# Optimal estimation of qubit mixed states with local measurements

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(Dated: December 20, 2005)

We study the estimation problem of a general mixed qubit state with protocols based on local measurements. We obtain the asymptotic bound of the fidelity for these settings and show that they do not attain the optimal joint measurement bound. We present an explicit protocol that uses classical communication in a very efficient way and saturates the local bound. We also analyze the case of mixed states known to lie on an equatorial plane of the Bloch sphere.

PACS numbers: 03.65Wj, 03.67.Hk, 03.65.Ta

## I. INTRODUCTION

The aim of quantum state estimation is to determine as accurately as possible the quantum state of a system. Contrary to classical states, linearity and the Heisenberg principle of Quantum Mechanics prevents the possibility to completely determine an unknown quantum state. Measurements provide partial information about the state at the price of destroying it. Thus, one must have a sample of identically prepared states upon which one can perform measurements and from their outcomes infer an approximation to the state. Only in the limit of having infinitely many copies of the state, one could get a perfect determination. With limited resources though, it is very relevant to know the performance of the estimation scheme, i.e. how well it approximates the true state as a function of the resources available.

State estimation is not just an issue of academic interest, almost any quantum estimation problem can be reduced to it. For instance, the secrecy capacity of quantum communication channels crucially depends on the ability to determine parameters of the state passing through the channel [1]. Although in theoretical analysis very often states are considered to be pure, in reality they tend to entangle with their environment and they are described by mixed rather than pure states.

The estimation of the most elementary system in Quantum Mechanics, the qubit, has been studied in many works, however quantitative results for mixed states are quite scarce [2–7]. This might be due to the greater complexity of mixed states —here one is dealing with the whole Poincaré sphere instead of just the surface— but also because there is a yet unresolved ambiguity on what is the natural choice for the uniform distribution of the radial parameter [8]. In a recent work [6] some of the authors presented the optimal estimation scheme of qubit mixed states given $N$ identical copies using the fidelity as a figure of merit. The results are rather independent of the above mentioned ambiguity. The expressions for large $N$, which provide very significant statistical information, were also obtained. This optimal scheme involves joint measurements on the sample of $N$ copies, i.e., quantum memories and coherent operations are required. However unrealistic it might seem, these protocols give us the absolute theoretical accuracy bounds for any other scheme.

In this paper we address the estimation of unknown qubit mixed states with schemes based only on local (we will also refer to them as separable) von Neumann measurements. These are realistic measurements that can be implemented with current technology. Our aim is to compare their performance with that of the optimal ones. We consider two cases, general mixed states —we call them 3D states— and the restricted case of states known to lay on an equatorial plane of the Poincaré sphere —2D states.

The paper is organized as follows. In the next section we introduce the notation used throughout the paper and review some previously known results relevant to our analysis. Specifically, we recall the optimal collective bound obtained in [6] and the results of plain tomography schemes discussed in [4]. The fidelity asymptotic behavior of the latter exhibits an atypical power law in $1/N$. We give an heuristic argument that helps to understand this result. The comparison of these two extreme schemes, may lead us to conclude that separable measurements are qualitatively worse than collective ones. In Section III we show that this is not the case and we derive a general bound for local measurements which reinstates the typical asymptotic behavior. In Section IV we present an explicit scheme based on local measurements that attains this bound. In this protocol one spends a fraction of the copies to get a rough estimation of the state orientation. Then one realigns the measurement devices for the remaining copies. The key ingredient of this scheme is the classical communication used in the realignment of the devices. Such schemes have been proposed earlier, for instance in [16], but only studied at a heuristic level. Our scheme is carefully tuned to guarantee asymptotically optimal fidelity. We end the paper with a brief summary and conclusions.

## II. PRELIMINARIES

A qubit mixed state is represented by a density matrix given by $\rho(\vec{r}) = (\mathbb{1} + \vec{r}\cdot\vec{\sigma})/2$, where $\vec{\sigma} = (\sigma_x, \sigma_y, \sigma_z)$ are the Pauli matrices and $\vec{r}$ is a point in the Poincaré sphere $\{\vec{r} : |\vec{r}| \leq 1\}$. We refer as orientation of the state the normalized vector $\vec{n} = \vec{r}/|\vec{r}|$ and as purity the modulus $r = |\vec{r}|$. The sample of $N$ copies is given by the tensor product $\rho^{\otimes N}$. We will drop the $\vec{r}$ dependence whenever to confusion arises.

A measurement is represented by a Positive Operator Measure Value (POVM) given by a set of operators $\mathcal{O} = \{O_\chi\}$, such that $O\chi \geq 0$ and $\sum_\chi O_\chi = \mathbb{1}$. In full generality these operators act in the Hilbert space of dimension $2^N$. A measurement is separable if $O_\chi = \prod_{k=1}^N O_{i_k}^{(k)}$. Notice that if only von Neumann measurements are considered, the outcomes can be labelled by a binary number $\chi = i_N i_{N-1} \ldots i_1$.

Upon obtaining an outcome $\chi$ one makes a guess for the state, $\rho(\vec{R}_\chi) \equiv \rho_\chi$. The quality of the guess is quantified by a suitable figure of merit. A widely used figure is the fidelity, defined as [9] $f(\vec{r}, \vec{R}_\chi) = \left(\mathrm{tr}\sqrt{\rho_\chi^{1/2}\rho\,\rho_\chi^{1/2}}\right)^2$. This quantity determines the maximum distinguishability between $\rho$ and $\rho_\chi$ that can be achieved by any measurement [10], it also gives the maximum overlap between all the purifications of the two mixed states [9, 11]. For qubits, the fidelity reads

$$f(\vec{r}, \vec{R}_\chi) = \frac{1 + \vec{r}\cdot\vec{R}_\chi + \sqrt{1-r^2}\sqrt{1-R_\chi^2}}{2}, \qquad (2.1)$$

where $\vec{r}$ and $\vec{R}_\chi$ are the Bloch vectors of the states $\rho$ and $\rho_\chi$ respectively, and $R = |\vec{R}|$.

The overall performance of the estimation protocol is given by the average fidelity over all possible outcomes and over the prior distribution. The probability of obtaining outcome $\chi$ if the state was $\rho$ is given by the Born rule

$$p(\chi|\vec{r}) = \mathrm{tr}\,[O_\chi \rho^{\otimes N}]. \qquad (2.2)$$

We denote the prior probability distribution by $d\rho$, whose particular form is discussed below. Hereafter we will refer to this average simply as the fidelity $F$, which reads

$$F = \sum_\chi \int d\rho\, f(\vec{r}, \vec{R}_\chi)\mathrm{tr}\,[O_\chi \rho^{\otimes N}] \equiv \frac{1}{2}(1+\Delta), \qquad (2.3)$$

where $\Delta = \langle \vec{r}\cdot\vec{R}_\chi + \sqrt{1-r^2}\sqrt{1-R_\chi^2}\rangle_{\chi\vec{r}}$, and $\langle\cdots\rangle_{\chi\vec{r}}$ denotes the average over the outcomes and the prior distribution. Finding the optimal scheme amounts to finding the measurement and the guess that maximizes Eq. (2.3), or equivalently $\Delta$.

Such scheme was found in [6] (see also [3]) for the case of uniform isotropic priors, i.e., for $d\rho = dr w(r) dn$, where $dn$ is the invariant measure on the 2-sphere when estimating a general qubit mixed state —the 3D states— and the corresponding measure on the circle for the equatorial states —the 2D states. The weight function $w(r)$ parametrize the family of isotropic priors and is a normalized positive function such that $\int_0^1 w(r) = 1$. Our results apply to very general classes of isotropic priors, however it is worth mentioning here that many arguments favor the choice of a Bures prior distribution, which is given by

$$w(r) = \begin{cases} \dfrac{4}{\pi}\dfrac{r^2}{\sqrt{1-r^2}} & 3\mathrm{D} \\ \dfrac{r}{\sqrt{1-r^2}} & 2\mathrm{D} \end{cases}. \qquad (2.4)$$

First, notice that Eq. (2.4) is precisely the volume element of the metric space induced by the fidelity (2.1). This metric belongs to the monotone Riemannian metrics, the type of metrics for which the distance decrease under coarse-graining [12]. Finally it has also been argued that it corresponds to maximal randomness of the signal states, and therefore describes an ensemble for which one has the minimal prior knowledge [13].

### A. Optimal scheme

We reproduce here the asymptotic expressions of the fidelity of the optimal scheme [6]. This provides us with the most relevant statistical information, which can be compared with the fidelity of local schemes in the same regime in an $N$-independent way.

For general states, 3D case, it was found that the asymptotic fidelity is (see also [14])

$$F^{3\mathrm{D}} = 1 - \frac{3+2\bar{r}}{4N} + o(1/N), \qquad (2.5)$$

where $\bar{r}$ stands for the mean purity over its prior distribution, i.e., $\bar{r} \equiv \int_0^1 dr\, w(r)\, r$. For the specific choice of a Bures distribution, Eq. (2.4), we have $\bar{r} = 2/(3\pi)$.

For 2D states, one has [6]:

$$F^{2\mathrm{D}} = 1 - \frac{1}{2N} + o(1/N), \qquad (2.6)$$

which is independent of the specific form of the prior, as long as is isotropic. In particular, it also applies to a Bures prior, Eq. (2.4).

### B. Tomography

We now move the other extreme and dwell with the most basic estimation scheme: tomography. Quantum state tomography has the nice property of being easily implementable with current technology. It tells us that asymptotically von Neumann measurements along two (three) fixed orthogonal directions, $x$, $y$, $(z)$ are sufficient to reconstruct a qubit mixed state. Any local von Neumann measurement is represented by two projectors $O(\pm\vec{m}) = (1 \pm \vec{m}\cdot\vec{\sigma})/2$, where $\vec{m}$ is a unit Bloch vector characterizing the measurement. After the sequence of measurements —$N/2(N/3)$ copies along each orthogonal direction— one obtains a set of outcomes $+1$ and $-1$ with relative frequencies $\alpha_i$ and $1-\alpha_i$, respectively ($i = x, y, (z)$). Carrying out the best data processing of the outcomes and considering the most physically relevant prior, the Bures prior, one obtains an asymptotic behavior of the fidelity for 2D and 3D given by [4]

$$F = 1 - \frac{\xi}{N^{3/4}} + o(1/N^{3/4}), \tag{2.7}$$

where $\xi$ is a constant whose specific value is not important for our discussion (obviously it has a different value for the 2D and 3D cases and $\xi_{2D} \leq \xi_{3D}$).

Observe the atypical asymptotic behavior $1 - F \sim 1/N^{3/4}$, instead of the ordinary $1 - F \sim 1/N$. This unusual behavior can be traced back to the contribution of the states near the surface of the Bloch sphere (the pure states). It can be understood as follows. Consider a shell $\mathcal{S}$ of the Bloch sphere of thickness $\delta$, $\mathcal{S} = \{\vec{r} : 1-\delta < r \leq 1\}$. The fidelity between two states inside $\mathcal{S}$, in the extreme case of both Bloch vectors pointing in the same direction, is $1 - f \propto \delta$. For a signal state not in the direction of one of the measurement axes, we cannot hope to reconstruct location *or length* of the Bloch vector to an accuracy better than $1/\sqrt{N}$. Thus $1-f$ will be of order $1/\sqrt{N}$ when the signal state is inside $\mathcal{S}$. Also, integrating (2.4) from $r = 1-\delta$ to $r = 1$, we see that the signal state has a probability of the order of $\sqrt{\delta}$ of being in $\mathcal{S}$. In particular, the signal state is within the critical distance of order $\delta \sim 1/\sqrt{N}$ with probability of order $\sqrt{\delta} \sim 1/N^{1/4}$. In this case, no processing of the data can result in $1-f$ between reconstructed and signal state of better order than $\delta \sim 1/\sqrt{N}$. The contribution to $1 - F$ from this increasingly thin outer shell of signal states is therefore of order $1/N^{3/4}$.

The comparison of the optimal bounds (2.5) and (2.6) with the tomographic ones (2.7), shows a distinctive qualitative difference in the accuracy that can be achieved in either approach. Tomography performs rather poorly, as is apparent from the atypical rate, $1/N^{3/4}$ in Eq. 2.7. This contrasts the pure state case where tomography performs very close to optimality [15]. Therefore, one might be tempted to conjecture that for mixed states this bad performance persists in any local measurement scheme. In the next section we show that this is not the case. We obtain a bound from statistical arguments that exhibits the $1/N$ asymptotic behavior of the fidelity of the optimal scheme. We then show in the following section an explicit local scheme attaining this bound. The key observation is that tomography does not exploit classical communication, i.e., the possibility to adapt the orientation of the measuring devices according to the information gathered from the previous outcomes.

## III. STATISTICAL BOUNDS

The fidelity, Eq. (2.1), averaged over all outcomes reads

$$\langle f(\vec{r}, \vec{R}_\chi)\rangle_\chi = 1 - \frac{1}{4}\mathrm{tr}\,[H(\vec{r})V_N(\vec{r}, \vec{\mathcal{R}}_{\mathcal{O}}, \mathcal{O})] + \ldots, \tag{3.1}$$

where (taking $r_1 = x,\ r_2 = y,\ r_3 = z$)

$$V_N(\vec{r}, \vec{\mathcal{R}}_{\mathcal{O}}, \mathcal{O})]_{ij} = \sum_\chi p(\chi|\vec{r})(R_{\chi i} - r_i)(R_{\chi j} - r_j) \tag{3.2}$$

is the mean square error and $H(\vec{r})$ is the quantum Fisher information matrix [recall that the conditional probabilites $p(\chi|\vec{r})$ are given by Eq. (2.2)]. The quantum Fisher information matrix can be calculated as the matrix with elements $H(\vec{r})_{ij} = \mathrm{Re}[\mathrm{tr}\,\rho(\vec{r})\lambda(\vec{r})_i\lambda(\vec{r})_j]$. The $\lambda$'s are called the symmetric logarithmic derivatives and can be defined as a self-adjoint solution to the equation $\partial_i\rho(\vec{r}) = \{\rho(\vec{r}), \lambda(\vec{r})_i\}/2$, where $\partial_i$ means partial derivative with respect to $r_i$ and $\{A, B\} = AB + BA$ is the anticommutator between $A$ and $B$.

From classical statistics arguments, one expects that for large $N$, a fixed measurement $\mathcal{O}$ and a reasonable estimator $\vec{R}_{\mathcal{O}}$, the mean square error will behave as

$$V_N(\vec{r}, \vec{\mathcal{R}}_{\mathcal{O}}, \mathcal{O})] = \frac{I(\vec{r}, \mathcal{O})^{-1}}{N} + o(1/N), \tag{3.3}$$

where $I(\vec{r}, \mathcal{O}) = \lim_{N\to\infty} I_N(\vec{r}, \mathcal{O})/N$, and $I_N(\vec{r}, \mathcal{O})$ is the Fisher information matrix based on a measurement $\mathcal{O}$ on $\rho(\vec{r})^{\otimes N}$. It can be computed to be

$$I_N(\vec{r}, \mathcal{O})_{ij} = \sum_\chi \frac{\partial_i p(\chi|\vec{r})\partial_j p(\chi|\vec{r})}{p(\chi|\vec{r})}. \tag{3.4}$$

Plugging (3.3) in (3.1) we get

$$\langle f(\vec{r}, \vec{R}_\chi)\rangle_\chi = 1 - \frac{1}{4N}\mathrm{Tr}\,[H(\vec{r})I(\vec{r}, \mathcal{O})^{-1}] + o(1/N). \tag{3.5}$$

If one restricts oneself to separable measurements, the following bound holds

$$\mathrm{Tr}\,[H(\vec{r})^{-1}I(\vec{r}, \mathcal{O})] \leq 1, \tag{3.6}$$

as proved in [16]. Hence, it is straightforward to check that

$$\mathrm{tr}\,[H(\vec{r})I(\vec{r}, \mathcal{O})^{-1}] \geq p^2, \tag{3.7}$$

where $p$ is the number of parameters (i.e. 2 for the 2D model and 3 for the 3D model). Using Eqs. (3.7) and (3.5) one gets that for any separable measurement the following bound must hold

$$\lim_{N\to\infty} N[1 - \langle f(\vec{r}, \vec{R}_\chi)\rangle] \geq \frac{p^2}{4} = \begin{cases} 1 & \text{2D} \\ 9/4 & \text{3D} \end{cases}. \tag{3.8}$$

This bound is expected to apply also when one averages over any prior. In fact a rigorous argument follows in exactly the same way as in appendix H of [6] using (3.6), and more general results of the same type are established in [17].

## IV. LOCC MEASUREMENT PROTOCOL

From our discussion in Sec. II B and the results of previous section, it is clear that to attain the bound (3.8) one should use some sort of classical communication. We take inspiration from the Gill-Massar approach [16], which uses this feature only once and is somehow the easiest scheme beyond fixed tomography. The idea is to spend a fraction of the copies to get a rough estimate for the orientation of the state, we call it $\vec{n}_0$. If this estimation were exact, surely the purity could be optimally inferred from equal von Neumann measurements along this direction. So it seems promising to consider the following scheme. One measures the remaining copies along three directions: $\vec{n}_0$, to estimate the purity, and two (or one in 2D) orthogonal directions to $\vec{n}_0$ to refine the guess of the orientation. The Gill-Massar approach leads to the same measurements but to a more complicated data-processing.

In more quantitative terms, we do plain tomography with a fraction of copies $N^\alpha \equiv N_0$ $(0<\alpha<1)$ for the orientation of the state [15] and obtain an estimate $\vec{n}_0$. The accuracy of $\vec{n}_0$ is quantified by the angle $\Theta$ between $\vec{n}$ and $\vec{n}_0$

$$\frac{\langle\Theta^2\rangle_0}{2} \approx 1-\langle C\rangle_0 = \frac{\xi(r)}{N_0} + o(1/N_0), \qquad (4.1)$$

where $C \equiv \cos\Theta$ and we use the shorthand notation $\langle\cdots\rangle_0$ to denote the average over the $N_0$ outcomes [i.e, $\langle\cdots\rangle_0 = \sum_{\chi_0} p(\vec{r}|\chi_o)(\cdots)$ and likewise below $\langle\cdots\rangle_{\vec{r}}$ will denote the average over the prior distribution $d\rho = dr\, w(r) d\Omega/(4\pi)$]. The function $\xi(r)$ can be computed to be $\xi(r) = 3(1/r^2 - 1/5)$ for 3D states and $\xi(r) = (1/r^2 - 1/4)$ for 2D states.

In the second step (to simplify the presentation we first consider only the 3D case an discuss the 2D case at the end) we perform again tomography along three *new* orthogonal axis $x$, $y$, $z$, where the $z$ axis is oriented along $\vec{n}_0$. We then measure each third of the remaining copies $N_1 \equiv (N-N_0)/3$ along each of these orthogonal directions. The relative frequencies of the outcomes are denoted by $\beta_x$, $\beta_y$ and $\alpha_z$. With the outcomes along the $z$ axis we estimate the purity to be $R = 2\alpha_z - 1$ and from the measurement outcomes of the other two directions we give a more precise estimate of the orientation

$$\vec{n}_1 = \sin\theta\cos\phi\,\vec{e}_x + \sin\theta\sin\phi\,\vec{e}_y + \cos\theta\,\vec{n}_0\,, \qquad (4.2)$$

where $\sin\theta = R^{-1}(s_x^2+s_y^2)^{1/2}$, $\tan\phi = s_y/s_x$ and $s_{x(y)} = 2\beta_{x(y)} - 1$. The average fidelity in a self-explanatory notation is $F = \langle f(\vec{r}, \vec{n}_1 R)\rangle_{0\,\vec{r}\,\beta\,\alpha}$. It is important to note that the frequencies $\beta_{x(y)}$ are binomially distributed as $\beta_{x(y)} \sim \mathrm{Bin}(N_1, (1 + r\, n_{x(y)})/2)$ and also that $\alpha_z$ is distributed as $\alpha_z \sim \mathrm{Bin}(N_1, (1+rC)/2)$, where we have defined $n_i = \vec{n}\cdot\vec{e}_i$, $(i = x, y)$.

Our task is to compute $F$, or equivalently $2F - 1 = \Delta \equiv \langle\Delta\rangle_{\vec{r}\,0\,\beta\,\alpha}$. We start by computing the first term of $\Delta$ in Eq. (2.3) without averaging over $\vec{r}$, $\Delta_1(\vec{r}) = \langle rR\,\vec{n}\cdot\vec{n}_1\rangle_{0\,\alpha\,\beta}$. This term can be written as a sum of two parts, $\Delta_1(\vec{r}) = \Delta_A(\vec{r}) + \Delta_B(\vec{r})$, with

$$\begin{aligned}\Delta_A(\vec{r}) &= r\langle CR\cos\theta\rangle_{0\,\alpha\,\beta} \qquad (4.3)\\ \Delta_B(\vec{r}) &= r\langle R(\sin\theta\cos\phi\, n_x + \sin\theta\sin\phi\, n_y)\rangle_{0\,\alpha\,\beta}\end{aligned}$$

Obviously, as a random variable and for large $N_1$, $R$ is normally distributed as $R \approx \mathrm{N}(rC, \sqrt{1-r^2C^2}/\sqrt{N_1})$ and we can write $R \approx rC$ for large $N$. Also the angle $\theta$ between $\vec{n}_0$ and $\vec{n}_1$ will be small for large $N$ ($\vec{n}_1$ it is just a refinement of $\vec{n}_0$), hence $\sin\theta \approx \theta$ and $\cos\theta \approx 1 - \theta^2/2$. Thus, we obtain $\Delta_A(\vec{r}) \approx r^2\langle C^2 - \langle s_x^2 + s_y^2\rangle_\beta / 2r^2\rangle_0$. The average $\langle s_x^2 + s_y^2\rangle_\beta$ can be trivially obtained from the moments $\langle\beta_{x(y)}\rangle_\beta$ and $\langle\beta_{x(y)}^2\rangle_\beta$ of of the binomial distributions. Finally, using the relation $n_x^2 + n_y^2 = 1 - C^2$ we get

$$\Delta_A(\vec{r}) \approx r^2\frac{3\langle C^2\rangle_0 - 1}{2} - \frac{1}{N_1}. \qquad (4.4)$$

The second term $\Delta_B(\vec{r})$ can be computed along the same lines simply by observing that $\sin\theta\cos\phi = s_x R^{-1}$ and $\sin\theta\sin\phi = s_y R^{-1}$. We get $\Delta_2(\vec{r}) \sim r^2\langle 1 - C^2\rangle_0$. Putting the two pieces together we have

$$\Delta_1(\vec{r}) = \frac{r^2}{2}\langle(1-C)^2\rangle_0 + r^2\langle C\rangle_0 - \frac{1}{N_1} + o(1/N_1) \qquad (4.5)$$

Observe that $r^2\langle C\rangle_0$ can be approximated by $\langle rR\rangle_{0\,\alpha}$ and that the term $\langle(1-C)^2\rangle_0$ is of order $o(N_0^{-2})$, see (4.1), which is subdominant if $\alpha > 1/2$. Therefore we obtain up to the order we are interested in

$$\Delta^{3\mathrm{D}} = \left\langle rR + \sqrt{1-r^2}\sqrt{1-R^2}\right\rangle_{0\,\vec{r}\,\alpha} - \frac{1}{N_1} + o(1/N) \qquad (4.6)$$

This is a gratifying result because the term in brackets can be identified as the average fidelity in the purity estimation problem discussed in [5], $F^{\mathrm{Purity}}$. Its asymptotic expression was already obtained there: $F^{\mathrm{Purity}} = 1 - 1/(2N_1) + \cdots$ with $\alpha > 1/2 + \kappa$, with $\kappa \geq 0$ (see [5] for a detailed justification of the value of $\kappa$, which depends on the prior $w(r)$, *e.g*, for a Bures prior we have $\alpha > 2/3$). Thus $\Delta^{3\mathrm{D}} = F^{\mathrm{Purity}} - 1/N_1 + \cdots$, and therefore

$$F^{3\mathrm{D}} = 1 - \frac{9}{4N} + o(1/N), \qquad (4.7)$$

which saturates the statistical bound Eq. (3.8).

Actually the above approximations break down at (near) $r = 0$. One can bound the contribution to the fidelity from a small ball at the origin and show that this is asymptotically negligeable for the values of $\alpha$ under consideration.

Let us finally address the 2D case, which can be solved along the same lines. Actually, the computation is even simpler than the 3D case. Here the guess of the direction is just

$$\vec{n}_1 = \sin\theta\,\vec{e}_x + \cos\theta\,\vec{n}_0 \qquad (4.8)$$

where $\theta \sim \sin\theta = R^{-1}(2\beta_x - 1)$. The analogous equation to (4.5) reads

$$\Delta_1(\vec{r}) = \frac{r^2}{2}\langle(1-C)^2\rangle_0 + r^2\langle C\rangle_0 - \frac{1}{2N_1} + \cdots . \quad (4.9)$$

i.e, the only difference with (4.5) is in the coeficient of the $1/N_1$ term. Hence $\Delta^{2D} = F^{\text{Purity}} - 1/(2N_1)$. Recall that the asymptotic expression of $F^{\text{Purity}}$ is the same for 2D and 3D states [5] and that here we have $N_1 = (N - N_0)/2$. We thus obtain

$$F^{2D} = 1 - \frac{1}{N} + o(1/N) . \quad (4.10)$$

We see that again the statistical bound is saturated with an estimation scheme based on local measurements, provided classical communication is used at least once.

## V. CONCLUSIONS

We have analyzed the estimation of an unknown mixed qubit state with schemes based on separable von Neumann measurements, which are the most relevant ones for practical purposes. We have obtained the asymptotic bound on the fidelity for such schemes using a statistical approach. It is readily seen that the rate at which the perfect determination limit can be attained is similar to the optimal schemes (which involve joint measurements) $F = 1 - \xi/N + \cdots$, however the accuracy is strictly lower. In particular, we have $\xi^{(2D)}_{\text{LOCC}} = 1 > \xi^{(2D)}_{\text{OPT}} = 1/2$ and $\xi^{(3D)}_{\text{LOCC}} = 9/4 > \xi^{(3D)}_{\text{OPT}} = (1 + 3\bar{r})/4$ (because $\bar{r} < 1$). These local bounds are independent on the prior distribution and are not referred to any particular scheme. Considering that tomography (which works very close to optimality for pure qubits) does not reproduce the $1/N$ behavior of the fidelity for a Bures prior, it is very relevant to find schemes that do saturate these local bounds. Finding a specific example of such schemes is one of the main results of our work. We have given a scheme, which is somehow the easiest extension of tomography, that embodies classical communication and indeed saturates the statistical bounds.

### Acknowledgments

We acknowledge financial support from Spanish Ministry of Science and Technology project BFM2002-02588, CIRIT project SGR-00185, Netherlands Organization for Scientific Research NWO project 613.003.047, the European Community projects QUPRODIS contract no. IST-2001-38877 and RESQ contract no IST-2001-37559.

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