcal{C}_{\mathbf{m}+\mathbf{k}}(z_l)=\varnothing\}}\bigg)$$
$$\times\sum_{\partial\mathbf{m}'=\partial\mathbf{k}'=\varnothing}\frac{w_{\mathcal{A}^{\rm c}\cap\mathcal{V}^{\rm c}_{\mathbf{m}+\mathbf{k}}}(\mathbf{m}')}{Z_{\mathcal{A}^{\rm c}\cap\mathcal{V}^{\rm c}_{\mathbf{m}+\mathbf{k}}}}\,\frac{w_{\mathcal{C}^{\rm c}\cap\mathcal{V}^{\rm c}_{\mathbf{m}+\mathbf{k}}}(\mathbf{k}')}{Z_{\mathcal{C}^{\rm c}\cap\mathcal{V}^{\rm c}_{\mathbf{m}+\mathbf{k}}}}\,\mathbb{1}_{\{z_1\underset{\mathbf{m}'+\mathbf{k}'}{\longleftrightarrow}z'_1\}}.\tag{4.66}$$

Since $z_1\ne z'_1\in\mathcal{C}\equiv\mathcal{C}_{\mathbf{n}}(y)$, they are connected via a nonzero alternating chain, which starts and ends with $\mathbf{m}'$-connected clusters (possibly with a single $\mathbf{m}'$-connected cluster), of mutually-disjoint $\mathbf{m}'$-connected clusters and mutually-disjoint $\mathbf{k}'$-connected clusters. Following the argument around (4.45)–(4.48), we can bound the second line of (4.66) by $\sum_{l\ge1}\big(\tilde G^2_\Lambda\big)^{*(2l-1)}(z_1,z'_1)$. By repeating this argument, (4.66) is bounded by

$$\prod_{i=1}^j\bigg(\sum_{l\ge1}\big(\tilde G^2_\Lambda\big)^{*(2l-1)}(z_i,z'_i)\bigg)\le\bigg(\prod_{i=1,j}\sum_{l\ge1}\big(\tilde G^2_\Lambda\big)^{*(2l-1)}(z_i,z'_i)\bigg)\bigg(\prod_{i=2}^{j-1}\big(\psi_\Lambda(z_i,z'_i)-\delta_{z_i,z'_i}\big)\bigg),\tag{4.67}$$

where, by convention, the empty product is regarded as 1.

Therefore, (4.65) is bounded by

$$\sum_{j\ge1}\sum_{\substack{z_1,\dots,z_j\\ z'_1,\dots,z'_j}}\bigg(\prod_{i=1,j}\sum_{l\ge1}\big(\tilde G^2_\Lambda\big)^{*(2l-1)}(z_i,z'_i)\bigg)\bigg(\prod_{i=2}^{j-1}\big(\psi_\Lambda(z_i,z'_i)-\delta_{z_i,z'_i}\big)\bigg)\sum_{\partial\mathbf{n}=y\vartriangle x}\frac{w_\Lambda(\mathbf{n})}{Z_\Lambda}\,\mathbb{1}_{\{y\underset{\mathbf{n}}{\overset{\mathcal{A}}{\longleftrightarrow}}x\}}$$
$$\times\sum_{T\ge j}\sum_{\vec b_T}\sum_{\mathcal{L}^{(j)}_{[0,T]}=\{s_it_i\}_{i=1}^j}\mathbb{1}_{H_{\mathbf{n};\vec b_T}(y,x)}\prod_{i=1}^j\mathbb{1}_{\{\mathcal{D}_{\mathbf{n};s_i}\ni z_i,\ \mathcal{D}_{\mathbf{n};t_i}\ni z'_i\}},\tag{4.68}$$

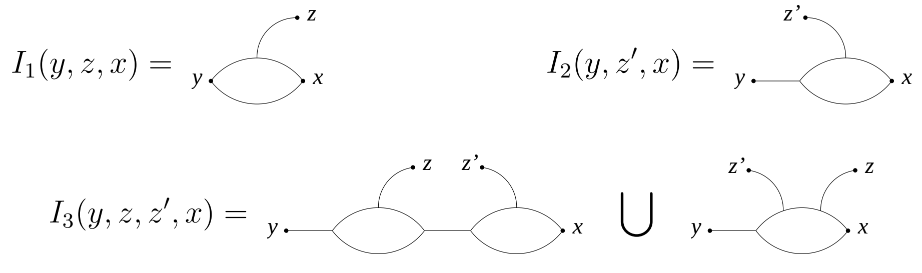

Figure 8: Schematic representations of $I_1(y,z,x)$, $I_2(y,z',x)$ and $I_3(y,z,z',x)$.

which depends only on a single current configuration, and hence we can apply the GHS idea to obtain its upper bound. To do so, we first simplify the second line of (4.68), which is, by definition, equal to the indicator of the event

$$\begin{aligned}
&\dot{\bigcup_{T\ge j}}\ \dot{\bigcup_{\vec b_T}}\ \dot{\bigcup_{\mathcal{L}^{(j)}_{[0,T]}=\{s_it_i\}_{i=1}^j}} \bigg\{H_{\mathbf{n};\vec b_T}(y,x)\cap\bigcap_{i=1}^j\big\{\mathcal{D}_{\mathbf{n};s_i}\ni z_i,\ \mathcal{D}_{\mathbf{n};t_i}\ni z_i'\big\}\bigg\}\\
&=\dot{\bigcup_{e_1,\dots,e_j}}\Bigg\{\dot{\bigcup_{T\ge j}}\ \dot{\bigcup_{\vec b_T}}\ \dot{\bigcup_{\substack{\mathcal{L}^{(j)}_{[0,T]}=\{s_it_i\}_{i=1}^j\\ b_{t_i+1}=e_{i+1}\ \forall i}}} \bigg\{H_{\mathbf{n};\vec b_T}(y,x)\cap\bigcap_{i=1}^j\big\{\mathcal{D}_{\mathbf{n};s_i}\ni z_i,\ \mathcal{D}_{\mathbf{n};t_i}\ni z_i'\big\}\bigg\}\Bigg\},
\end{aligned}\tag{4.69}$$

where, by convention, $t_0=0$. On the left-hand side, the first two unions are for the number and location of pivotal bonds for $y \underset{\mathbf{n}}{\longleftrightarrow} x$, while the third union identifies the double connections that are associated to z_i, z_i' for $i=1,\dots,j$. Let (see Figure 8)

$$I_1(y,z,x)=\{y \underset{\mathbf{n}}{\Longleftrightarrow} x,\ y \underset{\mathbf{n}}{\longleftrightarrow} z\},\qquad I_2(y,z',x)=\bigcup_u\big\{\{y \underset{\mathbf{n}}{\longleftrightarrow} u\}\circ I_1(u,z',x)\big\},\tag{4.70}$$

$$I_3(y,z,z',x)=\bigcup_u\Big\{\{I_2(y,z,u)\circ I_2(u,z',x)\}\cup\big\{\{y \underset{\mathbf{n}}{\longleftrightarrow} u\}\circ\{I_1(u,z,x)\cap I_1(u,z',x)\}\big\}\Big\}.\tag{4.71}$$

Then, (4.69) becomes a subset of

$$\begin{aligned}
\dot{\bigcup_{e_1,\dots,e_j}}\bigg\{&\Big\{I_1(y,z_1,\underline{e}_1)\circ I_3(\overline{e}_1,z_2,z_1',\underline{e}_2)\circ\cdots\circ I_3(\overline{e}_{j-1},z_j,z_{j-1}',\underline{e}_j)\circ I_2(\overline{e}_j,z_j',x)\Big\}\\
&\cap\bigcap_{i=1}^j\{n_{e_i}>0,\ e_i\text{ is pivotal for }y \underset{\mathbf{n}}{\longleftrightarrow} x\}\bigg\}.
\end{aligned}\tag{4.72}$$

To bound the sum over $\mathbf{n}$ in (4.68) using the GHS idea, we further consider an event that contains (4.72) as a subset. Without loss generality, we can assume that $y\ne\underline{e}_1$, $\overline{e}_{i-1}\ne\underline{e}_i$ for $i=2,\dots,j$, and $\overline{e}_j\ne x$; otherwise, the argument below can be simplified. Since every n_{e_i} is an odd integer, to consider each event I_i in (4.70)–(4.71) individually, we can assume that y and x are the only sources of $\partial\mathbf{n}$. On $I_1(y,z,x)$, according to the observation in

Step (i) described below (4.20), we have two edge-disjoint connections from y to z, one of which may go through x, and another edge-disjoint connection from y to x. Therefore,

$$I_1(y,z,x) \subset \big\{{}^\exists\omega_1,\omega_2 \in \Omega^{\mathbf{n}}_{y\to z}\ {}^\exists\omega_3 \in \Omega^{\mathbf{n}}_{y\to x} \text{ such that } \ \omega_i \cap \omega_l = \varnothing\ (i \neq l)\big\}. \tag{4.73}$$

Similarly, we have (cf., Figure 8)

$$I_2(y,z',x) \subset \big\{{}^\exists\omega_1,\omega_2 \in \Omega^{\mathbf{n}}_{x\to z'}\ {}^\exists\omega_3 \in \Omega^{\mathbf{n}}_{y\to x} \text{ such that } \ \omega_i \cap \omega_l = \varnothing\ (i \neq l)\big\}. \tag{4.74}$$

On $I_3(y,z,z',x)$, there are three edge-disjoint paths, from y to z, from z to z', and from z' to x. This is not so difficult to be seen from $\bigcup_u\{I_2(y,z,u) \circ I_2(u,z',x)\}$ in (4.71). For the remaining event in (4.71), take a look at the last picture in Figure 8 for one of the worst topological situation to extract such three edge-disjoint paths. Since there are at least three edge-disjoint paths between u and x, say, ζ_1,ζ_2 and ζ_3, we can go from y to z via ζ_1 and a part of ζ_2, and go from z to z' via the middle part of ζ_2, and then go from z' to x via the remaining part of ζ_2 and ζ_3. The other cases can be dealt with similarly. As a result, we have

$$I_3(y,z,z',x) \subset \big\{{}^\exists\omega_1 \in \Omega^{\mathbf{n}}_{y\to z}\ {}^\exists\omega_2 \in \Omega^{\mathbf{n}}_{z\to z'}\ {}^\exists\omega_3 \in \Omega^{\mathbf{n}}_{z'\to x} \text{ such that } \ \omega_i \cap \omega_l = \varnothing\ (i \neq l)\big\}. \tag{4.75}$$

Since

$$\bigcup_e \Big\{\big\{\{{}^\exists\omega \in \Omega^{\mathbf{n}}_{z\to \underline{e}}\} \circ \{{}^\exists\omega \in \Omega^{\mathbf{n}}_{\overline{e}\to z'}\}\big\} \cap \{n_e > 0\}\Big\} \subset \{{}^\exists\omega \in \Omega^{\mathbf{n}}_{z\to z'}\}, \tag{4.76}$$

the event (4.72) is a subset of

$$\tilde{I}^{(j)}_{\vec{z}_j,\vec{z}'_j}(y,x) = \left\{\begin{array}{c} {}^\exists\omega_1,\omega_2 \in \Omega^{\mathbf{n}}_{z_1\to y}\ {}^\exists\omega_3 \in \Omega^{\mathbf{n}}_{y\to z_2}\ {}^\exists\omega_4 \in \Omega^{\mathbf{n}}_{z_2\to z'_1}\ {}^\exists\omega_5 \in \Omega^{\mathbf{n}}_{z'_1\to z_3}\cdots \\ \cdots {}^\exists\omega_{2j} \in \Omega^{\mathbf{n}}_{z_j\to z'_{j-1}}\ {}^\exists\omega_{2j+1} \in \Omega^{\mathbf{n}}_{z'_{j-1}\to x}\ {}^\exists\omega_{2j+2},\omega_{2j+3} \in \Omega^{\mathbf{n}}_{x\to z'_j} \\ \text{such that } \ \omega_i \cap \omega_l = \varnothing\ (i \neq l) \end{array}\right\}, \tag{4.77}$$

where $\vec{z}^{(\prime)}_j = (z^{(\prime)}_1,\dots,z^{(\prime)}_j)$. Therefore, the sum over $\mathbf{n}$ in (4.68) is bounded by

$$\sum_{\partial\mathbf{n}=y\triangle x} \frac{w_\Lambda(\mathbf{n})}{Z_\Lambda}\, \mathbb{1}_{\{y \xleftrightarrow[\mathbf{n}]{\mathcal{A}} x\}}\, \mathbb{1}_{\tilde{I}^{(j)}_{\vec{z}_j,\vec{z}'_j}(y,x)}. \tag{4.78}$$

Now, we apply the GHS idea to bound (4.78). For the moment, we ignore the first indicator in (4.78) and consider the contribution from the second one. Without losing generality, we assume that the sites y,x,z_i,z'_i for $i=1,\dots,j$ are all different. Since there are $2j+3$ edge-disjoint paths on $\mathbb{G}_{\mathbf{n}}$ as in (4.77), we first multiply (4.78) by $(\frac{Z_\Lambda}{Z_\Lambda})^{2j+2}$ following Step (ii) of the strategy described in Section 4.1. Overlapping these $2j+3$ current configurations and using Lemma 4.2 with $k=2j+2$, $\mathcal{V}=\{y,x\}$ and so on, we obtain

$$\sum_{\partial\mathbf{n}=y\triangle x} \frac{w_\Lambda(\mathbf{n})}{Z_\Lambda}\, \mathbb{1}_{\tilde{I}^{(j)}_{\vec{z}_j,\vec{z}'_j}(y,x)} \le \langle\varphi_{z_1}\varphi_y\rangle^2_\Lambda\, \langle\varphi_y\varphi_{z_2}\rangle_\Lambda\, \langle\varphi_{z_2}\varphi_{z'_1}\rangle_\Lambda\, \langle\varphi_{z'_1}\varphi_{z_3}\rangle_\Lambda \cdots$$
$$\cdots \langle\varphi_{z_j}\varphi_{z'_{j-1}}\rangle_\Lambda\, \langle\varphi_{z'_{j-1}}\varphi_x\rangle_\Lambda\, \langle\varphi_x\varphi_{z'_j}\rangle^2_\Lambda. \tag{4.79}$$

Therefore, (4.68) (= (4.78)) without $\mathbb{1}_{\{y \xleftrightarrow[\mathbf{n}]{\mathcal{A}} x\}}$ is bounded by

$$\sum_{j\geq 1}\sum_{\substack{z_2,\dots,z_j\\ z'_1,\dots,z'_{j-1}}}\bigg(\prod_{i=2}^{j-1}\big(\psi_\Lambda(z_i,z'_i)-\delta_{z_i,z'_i}\big)\bigg)\bigg(\sum_{z_1}\langle\varphi_y\varphi_{z_1}\rangle_\Lambda^2\sum_{l\geq 1}\big(\tilde G_\Lambda^2\big)^{*(2l-1)}(z_1,z'_1)\bigg) \tag{4.80}$$
$$\times\langle\varphi_y\varphi_{z_2}\rangle_\Lambda\cdots\langle\varphi_{z'_{j-1}}\varphi_x\rangle_\Lambda\bigg(\sum_{z'_j}\langle\varphi_x\varphi_{z'_j}\rangle_\Lambda^2\sum_{l\geq 1}\big(\tilde G_\Lambda^2\big)^{*(2l-1)}(z_j,z'_j)\bigg)\leq\sum_{j\geq 1}P_\Lambda^{(j)}(y,x).$$

If $\mathbb{1}_{\{y \xleftrightarrow[\mathbf{n}]{\mathcal{A}} x\}}$ is present in the above argument, then at least one of the paths ω_i for $i=3,\dots,2j+1$ has to go through $\mathcal{A}$. For example, if ω_3 goes through $\mathcal{A}$, then we can split it into two edge-disjoint paths at some $u\in\mathcal{A}$, such as $\omega'_3\in\Omega^{\mathbf{N}}_{y\to u}$ and $\omega''_3\in\Omega^{\mathbf{N}}_{u\to z_2}$. The contribution from this case is bounded, by following the same argument as above, by (4.79) with $\langle\varphi_y\varphi_{z_2}\rangle_\Lambda$ being replaced by $\sum_{u\in\mathcal{A}}\langle\varphi_y\varphi_u\rangle_\Lambda\langle\varphi_u\varphi_{z_2}\rangle_\Lambda$. Bounding the other $2j-2$ cases similarly and summing these bounds over $j\geq 1$, we obtain that (4.68) is bounded by $\sum_{u\in\mathcal{A}}\sum_{j\geq 1}P'^{(j)}_{\Lambda;u}(y,x)$. This, together with (4.51), completes the proof of the bound on $\Theta'_{y,x;\mathcal{A}}$ in (4.32). □

Contribution to $\Theta''_{y,x,v;\mathcal{A}}$ from $\mathbb{1}_{\{y \underset{\mathbf{m}+\mathbf{n}}{\Longleftrightarrow} x\}\setminus\{y \underset{\mathbf{n}}{\Longleftrightarrow} x\}}$ of (4.49). By using (4.61), the contribution to $\Theta''_{y,x,v;\mathcal{A}}$ can be written, similarly to (4.62), as

$$\sum_{T\geq 1}\sum_{\vec b_T}\sum_{\substack{\partial\mathbf{m}=\varnothing\\ \partial\mathbf{n}=y\triangle x}}\frac{w_{\mathcal{A}^{\rm c}}(\mathbf{m})}{Z_{\mathcal{A}^{\rm c}}}\frac{w_\Lambda(\mathbf{n})}{Z_\Lambda}\mathbb{1}_{\{y \xleftrightarrow[\mathbf{m}+\mathbf{n}]{\mathcal{A}} x\}\cap H_{\mathbf{n};\vec b_T}(y,x)\cap\{\{y \underset{\mathbf{m}+\mathbf{n}}{\Longleftrightarrow} x\}\setminus\{y \underset{\mathbf{n}}{\Longleftrightarrow} x\}\}\cap\{y \underset{\mathbf{m}+\mathbf{n}}{\longleftrightarrow} v\}}. \tag{4.81}$$

To bound this, we will also use a similar expression to (4.65). We split (4.81) depending on whether or not there is a "cluster" that corresponds to $\mathcal{C}_{\mathbf{m}+\mathbf{k}}(z_i)$ in (4.65) for some $i\in\{1,\dots,j\}$ such that $v\in\mathcal{C}_{\mathbf{m}+\mathbf{k}}(z_i)$.

(i) If there is such a cluster, then we use $\mathbb{1}_{\{y \xleftrightarrow[\mathbf{m}+\mathbf{n}]{\mathcal{A}} x\}\cap\{\partial\mathbf{n}=y\triangle x\}}\leq\mathbb{1}_{\{y \xleftrightarrow[\mathbf{n}]{\mathcal{A}} x\}\cap\{\partial\mathbf{n}=y\triangle x\}}$, as in (4.65), to bound the contribution from this case to (4.81) by

$$\sum_{T\geq 1}\sum_{\vec b_T}\sum_{\partial\mathbf{n}=y\triangle x}\frac{w_\Lambda(\mathbf{n})}{Z_\Lambda}\mathbb{1}_{\{y \xleftrightarrow[\mathbf{n}]{\mathcal{A}} x\}\cap H_{\mathbf{n};\vec b_T}(y,x)}\sum_{j=1}^{T}\sum_{\{s_it_i\}_{i=1}^j\in\mathcal{L}^{(j)}_{[0,T]}}\sum_{\substack{z_1,\dots,z_j\\ z'_1,\dots,z'_j}}\bigg(\prod_{i=1}^{j}\mathbb{1}_{\{z_i\in\mathcal{D}_{\mathbf{n};s_i},\ z'_i\in\mathcal{D}_{\mathbf{n};t_i}\}}\bigg)$$
$$\times\sum_{\substack{\partial\mathbf{m}=\varnothing\\ \partial\mathbf{k}=\varnothing}}\frac{w_{\mathcal{A}^{\rm c}}(\mathbf{m})}{Z_{\mathcal{A}^{\rm c}}}\frac{w_{\mathcal{C}_{\mathbf{n}}(y)^{\rm c}}(\mathbf{k})}{Z_{\mathcal{C}_{\mathbf{n}}(y)^{\rm c}}}\bigg(\prod_{i=1}^{j}\mathbb{1}_{\{z_i \underset{\mathbf{m}+\mathbf{k}}{\longleftrightarrow} z'_i\}}\bigg)\bigg(\prod_{i\neq l}\mathbb{1}_{\{\mathcal{C}_{\mathbf{m}+\mathbf{k}}(z_i)\cap\mathcal{C}_{\mathbf{m}+\mathbf{k}}(z_l)=\varnothing\}}\bigg)\sum_{i=1}^{j}\mathbb{1}_{\{\mathcal{C}_{\mathbf{m}+\mathbf{k}}(z_i)\ni v\}}. \tag{4.82}$$

Consider the contribution from $\mathbb{1}_{\{\mathcal{C}_{\mathbf{m}+\mathbf{k}}(z_1)\ni v\}}$. Then, as in (4.66), by abbreviating $\mathcal{C}_{\mathbf{n}}(y)$ to $\mathcal{C}$ and conditioning over $\mathcal{V}_{\mathbf{m}+\mathbf{k}}\equiv\dot\bigcup_{i\geq 2}\mathcal{C}_{\mathbf{m}+\mathbf{k}}(z_i)$, the contribution to the second line of (4.82) equals

$$\sum_{\partial\mathbf{m}=\partial\mathbf{k}=\varnothing} \frac{w_{\mathcal{A}^{\rm c}}(\mathbf{m})}{Z_{\mathcal{A}^{\rm c}}} \frac{w_{\mathcal{C}^{\rm c}}(\mathbf{k})}{Z_{\mathcal{C}^{\rm c}}} \bigg(\prod_{i=2}^{j} \mathbb{1}_{\{z_i \underset{\mathbf{m}+\mathbf{k}}{\longleftrightarrow} z_i'\}}\bigg)\bigg(\prod_{\substack{i,l\geq 2\\ i\neq l}} \mathbb{1}_{\{\mathcal{C}_{\mathbf{m}+\mathbf{k}}(z_i)\cap\mathcal{C}_{\mathbf{m}+\mathbf{k}}(z_l)=\varnothing\}}\bigg)$$
$$\times \sum_{\partial\mathbf{m}'=\partial\mathbf{k}'=\varnothing} \frac{w_{\mathcal{A}^{\rm c}\cap\mathcal{V}^{\rm c}_{\mathbf{m}+\mathbf{k}}}(\mathbf{m}')}{Z_{\mathcal{A}^{\rm c}\cap\mathcal{V}^{\rm c}_{\mathbf{m}+\mathbf{k}}}} \frac{w_{\mathcal{C}^{\rm c}\cap\mathcal{V}^{\rm c}_{\mathbf{m}+\mathbf{k}}}(\mathbf{k}')}{Z_{\mathcal{C}^{\rm c}\cap\mathcal{V}^{\rm c}_{\mathbf{m}+\mathbf{k}}}} \mathbb{1}_{\{z_1 \underset{\mathbf{m}'+\mathbf{k}'}{\longleftrightarrow} z_1'\}} \mathbb{1}_{\{z_1 \underset{\mathbf{m}'+\mathbf{k}'}{\longleftrightarrow} v\}}. \tag{4.83}$$

If the last indicator is absent, then, as discussed below (4.66), the second line of (4.83) is bounded by a "chain of bubbles" $\sum_{l\geq 1}\big(\tilde{G}^2_\Lambda\big)^{*(2l-1)}(z_1,z_1')$. If $\mathbb{1}_{\{z_1 \underset{\mathbf{m}'+\mathbf{k}'}{\longleftrightarrow} v\}}=1$, then one of the bubbles has an extra vertex v' that is connected to v with another chain of bubbles $\psi_\Lambda(v',v)$. (This argument can be made precise following the argument around (4.45)–(4.48).) That is, the effect of the last indicator in (4.83) is to replace one of the $\tilde{G}_\Lambda$'s in the bound, say, $\tilde{G}_\Lambda(a,a')$, by $\sum_{v'}\tilde{G}_\Lambda(a,v')\,\langle\varphi_{v'}\varphi_{a'}\rangle_\Lambda\psi_\Lambda(v',v)$. The remaining terms in (4.82)–(4.83) can be estimated in the same way as before.

(ii) If there are no such clusters containing v, i.e., $v\notin\mathcal{C}_{\mathbf{m}+\mathbf{k}}(z_i)$ for all $i\in\{1,\dots,j\}$, then there is a $v'\in\mathcal{D}_{\mathbf{n};l}$ for some $l\in\{0,\dots,T\}$ such that $v'\underset{\mathbf{m}+\mathbf{k}}{\longleftrightarrow} v$ and $\mathcal{C}_{\mathbf{m}+\mathbf{k}}(v')\cap\mathcal{C}_{\mathbf{m}+\mathbf{k}}(z_i)=\varnothing$ for all i. In addition, since all connections on $\mathcal{C}_{\mathbf{n}}(y)\cup\dot{\bigcup}_{i=1}^{j}\mathcal{C}_{\mathbf{m}+\mathbf{k}}(z_i)$ from y to x have to go through $\mathcal{A}$, there is an $i\in\{1,\dots,j\}$ such that $z_i \overset{\mathcal{A}}{\underset{\mathbf{m}+\mathbf{k}}{\longleftrightarrow}} z_i'$. Therefore, the contribution from this case to (4.81) is bounded by

$$\sum_{T\geq 1}\sum_{\vec{b}_T}\sum_{\partial\mathbf{n}=y\triangle x}\frac{w_\Lambda(\mathbf{n})}{Z_\Lambda}\mathbb{1}_{H_{\mathbf{n};\vec{b}_T}(y,x)}\sum_{j=1}^{T}\sum_{\{s_it_i\}_{i=1}^{j}\in\mathcal{L}^{(j)}_{[0,T]}}\sum_{\substack{v',z_1,\dots,z_j\\ z_1',\dots,z_j'}}\bigg(\prod_{i=1}^{j}\mathbb{1}_{\{z_i\in\mathcal{D}_{\mathbf{n};s_i},\ z_i'\in\mathcal{D}_{\mathbf{n};t_i}\}}\bigg)\sum_{l=0}^{T}\mathbb{1}_{\{v'\in\mathcal{D}_{\mathbf{n};l}\}}$$
$$\times\sum_{\substack{\partial\mathbf{m}=\varnothing\\ \partial\mathbf{k}=\varnothing}}\frac{w_{\mathcal{A}^{\rm c}}(\mathbf{m})}{Z_{\mathcal{A}^{\rm c}}}\frac{w_{\mathcal{C}_{\mathbf{n}}(y)^{\rm c}}(\mathbf{k})}{Z_{\mathcal{C}_{\mathbf{n}}(y)^{\rm c}}}\bigg(\prod_{i=1}^{j}\mathbb{1}_{\{z_i\underset{\mathbf{m}+\mathbf{k}}{\longleftrightarrow}z_i'\}}\bigg)\bigg(\prod_{i\neq i'}\mathbb{1}_{\{\mathcal{C}_{\mathbf{m}+\mathbf{k}}(z_i)\cap\mathcal{C}_{\mathbf{m}+\mathbf{k}}(z_{i'})=\varnothing\}}\bigg)\bigg(\sum_{i=1}^{j}\mathbb{1}_{\{z_i\overset{\mathcal{A}}{\underset{\mathbf{m}+\mathbf{k}}{\longleftrightarrow}}z_i'\}}\bigg)$$
$$\times\,\mathbb{1}_{\{v'\underset{\mathbf{m}+\mathbf{k}}{\longleftrightarrow}v\}}\prod_{i=1}^{j}\mathbb{1}_{\{\mathcal{C}_{\mathbf{m}+\mathbf{k}}(v')\cap\mathcal{C}_{\mathbf{m}+\mathbf{k}}(z_i)=\varnothing\}}. \tag{4.84}$$

The contribution from the indicators in the third line is bounded, by the conditioning-over-clusters argument, by $\psi_\Lambda(v',v)$. Then, as in (4.66) and (4.83), by abbreviating $\mathcal{C}_{\mathbf{n}}(y)$ to $\mathcal{C}$ and conditioning over $\mathcal{I}_{\mathbf{m}+\mathbf{k}}(i)\equiv\dot{\bigcup}_{i'\neq i}\mathcal{C}_{\mathbf{m}+\mathbf{k}}(z_{i'})$, the second line of (4.84) is bounded by

$$\sum_{i=1}^{j}\sum_{\partial\mathbf{m}=\partial\mathbf{k}=\varnothing}\frac{w_{\mathcal{A}^{\rm c}}(\mathbf{m})}{Z_{\mathcal{A}^{\rm c}}}\frac{w_{\mathcal{C}^{\rm c}}(\mathbf{k})}{Z_{\mathcal{C}^{\rm c}}}\bigg(\prod_{i'\neq i}\mathbb{1}_{\{z_{i'}\underset{\mathbf{m}+\mathbf{k}}{\longleftrightarrow}z_{i'}'\}}\bigg)\bigg(\prod_{\substack{i',i''\neq i\\ i'\neq i''}}\mathbb{1}_{\{\mathcal{C}_{\mathbf{m}+\mathbf{k}}(z_{i'})\cap\mathcal{C}_{\mathbf{m}+\mathbf{k}}(z_{i''})=\varnothing\}}\bigg)$$
$$\times\sum_{\partial\mathbf{m}'=\partial\mathbf{k}'=\varnothing}\frac{w_{\mathcal{A}^{\rm c}\cap\mathcal{I}_{\mathbf{m}+\mathbf{k}}(i)^{\rm c}}(\mathbf{m}')}{Z_{\mathcal{A}^{\rm c}\cap\mathcal{I}_{\mathbf{m}+\mathbf{k}}(i)^{\rm c}}}\frac{w_{\mathcal{C}^{\rm c}\cap\mathcal{I}_{\mathbf{m}+\mathbf{k}}(i)^{\rm c}}(\mathbf{k}')}{Z_{\mathcal{C}^{\rm c}\cap\mathcal{I}_{\mathbf{m}+\mathbf{k}}(i)^{\rm c}}}\mathbb{1}_{\{z_i\overset{\mathcal{A}}{\underset{\mathbf{m}'+\mathbf{k}'}{\longleftrightarrow}}z_i'\}}. \tag{4.85}$$

If we ignore "through $\mathcal{A}$" in the last indicator, then, as discussed above, the second line of (4.85) is bounded by $\sum_{l\ge1}\big(\tilde G_\Lambda^2\big)^{*(2l-1)}(z_i,z_i')$. However, because of this "through $\mathcal{A}$"-condition, one of the $\tilde G_\Lambda$'s in the bound, say, $\tilde G_\Lambda(a,a')$, is replaced by $\sum_{u\in\mathcal{A}}\tilde G_\Lambda(a,u)\,\langle\varphi_u\varphi_{a'}\rangle_\Lambda$. Then, the summand of $\sum_{i=1}^j$ in the first line of (4.85) is bounded by a product of "chains of bubbles", similarly to (4.67).

Therefore, (4.84) is bounded by

$$\sum_{j\ge1}\sum_{\substack{v',z_1,\dots,z_j\\ z_1',\dots,z_j'}}\psi_\Lambda(v',v)\times\Big(\text{bounds described below (4.85)}\Big)$$
$$\times\sum_{\partial\mathbf{n}=y\triangle x}\frac{w_\Lambda(\mathbf{n})}{Z_\Lambda}\sum_{T\ge j}\sum_{\vec b_T}\sum_{\{s_it_i\}_{i=1}^j\in\mathcal{L}_{[0,T]}^{(j)}}\sum_{l=0}^{T}\mathbb{1}_{H_{\mathbf{n};\vec b_T}(y,x)}\,\mathbb{1}_{\{\mathcal{D}_{\mathbf{n};l}\ni v'\}}\prod_{i=1}^j\mathbb{1}_{\{\mathcal{D}_{\mathbf{n};s_i}\ni z_i,\ \mathcal{D}_{\mathbf{n};t_i}\ni z_i'\}}. \tag{4.86}$$

The desired bound on the second line can be obtained by reproducing the argument between (4.68) and (4.80), and we refrain from giving its details.

Summarizing the above (i) and (ii), we finally obtain that (4.81) is bounded by

$$\sum_{j\ge1}\sum_{u\in\mathcal{A}}P''^{(j)}_{\Lambda;u,v}(y,x). \tag{4.87}$$

This, together with (4.60), completes the proof of the bound on $\Theta''_{y,x,v;\mathcal{A}}$ in (4.32). □

4.3 Proof of Proposition 2.4

In this subsection, we prove Proposition 2.4 using Proposition 4.1 and the following:

Proposition 4.5. *(i) Let $a\ge b>0$ and $a+b>d$. There is a $C=C(a,b,d)$ such that*

$$\sum_y\frac{1}{|\!|\!|y-v|\!|\!|^a}\,\frac{1}{|\!|\!|x-y|\!|\!|^b}\le\frac{C}{|\!|\!|x-v|\!|\!|^{(a\wedge d+b)-d}}. \tag{4.88}$$

(ii) Let $\frac12 d<q<d$. There is a $C'=C'(d,q)$ such that

$$\sum_z\frac{1}{|\!|\!|x-z|\!|\!|^q}\,\frac{1}{|\!|\!|x'-z|\!|\!|^q}\,\frac{1}{|\!|\!|z-y|\!|\!|^q}\,\frac{1}{|\!|\!|z-y'|\!|\!|^q}\le\frac{C'}{|\!|\!|x-y|\!|\!|^q|\!|\!|x'-y'|\!|\!|^q}. \tag{4.89}$$

Proof. The inequality (4.88) is identical to [14, Proposition 1.7(i)]. We use this to prove (4.89). By the triangle inequality, we have $\frac12|\!|\!|x^{(\prime)}-y^{(\prime)}|\!|\!|\le|\!|\!|x^{(\prime)}-z|\!|\!|\vee|\!|\!|z-y^{(\prime)}|\!|\!|$. Suppose that $|\!|\!|x-z|\!|\!|\le|\!|\!|z-y|\!|\!|$ and $|\!|\!|x'-z|\!|\!|\le|\!|\!|z-y'|\!|\!|$. Then, by (4.88) with $a=b=q$, the contribution from this case is bounded by

$$\frac{2^{2q}}{|\!|\!|x-y|\!|\!|^q|\!|\!|x'-y'|\!|\!|^q}\sum_z\frac{1}{|\!|\!|x-z|\!|\!|^q}\,\frac{1}{|\!|\!|x'-z|\!|\!|^q}\le\frac{2^{2q}C|\!|\!|x-x'|\!|\!|^{d-2q}}{|\!|\!|x-y|\!|\!|^q|\!|\!|x'-y'|\!|\!|^q}, \tag{4.90}$$

where $|\!|\!|x-x'|\!|\!|^{d-2q}\le1$, due to $d-2q<0$. The other three possible cases can be estimated similarly (see Figure 9(a)). This completes the proof. □

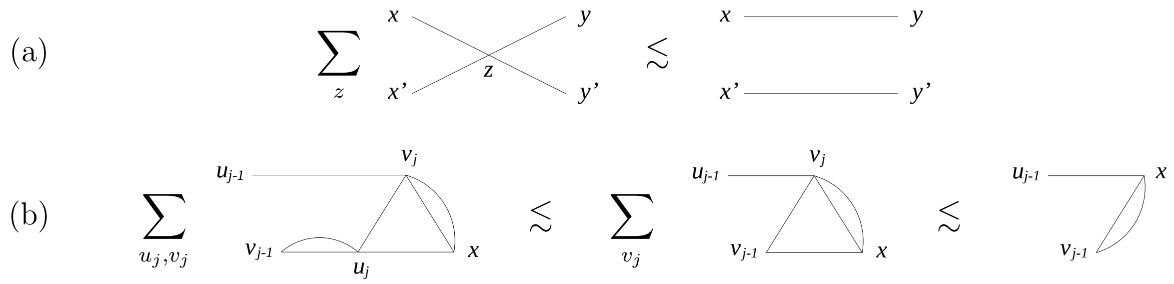

Figure 9: (a) A schematic representation of (4.89). (b) A schematic representation of (4.106), which is a result of successive application of (4.89), with $x = x'$ or $y = y'$.

Before going into the proof of Proposition 2.4, we summarize a few prerequisites. Recall that (4.11)–(4.12) involve $\tilde{G}_\Lambda$, and note that

$$\langle \varphi_o \varphi_x \rangle_\Lambda^3 \le \delta_{o,x} + \tilde{G}_\Lambda(o,x)^3. \tag{4.91}$$

We first show

$$\tilde{G}_\Lambda(o,x) \le \frac{O(\theta)}{\|\!|x|\!\|^q}, \qquad \sum_{b:\underline{b}=o} \tau_b \left(\delta_{\overline{b},x} + \tilde{G}_\Lambda(\overline{b},x)\right) \le \frac{O(\theta)}{\|\!|x|\!\|^q}. \tag{4.92}$$

Proof. By (2.23), we have

$$\tilde{G}_\Lambda(o,x) = \tau D(x) + \sum_{y \ne o,x} \tau D(y) \, \langle \varphi_y \varphi_x \rangle_\Lambda \le 2D(x) + \sum_{y \ne o,x} 2D(y) \, G(x-y), \tag{4.93}$$

where, and from now on without stating explicitly, we use the second Griffiths inequality and the translation invariance of $G(x)$. Using the definition (2.19) and the assumption in the statement of Proposition 2.4 that θL^{d-q}, with $q < d$, is bounded away from zero, we obtain

$$D(x) \le O(L^{-d}) \mathbb{1}_{\{0 < \|x\|_\infty \le L\}} \le \frac{O(L^{-d+q})}{\|\!|x|\!\|^q} \le \frac{O(\theta)}{\|\!|x|\!\|^q}. \tag{4.94}$$

For the last term in (4.93), we consider the cases of $|x| \le 2\sqrt{d}L$ and $|x| \ge 2\sqrt{d}L$ separately.

When $|x| \le 2\sqrt{d}L$, we use (4.94), (2.23) and (4.88) with $\frac{1}{2}d < q < d$ to obtain

$$\sum_{y \ne o,x} D(y) \, G(x-y) \le \sum_y \frac{O(L^{-d+q})}{\|\!|y|\!\|^q} \frac{\theta}{\|\!|x-y|\!\|^q} \le \frac{O(\theta L^{-d+q})}{\|\!|x|\!\|^{2q-d}} \le \frac{O(\theta)}{\|\!|x|\!\|^q}. \tag{4.95}$$

When $|x| \ge 2\sqrt{d}L$, we use the triangle inequality $|x-y| \ge |x| - |y|$ and the fact that $D(y)$ is nonzero only when $0 < \|y\|_\infty \le L$ (so that $|y| \le \sqrt{d}\|y\|_\infty \le \sqrt{d}L \le \frac{1}{2}|x|$). Then, we obtain

$$\sum_{y \ne o,x} D(y) \, G(x-y) \le \sum_y D(y) \frac{2^q \theta}{\|\!|x|\!\|^q} = \frac{2^q \theta}{\|\!|x|\!\|^q}. \tag{4.96}$$

This completes the proof of the first inequality in (4.92). The second inequality can be proved similarly. □

By repeated use of (4.92) and (4.88) with $a = b = 2q$ (or (4.89) with $x = x'$ and $y = y'$), we obtain

$$\psi_\Lambda(v', v) \le \delta_{v',v} + \frac{O(\theta^2)}{\|v - v'\|^{2q}}. \tag{4.97}$$

Together with the naive bound $G(x) \le O(1)\|x\|^{-q}$ (cf., (2.23)), we also obtain

$$\begin{aligned}\sum_{v'} G(v'-y)\,G(z-v')\,\psi_\Lambda(v',v) &\le G(v-y)\,G(z-v) + \sum_{v'} \frac{O(\theta^2)}{\|v'-y\|^q\|z-v'\|^q\|v-v'\|^{2q}} \\ &\le \frac{O(1)}{\|v-y\|^q\|z-v\|^q},\end{aligned} \tag{4.98}$$

where we have used (4.89) (with $x = x'$ or $y = y'$). The $O(1)$ term in the right-hand side is replaced by $O(\theta)$ or $O(\theta^2)$ depending on the number of replacement of G by $\tilde{G}_\Lambda$ in the left-hand side.

Proof of Proposition 2.4. Since (4.91)–(4.92) immediately imply the bound on $\pi_\Lambda^{(0)}(x)$, it suffices to prove the bounds on $\pi_\Lambda^{(i)}(x)$ for $i \ge 1$. To do so, we first estimate the building blocks of the diagrammatic bound (4.13): $\sum_{b:\underline{b}=y} \tau_b\, Q'_{\Lambda;u}(\overline{b}, x)$ and $\sum_{b:\underline{b}=y} \tau_b\, Q''_{\Lambda;u,v}(\overline{b}, x)$.

Recall (4.8)–(4.12). First, by using $G(x) \le O(1)\|x\|^{-q}$ and (4.98), we obtain

$$P'^{(0)}_{\Lambda;u}(y,x) \le \frac{O(1)}{\|x-y\|^{2q}\|u-y\|^q\|x-u\|^q}, \tag{4.99}$$

$$P''^{(0)}_{\Lambda;u,v}(y,x) \le \frac{O(1)}{\|x-y\|^q\|u-y\|^q\|x-u\|^q\|v-y\|^q\|x-v\|^q}. \tag{4.100}$$

We will show at the end of this subsection that, for $j \ge 1$,

$$P'^{(j)}_{\Lambda;u}(y,x) \le \frac{O(j)\,O(\theta^2)^j}{\|x-y\|^{2q}\|u-y\|^q\|x-u\|^q}, \tag{4.101}$$

$$P''^{(j)}_{\Lambda;u,v}(y,x) \le \frac{O(j^2)\,O(\theta^2)^j}{\|x-y\|^q\|u-y\|^q\|x-u\|^q\|v-y\|^q\|x-v\|^q}. \tag{4.102}$$

As a result, $P'^{(0)}_{\Lambda;u}(y,x)$ (resp., $P''^{(0)}_{\Lambda;u,v}(y,x)$) is the leading term of $P'_{\Lambda;u}(y,x)$ (resp., $P''_{\Lambda;u,v}(y,x)$), which thus obeys the same bound as in (4.99) (resp., (4.100)), with a different constant in $O(1)$. Combining these bounds with (4.92) and (4.98) (with both G in the left-hand side being replace by $\tilde{G}_\Lambda$), and then using (4.89), we obtain

$$\sum_{b:\underline{b}=y} \tau_b\, Q'_{\Lambda;u}(\overline{b},x) \le \sum_z \frac{O(\theta)}{\|z-y\|^q\|x-z\|^{2q}\|u-z\|^q\|x-u\|^q} \le \frac{O(\theta)}{\|x-y\|^q\|x-u\|^{2q}}, \tag{4.103}$$

$$\begin{aligned}\sum_{b:\underline{b}=y} \tau_b\, Q''_{\Lambda;u,v}(\overline{b},x) &\le \sum_z \bigg(\frac{O(\theta)}{\|z-y\|^q\|x-z\|^q\|u-z\|^q\|x-u\|^q\|v-z\|^q\|x-v\|^q} \\ &\qquad + \frac{O(\theta^2)}{\|v-y\|^q\|z-v\|^q\|x-z\|^{2q}\|u-z\|^q\|x-u\|^q}\bigg) \\ &\le \frac{O(\theta)}{\|v-y\|^q\|x-v\|^q\|x-u\|^{2q}}.\end{aligned} \tag{4.104}$$

This completes the bounds on the building blocks.

Now, we prove the bounds on $\pi_\Lambda^{(j)}(x)$ for $j \geq 1$. For the bounds on $\pi_\Lambda^{(j)}(x)$ for $j \geq 2$, we simply apply (4.99) and (4.103)–(4.104) to the diagrammatic bound (4.13). Then, we obtain

$$\pi_\Lambda^{(j)}(x) \leq \sum_{\substack{u_1,\dots,u_j \\ v_1,\dots,v_j}} \frac{O(1)}{\|u_1\|^{2q}\|v_1\|^q\|u_1-v_1\|^q} \Bigg(\prod_{i=1}^{j-1} \frac{O(\theta)}{\|v_{i+1}-u_i\|^q\|u_{i+1}-v_{i+1}\|^q\|u_{i+1}-v_i\|^{2q}}\Bigg) \times \frac{O(\theta)}{\|x-u_j\|^q\|x-v_j\|^{2q}} \qquad (j \geq 2). \tag{4.105}$$

First, we consider the sum over u_j and v_j. By successive application of (4.89) (with $x = x'$ or $y = y'$), we obtain (see Figure 9(b))

$$\begin{aligned}
&\sum_{v_j}\sum_{u_j} \frac{O(\theta)}{\|v_j-u_{j-1}\|^q\|u_j-v_j\|^q\|u_j-v_{j-1}\|^{2q}} \frac{O(\theta)}{\|x-u_j\|^q\|x-v_j\|^{2q}} \\
&\leq \sum_{v_j} \frac{O(\theta^2)}{\|v_j-u_{j-1}\|^q\|v_{j-1}-v_j\|^q\|x-v_{j-1}\|^q\|x-v_j\|^{2q}} \leq \frac{O(\theta^2)}{\|x-u_{j-1}\|^q\|x-v_{j-1}\|^{2q}}.
\end{aligned} \tag{4.106}$$

This implies that the right-hand side of (4.105) is bounded by itself with j being replaced by $j-1$, multiplied by an extra $O(\theta)$. Therefore, by repeating the application of (4.89), we end up with

$$\pi_\Lambda^{(j)}(x) \leq \sum_{v_1}\sum_{u_1} \frac{O(1)}{\|u_1\|^{2q}\|v_1\|^q\|u_1-v_1\|^q} \frac{O(\theta)^j}{\|x-u_1\|^q\|x-v_1\|^{2q}} \leq \frac{O(\theta)^j}{\|x\|^{3q}}. \tag{4.107}$$

For the bound on $\pi_\Lambda^{(1)}(x)$, instead of using (4.99), we use

$$P'^{(0)}_{\Lambda;v}(o,u) = \delta_{o,u}\delta_{o,v} + (1-\delta_{o,u}\delta_{o,v})\, P'^{(0)}_{\Lambda;v}(o,u) \leq \delta_{o,u}\delta_{o,v} + \frac{O(\theta^2)}{\|u\|^{2q}\|v\|^q\|u-v\|^q}, \tag{4.108}$$

where $O(\theta)^2$ arises when $u = o$ and $v \neq o$, and is due to (4.4) and (4.92). In addition, instead of using (4.103), we use

$$\begin{aligned}
\sum_{b:\underline{b}=u} \tau_b\, Q'_{\Lambda;v}(\overline{b},x) &\leq \sum_z \frac{O(\theta)}{\|z-u\|^q}\Bigg(\delta_{z,v}\delta_{z,x} + (1-\delta_{z,v}\delta_{z,x})\, P'^{(0)}_{\Lambda;v}(z,x) + \sum_{j\geq 1} P'^{(j)}_{\Lambda;v}(z,x)\Bigg) \\
&\leq \frac{O(\theta)}{\|x-u\|^q}\,\delta_{v,x} + \sum_z \frac{O(\theta^3)}{\|z-u\|^q\|x-z\|^{2q}\|v-z\|^q\|x-v\|^q} \\
&\leq \frac{O(\theta)}{\|x-u\|^q}\,\delta_{v,x} + \frac{O(\theta^3)}{\|x-u\|^q\|x-v\|^{2q}},
\end{aligned} \tag{4.109}$$

due to (4.92), (4.101) and (4.108). Applying (4.108)–(4.109) to (4.13) and then using (4.89), we end up with

$$\begin{aligned}
\pi_\Lambda^{(1)}(x) &\leq O(\theta)\,\delta_{o,x} + \frac{O(\theta^3)}{\|x\|^{3q}} + \sum_{u,v} \frac{O(\theta^2)}{\|u\|^{2q}\|v\|^q\|u-v\|^q}\Bigg(\frac{O(\theta)\,\delta_{v,x}}{\|x-u\|^q} + \frac{O(\theta^3)}{\|x-u\|^q\|x-v\|^{2q}}\Bigg) \\
&\leq O(\theta)\,\delta_{o,x} + \frac{O(\theta^3)}{\|x\|^{3q}}.
\end{aligned} \tag{4.110}$$

To complete the proof of Proposition 2.4, it remains to show (4.101)–(4.102). To do so, we first recall $P^{(j)}_\Lambda(y,x)$ in (4.6)–(4.7) and the definition of $P'^{(j)}_{\Lambda;u}(y,x)$ and $P''^{(j)}_{\Lambda;u,v}(y,x)$ described below (4.7). Then, by recalling the pictures in the second line of Figure 5, (4.101)–(4.102) for $j=1$ follow from (2.23), (4.97)–(4.98) and the following bound on $\psi_\Lambda(y,x)-\delta_{y,x}$ with one of the consisting $\tilde G_\Lambda$, say, $\tilde G_\Lambda(u_{k-1},u_k)$, being replaced by $\sum_{v'}\tilde G_\Lambda(u_{k-1},v')\,\langle\varphi_{v'}\varphi_{u_k}\rangle_\Lambda\,\psi_\Lambda(v',v)$:

$$\begin{aligned}
&\sum_{l=1}^\infty\sum_{k=1}^l\sum_{\substack{u_0,\dots,u_l,v'\\u_0=y,\,u_l=x}}\bigg(\prod_{\substack{i=1,\dots,l\\i\neq k}}\tilde G_\Lambda(u_{i-1},u_i)^2\bigg)\,\tilde G_\Lambda(u_{k-1},u_k)\,\tilde G_\Lambda(u_{k-1},v')\,\langle\varphi_{v'}\varphi_{u_k}\rangle_\Lambda\,\psi_\Lambda(v',v)\\
&\le\sum_{l=1}^\infty\sum_{k=1}^l\sum_{\substack{u_0,\dots,u_l,v'\\u_0=y,\,u_l=x}}\bigg(\prod_{\substack{i=1,\dots,l\\i\neq k}}\frac{O(\theta^2)}{|||u_i-u_{i-1}|||^{2q}}\bigg)\frac{O(\theta^2)}{|||u_k-u_{k-1}|||^q|||v-u_{k-1}|||^q|||u_k-v|||^q}\\
&\le\sum_{l=1}^\infty\frac{O(l)\,O(\theta^2)^l}{|||x-y|||^q|||v-y|||^q|||x-v|||^q}\;\le\frac{O(\theta^2)}{|||x-y|||^q|||v-y|||^q|||x-v|||^q},
\end{aligned}\tag{4.111}$$

where we have used (4.89).

For (4.101)–(4.102) with $j\ge2$, we first note that $P^{(j)}_\Lambda(y,x)$ is bounded, by using (4.97), as

$$\begin{aligned}
P^{(j)}_\Lambda(y,x)\le\sum_{\substack{v_2,\dots,v_j\\v'_1,\dots,v'_{j-1}}}&\frac{O(\theta^2)}{|||v'_1-y|||^{2q}|||v_2-y|||^q|||v'_1-v_2|||^q}\bigg(\prod_{i=2}^{j-1}\frac{O(\theta^2)}{|||v'_i-v_i|||^{2q}|||v_{i+1}-v'_{i-1}|||^q|||v'_i-v_{i+1}|||^q}\bigg)\\
&\times\frac{O(\theta^2)}{|||x-v_j|||^{2q}|||x-v'_{j-1}|||^q}\qquad(j\ge2).
\end{aligned}\tag{4.112}$$

By definition, the bound on $P'^{(j)}_{\Lambda;u}(y,x)$ is obtained by "embedding u" in one of the $2j-1$ factors of $|||\cdots|||^q$ (not $|||\cdots|||^{2q}$) and then summing over all these $2j-1$ choices. For example, the contribution from the case in which $|||v_2-y|||^q$ is replaced by $|||u-y|||^q|||v_2-u|||^q$ is bounded, similarly to (4.107), by

$$\begin{aligned}
&\sum_{v_2,v'_1}\frac{O(\theta^2)}{|||v'_1-y|||^{2q}|||u-y|||^q|||v_2-u|||^q|||v'_1-v_2|||^q}\,\frac{O(\theta^2)^{j-1}}{|||x-v'_1|||^q|||x-v_2|||^{2q}}\\
&\le\sum_{v'_1}\frac{O(\theta^2)^j}{|||v'_1-y|||^{2q}|||u-y|||^q|||x-u|||^q|||x-v'_1|||^{2q}}\le\frac{O(\theta^2)^j}{|||x-y|||^{2q}|||u-y|||^q|||x-u|||^q}.
\end{aligned}\tag{4.113}$$

The other $2j-2$ contributions can be estimated in a similar way, with the same form of the bound. This completes the proof of (4.101).

With the help of (4.111), the bound on $P''^{(j)}_{\Lambda;u,v}(y,x)$ is also obtained by "embedding u and v" in one of the $2j-1$ factors of $|||\cdots|||^q$ and one of the j factors of $|||\cdots|||^{2q}$ in (4.112), and then summing over all these combinations. For example, the contribution from the case in which $|||v_2-y|||^q$ and $|||v'_1-y|||^{2q}$ in (4.112) are replaced, respectively, by $|||u-y|||^q|||v_2-u|||^q$

and $\|v_1'-y\|^q\|v-y\|^q\|v_1'-v\|^q$, is bounded by

$$\sum_{v_2,v_1'} \frac{O(\theta^2)}{\|v_1'-y\|^q\|v-y\|^q\|v_1'-v\|^q\|u-y\|^q\|v_2-u\|^q\|v_1'-v_2\|^q} \frac{O(\theta^2)^{j-1}}{\|x-v_1'\|^q\|x-v_2\|^{2q}}$$
$$\leq \sum_{v_1'} \frac{O(\theta^2)^j}{\|v_1'-y\|^q\|v-y\|^q\|v_1'-v\|^q\|u-y\|^q\|x-u\|^q\|x-v_1'\|^{2q}}$$
$$\leq \frac{O(\theta^2)^j}{\|x-y\|^q\|u-y\|^q\|x-u\|^q\|v-y\|^q\|x-v\|^q}. \tag{4.114}$$

The other $2(j-1)^2$ contributions can be estimated similarly, with the same form of the bound. This completes the proof of (4.102) and thus Proposition 2.4. □

Acknowledgements

First of all, I am grateful to Masao Ohno for having drawn my attention to the subject of this paper. I would like to thank Takashi Hara for stimulating discussions and his hospitality during my visit to Kyushu University in December 2004 and April 2005. I would also like to thank Aernout van Enter for valuable discussions on the previous results. Special thanks go to Remco van der Hofstad for his support in various aspects. This work was supported in part by the Netherlands Organization for Scientific Research (NWO) and in part by a Postdoctoral Fellowship of EURANDOM, the Netherlands.

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