

STOCHASTIC DECOMPOSITION OF THE $M/G/\infty$ QUEUE IN A RANDOM ENVIRONMENT

BERNARDO D'AURIA

ABSTRACT. We prove a stochastic decomposition formula for the number of customers in an $M/G/\infty$ system with service speeds depending on the state of an independent random environment. In addition, we present a new technique to analyze this kind of queues and apply it to solve some examples of $M/M/\infty$ queues in random environment.

1. INTRODUCTION

Stochastic decomposition formulas were obtained for the $M/G/1$; for example the $M/G/1$ queue with vacations has been studied by Fuhrmann [1984], Fuhrmann and Cooper [1985] and Levy and Yechiali [1975]. Kella and Whitt [1991] generalized the stochastic decomposition to the case when the input process is a Lévy process with non negative jumps. In this paper we focus our attention on the $M/G/\infty$ system. This system has the nice property that the customers' lifetimes are mutually independent random variables. This makes it possible to find stochastic decompositions in a more general setting than the $M/G/1$ case. In Baykal-Gursoy and Xiao [2004] the special case of the $M/M/\infty$ queue in an ON-OFF Markovian random environment has been analyzed and an explicit expression for the stationary probability distribution function of the number of customers in the system has been obtained. In this paper we generalize the stochastic decomposition to the $M/G/\infty$ queue and in addition we show that it is not necessary to require that the random environment is Markovian. The environment can indeed be a general stochastic process as long as it is independent of the queueing process. Finally we study some examples of $M/M/\infty$ queues in a random environment for which we are able to compute explicitly the stationary probability distribution function for the additional number of customers in the system with respect to the isolated one. One example is the queue already studied in Baykal-Gursoy and Xiao [2004], we obtain for it a slightly different decomposition by using a different technique that we believe gives better probabilistic understanding of the analyzed queueing process.

The paper is organized as follows. In Section 2 we introduce the model, we prove stability and derive the general stochastic decomposition formula. In Section 3 we look at the special case when the service times are exponentially distributed.

2. $M/G/\infty$ CASE

We consider an $M/G/\infty$ queue where customers arrive according to a Poisson process with constant rate λ and require independent service, σ , that is distributed according to a general probability function $G(d\sigma)$. The queue is supposed to operate in a random environment $\Gamma(t)$

Date: January 11, 2006.

2000 Mathematics Subject Classification. Primary 60K25; secondary 60K37.

Key words and phrases. $M/G/\infty$ queue, random environment, stochastic decomposition.

The author acknowledges Onno J. Boxma for the interesting and stimulating discussions about the present topic.

that is a non-negative stochastic process independent of the arrival process. We denote the space of the sample paths of $\Gamma(t)$ by \mathcal{G} . We assume that at time t all servers operate at the same speed, $\nu(\Gamma(t))$, that is a function of the random environment. In particular we suppose that $\nu : [0, \infty) \rightarrow [0, 1]$. Before proving the stochastic decomposition formula for the M/G/ ∞ queue, we first prove the stability of the queue in a more general setting. We suppose that the queue is of G/G/ ∞ kind, whose input is a stationary marked point process with intensity λ , and whose service speeds depend on the stationary and ergodic process $\Gamma(t)$. The number of customers in the system is denoted by N .

Theorem 2.1. *Consider a G/G/ ∞ queue with service rate depending on a function $\nu : [0, \infty) \rightarrow [0, 1]$ of a stationary and ergodic stochastic process $\Gamma(t)$, independent of the arrival process. Suppose that $\tilde{\nu} := \mathbb{E}[\nu(\Gamma(0))] > 0$. If $\lambda < \infty$ and σ is integrable, then*

$$(2.1) \quad P\{N < \infty\} = 1$$

and therefore, in this sense, the queue is stable.

Proof. In order to prove (2.1) we first show that for almost every sample path the first moment of N is finite. Let us denote by $N|_\gamma$ the restriction of the random variable N to the set $\{\Gamma = \gamma\}$, then we have

$$(2.2) \quad N|_\gamma = \sum_{n \in \mathbb{Z}} f_\gamma(T_n, \sigma_n),$$

where $\{T_n\}_{n \in \mathbb{Z}}$ is the sequence of the arrival points and the function f_γ is given by

$$f_\gamma(t, \sigma) = \begin{cases} 1, & t < 0 \text{ and } \sigma > \int_{-t}^0 \nu(\gamma(\tau)) d\tau; \\ 0, & \text{otherwise.} \end{cases}$$

In the following to write formulas in a more compact way we denote by $F_\gamma(t) = \int_t^0 \nu(x(\tau)) d\tau$. Let us first prove that $\mathbb{E}[N|_\gamma] < \infty$ for every $\gamma \in \mathcal{G}$. By Campbell's formula [cf. Baccelli and Brémaud, 2003], we have

$$\mathbb{E}[N|_\gamma] = \lambda \int_{-\infty}^0 \left(\int_{F_\gamma(-t)}^{\infty} G\{\sigma \in d\sigma\} \right) dt = \lambda \int_{-\infty}^0 \bar{G}(F_\gamma(-t)) dt,$$

where $\bar{G}(s) := 1 - G(s)$ denotes the tail probability of the service times. For the ergodicity of $\Gamma(t)$ we have P-a.s. that

$$\lim_{t \rightarrow \infty} F_\gamma(-t)/t = \tilde{\nu},$$

so for fixed $0 < \epsilon < \tilde{\nu}$, there exist $t_\gamma > 0$ s.t. $F_\gamma(-t) \geq (\tilde{\nu} - \epsilon)t$ for every $t > t_\gamma$. It follows that

$$(2.3) \quad \mathbb{E}[N|_\gamma] \leq \lambda t_\gamma + \lambda \int_{-\infty}^0 \bar{G}((\tilde{\nu} - \epsilon)t) dt = \lambda t_\gamma + \frac{\lambda \mathbb{E}[\sigma]}{\tilde{\nu} - \epsilon} < \infty.$$

Finally to prove that $P\{N < \infty\} = 1$ we have

$$\mathbb{E}[1\{N < \infty\}] = \mathbb{E}[\mathbb{E}[1\{N < \infty\} | \Gamma]],$$

with $1\{\cdot\}$ the indicator function of the set $\{\cdot\}$. The proof follows from the fact that Equation (2.3) implies P-a.s. that $\mathbb{E}[1\{N < \infty\} | \Gamma] = \mathbb{E}[1\{N|_\Gamma < \infty\}] = 1$. \square

In the sequel we restrict our attention to the M/G/∞ queue where the arrival process is a Marked Poisson process. It implies that given the sample path of the environment the arrival process induces a Poisson random measure on the space $\mathbb{R} \times \mathbb{R}^+$. The property of this measure, i.e. that the numbers of points belonging to either one of two disjoint sets are independent, implies the required stochastic independence. The next theorem more formally restates this idea.

Theorem 2.2. *Consider an M/G/∞ system with service rate depending via a function $\nu : [0, \infty) \rightarrow [0, 1]$ on a stationary and ergodic stochastic process $\Gamma(t)$, independent of the arrival process. Suppose that $\tilde{\nu} := \mathbb{E}[\nu(\Gamma(0))] > 0$, then the following stochastic decomposition holds:*

$$(2.4) \quad N \stackrel{d}{=} N_1 + N_\nu,$$

with N_1 and N_ν two independent and non negative random variables. N_1 is the number of customers in a stationary M/G/∞ queue with service speed constant equal to 1, i.e. it is a Poisson distributed r.v. with parameter $\lambda \mathbb{E}[\sigma]$. N_ν is instead a Randomized Poisson (RP) random variable.

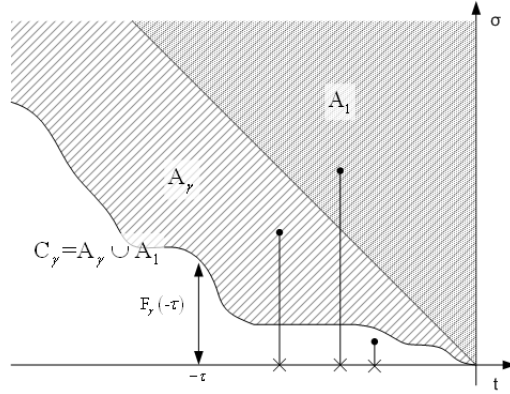


FIGURE 1. Area decomposition.

Proof. If we fix the sample path for the environment process, $\gamma \in \mathcal{G}$, according to (2.2), we have that the stationary number of customers in the queue $N_{|\gamma}$ is given by the Poisson Random Measure (PRM) $\mathcal{N}\{\cdot\}$, induced by the arrival Marked Poisson process [cf. Baccelli and Brémaud, 2003], evaluated on the set $C_\gamma = \{(t, \sigma) : t < 0, \sigma \geq F_\gamma(-t)\}$, as shown in Figure 1. We denote by $\lambda\{\cdot\} = \mathbb{E}[\mathcal{N}\{\cdot\}]$ the intensity measure of the PRM \mathcal{N} , and by $|A|$ the measure of the set A under the intensity measure λ , i.e. $|A_\Gamma| := \lambda\{A_\Gamma\}$. In our case we have $\lambda\{dt \times d\sigma\} = \lambda dt G(d\sigma)$. Since $\nu(\gamma) \leq 1$ we can always decompose $C_\gamma = A_1 \cup A_\gamma$ where $A_1 = \{(t, \sigma) : t < 0, \sigma \geq -t\}$ and $A_\gamma = A_1^c \cap C_\gamma$. From the properties of the Poisson random measure, since A_1 and A_γ are disjoint sets, the random variables $\mathcal{N}\{A_1\}$ and $\mathcal{N}\{A_\gamma\}$ are independent and Poisson distributed with parameters respectively $|A_1|$ and $|A_\gamma|$.

Since A_1 does not depend on γ and $\forall \gamma \in \mathcal{G}$ the set $A_\gamma \subset A_1^c$, the independence can be extended to $N_1 = \mathcal{N}\{A_1\}$ and $N_\nu = \mathcal{N}\{A_\Gamma\}$. Indeed by looking at the characteristic functions we have

$$\mathbb{E}[z^{N_1+N_\nu}] = \mathbb{E}[\mathbb{E}[z^{N_1+N_\nu} | \Gamma]] = \mathbb{E}[\mathbb{E}[z^{N_1}] \mathbb{E}[z^{N_\nu} | \Gamma]] = \mathbb{E}[z^{N_1}] \mathbb{E}[z^{N_\nu}],$$

and we finally get the following stochastic decomposition

$$N = N_1 + N_\nu$$

with $N_1 \sim \text{Poisson}(\lambda \mathbb{E}[\sigma])$. N_ν is then a Randomized Poisson random variable whose parameter is the random variable $|A_\Gamma|$. \square

Remark 2.3. *To have a practical interpretation of the last theorem, we can just look at the customer in the system as belonging to two classes, the ones whose service time is such that they would have been in the system despite the slowing down of service speed, and the rest. As the latter are completely independent in the arrival times and service amounts from the former, we get the stochastic decomposition.*

3. M/M/ ∞ IN AN ON-OFF RANDOM ENVIRONMENT

In this section we focus our attention on the M/M/ ∞ case where the random environment is a general renewal ON-OFF process. We shall derive an integral equation for the generating function of the excess customer r.v. N_ν .

We assume that the process $\Gamma(t)$ is ON-OFF alternatively assuming values respectively 1 and 0. We denote by H a general ON period, and we assume it is distributed according to distribution F_H , and by L the general OFF period with distribution F_L . The customer service times are independent exponential random variables with parameter μ . We assume that the rate function is given by

$$\nu(\gamma) = \begin{cases} \beta, & \gamma = 0; \\ 1, & \gamma = 1, \end{cases}$$

with $0 \leq \beta < 1$. When $\beta = 0$ we have the special case when all servers periodically break down for a random time while still keeping the customers in service, when $\beta > 0$ they periodically slow down from speed μ to speed $\beta\mu$.

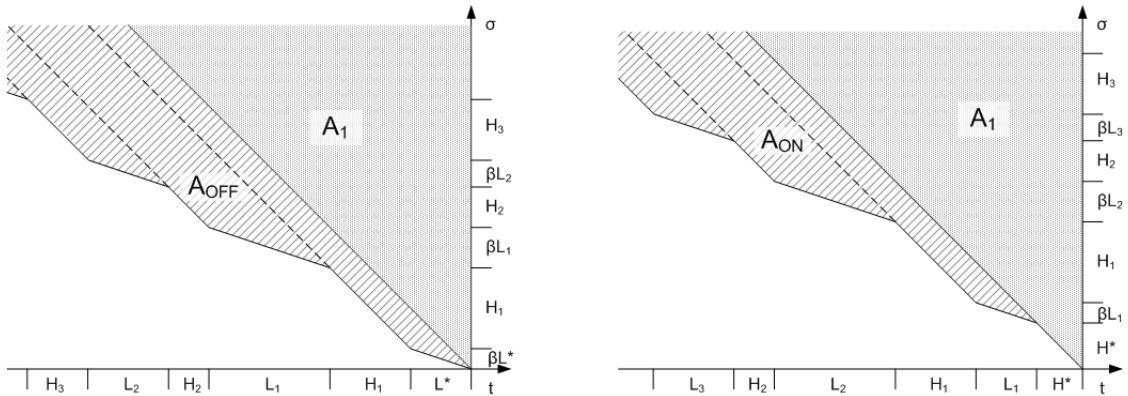


FIGURE 2. Areas for ON-OFF Random Environment.

When Γ is an ON-OFF process the set A_Γ has the following expression

$$(3.1) \quad A_\Gamma = A_{\text{ON}} 1\{\Gamma(0) = 1\} + A_{\text{OFF}} 1\{\Gamma(0) = 0\}$$

where

$$\begin{aligned}
A_{\text{ON}} &= \bigcup_{k=0}^{\infty} \left\{ -S_{k+1} \leq \frac{\sigma+t}{1-\beta} < -S_k; \sigma + \beta t > (1-\beta)(H^* + T_k) \right\} \\
A_{\text{OFF}} &= \left\{ -L^* \leq \frac{\sigma+t}{1-\beta} < 0; \sigma + \beta t > 0 \right\} \cup \\
&\quad \bigcup_{k=0}^{\infty} \left\{ -(L^* + S_{k+1}) \leq \frac{\sigma+t}{1-\beta} < -(L^* + S_k); \sigma + \beta t > (1-\beta)T_{k+1} \right\}.
\end{aligned}$$

In the previous expressions, L^* denotes a residual lifetime for an OFF period, distributed according to $F_L^*(x) = r \int_0^x \bar{F}_L(y) dy$ and H^* denotes a residual lifetime for an ON period distributed according to $F_H^*(x) = f \int_0^x \bar{F}_H(y) dy$. The random variables $\{S_k\}_k$ and $\{T_k\}_k$ are defined recursively, i.e. $S_{k+1} = S_k + L_k$, $S_0 = 0$ and $T_{k+1} = T_k + H_k$, $T_0 = 0$. A graphical interpretation of the sets A_{ON} and A_{OFF} is given in Figure 2.

By changing variables $x = \beta/(1-\beta)(\sigma+t)$, $y = \sigma$, we have that the given expressions simplify to

$$\begin{aligned}
A_{\text{ON}} &= \bigcup_{k=0}^{\infty} \{ -\beta S_{k+1} \leq x < -\beta S_k; y + x > H^* + T_k \} \\
A_{\text{OFF}} &= \{ -\beta L^* \leq x < 0; y + x > 0 \} \cup \\
&\quad \bigcup_{k=0}^{\infty} \{ -\beta(L^* + S_{k+1}) \leq x < -\beta(L^* + S_k); y + x > T_{k+1} \}.
\end{aligned}$$

The intensity measure in the plane $(x, y) \in \mathbb{R} \times \mathbb{R}^+$ is given by $\lambda' \{dx \times dy\} = \frac{1-\beta}{\beta} \lambda dx G(dy)$.

Figure 3 shows the random sets A_{ON} and A_{OFF} in the (x, y) space.

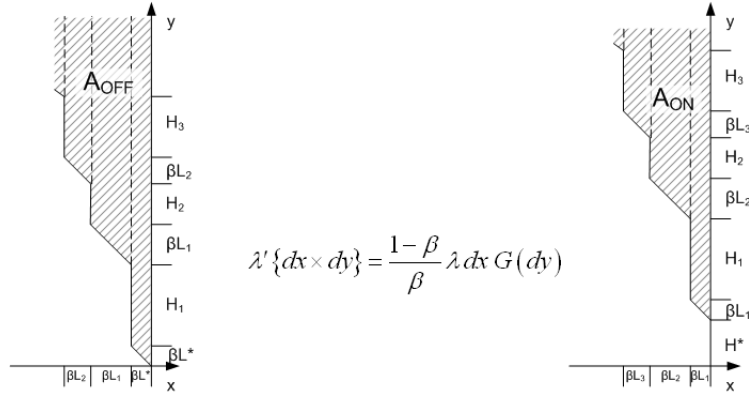


FIGURE 3. Areas for ON-OFF Random Environment.

If we focus now on the case with exponential service times, i.e. $G(dy) = \mu e^{-\mu y} dy$, the intensity measure simplifies and we get the following important property whose proof is trivial.

Lemma 3.1. *Given the transformation $\mathcal{T}_{s,t}$ that translates the set $A \subset \mathbb{R} \times \mathbb{R}^+$ in the set $\mathcal{T}_{s,t}A = \{(x, y) : (x + s, y - t) \in A\}$, in the case of $G(dy) = \mu e^{-\mu y} dy$, we have that*

$$(3.2) \quad |\mathcal{T}_{s,t}A| = e^{-\mu t} |A|.$$

If we define

$$\tilde{A}_{\text{ON}} = \bigcup_{k=0}^{\infty} \{-\beta S_{k+1} \leq x < -\beta S_k; y + x > T_k + 1\}$$

we can easily realize that

$$\begin{aligned} A_{\text{ON}} &\stackrel{\text{d}}{=} \{-\beta L \leq x < 0; y + x > H^*\} \cup \mathcal{T}_{\beta L, H^* + \beta L} \tilde{A}_{\text{ON}} \\ A_{\text{OFF}} &\stackrel{\text{d}}{=} \{-\beta L^* \leq x < 0; y + x > 0\} \cup \mathcal{T}_{\beta L^*, \beta L^*} \tilde{A}_{\text{ON}}. \end{aligned}$$

Hence we readily get the following

$$\begin{aligned} |A_{\text{ON}}| &\stackrel{\text{d}}{=} e^{-\mu H^*} \frac{\lambda}{\mu} \frac{1-\beta}{\beta} (1 - e^{-\beta \mu L}) + e^{-\mu(H^* + \beta L)} |\tilde{A}_{\text{ON}}| \\ |A_{\text{OFF}}| &\stackrel{\text{d}}{=} \frac{\lambda}{\mu} \frac{1-\beta}{\beta} (1 - e^{-\beta \mu L^*}) + e^{-\beta \mu L^*} |\tilde{A}_{\text{ON}}|, \end{aligned}$$

and, cf. Figure 4,

$$|\tilde{A}_{\text{ON}}| \stackrel{\text{d}}{=} e^{-\mu H} \frac{\lambda}{\mu} \frac{1-\beta}{\beta} (1 - e^{-\beta \mu L}) + e^{-\mu(H + \beta L)} |\tilde{A}_{\text{ON}}|.$$

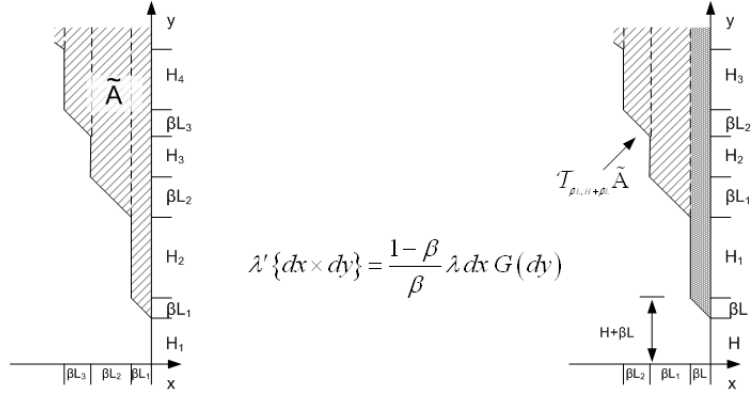


FIGURE 4. Area \tilde{A}_{ON} .

Define $\phi_{\text{ON}}(s) = \mathbb{E}[e^{-s|A_{\text{ON}}|}]$, $\phi_{\text{OFF}}(s) = \mathbb{E}[e^{-s|A_{\text{OFF}}|}]$, and $\tilde{\phi}_{\text{ON}}(s) = \mathbb{E}[e^{-s|\tilde{A}_{\text{ON}}|}]$, the characteristic functions of the corresponding random areas. By using the previous relations we can directly get the following result.

Theorem 3.2. *In the case of exponential service times, the characteristic functions of the random sets A_{ON} and A_{OFF} satisfy the following equations*

$$(3.3) \quad \phi_{\text{ON}}(s) = \mathbb{E} \left[\tilde{\phi}_{\text{ON}} \left(s e^{-\mu(H^* + \beta L)} \right) e^{-s \frac{\lambda}{\mu} \frac{1-\beta}{\beta} e^{-\mu H^*} (1 - e^{-\beta \mu L})} \right]$$

$$(3.4) \quad \phi_{\text{OFF}}(s) = \mathbb{E} \left[\tilde{\phi}_{\text{ON}} \left(s e^{-\mu \beta L^*} \right) e^{-s \frac{\lambda}{\mu} \frac{1-\beta}{\beta} (1 - e^{-\beta \mu L^*})} \right]$$

where $\tilde{\phi}_{\text{ON}}$ satisfies the following integral equation

$$(3.5) \quad \tilde{\phi}_{\text{ON}}(s) = \mathbb{E} \left[\tilde{\phi}_{\text{ON}} \left(se^{-\mu(H+\beta L)} \right) e^{-s \lambda \frac{1-\beta}{\mu}} e^{-\mu H (1-e^{-\beta \mu L})} \right].$$

Remark 3.3. It is worth noticing that when the random variable H is exponentially distributed we have that $H^* \stackrel{d}{=} H$ and hence $A_{\text{ON}} \stackrel{d}{=} \tilde{A}_{\text{ON}}$.

Remark 3.4. Modifying the definition of \tilde{A}_{ON} and requiring that it starts with a regular OFF period instead of a regular ON period, it is possible to define an additional random area, \tilde{A}_{OFF} , whose characteristic function satisfies the following recursive equation

$$(3.6) \quad \tilde{\phi}_{\text{OFF}}(s) = \mathbb{E} \left[\tilde{\phi}_{\text{OFF}} \left(se^{-\mu(H+\beta L)} \right) e^{-s \lambda \frac{1-\beta}{\mu}} (1-e^{-\beta \mu L}) \right].$$

As noticed in the previous remark, in case L is exponentially distributed we get that $A_{\text{OFF}} \stackrel{d}{=} \tilde{A}_{\text{OFF}}$, and hence $\phi_{\text{OFF}}(s)$ would satisfy (3.6) as well.

Remark 3.5. All the computation done to get the results of Theorem 3.2 can be easily restated in algebraic terms instead of using the chosen set arguments. It would be enough to express the measures of the random areas in the following ways,

$$\begin{aligned} |A_{\text{ON}}| &= \sum_{h=0}^{\infty} \lambda \frac{1-\beta}{\beta} \int_{\beta S_h}^{\beta S_{h+1}} P\{\sigma > x + H^* + T_h\} dx \\ |A_{\text{OFF}}| &= \lambda \frac{1-\beta}{\beta} \int_0^{\beta L^*} P\{\sigma > x\} dx + \sum_{h=0}^{\infty} \lambda \frac{1-\beta}{\beta} \int_{\beta L^* + \beta S_h}^{\beta L^* + \beta S_{h+1}} P\{\sigma > x + T_{h+1}\} dx \\ |\tilde{A}_{\text{ON}}| &= \sum_{h=0}^{\infty} \lambda \frac{1-\beta}{\beta} \int_{\beta S_h}^{\beta S_{h+1}} P\{\sigma > x + T_{h+1}\} dx. \end{aligned}$$

Having obtained an integral equation for $\tilde{\phi}_{\text{ON}}$, in the sequel we are going to study two special cases where it can be explicitly solved. The first refers to the case L has a general distribution while H is exponentially distributed and $\beta = 0$, the second one deals with the case both L and H are exponentially distributed.

3.1. $\beta = 0$ and H exponential. In the following we are going to study the characteristic function $\phi(z) = \mathbb{E}[z_{\nu}^N]$ of the stationary number of supplemental customers in the system in a Random Environment with respect to the isolated system. We are going to suppose that $\beta = 0$ and $P\{H > x\} = e^{-fx}$. We define $\psi(s) = \mathbb{E}[e^{-sL}]$ the characteristic function for the general OFF period L with $\mathbb{E}[L] = r^{-1}$, and $\psi^*(s) = r \frac{1-\psi(s)}{s}$ the one of the residual lifetime L^* .

Theorem 3.6. The number of excess customers in the system, N_{ν} , at equilibrium has the form

$$(3.7) \quad N_{\nu} \stackrel{d}{=} B Y_1 + (1-B)(Y_2 + X)$$

where B , X , Y_1 and Y_2 are positive and independent random variables. B is Bernoulli with parameter $r/(r+f)$, Y_1 and Y_2 are RP's with ch.f. $R(z) = e^{\frac{f}{r}} \frac{1}{r} \int_0^{\lambda(1-z)} \psi^*(s) ds$ and X is RP with ch.f. $\psi^*(\lambda(1-z))$.

Proof. By substituting $\beta = 0$ in Equation (3.5) we have

$$\tilde{\phi}_{\text{ON}}(s) = \mathbb{E} \left[\tilde{\phi}_{\text{ON}}(se^{-\mu H}) \psi(\lambda se^{-\mu H}) \right] = f \int_0^{\infty} \tilde{\phi}_{\text{ON}}(se^{-\mu h}) \psi(\lambda se^{-\mu h}) e^{-fh} dh,$$

and using the variable changing $y = se^{-\mu h}$ we get

$$s^{\frac{f}{\mu}} \tilde{\phi}_{\text{ON}}(s) = \frac{f}{\mu} \int_0^s \tilde{\phi}_{\text{ON}}(y) \psi(\lambda y) \frac{dy}{y}.$$

Differentiating both sides we have

$$\frac{\tilde{\phi}'_{\text{ON}}(s)}{\tilde{\phi}_{\text{ON}}(s)} = -\frac{f}{\mu} \frac{1 - \psi(\lambda s)}{s},$$

that together with the initial condition $\tilde{\phi}_{\text{ON}}(0) = 1$ gives the following solution

$$\tilde{\phi}_{\text{ON}}(s) = e^{-\frac{f}{\mu} \int_0^{\lambda s} \frac{1 - \psi(y)}{y} dy} = e^{-\frac{f}{\mu} \int_0^{\lambda s} \psi^*(y) dy}.$$

Finally Equation (3.7) is obtained via Equation (3.1) noticing that $\phi_{\text{ON}}(s) = \tilde{\phi}_{\text{ON}}(s)$ and that Equation (3.4) for $\beta = 0$ factorizes in

$$\phi_{\text{OFF}}(s) = \phi_{\text{ON}}(s) \psi^*(\lambda s).$$

□

3.2. L and H exponential. In the sequel we suppose that both L and H are exponentially distributed with mean respectively r^{-1} and f^{-1} . For this case Baykal-Gursoy and Xiao [2004] already obtained a similar decomposition by using Markov chains arguments. The method we use here is to compute the random variable $|A_{\Gamma}|$. In the following by truncated beta random variable $\text{TB}(a, b, c)$ we mean a positive random variable with density

$$f_{\text{TB}}(x) = \frac{(x/c)^{a-1} (1-x/c)^{b-1}}{c B(a, b)},$$

with $B(a, b) = \Gamma(a)\Gamma(b)/\Gamma(b+a)$ the Beta function. In Baykal-Gursoy and Xiao [2004], it is shown that the characteristic function of a r.v. $Z \sim \text{TB}(a, b, c)$ is given by

$$\mathbb{E} [e^{-sZ}] = M(a, b+a, -cs)$$

where the function $y(x) = M(a, b, x)$ is called Kummer's function [Abramovits and Stegun, 1964] and it is one solution of Kummer's differential equation

$$xy''(x) + (b-x)y'(x) - ay(x) = 0.$$

Theorem 3.7. *The number of excess customers in the system, N_{ν} , at equilibrium has the form*

$$(3.8) \quad N_{\nu} \stackrel{\text{d}}{=} B Y_1 + (1-B)(Y_2 + X)$$

where B , X , Y_1 and Y_2 are positive and independent random variables. B is Bernoulli with parameter $r/(r+f)$, Y_1 and Y_2 are Randomized Poisson whose parameters are Truncated Beta distributions respectively $\text{TB}\left(\frac{f}{\mu}, \frac{r}{\beta\mu} + 1, \frac{\lambda}{\mu} \frac{1-\beta}{\beta}\right)$ and $\text{TB}\left(\frac{f}{\mu} + 1, \frac{r}{\beta\mu}, \frac{\lambda}{\mu} \frac{1-\beta}{\beta}\right)$.

Proof. Starting from Equation (3.5) and using the fact that H is exponentially distributed we get

$$\begin{aligned} \phi_{\text{ON}}(s) &= f \int_0^{\infty} \mathbb{E} \left[\phi_{\text{ON}} \left(se^{-\mu h} e^{-\mu\beta L} \right) e^{se^{-\mu h} \frac{\lambda}{\mu} \frac{1-\beta}{\beta} e^{-\mu\beta L}} \right] e^{-se^{-\mu h} \frac{\lambda}{\mu} \frac{1-\beta}{\beta}} e^{-fh} dh \\ &= \frac{f}{\mu} \int_0^s \mathbb{E} \left[\phi_{\text{ON}} \left(ye^{-\mu\beta L} \right) e^{y \frac{\lambda}{\mu} \frac{1-\beta}{\beta} e^{-\mu\beta L}} \right] e^{-y \frac{\lambda}{\mu} \frac{1-\beta}{\beta}} \left(\frac{y}{s} \right)^{\frac{f}{\mu}} \frac{dy}{y}. \end{aligned}$$

By differentiating and simplifying we obtain the following differential equation

$$e^{s \frac{\lambda(1-\beta)}{\mu}} \left[\phi_{\text{ON}}(s) + \frac{\mu}{f} s \phi'_{\text{ON}}(s) \right] = \mathbb{E} \left[e^{\frac{\lambda(1-\beta)}{\mu} s e^{-\mu\beta L}} \phi_{\text{ON}} \left(s e^{-\mu\beta L} \right) \right].$$

By repeating the same procedure for the random variable L we finally get the following differential equation

$$s \phi''_{\text{ON}}(s) + \phi'_{\text{ON}}(s) \left[\left(\frac{f}{\mu} + \frac{r}{\beta\mu} + 1 \right) + \frac{\lambda(1-\beta)}{\mu\beta} s \right] + \phi_{\text{ON}}(s) \frac{f\lambda(1-\beta)}{\mu\mu\beta} = 0.$$

By expressing the last equation in terms of the function $\phi_{\text{ON}} \left(- \left(\frac{\lambda(1-\beta)}{\mu\beta} \right)^{-1} s \right)$, it reduces into Kummer's differential equation and hence using the condition $\phi_{\text{ON}}(0) = 1$ we obtain that the solution is

$$\phi_{\text{ON}}(s) = M \left(\frac{f}{\mu}, \frac{f}{\mu} + \frac{r}{\beta\mu} + 1, -\frac{\lambda(1-\beta)}{\mu\beta} s \right).$$

As for $\phi_{\text{OFF}}(s)$, we use the recursion equation (3.6) and carrying out the same computations as before we finally get the following differential equation

$$s \phi''_{\text{OFF}}(s) + \phi'_{\text{OFF}}(s) \left[\left(\frac{f}{\mu} + \frac{r}{\mu\beta} + 1 \right) + \frac{\lambda(1-\beta)}{\mu\beta} s \right] + \phi_{\text{OFF}}(s) \left(\frac{f}{\mu} + 1 \right) \frac{\lambda(1-\beta)}{\mu\beta} = 0$$

whose solution is

$$\phi_{\text{OFF}}(s) = M \left(\frac{f}{\mu} + 1, \frac{f}{\mu} + \frac{r}{\beta\mu} + 1, -\frac{\lambda(1-\beta)}{\mu\beta} s \right),$$

and the proof follows from the decomposition equation (3.1). \square

REFERENCES

- M. Abramovits and I.A. Stegun. *Handbook of Mathematical Functions*. Number 55. 2nd edition, 1964.
- F. Baccelli and P. Brémaud. *Elements of Queueing Theory. Palm–Martingale Calculus and Stochastic Recurrences*. Springer–Verlag, Berlin, 2003.
- M. Baykal-Gursoy and W. Xiao. Stochastic decomposition in $M/M/\infty$ queues with markov-modulated service rates. *Queueing Systems*, 48:75–88, 2004.
- S.W. Fuhrmann. A note on the M/G/1 queue with server vacations. *Oper. Res.*, 32:1368–1373, 1984.
- S.W. Fuhrmann and R.B. Cooper. Stochastic decomposition in the M/G/1 queue with generalized vacations. *Oper. Res.*, 33:1117–1129, 1985.
- O. Kella and W. Whitt. Queues with Server Vacations and Lévy Processes with Secondary Jump Input. *Ann. Appl. Probab.*, 1(1):104–117, Feb 1991.
- Y. Levy and U. Yechiali. Utilization of idle time in an M/G/1 queueing system. *Managm. Sci.*, 22:202–211, 1975.