# POLYMER PINNING AT AN INTERFACE

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Abstract. We consider a (1+1)-dimensional hydrophobic homopolymer, in interaction with an oil-water interface. In $\mathbb{Z}^2$, the interface is modelled by the $x$ axis, the oil is above, the water is below, and the polymer configurations are given by a simple random walk $(S_i)_{i\geq 0}$. The hydrophobicity of each monomer tends to delocalize the polymer in the upper half plane, through a reward $h > 0$ for each monomer in the oil and a penalty $-h < 0$ for each monomer in the water. On the other hand, the chain receives a random reward (or penalty) when crossing the interface, depending on a local random charge attached to the interface. At site $i$ this reward is $\beta(1 + s\zeta_i)$, where $(\zeta_i)_{i\geq 1}$ is a sequence of i.i.d. centered random variables, and $s \geq 0$, $\beta \geq 0$. Since the reward is positive on the average, the interface attracts the polymer and a localization effect may arise. We transform the measure of each trajectory with the hamiltonian $\beta \sum_{i=1}^{N}(1+s\zeta_i)\mathbf{1}_{\{S_i=0\}}+h\sum_{i=1}^{N}\text{sign}(S_i)$, and study the critical curve $h_c^s(\beta)$ that separates the $(\beta, h)$-plane into a localized and a delocalized phase for $s$ fixed.

It is not difficult to show that $h_c^s(\beta) \geq h_c^0(\beta)$ for all $s \geq 0$ with the former explicitly computable. In this article we give a method to improve in a quantitative way this lower bound. To that aim, we transform the strategy developed by Bolthausen and den Hollander in [4], by taking into account the fact that the chain can target the sites where it comes back to the origin. The improved lower bound is interesting even for the case where only the interaction at the interface is active, i.e., for the pure pinning model. Our bound improves an earlier bound of Alexander and Sidoravicius in [1].

*Keywords: Polymers, localization-delocalization transition, pinning, random walk, wetting.*
*AMS 2000 subject classifications: 82B41, 60K35, 60K37*

## 1. Introduction

1.1. **The model.** Let $S = (S_n)_{n\geq 0}$ be a simple symmetric random walk starting at 0, i.e., $S_0 = 0$ , $S_n = \sum_{i=1}^{n} X_i$, where $\{X_i\}_{i\geq 1}$ are i.i.d. random variables such that $P(X_1 = \pm 1) = 1/2$. Let $\Lambda_i = \text{sign}(S_i)$ if $S_i \neq 0$, $\Lambda_i = \Lambda_{i-1}$ otherwise. Let $\{\zeta_i\}_{i\geq 1}$ be i.i.d. random variables, non a.s. equal to 0, such that $\mathbb{E}(\zeta_1) = 0$ and $\mathbb{E}\left(e^{\lambda|\zeta_1|}\right) < \infty$ for every $\lambda > 0$.

For $h \geq 0$, $s \geq 0$ and for every trajectory $S$ of the random walk, we define the hamiltonian

$$H_{N,\beta,h}^{\zeta,s}(S) = \beta \sum_{i=1}^{N} (1 + s\zeta_i)\, \mathbf{1}_{\{S_i=0\}} + h \sum_{i=1}^{N} \Lambda_i, \tag{1.1}$$

and the probability measure $P_{N,\beta,h}^{\zeta,s}$

$$\frac{dP_{N,\beta,h}^{\zeta,s}}{dP}(S) = \frac{\exp\left(H_{N,\beta,h}^{\zeta,s}(S)\right)}{Z_{N,\beta,h}^{\zeta,s}} \tag{1.2}$$

*Date*: February 7, 2006.

with the partition function

$$Z_{N,\beta,h}^{\zeta,s} = E\big( \exp \big( H_{N,\beta,h}^{\zeta,s}(S) \big) \big). \tag{1.3}$$

The law $P_{N,\beta,h}^{\zeta,s}$ is called the polymer measure of size $N$. Under this measure, two types of trajectories seem to be favoured: the localized trajectories that come back often to the origin to receive a positive pinning reward along the $x$ axis, on the other hand, the delocalized trajectories that spend almost all the time in the upper half plane. The latter are favoured at the same time by the second term of the hamiltonian and by the fact that they are much more numerous than the former. Thus, a competition between these two possible behaviors arises.

1.2. **Free energy.** To decide, at fixed parameters, if the system is localized or not, we introduce the free energy, denoted by $\Psi^s(\beta, h)$, and defined by

$$\Psi^s(\beta, h) = \lim_{N\to\infty} \frac{1}{N} \log Z_{N,\beta,h}^{\zeta,s}.$$

This limit is non-random and occurs $\mathbb{P}$ almost surely in $\zeta$ and $\mathbb{L}^1$. The proof of this convergence is similar to the one given in [8] or [4]. For this reason, we do not detail it in this article.

The free energy can be bounded from below by computing its restriction to the subset $D_N$ defined by $D_N = \{S : S_i > 0 \ \forall\ i \in \{1, \ldots, N\}\}$. For each trajectory of $D_N$, the hamiltonian is equal to $hN$, because the chain stays in the upper half plane and never comes back to the interface. Moreover, $P(D_N) \sim c/N^{1/2}$ as $N \to \infty$. Hence,

$$\Psi^s(\beta, h) \geq \liminf_{N\to\infty} \frac{1}{N} \log E\left( e^{hN} \mathbf{1}_{\{D_N\}} \right) \geq h + \liminf_{N\to\infty} \frac{\log (P(D_N))}{N} \geq h,$$

and so the free energy is larger than or equal to $h$. We will say that the polymer is delocalized if $\Psi^s(\beta, h) = h$ (because then the trajectories of $D_N$ give us the whole free energy) and delocalized if $\Psi^s(\beta, h) > h$.

This separation between the localized and delocalized regimes seems a bit crude. Indeed, many trajectories that come back only a few times to the origin, and spend almost all the time in the upper half plane, should also be called delocalized. Thus, taking only into account the trajectories of $D_N$ could be insufficient. However, the convexity of the free energy ensures that throughout the localized phase the chain comes back to the interface in a positive density of sites. Another result helps us to understand the localization phenomenon. This result is due to Sinai [17], and we can adapt it to our pinning model to control the vertical displacement of the chain in the localized area. To that aim, we transform the hamiltonian to $\beta \sum_{i=1}^{N} (1 + s\zeta_{N-i}) \mathbf{1}_{\{S_i=0\}} + h \sum_{i=1}^{N} \Lambda_i$. Thus, the disorder is fixed in the neighborhood of $S_N$, while the free energy is not modified. Then, for $\Psi^s(\beta, h) > 0$ and $\epsilon > 0$, we can show that, $\mathbb{P}$ almost surely in $\zeta$, there exists a finite constant $C_\zeta^\epsilon > 0$ such that, for every $L \geq 0$ and $N \geq 0$,

$$P_{N,\beta,h}^{\zeta,s}(|S_N| > L) \leq C_\zeta^\epsilon \exp\left( -\left( \Psi^s(\beta, h) - \epsilon \right) L \right).$$

This result cannot hold if we keep the original hamiltonian, because the disorder is not fixed close to $S_N$. Therefore, $\mathbb{P}$ almost surely in $\zeta$, we meet arbitrary long stretches of negative rewards, which push $S_N$ far away from the interface.

Some pathwise results have been proved in the delocalized area. In our case, we can use the method developed in the last part of [3] to prove that $\mathbb{P}$ almost surely in $\zeta$, and for every $K > 0$,

$$\lim_{N\to\infty} E^{\zeta,s}_{N,\beta,h}\left(\sharp\{i \in \{1,\dots,N\} : S_i > K\}/N\right) = 1.$$

These results allow us to understand more deeply what localization and delocalization mean.

1.3. **Simplification of the model.** We transform the hamiltonian to simplify the localization condition. To that aim, we notice that

$$\Psi^s(\beta,h) - h = \lim_{N\to\infty} \frac{1}{N} \log\left( E\left( \exp\left(\beta \sum_{i=1}^N (1+s\zeta_i)\mathbf{1}_{\{S_i=0\}} + h\sum_{i=1}^N (\Lambda_i - 1)\right)\right)\right)$$

and we define $\Phi^s(\beta,h) = \Psi^s(\beta,h) - h$. The delocalization condition becomes $\Phi^s(\beta,h) = 0$ and the localization condition $\Phi^s(\beta,h) > 0$. Finally, we set $\Delta_i = 1$ if $\Lambda_i = -1$ and $\Delta_i = 0$ if $\Lambda_i = 1$. Then the hamiltonian becomes

$$H^{\zeta,s}_{N,\beta,h}(S) = \beta \sum_{i=1}^N (1+s\zeta_i)\,\mathbf{1}_{\{S_i=0\}} - 2h\sum_{i=1}^N \Delta_i,$$

and we keep $Z^{\zeta,s}_{N,\beta,h} = E\big(e^{H^{\zeta,s}_{N,\beta,h}}\big)$. Thus, we obtain

$$\Phi^s(\beta,h) = \lim_{N\to\infty}\frac{1}{N}\log Z^{\zeta,s}_{N,\beta,h}.$$

The function $\Phi^s$ is convex and continuous in both variables, non-decreasing in $\beta$ and non-increasing in $h$.

## 2. Motivation and Preview

2.1. **Physical motivation.** Systems of random walk attracted by a potential at an interface are closely studied at this moment (see [8]). One of the major issue in the subject consists in understanding better the influence of a random potential compared to a constant one (with the same expectation). Indeed, while it seems intuitively clear that a random potential has a stronger power of attraction than a constant one, it is much less obvious how to quantify this difference.

In this article, we consider a potential at the interface together with the fact that the polymer prefers lying in the upper half plane than in the lower half plane. Such a type of system has been studied numerically in [11] and describes, for instance, a hydrophobic homopolymer at an interface between oil and water. Close to this interface, some very small droplets of a third solvent (microemulsions) are placed. These droplets have a strong capacity of attraction on the monomers composing our chain. Thus, the pinning rewards that the chain can receive when it comes back to the origin represent the attractive emulsions that the polymer touches close to the interface.

2.2. **Preview.** In this article, we investigate new strategies of localization for the polymer, consisting in targeting the sites where it comes back to the interface. We find an explicit lower bound on the critical curve that lies strictly above the non-random one.

Our result covers, as a limit case when $h$ tends to infinity, the wetting transition model. Indeed, in the last ten years the wetting problem, i.e., the case of a polymer interacting with an (impenetrable) interface, has attracted a lot of interest, because it can be seen as a Poland-Scheraga model of the DNA strand (see [16], [7]). The localization transition with a constant disorder occurs for the pinning reward $\log 2$, and several open questions are linked with the effect of a small random perturbation added to the reward $\log 2$. Moreover, with the constant pinning reward $\log 2$, the simple random walk conditioned to stay positive has the same law as the reflected random walk (see [10]). That is why, to study the wetting model around the pinning reward $\log 2$, it suffices to consider the pure pinning model, i.e., a reflected random walk pinned at the origin by small random variables.

This pure pinning model has been closely studied. For example, in [12] a particular type of positive potential has been considered and a criterium has been given to decide for every disorder realization whether it localizes the polymer or not. But a very difficult question consists in estimating, for small $s$, the critical delocalization average $u_c(s)$ of a pinning potential $\{-u+s\zeta_i\}_{i\geq 1}$, where $\{\zeta_i\}_{i\geq 1}$ are i.i.d., centered and of variance 1 (i.e., $Var(-u+s\zeta_i)=s^2$). The annealed critical curve, denoted by $u_a(s)$, is an upper bound of $u_c(s)$ and verifies

$$u_a(s)=\log E\left(\exp(s\zeta_i)\right)=(1+o(1))s^2/2 \quad \text{when } s \text{ tends to } 0.$$

Moreover, $u_a(s)$ is equal to $s^2/2$ when $\zeta_i$ follows an $N(0,1)$ law.

In the last 20 years, there has been a lot of activity on this question, mostly from the physics side, and it is now widely believed that $u_c(s)$ behaves as $s^2/2$. But it is still an open question wether $u_c(s)=s^2/2$ (see [6]) for $s$ small or $u_c(s)<s^2/2$ for all $s$ (see [5] or [13]).

However, on the mathematics side the only rigorous fact that has been proved is in [1], where Alexander and Sidoravicius have studied a general class of random walks pinned either by an interface between two solvents or by an impenetrable wall. If we apply their results in our case, we obtain that the quenched quantity $u_c(s)$ is strictly larger than the non-disordered one $u_c(0)$. In this paper, we develop a new localization strategy, which allows us to go further, by giving a lower bound of $u_c(s)$ which has the same scale as the annealed upper bound for $s$ small (i.e. $-cs^2$ with $c>0$).

## 3. Critical curve

In this article, we are particularly interested in the critical curve of the system, namely, the curve that divides the ($h$,$\beta$)-plane into a localized and a delocalized phase. Before defining this curve precisely, it is helpful to consider the non-disordered case ($s=0$), which is easier to understand and gives a good intuition of what happens in the disordered case ($s\neq 0$).

### 3.1. **Non-disordered case (Proposition 1).**

Above the critical curve the system is delocalized, and below localized. In appendix B, we compute the equation of this curve when $s=0$. We obtain

$$\begin{aligned} h_c^0:\ & [0,\log 2)\to\mathbb{R} \\ & \beta\longrightarrow h_c^0(\beta)=-\frac{1}{4}\log\left(1-4\left(1-e^{-\beta}\right)^2\right). \end{aligned} \tag{3.1}$$

This curve is increasing, convex and tends to $\infty$ when $\beta$ tends to $\log 2$ from the left. When $\beta\geq\log 2$ the system is always localized. In fact, when $h$ is chosen large, the free energy is strictly positive. That is why this critical curve is only defined on $[0,\log 2)$ (see Fig 1).

Our first result concerns $s\neq 0$ and shows that the critical curve has a form that is qualitatively similar to (3.1).

**Proposition 1.** *For $s\geq 0$ and $\beta\geq 0$ the following properties are verified.*

i) *There exists* $h_c^s(\beta) \in [0, +\infty]$ *such that*

$$\Phi^s(\beta, h) > 0 \quad \textit{if} \quad h < h_c^s(\beta),$$
$$\Phi^s(\beta, h) = 0 \quad \textit{if} \quad h \geq h_c^s(\beta).$$

ii) *The function* $\beta \to h_c^s(\beta)$ *is convex and increasing.*

iii) *For* $s \geq 0$ *there exists* $\beta_0(s) \in (0, \infty]$ *such that* $h_c^s(\beta) < +\infty$ *when* $\beta < \beta_0(s)$ *and* $h_c^s(\beta) = +\infty$ *when* $\beta > \beta_0(s)$.

iv) *The non-disordered critical curve* $h_c^0(\beta)$ *is a lower bound for* $h_c^s(\beta)$.

v) $\beta_0(s) \leq \beta_0(0) = \log 2$.

**Remark 1.** The case $\beta = \beta_0(s)$ remains open. More precisely, two different behaviors of the curve may occur. Either $\lim_{\beta \to \beta_0^-(s)} h_c(\beta) = +\infty$, or there exists $h_0^s < \infty$ such that $\lim_{\beta \to \beta_0^-(s)} h_c(\beta) = h_0^s$. In the latter case, by continuity of $\Phi^s$ in $\beta$, we obtain $\Phi(\beta_0(s), h_0^s) = 0$ and $h_c(\beta_0(s)) = h_0^s$.

3.2. **Annealed case.** We obtain an upper bound of $h_c^s(\beta)$, as usual, by computing the annealed free energy. This is, by Jensen's inequality, an upper bound on the quenched free energy. The annealed system gives a critical curve ($\beta \to h_c^{an,s}(\beta)$), which is an upper bound on the quenched critical curve. The annealed free energy is given by

$$\Phi_{ann.}^s(h, \beta) = \lim_{N \to \infty} \frac{1}{N} \log E\mathbb{E}\left(\exp\left(\beta \sum_{i=1}^{N} (1 + s\zeta_i) \mathbf{1}_{\{S_i = 0\}} - 2h \sum_{i=1}^{N} \Delta_i\right)\right).$$

We integrate over $\mathbb{P}$ to obtain

$$\begin{aligned}\Phi_{ann.}^s(h, \beta) &= \lim_{N \to \infty} \frac{1}{N} \log E\left(\exp\left(\left(\beta + \log \mathbb{E}(e^{\beta s \zeta_1})\right) \sum_{i=1}^{N} \mathbf{1}_{\{S_i = 0\}} - 2h \sum_{i=1}^{N} \Delta_i\right)\right)\\ &= \Phi^0\big(h, \beta + \log \mathbb{E}(e^{\beta s \zeta_1})\big). \end{aligned} \tag{3.2}$$

Finally, we denote by $\beta_{an}^s$ the unique solution of $\beta + \log \mathbb{E}(e^{\beta s \zeta_1}) = \log 2$, and for $\beta \in [0, \beta_{an}^s)$ we obtain $h_c^{an,s}(\beta) = h_c^0\left(\beta + \log \mathbb{E}\left(e^{\beta s \zeta_1}\right)\right)$ (see Fig 1).

**Remark 2.** We notice that $h_c^{an,s}(\beta)$ and $h_c^0(\beta)$ are both equal to $\beta^2(1 + o(1))$ when $\beta$ tends to 0.

3.3. **Disordered model.** Up to now, three types of localization strategy have been used to find lower bounds on the quenched critical curve. The first one consists in computing the free energy on a particular subset of trajectories, i.e., trajectories that come back often to the interface ([2]). The second strategy consists in transforming (by using Radon-Nikodym derivatives) the law of the excursions out of the origin. Bolthausen and den Hollander have used this second method in [4], to constrain the chain to come back to the origin in a positive density of sites. Finally, in the same spirit as the work of Alexander and Sidoravicius ([1]), we use a third strategy which goes further than the former one, by making the chain choose, at each excursion, a law adapted to the local disorder.

Proposition 1 tells that $h_c^s(\beta) = \infty$ when $s \geq 0$ and $\beta \geq \log 2$. Therefore, the critical curve is not defined after $\log 2$. For this reason, we will only consider the case $\beta \leq \log 2$.

**Theorem 2.** *If $Var(\zeta_1) \in (0, \infty)$, then there exist $c_1 > 0$, $c_2 > 0$ such that, for every $s \le c_1$ and $\beta \in [0, \log 2 - c_2 s^2 \beta^2)$,*

$$h_c^s(\beta) \ge -\frac{1}{4} \log \left(1 - 4\left(1 - e^{-\beta - c_2 s^2 \beta^2}\right)^2\right) = m^s(\beta).$$

On Fig. 1 below, we draw the curves which we have mentioned up to now.

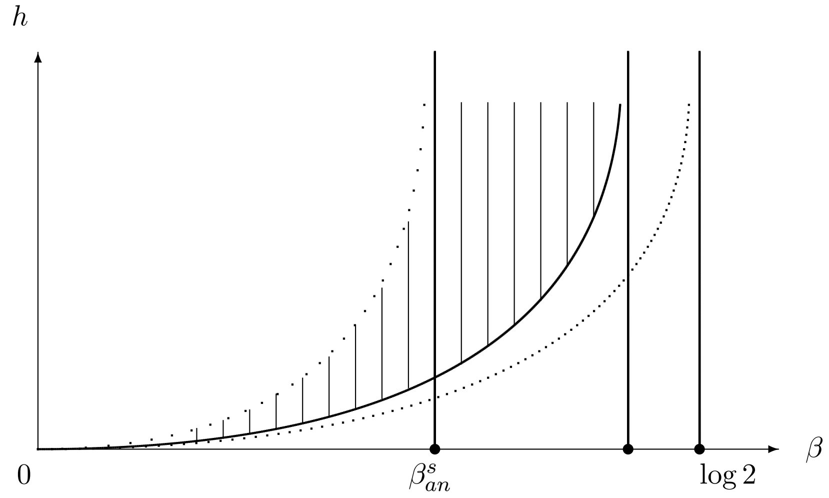


Fig. 1: . . . . . . . $h_c^{an,s}(\beta)$

$m^s(\beta)$ possible location of $h_c^s(\beta)$

............ $h_c^0(\beta)$

**Remark 3.** In the proof of Theorem 2, we restrict to $\mathbb{P}(\zeta_1 > 0) = 1/2$ and $\mathbb{E}\left(\zeta_1 \mathbf{1}_{\{\zeta_1 > 0\}}\right) = 1$. In this case, $c_1 = 1$ and $c_2 = 1/(5 \times 2^{14})$. With other conditions on $\mathbb{P}(\zeta_1 > 0)$ and $\mathbb{E}\left(\zeta_1 \mathbf{1}_{\{\zeta_1 > 0\}}\right)$, the constants $c_1$ and $c_2$ would have to be chosen differently, but the strategy to obtain the lower bound still works.

## 4. Pure pinning and wetting model

The pure pinning model is different from the previous one. The $h-$term is removed, and the rewards at the interface take the form $-u + s\zeta_i$ with $u \ge 0$. The corresponding hamiltonian is

$$H_{N,s}^{\zeta,u}(S) = \sum_{i=1}^{N} (-u + s\zeta_i) \mathbf{1}_{\{S_i = 0\}}.$$

The localization and delocalization conditions associated to the free energy remain the same. We obtain a critical $u$ denoted by $u_c(s)$, such that the system is localized when $u < u_c(s)$ and delocalized when $u \ge u_c(s)$.

For this model, the annealed model gives an upper bound on $u_c(s)$, denoted by $u_c^{an}(s)$. If $Var(\zeta_1) = 1$, then this annealed upper bound satisfies $u_c^{an}(s) = (1 + o(1))s^2/2$ when $s \to 0$. A corollary of Theorem 2 gives a lower bound on $u_c(s)$, which has the same scale (i.e., $cs^2$ as $s \to 0$).

**Corollary 3.** *If $Var(\zeta_1) \in (0, \infty)$, then there exist $c_3, c_4 > 0$ such that, for every $s \le c_3$,*

$$u_c(s) \ge c_4 s^2.$$

**Remark 4.** The values of $c_3$ and $c_4$ depend on the law of $\zeta_1$. In the proof of Corollary 3, we will consider the conditions of Remark 3 concerning $\zeta_1$. In this case, $c_3 = \log 2$ and $c_4 = 1/(5 \times 2^{16})$.

## 5. Proof of theorem and proposition

5.1. **Proof of Proposition** 1. The proof of parts i)-v) are given below.

**i)** For $\beta \ge 0$ and $s \ge 0$, let $J_\beta^s = \{h \ge 0 \,:\, \Phi^s(\beta, h) = 0\}$. Let $h_c^s(\beta)$ be the infimum of $J_\beta^s$. Recall that $\Phi$ is positive, continuous, and non-increasing in $h$. Hence, $J_\beta^s = [h_c^s(\beta), +\infty)$ and i) is proved.

**iii)** The function $\Phi$ is convex in $\beta$, positive, and $\Phi^s(0, h) = 0$ for every $h \ge 0$. Therefore, $\Phi$ is non-decreasing in $\beta$, and $h_c^s(\beta)$ is non-decreasing. If we define $\beta_0(s) = \sup\{\beta \ge 0 : J_\beta^s \neq \emptyset\}$, then the annealed computation gives $\beta_0(s) > 0$. Indeed, $J_{ann.\beta}^s \subset J_\beta^s$ because $\Phi^s(h, \beta) \le \Phi_{ann.}^s(h, \beta)$. Thus, $\beta_0(s) \ge \beta_{an}^s > 0$ and iii) is proved.

**iv)** We want to show that $h_c^s(\beta) \ge h_c^0(\beta)$ when $s \ge 0$. To that aim, we prove that $\Phi^s(\beta, h) > 0$ when $s \ge 0$, $\beta \ge 0$ and $h < h_c^0(\beta)$. For $\beta$ and $h$ fixed, $\Phi^s(\beta, h)$ is convex in $s$, because it is the limit as $N \to \infty$ of $\Phi_N^s(\beta, h) = \mathbb{E}\big(1/N \log E\big(\big(\exp\big(H_{N,\beta,h}^{\zeta,s}\big)\big)\big)$, which is convex in s. Moreover, for every $N > 0$, $\Phi_N^s(\beta, h)$ can be differentiated w.r.t. $s$. This gives

$$\frac{\partial \Phi_N^s(\beta, h)}{\partial s} = \frac{1}{N}\mathbb{E}\left(\frac{E\left(\beta \sum_{i=1}^N \zeta_i \mathbf{1}_{\{S_i=0\}} \exp\left(H_{N,\beta,h}^{\zeta,s}\right)\right)}{E\left(\exp\left(H_{N,\beta,h}^{\zeta,s}\right)\right)}\right).$$

But, when $s = 0$, the hamiltonian does not depend on the disorder $\zeta$. Therefore, by the Fubini-Tonelli Theorem and the fact that $\mathbb{E}(\zeta_i) = 0$, we can write

$$\left.\frac{\partial \Phi_N^s(\beta, h)}{\partial s}\right|_{s=0} = \frac{1}{N}\frac{E\left(\beta \sum_{i=1}^N \mathbb{E}(\zeta_i)\, \mathbf{1}_{\{S_i=0\}} \exp\left(H_{N,\beta,h}^{\zeta,0}\right)\right)}{E\left(\exp\left(H_{N,\beta,h}^{\zeta,0}\right)\right)} = 0.$$

Hence, the convergence of $\Phi_N$ towards $\Phi$ and their convexity allows us to write

$$\left.\frac{\partial_{right}\Phi^s(\beta, h)}{\partial s}\right|_{s=0} \ge \lim_{N\to\infty} \left.\frac{\partial_{right}\Phi_N^0(\beta, h)}{\partial s}\right|_{s=0} = 0.$$

Thus, by convexity in $s$, we can assert that $\Phi^s(\beta, h)$ is non-decreasing in $s$. Hence, $s \ge 0$ implies $\Phi^s(\beta, h) \ge \Phi^0(\beta, h) > 0$. That is why $h_c^s(\beta) \ge h_c^0(\beta)$, and iv) is verified.

**v)** This is a direct consequence of iv).

**ii)** We want to prove that $h_c^s(\beta)$ is convex, and therefore continuous on $[0,\beta_0(s))$. To prove convexity, we let $0<a<b$ and $\lambda\in[0,1]$. Then, since

$$H^{\zeta,s}_{N,\ \lambda a+(1-\lambda)b,\ \lambda h_c^s(a)+(1-\lambda)h_c^s(b)}=H^{\zeta,s}_{N,\ \lambda a,\ \lambda h_c^s(a)}+H^{\zeta,s}_{N,\ (1-\lambda)b,\ (1-\lambda)h_c^s(b)},$$

the Hölder inequality gives

$$\frac{1}{N}\log E\left(\exp\left(Z^{\zeta,s}_{N,\ \lambda(a,h_c^s(a))+(1-\lambda)(b,h_c^s(b))}\right)\right)\leq\frac{\lambda}{N}\log E\left(\exp\left(Z^{\zeta,s}_{N,\ a,\ h_c^s(a)}\right)\right)+\frac{1-\lambda}{N}\log E\left(\exp\left(Z^{\zeta,s}_{N,\ b,\ h_c^s(b)}\right)\right). \quad (5.1)$$

Therefore, if $N\to\infty$, the r.h.s. of (5.1) tends to zero, because, by continuity of $\Phi$ in $h$, $\Phi(a,h_c^s(a))=\Phi(b,h_c^s(b))=0$. Hence,

$$\Phi^s(\lambda a+(1-\lambda)b\,,\,\lambda h_c^s(a)+(1-\lambda)h_c^s(b))=0,$$

and

$$h_c^s(\lambda a+(1-\lambda)b)\leq\lambda h_c^s(a)+(1-\lambda)h_c^s(b).$$

This completes the proof of the first part of ii). To get the second part of ii), we show that $h_c^s(\beta)$ is increasing in $\beta$. Indeed, since $h_c^s(0)=0$ and $h_c^s(\beta)\geq h_c^0(\beta)>0$ for $\beta>0$, the convexity of $h_c^s(\beta)$ gives us the result.

5.2. **Proof of Theorem** 2. In the following we consider $h>0$, $\beta\leq\log 2$, $\mathbb{P}(\zeta_1>0)=1/2$, $\mathbb{E}\left(\zeta_1\mathbf{1}_{\{\zeta_1>0\}}\right)=1$ and $s\leq 1$.

**STEP I: Transformation of the excursion law.**

**Definition 4.** *From now on, we denote by $i_j$ the site of the $j^{th}$ return to the origin. Thus, $i_0=0$ and $i_j=\inf\{i>i_{j-1}:S_i=0\}$. Let $\tau_j$ be the length of the $j^{th}$ excursion away of the origin, i.e., $\tau_j=i_j-i_{j-1}$. Also, let $l_N$ be the number of returns to the origin before time $N$.*

By independence of the excursion signs, we can rewrite the partition function as

$$H_N=E\Bigg(\exp\left(\beta s\sum_{j=1}^{l_N}\zeta_{i_j}\right)\exp(\beta l_N)\prod_{j=1}^{l_N}\left(\frac{1+\exp(-2h\tau_j)}{2}\right)\times\frac{1+\exp\left(-2h\left(N-i_{l_N}\right)\right)}{2}\Bigg). \quad (5.2)$$

We want to transform the law of the excursions away of the origin to constrain the chain to come back to zero in a positive density of sites. For that, we introduce $P^{\beta}_{\alpha,h}$, the law of a homogeneous positive recurrent Markov process. Its excursions law is given by

$$\forall\, n\in\mathbb{N}\backslash\{0\}\qquad P^{\beta}_{\alpha,h}(\tau_1=2n)=\left(\frac{1+\exp(-4hn)}{2}\right)\alpha^{2n}\frac{P(\tau=2n)}{H^{\beta}_{\alpha,h}}\exp(\beta), \quad (5.3)$$

with

$$H^{\beta}_{\alpha,h}=\sum_{i=1}^{\infty}\frac{\exp(-4hi)+1}{2}e^{\beta}\alpha^{2i}P(\tau=2i)=e^{\beta}\Bigg(1-\frac{\sqrt{1-\alpha^2}+\sqrt{1-e^{-4h}\alpha^2}}{2}\Bigg). \quad (5.4)$$

We notice that the term inside the expectation of (5.2) only depends on $l_N$ and on the positions of the returns to the origin, i.e., $i_1, \dots, i_{l_N}$. Therefore, we can rewrite $H_N$ as an expectation under $P^{\beta}_{\alpha,h}$, because we know the Radon-Nikodym derivative $dP/dP^{\beta}_{\alpha,h}(\{i_1,\dots,i_{l_N}\})$. Hence, $H_N$ becomes

$$H_N = E^{\beta}_{\alpha,h}\left(\exp\left(\beta s \sum_{j=1}^{l_N} \zeta_{i_j}\right)\prod_{j=1}^{l_N} \frac{H^{\beta}_{\alpha,h}}{\alpha^{\tau_j}} \left(\frac{1+e^{-2h\left(N-i_{l_N}\right)}}{2}\right) \frac{P\left(\tau \geq N - i_{l_N}\right)}{P^{\beta}_{\alpha,h}\left(\tau \geq N - i_{l_N}\right)}\right).$$

Next we aim at transforming the excursion law again, so that the chain comes back more often in sites where the pinning reward is large. Indeed, we want the chain to take into account its local environment. For that, we define $P^{\beta,\zeta,\alpha_1}_{\alpha,h}$ the law of a non-homogenous Markov process, that depends on the environment. Its excursion law is defined as follow. Let

$$\alpha_1 < \frac{1-P^{\beta}_{\alpha,h}(\tau=2)}{P^{\beta}_{\alpha,h}(\tau=2)} \text{ such that } \mu_1 = 1-\alpha_1 \frac{P^{\beta}_{\alpha,h}\left(\tau=2\right)\big)}{\big(1-P^{\beta}_{\alpha,h}\left(\tau=2\right)\big)} > 0,$$

and let

$$\begin{aligned} P^{\beta,\zeta,\alpha_1}_{\alpha,h}\left(\tau=2\right) &= P^{\beta}_{\alpha,h}\left(\tau=2\right)\left(1+\alpha_1\right)^{\mathbf{1}_{\{\zeta_2>0\}}} \\ P^{\beta,\zeta,\alpha_1}_{\alpha,h}\left(\tau=2r\right) &= P^{\beta}_{\alpha,h}\left(\tau=2r\right)\mu_1^{\mathbf{1}_{\{\zeta_2>0\}}} \text{ for } r \geq 2. \end{aligned} \tag{5.5}$$

Under the law of this process, if the chain comes back to the origin at time i, then the law of the following excursion is $P^{\beta,\zeta_{i+\cdot},\alpha_1}_{\alpha,h}$. Thus, the chain checks whether the reward at time $i+2$ is positive or negative. If $\zeta_{i+2} \geq 0$, then the probability to come back to zero at time $i+2$ increases. Else it remains the same.

With this new process we can write

$$\begin{aligned} H_N = E^{\beta,\zeta,\alpha_1}_{\alpha,h}\Bigg(\exp\left(\beta s \sum_{j=1}^{l_N} \zeta_{i_j}\right)\prod_{j=1}^{l_N}\left(\frac{H^{\beta}_{\alpha,h}}{\alpha^{\tau_j}}\right)\left(\frac{1}{2}+\frac{e^{-2h\left(N-i_{l_N}\right)}}{2}\right) \\ \times \prod_{j=1}^{l_N}\left(\frac{P^{\beta}_{\alpha,h}\left(\tau_j\right)}{P^{\beta,\zeta_{i_{j-1}+\cdot},\alpha_1}_{\alpha,h}\left(\tau_j\right)}\right)\frac{P\left(\tau \geq N-i_{l_N}\right)}{P^{\beta,\zeta_{i_{l_N}+\cdot},\alpha_1}_{\alpha,h}\left(\tau \geq N-i_{l_N}\right)}\Bigg) \end{aligned}$$

$$\begin{aligned} \geq E^{\beta,\zeta,\alpha_1}_{\alpha,h}\Bigg(\exp\left(\beta s\sum_{j=1}^{l_N}\zeta_{i_j}\right)\left(H^{\beta}_{\alpha,h}\right)^{l_N} \\ \times \frac{1}{2}\prod_{j=1}^{l_N}\left(\frac{P^{\beta}_{\alpha,h}(\tau_j)}{P^{\beta,\zeta_{i_{j-1}+\cdot},\alpha_1}_{\alpha,h}\left(\tau_j\right)}\right)P\left(\tau \geq N-i_{l_N}\right)\Bigg). \end{aligned}$$

We apply Jensen's inequality to obtain

$$\begin{aligned}\mathbb{E}\left(\frac{1}{N}\log H_N\right) \geq& \frac{\beta s}{N}\ \mathbb{E}E_{\alpha,h}^{\beta,\zeta,\alpha_1}\left(\sum_{j=1}^{l_N}\zeta_{i_j}\right)+\log\left(H_{\alpha,h}^{\beta}\right)\mathbb{E}E_{\alpha,h}^{\beta,\zeta,\alpha_1}\left(\frac{l_N}{N}\right)+\frac{1}{N}\log\left(\frac{1}{2}\right)\\ &+\frac{1}{N}\mathbb{E}E_{\alpha,h}^{\beta,\zeta,\alpha_1}\left(\sum_{j=1}^{l_N}\log\left(\frac{P_{\alpha,h}^{\beta}(\tau_j)}{P_{\alpha,h}^{\beta,\zeta_{i_{j-1}+\cdot},\alpha_1}(\tau_j)}\right)\right)+\frac{\log\left(P\left(\tau\geq N\right)\right)}{N}.\end{aligned} \tag{5.6}$$

At this stage, we can divide the lower bound of (5.6) in two parts. The first part (called $E_1(N)$) is a positive energetic term, corresponding to the additional reward that the chain can expect by coming back often in "high reward" sites, namely,

$$E_1(N)=\frac{\beta s}{N}\ \mathbb{E}E_{\alpha,h}^{\beta,\zeta,\alpha_1}\left(\sum_{j=1}^{l_N}\zeta_{i_j}\right).$$

The second part (called $E_2(N)$) is a negative entropic term, because the measure transformations we performed have an entropic cost, namely,

$$\begin{aligned}E_2(N)=&\log\left(H_{\alpha,h}^{\beta}\right)\mathbb{E}E_{\alpha,h}^{\beta,\zeta,\alpha_1}\left(\frac{l_N}{N}\right)+\frac{1}{N}\log\left(\frac{1}{2}\right)\\ &+\frac{1}{N}\mathbb{E}E_{\alpha,h}^{\beta,\zeta,\alpha_1}\left(\sum_{j=1}^{l_N}\log\left(\frac{P_{\alpha,h}^{\beta}(\tau_j)}{P_{\alpha,h}^{\beta,\zeta_{i_{j-1}+\cdot},\alpha_1}(\tau_j)}\right)\right)+\frac{1}{N}\log\left(P\left(\tau\geq N\right)\right).\end{aligned}$$

**STEP II: Energy term computation.**

Notice that

$$\begin{aligned}\sum_{j=1}^{l_N}\zeta_{i_j}=&\sum_{i=0}^{N-2}\zeta_{i+2}\,\mathbf{1}_{\{S_i=0\}}\,\mathbf{1}_{\{S_{i+2}=0\}}\\ &+\sum_{k=3}^{N}\sum_{s=0}^{N-k}\zeta_{s+k}\,\mathbf{1}_{\{S_s=0\}}\,\mathbf{1}_{\{S_i\neq 0\ \forall\, i\in\{s+1,\dots,s+k-1\}\}}\mathbf{1}_{\{S_{s+k}=0\}}.\end{aligned} \tag{5.7}$$

Let $A=\sum_{i=0}^{N-2}\zeta_{i+2}\,\mathbf{1}_{\{S_i=0\}}\,\mathbf{1}_{\{S_{i+2}=0\}}$ and

$B=\sum_{k=3}^{N}\sum_{s=0}^{N-k}\zeta_{s+k}\,\mathbf{1}_{\{S_s=0\}}\,\mathbf{1}_{\{S_i\neq 0\ \forall\, i\in\{s+1,\dots,s+k-1\}\}}\mathbf{1}_{\{S_{s+k}=0\}}$.

We compute separately the contributions of $A$ and $B$. We begin with

$$\mathbb{E}E_{\alpha,h}^{\beta,\zeta,\alpha_1}(B)=\sum_{k=3}^{N}\sum_{s=0}^{N-k}\mathbb{E}E_{\alpha,h}^{\beta,\zeta,\alpha_1}\left(\zeta_{s+k}\,\mathbf{1}_{\{S_s=0\}}\,\mathbf{1}_{\{S_i\neq 0\ \forall\, i\in\{s+1,\dots,s+k-1\}\}}\mathbf{1}_{\{S_{s+k}=0\}}\right).$$

By the Markov property,

$$\begin{aligned}\mathbb{E}E_{\alpha,h}^{\beta,\zeta,\alpha_1}(B)=&\sum_{k=3}^{N}\sum_{s=0}^{N-k}\mathbb{E}\left(\mathbf{1}_{\{\zeta_{s+2}>0\}}E_{\alpha,h}^{\beta,\zeta,\alpha_1}\left(\mathbf{1}_{\{S_s=0\}}\right)P_{\alpha,h}^{\beta}(k)\,\mu_1\,\zeta_{s+k}\right)\\ &+\mathbb{E}\left(\mathbf{1}_{\{\zeta_{s+2}\leq 0\}}E_{\alpha,h}^{\beta,\zeta,\alpha_1}\left(\mathbf{1}_{\{S_s=0\}}\right)P_{\alpha,h}^{\beta}(k)\ \zeta_{s+k}\right).\end{aligned}$$

But $E^{\beta,\zeta,\alpha_1}_{\alpha,h}\left(\mathbf{1}_{\{S_s=0\}}\right)$ only depends on $\{\zeta_1,\zeta_2,\ldots,\zeta_s\}$, and the $\{\zeta_i\}_{i\geq 1}$ are independent and centered. For this reason, and since $k\geq 3$ we have $\mathbb{E}E^{\beta,\zeta,\alpha_1}_{\alpha,h}(B)=0$.

The contribution of part A in (5.7) is given by

$$\begin{aligned}\mathbb{E}E^{\beta,\zeta,\alpha_1}_{\alpha,h}(A) &= \sum_{i=0}^{N-2}\mathbb{E}\left(E^{\beta,\zeta,\alpha_1}_{\alpha,h}\left(\mathbf{1}_{\{S_i=0\}}\right)P^{\beta}_{\alpha,h}(2)\left(1+\alpha_1\right)\zeta_{i+2}\mathbf{1}_{\{\zeta_{i+2}>0\}}\right)\\ &\quad+\sum_{i=0}^{N-2}\mathbb{E}\left(E^{\beta,\zeta,\alpha_1}_{\alpha,h}\left(\mathbf{1}_{\{S_i=0\}}\right)P^{\beta}_{\alpha,h}(2)\,\zeta_{i+2}\mathbf{1}_{\{\zeta_{i+2}\leq 0\}}\right)\\ &=\alpha_1 P^{\beta}_{\alpha,h}(2)\,\mathbb{E}\left(\zeta_1\,\mathbf{1}_{\{\zeta_1\}>0}\right)\mathbb{E}E^{\beta,\zeta,\alpha_1}_{\alpha,h}\left(\sharp\{i\in\{0,\ldots,N-2\}:S_i=0\}\right).\end{aligned}$$

Therefore, the contribution of this energy term is

$$\begin{aligned}E_1(N)=&\beta s\alpha_1 P^{\beta}_{\alpha,h}(2)\,\frac{\mathbb{E}E^{\beta,\zeta,\alpha_1}_{\alpha,h}\left(\sharp\{i\in\{0,\ldots,N-2\}:S_i=0\}\right)}{N}\\ \geq&\beta s\alpha_1 P^{\beta}_{\alpha,h}(2)\,\frac{\mathbb{E}E^{\beta,\zeta,\alpha_1}_{\alpha,h}(l_N)}{N}.\end{aligned}\tag{5.8}$$

**STEP III: Computation of the entropic term.**

Notice that the terms $1/N\log(P(\tau\geq N))$ and $1/N\log(1/2)$ tend to 0 as $N\to\infty$, independently of all the other parameters. Hence, if we denote by $R_N$ the quantity $1/N\log\left(P\left(\tau\geq N\right)\right)+1/N\log\left(1/2\right)$, then we can write

$$E_2(N)=\frac{S_N}{N}+\log\left(H^{\beta}_{\alpha,h}\right)\mathbb{E}E^{\beta,\zeta,\alpha_1}_{\alpha,h}\left(\frac{l_N}{N}\right)+R_N,$$

with

$$S_N=\mathbb{E}E^{\beta,\zeta,\alpha_1}_{\alpha,h}\left(\sum_{j=1}^{l_N}\log\left(\frac{P^{\beta}_{\alpha,h}(\tau_j)}{P^{\beta,\zeta_{i_{j-1}+\cdot},\alpha_1}_{\alpha,h}(\tau_j)}\right)\right).\tag{5.9}$$

The definitions (5.3) and (5.5) of $P^{\beta,\zeta_{i_{j-1}+\cdot},\alpha_1}_{\alpha,h}$ and $P^{\beta}_{\alpha,h}$ immediately give

$$\begin{aligned}S_N=&-\mathbb{E}E^{\beta,\zeta,\alpha_1}_{\alpha,h}\left(\sum_{j=1}^{l_N}\mathbf{1}_{\{\zeta_{i_{j-1}+2}>0\}}\left(\mathbf{1}_{\{\tau_j=2\}}\log\left(1+\alpha_1\right)+\mathbf{1}_{\{\tau_j>2\}}\log\left(\mu_1\right)\right)\right)\\ =&-\sum_{i=0}^{N-2}\mathbb{E}\left(E^{\beta,\zeta,\alpha_1}_{\alpha,h}\left(\mathbf{1}_{\{S_i=0\}}\,\mathbf{1}_{\{S_{i+2}=0\}}\right)\mathbf{1}_{\{\zeta_{i+2}>0\}}\log\left(1+\alpha_1\right)\right)\\ &-\sum_{k=3}^{N}\sum_{s=0}^{N-k}\mathbb{E}\left(E^{\beta,\zeta,\alpha_1}_{\alpha,h}\left(\mathbf{1}_{\{S_s=0\}}\,\mathbf{1}_{\{S_i\neq 0\ \forall\, i\in\{s+1,\ldots,s+k-1\}\}}\,\mathbf{1}_{\{S_{s+k}=0\}}\right)\right.\\ &\hspace{20em}\left.\times\,\mathbf{1}_{\{\zeta_{s+2}>0\}}\log\left(\mu_1\right)\right).\end{aligned}$$

By the Markov property, we can write

$$\mathbf{1}_{\{\zeta_{i+2}>0\}}\,E^{\beta,\zeta,\alpha_1}_{\alpha,h}\left(\mathbf{1}_{\{S_i=0\}}\mathbf{1}_{\{S_{i+2}=0\}}\right)=\mathbf{1}_{\{\zeta_{i+2}>0\}}\,E^{\beta,\zeta,\alpha_1}_{\alpha,h}\left(\mathbf{1}_{\{S_i=0\}}\right)\left(1+\alpha_1\right)P^{\beta}_{\alpha,h}(2),$$

and we notice that $E^{\beta,\zeta,\alpha_1}_{\alpha,h}\left(\mathbf{1}_{\{S_i=0\}}\right)$ is independent of $\zeta_{i+2}$ and $\mathbb{P}\left(\zeta_{i+2}>0\right)=1/2$. Hence,

$$\begin{aligned}S_N=&-\frac{P^{\beta}_{\alpha,h}(2)}{2}(1+\alpha_1)\log(1+\alpha_1)\,\mathbb{E}E^{\beta,\zeta,\alpha_1}_{\alpha,h}(l_{N-2})\\&-\sum_{k=3}^{N}\frac{\mu_1\log(\mu_1)}{2}\,P^{\beta}_{\alpha,h}(k)\ \mathbb{E}E^{\beta,\zeta,\alpha_1}_{\alpha,h}(l_{N-k}).\end{aligned}$$

Finally, the entropic contribution is

$$\begin{aligned}E_2(N)=&\log\left(H^{\beta}_{\alpha,h}\right)\mathbb{E}E^{\beta,\zeta,\alpha_1}_{\alpha,h}\left(\frac{l_N}{N}\right)-\frac{P^{\beta}_{\alpha,h}(2)}{2}(1+\alpha_1)\log(1+\alpha_1)\,\mathbb{E}E^{\beta,\zeta,\alpha_1}_{\alpha,h}\left(\frac{l_{N-2}}{N}\right)\\&-\sum_{k=3}^{N}\frac{\mu_1\log(\mu_1)}{2}\,P^{\beta}_{\alpha,h}(k)\ \mathbb{E}E^{\beta,\zeta,\alpha_1}_{\alpha,h}\left(\frac{l_{N-k}}{N}\right)+R_N,\end{aligned}\tag{5.10}$$

and (5.8) and (5.10) give us a lower bound of formula (5.6) of the form

$$\mathbb{E}\Big(\frac{1}{N}\log(H_N)\Big)\geq E_1(N)+E_2(N).\tag{5.11}$$

**STEP IV: Estimation of $H^{\beta}_{\alpha,h}$ and choice of $\alpha$ and $\alpha_1$.**

Next we want to evaluate $H^{\beta}_{\alpha,h}$ with the expression of (5.4), namely,

$$H^{\beta}_{\alpha,h}=e^{\beta}\left(1-\frac{\sqrt{1-\alpha^2}+\sqrt{1-e^{-4h}\alpha^2}}{2}\right).$$

To compare $\log\left(H^{\beta}_{\alpha,h}\right)$ with the other terms of (5.11), we denote $\alpha^2=1-c\alpha_1^2$, with $c>0$ and $\sqrt{c}\alpha_1\leq 1$. In this way, we obtain

$$\begin{aligned}H^{\beta}_{\alpha,h}&=e^{\beta}\left(1-\frac{\sqrt{1-e^{-4h}}}{2}+\frac{\sqrt{1-e^{-4h}}-\sqrt{1-e^{-4h}\left(1-c\alpha_1^2\right)}-\sqrt{c}\alpha_1}{2}\right)\\&=e^{\beta}\left(1-\frac{\sqrt{1-e^{-4h}}}{2}\right)\left[1+\frac{\sqrt{1-e^{-4h}}\left(1-\sqrt{1+\frac{ce^{-4h}\alpha_1^2}{1-e^{-4h}}}\right)-\sqrt{c}\alpha_1}{2-\sqrt{1-e^{-4h}}}\right].\end{aligned}$$

Since $\sqrt{1+x}\leq 1+x/2$ for $x\in(-1,+\infty)$, and since $2-\sqrt{1-e^{-4h}}\geq 1$, we obtain

$$\log\left(H^{\beta}_{\alpha,h}\right)\geq\log\left(e^{\beta}\left(1-\frac{\sqrt{1-e^{-4h}}}{2}\right)\right)+\log\left(1-\sqrt{c}\alpha_1-\frac{c\alpha_1^2e^{-4h}}{2\sqrt{1-e^{-4h}}}\right).$$

As $\sqrt{c}\alpha_1\leq 1$, we can bound from above the term

$$\sqrt{c}\alpha_1+\frac{c\alpha_1^2e^{-4h}}{2\sqrt{1-e^{-4h}}}=\sqrt{c}\alpha_1\left(1+\frac{\sqrt{c}\alpha_1e^{-4h}}{2\sqrt{1-e^{-4h}}}\right)\leq\sqrt{c}\alpha_1\left(1+\frac{1}{2\sqrt{1-e^{-4h}}}\right).\tag{5.12}$$

To continue this computation, we need to choose precise values for $\alpha_1$ and $c$. That is why, recalling that $(\alpha^2=1-c\alpha_1^2)$, we denote

$$\alpha_1=\beta s/\left(5\times 2^8\right)\qquad\sqrt{c}=\beta s/\left(3\times 2^4\left(1+\frac{1}{2\sqrt{1-e^{-4h}}}\right)\right).\tag{5.13}$$

Notice that $\log(1-x) \geq -3x/2$ if $x \in [0,1/3]$, and since $\beta s \leq \log(2)$ the r.h.s. of (5.12) satisfies

$$\sqrt{c}\alpha_1\left(1+\frac{1}{2\sqrt{1-e^{-4h}}}\right) \leq \frac{\beta^2 s^2}{15\times 2^{12}} \leq \frac{1}{3}.$$

Hence $\log\left(H^{\beta}_{\alpha,h}\right)$ becomes

$$\begin{aligned}\log\left(H^{\beta}_{\alpha,h}\right) &\geq \log\left(e^{\beta}\left(1-\frac{\sqrt{1-e^{-4h}}}{2}\right)\right) - \frac{3}{2}\sqrt{c}\alpha_1\left(1+\frac{1}{2\sqrt{1-e^{-4h}}}\right)\\ &\geq \log\left(e^{\beta}\left(1-\frac{\sqrt{1-e^{-4h}}}{2}\right)\right) - \frac{\beta^2 s^2}{5\times 2^{13}}.\end{aligned}$$

Then, since $\log(1+\alpha_1) \leq \alpha_1$, we can rewrite (5.6) as

$$\begin{aligned}\mathbb{E}\left(\frac{1}{N}\log\left(H_N\right)\right) \geq &\left[\beta s\alpha_1 P^{\beta}_{\alpha,h}\left(2\right) - \frac{1}{2}P^{\beta}_{\alpha,h}\left(2\right)\left(1+\alpha_1\right)\alpha_1\right.\\ &\left.+\log\left(e^{\beta}\left(1-\frac{\sqrt{1-e^{-4h}}}{2}\right)\right) - \frac{\beta^2 s^2}{5\times 2^{13}}\right]\mathbb{E}\left(E^{\beta,\zeta,\alpha_1}_{\alpha,h}\left(\frac{l_N}{N}\right)\right)\\ &-\sum_{k=3}^{N} P^{\beta}_{\alpha,h}\left(k\right)\frac{\mu_1\log\left(\mu_1\right)}{2}\mathbb{E}\left(E^{\beta,\zeta,\alpha_1}_{\alpha,h}\left(\frac{l_{N-k}}{N}\right)\right) + R_N. \qquad (5.14)\end{aligned}$$

**STEP V: Intermediate computation.**

In the following steps, we need some inequalities on $P^{\beta}_{\alpha,h}$ and $H^{\beta}_{\alpha,h}$. As $\beta s \leq \log 2$, the equations in (5.13) show that $\alpha_1\sqrt{c} \in [0,1/4]$. Therefore, $\alpha^2 = 1 - c\alpha_1^2 \geq 1 - 1/2^4 \geq 3/4$, and we can bound from above and below the quantity $H^{\beta}_{\alpha,h}$ (introduced in (5.4))

$$e^{\beta} \geq H^{\beta}_{\alpha,h} \geq e^{\beta}\left(1-\frac{\sqrt{c}\alpha_1}{2}-\frac{1}{2}\right) \geq \frac{3e^{\beta}}{8}.$$

At this stage, we need to bound from above and below the quantity $P^{\beta}_{\alpha,h}\left(2\right)$, which has been defined in (5.3). With the previous inequalities, we have $e^{\beta}/H^{\beta}_{\alpha,h} \geq 1$ and $\sqrt{1-\alpha^2} \leq 1/4$. Thus,

$$\begin{aligned}P^{\beta}_{\alpha,h}\left(2\right) = 1 - \sum_{i=2}^{\infty} P^{\beta}_{\alpha,h}\left(2i\right) &\leq 1 - \sum_{i=2}^{\infty}\frac{1}{2}\alpha^{2i}P(\tau=2i)\\ &= 1 - \frac{1}{2}\left(1-\sqrt{1-\alpha^2}-\frac{\alpha^2}{2}\right) \leq \frac{7}{8}, \qquad (5.15)\end{aligned}$$

and

$$\frac{1}{8} = \frac{1}{4}\times\frac{e^{\beta}}{2e^{\beta}} \leq P^{\beta}_{\alpha,h}\left(2\right). \qquad (5.16)$$

Finally, with (5.15) and (5.16), we notice that

$$\frac{1}{8} \leq 1 - P^{\beta}_{\alpha,h}\left(2\right) \quad \text{and} \quad \frac{1}{7} \leq \frac{P^{\beta}_{\alpha,h}\left(2\right)}{1-P^{\beta}_{\alpha,h}\left(2\right)} \leq 7. \qquad (5.17)$$

Hence, the condition $\alpha_1 < P^{\beta}_{\alpha,h}\left(\tau=2\right)/\left(1-P^{\beta}_{\alpha,h}\left(\tau=2\right)\right)$ is obviously satisfied.

**STEP VI: Conclusion**

In (5.14), we still have to calculate the term

$$\sum_{k=3}^{N} P_{\alpha,h}^{\beta}(k)\,\mathbb{E}\left(E_{\alpha,h}^{\beta,\zeta,\alpha_1}\left(\frac{l_{N-k}}{N}\right)\right).$$

If $N \geq N_0$, then

$$\begin{aligned}\sum_{k=3}^{N} P_{\alpha,h}^{\beta}(k)\,\mathbb{E}\left(E_{\alpha,h}^{\beta,\zeta,\alpha_1}\left(\frac{l_{N-k}}{N}\right)\right) \geq & P_{\alpha,h}^{\beta}(\{3,\dots,N_0\})\mathbb{E}E_{\alpha,h}^{\beta,\zeta,\alpha_1}\left(\frac{l_{N-N_0}}{N}\right)\\ \geq & \left(1-P_{\alpha,h}^{\beta}(2)\right)\mathbb{E}E_{\alpha,h}^{\beta,\zeta,\alpha_1}\left(\frac{l_N}{N}\right)-\frac{N_0}{N}\\ & -P_{\alpha,h}^{\beta}(\{N_0+1,\dots,\infty\})\,\mathbb{E}E_{\alpha,h}^{\beta,\zeta,\alpha_1}\left(\frac{l_N}{N}\right),\end{aligned}$$

and equation (5.14) becomes

$$\begin{aligned}\mathbb{E}\left(\frac{1}{N}\log(H_N)\right) \geq & \Bigg[\beta s\alpha_1 P_{\alpha,h}^{\beta}(2)-\frac{1}{2}P_{\alpha,h}^{\beta}(2)(1+\alpha_1)\,\alpha_1-\frac{\beta^2 s^2}{5\times 2^{13}}\\ & +\log\left(e^{\beta}\left(1-\frac{\sqrt{1-e^{-4h}}}{2}\right)\right)-\left(1-P_{\alpha,h}^{\beta}(2)\right)\frac{\mu_1\log(\mu_1)}{2}\\ & +P_{\alpha,h}^{\beta}(\{N_0+1,\dots,\infty\})\,\frac{\mu_1\log(\mu_1)}{2}\Bigg]\mathbb{E}\left(E_{\alpha,h}^{\beta,\zeta,\alpha_1}\left(\frac{l_N}{N}\right)\right)\\ & +\frac{N_0}{N}\frac{\mu_1\log(\mu_1)}{2}+R_N. \end{aligned}\tag{5.18}$$

With (5.13) and (5.16), we can now bound from below

$$\beta s\alpha_1 P_{\alpha,h}^{\beta}(2) \geq \frac{\beta s}{2^3}\frac{\beta s}{5\times 2^8}=\frac{\beta^2 s^2}{5\times 2^{11}}.$$

Moreover, $\mu_1 = 1-(\alpha_1 P_{\alpha,h}^{\beta}(2)/\big(1-P_{\alpha,h}^{\beta}(2)\big)$ and $-\log(1-x)\geq x$ for $x\in[0,1)$. Therefore, we obtain

$$-\frac{1-P_{\alpha,h}^{\beta}(2)}{2}\mu_1\log(\mu_1) \geq \frac{\alpha_1 P_{\alpha,h}^{\beta}(2)}{2}-\frac{\alpha_1^2 P_{\alpha,h}^{\beta}(2)^2}{2\big(1-P_{\alpha,h}^{\beta}(2)\big)}.$$

In (5.16) and (5.17) we had $P_{\alpha,h}^{\beta}(2)\leq 7/8$ and $P_{\alpha,h}^{\beta}(2)/\big(2\big(1-P_{\alpha,h}^{\beta}(2)\big)\big)\leq 7/2$. Therefore,

$$-\frac{1-P_{\alpha,h}^{\beta}(2)}{2}\mu_1\,\log\mu_1 \;\geq\; \frac{\alpha_1 P_{\alpha,h}^{\beta}(2)}{2}-\frac{7^2\alpha_1^2}{2^4}\geq\frac{\alpha_1 P_{\alpha,h}^{\beta}(2)}{2}-4\alpha_1^2.$$

In that way, the inequality in (5.18) can be written as

$$\begin{aligned}\mathbb{E}\left(\frac{1}{N}\log\left(H_N\right)\right) \geq &\left[\frac{\beta^2 s^2}{5\times 2^{12}} - \frac{1}{2}P^{\beta}_{\alpha,h}\left(2\right)\left(1+\alpha_1\right)\alpha_1 + \frac{\alpha_1 P^{\beta}_{\alpha,h}\left(2\right)}{2} - 4\alpha_1^2\right.\\ &+ \log\left(e^{\beta}\left(1-\frac{\sqrt{1-e^{-4h}}}{2}\right)\right)\\ &\left.+ P^{\beta}_{\alpha,h}\left(\left\{N_0+1,\dots,\infty\right\}\right)\frac{\mu_1\log\left(\mu_1\right)}{2}\right]\mathbb{E}\left(E^{\beta,\zeta,\alpha_1}_{\alpha,h}\left(\frac{l_N}{N}\right)\right)\\ &+ \frac{N_0}{N}\mu_1\log\mu_1 + R_N. \qquad (5.19)\end{aligned}$$

By (5.17) and (5.16), we know that $P^{\beta}_{\alpha,h}(2) \leq 7/8$ and $P^{\beta}_{\alpha,h}\left(2\right)/\left(1-P^{\beta}_{\alpha,h}\left(2\right)\right) \leq 7$. Thus, we have the inequalities

$$-\frac{1}{2}P^{\beta}_{\alpha,h}\left(2\right)\left(1+\alpha_1\right)\alpha_1 + \frac{\alpha_1 P^{\beta}_{\alpha,h}\left(2\right)}{2} - 4\alpha_1^2 \geq -5\alpha_1^2 \geq -\frac{\beta^2 s^2}{5\times 2^{16}}, \qquad (5.20)$$

$$\text{and} \qquad \frac{\alpha_1 P^{\beta}_{\alpha,h}\left(2\right)}{1-P^{\beta}_{\alpha,h}\left(2\right)} \leq 7\alpha_1 = \frac{7\beta s}{5\times 2^8} < \frac{1}{3}.$$

Since $\mu_1 \leq 1$ and $\log\left(1-x\right) \geq -3x/2$ for $x \in [0,1/3]$, the second inequality of (5.20) allows us to bound from below

$$\mu_1\log\mu_1 \geq -\frac{3}{2}\,\frac{P^{\beta}_{\alpha,h}\left(2\right)}{1-P^{\beta}_{\alpha,h}\left(2\right)}\,\alpha_1 \geq -\frac{21\beta s}{5\times 2^9} \geq -1.$$

Then, (5.19) becomes

$$\begin{aligned}\mathbb{E}\left(\frac{1}{N}\log\left(H_N\right)\right) \geq &\left[\frac{\beta^2 s^2}{5\times 2^{13}} + \log\left(e^{\beta}\left(1-\frac{\sqrt{1-e^{-4h}}}{2}\right)\right)\right.\\ &\left.- P^{\beta}_{\alpha,h}\left(\left\{N_0+1,\dots,\infty\right\}\right)\right]\mathbb{E}\left(E^{\beta,\zeta,\alpha_1}_{\alpha,h}\left(\frac{l_N}{N}\right)\right) - \frac{N_0}{N} + R_N. \qquad (5.21)\end{aligned}$$

As proved in Appendix A.1, $P^{\beta}_{\alpha,h}\left(\left\{N_0+1,\dots,\infty\right\}\right)$ tends to zero as $N_0 \to \infty$, independently of $h \geq 0$. Therefore, for $N_0$ large enough and for all $h > 0$,

$$P^{\beta}_{\alpha,h}\left(\left\{N_0+1,\dots,\infty\right\}\right) \leq \frac{\beta^2 s^2}{5\times 2^{14}}.$$

If we denote $q\left(s\right) = \frac{\beta^2 s^2}{5\times 2^{14}}$, then, for $N \geq N_0$ and $h > 0$, (5.21) gives

$$\mathbb{E}\left(\frac{1}{N}\log\left(H_N\right)\right) \geq \left[q\left(s\right) + \log\left(e^{\beta}\left(1-\frac{\sqrt{1-e^{-4h}}}{2}\right)\right)\right]\mathbb{E}\left(E^{\beta,\zeta,\alpha_1}_{\alpha,h}\left(\frac{l_N}{N}\right)\right) + R^{N_0}_N \qquad (5.22)$$

with $R^{N_0}_N = R_N - N_0/N$. As proved in appendix A.2, $\mathbb{E}\left(E^{\beta,\zeta,\alpha_1}_{\alpha,h}\left(l_N/N\right)\right) \geq \mathbb{E}\left(E^{\beta}_{\alpha,h}\left(l_N/N\right)\right)$ for every $N \geq 1$. If we denote by $h_0\left(\beta\right)$ the only solution of

$$\log\left(e^{\beta}\left(1-\sqrt{1-e^{-4h_o(\beta)}}/2\right)\right) = -q\left(s\right),$$

then, for every $h < h_0(\beta)$ and $N \geq N_0$, we have

$$\mathbb{E}\left(\frac{1}{N}\log\left(H_N\right)\right) \geq \left[q\left(s\right) + \log\left(e^{\beta}\left(1 - \frac{\sqrt{1-e^{-4h}}}{2}\right)\right)\right] \mathbb{E}\left(E^{\beta}_{\alpha,h}\left(\frac{l_N}{N}\right)\right) + R^{N_0}_N.$$

Consequently,

$$\Phi^s\left(\beta, h\right) \geq \left[q\left(s\right) + \log\left(e^{\beta}\left(1 - \frac{\sqrt{1-e^{-4h}}}{2}\right)\right)\right] \times \liminf_{N\to\infty} \mathbb{E}\left(E^{\beta}_{\alpha,h}\left(\frac{l_N}{N}\right)\right).$$

Notice also that $\liminf_{N\to\infty} \mathbb{E}\big(E^{\beta}_{\alpha,h}\left(l_N/N\right)\big) > 0$ (because $\alpha \in (0,1)$). Hence, for every $\beta$ in $[0, \log(2) - q_s)$, $h_0(\beta)$ is a lower bound for $h_c(\beta)$, namely,

$$h_c\left(\beta\right) \geq h_0\left(\beta\right) = -\frac{1}{4}\log\left(1 - 4\left(1 - e^{-\beta - q(s)}\right)^2\right).$$

This completes the proof of Theorem 2.

**Remark 5.** The precise value of $c_2 = 1/\left(5 \times 2^{14}\right)$ could certainly be improved, by using more complicated laws of return to the origin. For instance, some laws that depend more deeply on the environment (by taking into account $\zeta_{i+2}, \zeta_{i+4}$, etc.). However, the computations would be more complicated, and our aim here is not to optimize the value of $c_1, c_2$ but rather to expose a simple strategy that improves the non-disordered lower bound of a term $cs^2\beta^2$ with $c > 0$.

5.3. **Proof of Corollary 3.** As shown just before in (5.22), there exists $N_0 \in \mathbb{N} \setminus \{0\}$ such that, for $h > 0$ and $N \geq N_0$,

$$\mathbb{E}\left(\frac{1}{N}\log E\left(\exp\left(\beta\sum_{i=1}^{N}\mathbf{1}_{\{S_i=0\}}\left(s\zeta_i + 1\right) - 2h\sum_{i=1}^{N}\Delta_i\right)\right)\right)$$
$$\geq \left[\frac{\beta^2 s^2}{5\times 2^{14}} + \log\left(e^{\beta}\left(1 - \frac{\sqrt{1-e^{-4h}}}{2}\right)\right)\right]\mathbb{E}\left(E^{\beta,\zeta,\alpha_1}_{\alpha,h}\left(\frac{l_N}{N}\right)\right) + R^{N_0}_N.$$

Moreover, in appendix A.2, we prove the following inequalities:

$$\mathbb{E}\left(E^{\beta,\zeta,\alpha_1}_{\alpha,h}\left(\frac{l_N}{N}\right)\right) \geq \mathbb{E}\left(E^{\beta}_{\alpha,h}\left(\frac{l_N}{N}\right)\right) \geq \mathbb{E}\left(E^{0}_{\alpha,\infty}\left(\frac{l_N}{N}\right)\right) > 0. \tag{5.23}$$

Thus, for $\beta, s$ and $N$ fixed, we let $h \to \infty$ and obtain

$$\mathbb{E}\left(\frac{1}{N}\log E\left(\exp\left(\beta\sum_{i=1}^{N}\mathbf{1}_{\{S_i=0\}}\left(s\zeta_i + 1\right)\right)\mathbf{1}_{\{S_i\geq 0, \forall\, i\in\{1,\dots,N\}\}}\right)\right)$$
$$\geq \left[\frac{\beta^2 s^2}{5\times 2^{14}} + \log\left(e^{\beta}\frac{1}{2}\right)\right]\mathbb{E}\left(E^{0}_{\alpha,\infty}\left(\frac{l_N}{N}\right)\right) + R^{N_0}_N.$$

Since $P\left(\{S_i \geq 0, \forall\, i \in \{1, \ldots, N\}\}\right) = (1 + o(1))\, D/N^{1/2}$ when $N \to \infty$ (with $D > 0$), the lower bound becomes

$$\mathbb{E}\left(\frac{1}{N}\log E\left(\exp\left(\beta\sum_{i=1}^{N}\mathbf{1}_{\{S_i=0\}}\left(s\zeta_i+1\right)\right)\middle|\{S_i \geq 0, \forall\, i \in \{1,\ldots,N\}\}\right)\right)$$
$$\geq \left[\frac{\beta^2 s^2}{5\times 2^{14}} + \log\left(e^{\beta}\frac{1}{2}\right)\right]\mathbb{E}\left(E^0_{\alpha,\infty}\left(\frac{l_N}{N}\right)\right) + K_N^{N_0}$$

with $K_N^{N_0} = R_N^{N_0} - 1/N \log\left(P\left(\{S_i \geq 0, \forall\, i \in \{1, \ldots, N\}\}\right)\right)$, so that it tends to 0 as $N \to \infty$ independently of all the other parameters. By [10], we can apply the fact that, for an odd number of steps, the random walk conditioned to stay positive, and pinned by $\log 2$ along the $x$ axis, becomes the reflected random walk. Indeed,

$$\frac{P_{\text{refl.RW}}}{P_{\text{RWcond.to be}\geq 0}}(S) = \frac{\exp\left((\log 2)\sum_{i=1}^{2N+1}\mathbf{1}_{\{S_i=0\}}\quad \mathbf{1}_{\{S_i\geq 0\ \forall\, i\in\{0,2N+1\}\}}\right)}{V_{2N+1}}.$$

The term $\frac{1}{N}\log V_N$ tends to 0 as $N \to \infty$. Hence, we denote $\beta = \log 2 - u$, and we obtain

$$\mathbb{E}\left(\frac{1}{2N+1}\log E\left(\exp\left(\log(2)\sum_{i=1}^{2N+1}\mathbf{1}_{\{S_i=0\}} + \sum_{i=1}^{2N+1}\mathbf{1}_{\{S_i=0\}}(-u+\beta s\zeta_i)\right)\right|\right.$$
$$\left.\left.\{S_i \geq 0, \forall\, i \leq 2N+1\}\right)\right) \geq \left[\frac{\beta^2 s^2}{5\times 2^{14}} - u\right]\mathbb{E}\left(E^0_{\alpha,\infty}\left(\frac{l_{2N+1}}{2N+1}\right)\right) + K_{2N+1}^{N_0}$$

and

$$\mathbb{E}\left(\frac{1}{2N+1}\log E\left(\exp\left(\sum_{i=1}^{2N+1}\mathbf{1}_{\{S_i=0\}}(-u+\beta s\zeta_i)\right)\right)\right) \geq$$
$$\left[\frac{\beta^2 s^2}{5\times 2^{14}} - u\right]\mathbb{E}\left(E^0_{\alpha,\infty}\left(\frac{l_{2N+1}}{2N+1}\right)\right) + K_{2N+1}^{N_0} + \frac{1}{2N+1}\log V_{2N+1}.$$

Let $N \to \infty$, and recall that $\beta = \log(2) - u$. Then

$$\lim_{N\to\infty}\mathbb{E}\left(\frac{1}{N}\log E\left(\exp\left(\sum_{i=1}^{N}\mathbf{1}_{\{S_i=0\}}(-u+\beta s\zeta_i)\right)\right)\right) \geq$$
$$\left[\frac{\beta^2 s^2}{5\times 2^{14}} - u\right]\lim_{N\to\infty}E^0_{\alpha,\infty}\left(\frac{l_N}{N}\right),$$

and, for $u \leq \log(2)/2$ (i.e., $\beta \geq (\log 2)/2$), we have

$$\lim_{N\to\infty}\mathbb{E}\left(\frac{1}{N}\log E\left(\exp\left(\sum_{i=1}^{N}\mathbf{1}_{\{S_i=0\}}(-u+\beta s\zeta_i)\right)\right)\right) \geq$$
$$\left[\frac{\log(2)^2 s^2}{5\times 2^{16}} - u\right]\lim_{N\to\infty}E^0_{\alpha,\infty}\left(\frac{l_N}{N}\right).$$

By convexity, the free energy $\Phi$, defined by

$$\Phi(u,v) = \lim_{N\to\infty} \mathbb{E}\left(\frac{1}{N}\log E\left(\exp\left(\sum_{i=1}^{N} \mathbf{1}_{\{S_i=0\}}(-u+v\zeta_i)\right)\right)\right),$$

is not decreasing in $v$. Therefore,

$$\Phi(u,\log(2)s) \geq \left[\frac{\log(2)^2 s^2}{5\times 2^{16}} - u\right] \lim_{N\to\infty} E^0_{\alpha,\infty}\left(\frac{l_N}{N}\right),$$

and, for $s\in[0,\log 2]$,

$$u_c(s) \geq \frac{s^2}{5\times 2^{16}}.$$

## 6. Appendix

6.1. **A.1.** We have to prove that $P^{\beta}_{\alpha,h}(\{N_0,\dots,+\infty\})$ tends to 0 as $N_0\to\infty$ independently of $h\geq 0$. To that aim, we bound the quantity in (5.3) as follows:

$$\begin{aligned} P^{\beta}_{\alpha,h}(\tau_1=2n) &= \left(\frac{1+\exp(-4hn)}{2}\right)\alpha^{2n}\frac{P(\tau=2n)}{H^{\beta}_{\alpha,h}}\exp(\beta) \\ &\leq \frac{\alpha^{2n}P(\tau=2n)}{\sum_{j=1}^{+\infty}\frac{1}{2}\alpha^{2j}P(\tau=2j)}. \end{aligned}$$

The r.h.s. of this inequality does not depend on $h$, and is the general term of a convergent series. Hence, we have uniform convergence in $h$.

6.2. **A.2.** We want to prove the inequalities of (5.23), i.e.,

$$\mathbb{E}\left(E^{\beta,\zeta,\alpha_1}_{\alpha,h}\left(\frac{l_N}{N}\right)\right) \geq \mathbb{E}\left(E^{\beta}_{\alpha,h}\left(\frac{l_N}{N}\right)\right) \geq \mathbb{E}\left(E^{0}_{\alpha,\infty}\left(\frac{l_N}{N}\right)\right). \tag{6.1}$$

For that, we recall a coupling theorem (see [14] or [15]):

**Theorem 5.** *$\mu_1$ and $\mu_2$ are two probability measures on $2\mathbb{N}\setminus\{0\}$. If, for every bounded and non-decreasing function $f$ defined on $2\mathbb{N}\setminus\{0\}$, $\mu_1(f)\leq\mu_2(f)$, then we define on the same probability space $(\Omega,P)$ two random variables $(T_1,T_2)$ of law $(\mu_1,\mu_2)$ such that, $P$ almost surely, $T_1\leq T_2$.*

**Remark 6.** We notice that, to satisfy the hypothesis of the theorem, it is enough to show that there exists an integer $i_0$ such that, $\mu_1(2i)\geq\mu_2(2i)$ for every $i\geq i_0$ and $\mu_1(2i)\leq\mu_2(2i)$ for every $i\geq i_0+1$. We can prove this easily by writing

$$\mu_2(f)-\mu_1(f) = \sum_{i=1}^{i_0}(\mu_2(2i)-\mu_1(2i))f(2i) + \sum_{i=i_0+1}^{\infty}(\mu_2(2i)-\mu_1(2i))f(2i).$$

As $f$ is non-decreasing, $f(2i)\geq f(2i_0)$ for every $i\geq i_0+1$, and $f(2i)\leq f(2i_0)$ for every $i\leq i_0$. Moreover, since $\mu_2(2i)-\mu_1(2i)$ is positive when $i\geq i_0+1$ and negative otherwise, we have the inequality

$$\begin{aligned}\mu_2(f)-\mu_1(f) &\geq f(2i_0)\sum_{i=1}^{i_0}\mu_2(2i)-\mu_1(2i)\ +f(2i_0)\sum_{i=i_0+1}^{\infty}\mu_2(2i)-\mu_1(2i)\\ &\geq -f(2i_0)\ (\mu_1-\mu_2)(\{2,\ldots,2i_0\})\\ &\qquad\qquad +f(2i_0)\ (\mu_2-\mu_1)(\{2(i_0+1),\ldots,\infty\}).\end{aligned}$$

Since $(\mu_2-\mu_1)\left(\{2(i_0+1),\ldots,\infty\}\right)=-(\mu_2-\mu_1)\left(\{2,\ldots,2i_0\}\right)$, we obtain

$$\mu_2(f)-\mu_1(f)\geq -f(2i_0)(\mu_1-\mu_2)\left(\{2,\ldots,2i_0\}\right)+f(2i_0)(\mu_1-\mu_2)\left(\{2,\ldots,2i_0\}\right)\geq 0.$$

This is why we can use Theorem 5 in this situation.

We want to apply this remark to the following probability measures on $2\mathbb{N}\setminus\{0\}$: $P^{0}_{\alpha,\infty}$, $P^{\beta}_{\alpha,h}$ and $P^{\beta,+,\alpha_1}_{\alpha,h}$, which is the law defined in (5.5) when $\zeta_2\geq 0$. For that, we compare $P^{\beta}_{\alpha,h}$ and $P^{\beta,+,\alpha_1}_{\alpha,h}$, which is easy because

$$\begin{aligned}P^{\beta,+,\alpha_1}_{\alpha,h}(\tau=2)&=P^{\beta}_{\alpha,h}(\tau=2)\,(1+\alpha_1)\\ P^{\beta,+,\alpha_1}_{\alpha,h}(\tau=2r)&=P^{\beta}_{\alpha,h}(\tau=2r)\,\mu_1\ \text{ for } r>2.\end{aligned}$$

Since $\alpha_1>0$ and $\mu_1<1$, we have the inequalities $P^{\beta,+,\alpha_1}_{\alpha,h}(\tau=2)>P^{\beta}_{\alpha,h}(\tau=2)$ and $P^{\beta,+,\alpha_1}_{\alpha,h}(\tau=2r)<P^{\beta}_{\alpha,h}(\tau=2r)$ for $r\geq 2$. Thus, Remark 6 tells us that we can use Theorem 5 and define on a probability space $(\Omega,P)$ a sequence of i.i.d. random variables $\left(T^1_i,T^2_i\right)_{i\geq 1}$ such that

- $P^{\beta,+,\alpha_1}_{\alpha,h}$ is the law of $T^1_i$ for every $i\geq 1$,
- $P^{\beta}_{\alpha,h}$ the law of $T^2_i$ for every $i\geq 1$,
- $P$ almost surely $T^1_i\leq T^2_i$ for every $i\geq 1$.

At this stage, for every fixed disorder $\zeta$, we define by recurrence another process $(T^3_i)_{i\geq 1}$ with

$$\begin{aligned}T^3_i&=T^2_i\quad\text{if}\quad \zeta_{T^3_1+\cdots+T^3_{i-1}+2}\geq 0\\ &=T^1_i\quad\text{if}\quad \zeta_{T^3_1+\cdots+T^3_{i-1}+2}<0.\end{aligned}$$

With these notations, $\left(T^2_i\right)_{i\geq 1}$ is the sequence of the excursion lengths of a random walk under the law $P^{\beta}_{\alpha,h}$, and $\left(T^3_i\right)_{i\geq 1}$ the one of a random walk under the law $P^{\beta,\zeta,\alpha_1}_{\alpha,h}$. By construction, $T^3_i\leq T^2_i$ for every $i\geq 1$. Thus, for $j=$ 2 or 3, we note $l^j_N=\max\{s\geq 1: T^j_1+\cdots+T^j_s\leq N\}$, and we have immediately that $l^3_N\geq l^2_N$ $P$ almost surely. Therefore, for every $\zeta$, we have

$$E^{\beta,\zeta,\alpha_1}_{\alpha,h}\left(\frac{l_N}{N}\right)=E_P\left(\frac{l^3_N}{N}\right)\geq E_P\left(\frac{l^2_N}{N}\right)=E^{\beta}_{\alpha,h}\left(\frac{l_N}{N}\right),$$

and, by integration with respect to $\zeta$, we obtain the l.h.s. of inequality (6.1).

To finish with these inequalities, we must show that the same argument allow us to compare $\mathbb{E}\left(E^{\beta}_{\alpha,h}\left(\frac{l_N}{N}\right)\right)$ and $\mathbb{E}\left(E^{0}_{\alpha,\infty}\left(\frac{l_N}{N}\right)\right)$. Indeed, we want to prove that Remark 6 also occurs. Recall that

$$P^{\beta}_{\alpha,h}(\tau_1 = 2n) = \left(\frac{1+\exp(-4hn)}{2}\right)\alpha^{2n}\frac{P(\tau = 2n)}{H^{\beta}_{\alpha,h}}\exp(\beta)$$

$$P^{0}_{\alpha,\infty}(\tau_1 = 2n) = \frac{\alpha^{2n}P(\tau = 2n)}{2H^{0}_{\alpha,\infty}}.$$

If we note

$$L_n = \frac{P^{\beta}_{\alpha,h}(\tau_1 = 2n)}{P^{0}_{\alpha,\infty}(\tau_1 = 2n)} = (1+\exp(-4hn))\frac{H^{0}_{\alpha,\infty}}{H^{\beta}_{\alpha,h}}\exp(\beta),$$

then we immediately notice that $L_n$ decreases with $n$, but we also have

$$\sum_{i=1}^{\infty} P^{\beta}_{\alpha,h}(\tau_1 = 2i) = \sum_{i=1}^{\infty} P^{0}_{\alpha,\infty}(\tau_1 = 2i) = 1.$$

Hence, there exists necessarily an $i_0$ in $\mathbb{N}\setminus\{0\}$ such that $P^{\beta}_{\alpha,h}(\tau_1 = 2i) \geq P^{0}_{\alpha,\infty}(\tau_1 = 2i)$ for $i \leq i_0$ and $P^{\beta}_{\alpha,h}(\tau_1 = 2i) \leq P^{0}_{\alpha,\infty}(\tau_1 = 2i)$ for $i > i_0$. This completes the proof.

6.3. **B.** First we recall a classical property, which tells us that we do not transform the free energy if we force the last monomer of the chain to touch the $x$ axis. This is proved for a different case in [4], but the same technique can be applied to our hamiltonian. Therefore, we can write

$$\Phi^0(h,\beta) = \lim_{N\to\infty}\mathbb{E}\left(\frac{1}{2N}\log E\left(\exp\left(\beta\sum_{i=1}^{2N}\mathbf{1}_{\{S_i=0\}} - 2h\sum_{i=1}^{2N}\Delta_i\right)\mathbf{1}_{\{S_{2N}=0\}}\right)\right).$$

We note $Z_{2N,\beta,h} = E\left(\exp\left(\beta\sum_{i=1}^{2N}\mathbf{1}_{\{S_i=0\}} - 2h\sum_{i=1}^{2N}\Delta_i\right)\mathbf{1}_{\{S_{2N}=0\}}\right)$, and we remark that $Z_{2N,\beta,h}$ can be rewritten as

$$\begin{aligned} Z_{2N,\beta,h} =& \sum_{j=1}^{N} E\left(e^{\beta j}e^{-2h\sum_{i=1}^{2N}\Delta_i}\mathbf{1}_{\{l_{2N}=j\}}\mathbf{1}_{\{S_{2N}=0\}}\right) \\ =& \sum_{j=1}^{N}\sum_{\substack{\bar{l}\in\mathbb{N}^{*j}\\ |\bar{l}|=N}}\prod_{i=1}^{j}\left(e^{\beta j}\ V_{h,l_j}\right) \end{aligned}$$

with $V_{h,l} = P(\tau = 2l)\left(e^{-4hl}+1\right)/2$. We aim at computing the generating function of $Z_{2N,\beta,h}$, called $\theta_h(z)$. This gives

$$\begin{aligned} \theta_h(z) =& \sum_{N=1}^{\infty} Z_{2N,\beta,h}z^{2N} = \sum_{N=1}^{\infty} z^{2N}\sum_{j=1}^{N} e^{\beta j}\sum_{\substack{\bar{l}\in\mathbb{N}^{*j}\\ |\bar{l}|=N}}\prod_{i=1}^{j} V_{h,l_j} \\ =& \sum_{j=1}^{\infty}\sum_{N=j}^{\infty}\sum_{\substack{\bar{l}\in\mathbb{N}^{*j}\\ |\bar{l}|=N}}\prod_{i=1}^{j}\left(e^{\beta}z^{2l_j}\ V_{h,l_j}\right) \\ =& \sum_{j=1}^{\infty}\left(\sum_{l=1}^{\infty} e^{\beta}z^{2l}\ V_{h,l}\right)^j = \sum_{j=1}^{\infty}\left(\sum_{l=1}^{\infty}\frac{P(\tau=2l)}{2}\left(1+e^{-4hl}\right)e^{\beta}z^{2l}\right)^j. \end{aligned}$$

Finally, since

$$\sum_{l=1}^{\infty} P(\tau = 2l) z^{2l} = 1 - \sqrt{1 - z^2},$$

we obtain

$$\theta_h(z) = \sum_{j=1}^{\infty} \left( \frac{e^\beta}{2} \left( 2 - \sqrt{1 - z^2} - \sqrt{1 - z^2 e^{-4h}} \right) \right)^j .$$

This series converges when $e^\beta\big(2 - \sqrt{1 - z^2} - \sqrt{1 - z^2 e^{-4h}}\big) < 2$, and if we denote by $R$ its convergence radius, then we have $\Phi(\beta, h) = -\log(R)$. That is why $\Phi(\beta, h) > 0$ if and only if $R < 1$. So, we can exclude that $(h, \beta)$ is on the critical curve if and only if, for $z = 1$, $e^\beta \left(2 - \sqrt{1 - z^2} - \sqrt{1 - z^2 e^{-4h}}\right) = 2$, i.e., $\sqrt{1 - e^{-4h}} = 2\big(1 - e^{-\beta}\big)$. This gives us the critical curve equation

$$h_c^0(\beta) = \frac{1}{4} \log \left( 1 - 4 \left( 1 - e^{-\beta} \right)^2 \right).$$

## Acknowledgments

I am grateful to Giambattista Giacomin, Roberto Fernández and Frank den Hollander for their precious help and suggestions.

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