Bounding a Random Environment for Two-dimensional Edge-reinforced Random Walk

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Abstract

We consider edge-reinforced random walk on the infinite two-dimensional lattice. The process has the same distribution as a random walk in a certain strongly dependent random environment, which can be described by random weights on the edges. In this paper, we show some decay properties of these random weights. Using these estimates, we derive bounds for some hitting probabilities of the edge-reinforced random walk. [1][2]

1 Introduction and results

1.1 Introduction

Edge-reinforced random walk on $\mathbb{Z}^2$ is the following model: Consider the two-dimensional integer lattice $\mathbb{Z}^2$ as a graph with edge set

$$E = \{\{x, y\} \in \mathbb{Z}^2 \times \mathbb{Z}^2 : \ |x - y| = 1\}. \tag{1.1}$$

Here, $|\cdot|$ denotes the euclidean norm. In particular, the edges are undirected. Fix a vertex $v_0 \in \mathbb{Z}^2$ and a positive number $a > 0$. A non-Markovian random walker starts in $X_0 = v_0$. At every discrete time $t \in \mathbb{N}_0$, it jumps from its current position X_t to a neighboring vertex X_{t+1} in $\mathbb{Z}^2$, $|X_{t+1} - X_t| = 1$. The law $P_{v_0,a}$ of the random walker is defined in terms of the time-dependent weights

$$w_e(t) = a + \sum_{s=0}^{t-1} 1_{e=\{X_s, X_{s+1}\}}, \quad e \in E. \tag{1.2}$$

The weight $w_e(t)$ of edge e at time t equals the number of traversals of e up to time t plus the initial weight a. The transition probability $P_{v_0,a}(\{X_t, X_{t+1}\} = e \mid X_0, \ldots, X_t)$ is proportional to $w_e(t)$ for all edges $e \ni X_t$:

$$P_{v_0,a}(\{X_t, X_{t+1}\} = e \mid X_0, \ldots, X_t) = \frac{w_e(t)}{\sum_{e' \ni X_t} w_{e'}(t)} 1_{e \ni X_t}. \tag{1.3}$$

This model was introduced by Diaconis [Dia88]. Pemantle [Pem88] examined the model on infinite trees; in particular, he showed that the process *on tree graphs* has the same

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distribution as a random walk in an *independent* random environment. As Pemantle [Pem88] remarks, Diaconis asked whether this process on $\mathbb{Z}^2$ and more generally on $\mathbb{Z}^d$, $d \geq 2$, is recurrent. This question is still open. For more information about the history of edge-reinforced random walk, see [MR05c].

In [MR05d], we have shown that the edge-reinforced random walk on any locally finite graph has the same distribution as a random walk in a random environment given by random, time-independent, strictly positive weights $(x_e)_e$ on the edges. For graphs where the edge-reinforced random walk is recurrent, a slightly different representation follows from the paper [DF80] by Diaconis and Freedman, which was written before edge-reinforced random walk was introduced. For finite graphs, Coppersmith and Diaconis [CD86] discovered an explicit, but complicated formula for the joint law of the fraction of time spent on the edges. In fact, this joint law just equals the law of the weights $(x_e)_e$; see [Rol03]. A proof of the formula describing the joint law of $(x_e)_e$ is published in [KR00]. The random environment is strongly dependent, unless the graph is a tree-graph.

On infinite ladders, for sufficiently large initial weights a, the random environment can be described as an infinite-volume Gibbs measure. It arises as an infinite-volume limit of finite-volume Gibbs measures; see [MR05b], [Rol05], [MR05a]. The finite-volume Gibbs measures are just a reinterpretation of the formula of Coppersmith and Diaconis in the version described by Keane and Rolles. This reinterpretation has been used in the above references to prove recurrence of the process on ladders and also to analyze the asymptotic behavior of the edge-reinforced random walk in more detail.

1.2 Infinite-volume results

In this paper, we use the formula of Coppersmith, Diaconis, Keane, and Rolles to examine decay properties of the weights $(x_e)_{e\in E}$ for the edge-reinforced random walk on $\mathbb{Z}^2$ with any initial weights $a > 0$. From these results, we deduce estimates for hitting probabilities of the edge-reinforced random walk.

All constants like $\beta(a), c_1, c_2, \ldots$ keep their meaning throughout the whole article.

For $v \in \mathbb{Z}^2$, let $\tau_v := \inf\{n \geq 1 : X_n = v\}$ denote the first time ≥ 1 when the random walker visits v.

Theorem 1.1 (Hitting probabilities for ERRW) *For all $a > 0$, there are $c_1(a) > 0$ and $\beta(a) > 0$, such that for all $v \in \mathbb{Z}^2 \setminus \{0\}$, the following hold:*

(a) The probability to reach the vertex v before returning to the vertex 0 is bounded by

$$P_{0,a}[\tau_v < \tau_0] \leq c_1(a)|v|^{-\beta(a)}. \tag{1.4}$$

(b) For all $n \geq 1$, the probability that the random walker visits the vertex v at time n satisfies the same bound

$$P_{0,a}[X_n = v] \leq c_1(a)|v|^{-\beta(a)}. \tag{1.5}$$

If we knew $\beta(a) > 1$, the estimate (1.4) would imply recurrence. However, our estimates yield only

$$\beta(a) = \left[1024e(1+4a)(1+(e \cdot \max\{\lceil 1/\sqrt{a}\rceil, 2\} \log 2)^{-1})\right]^{-1}. \tag{1.6}$$

In particular, $\beta(a)$ is a decreasing function of a with

$$\beta(a) \stackrel{a\to 0}{\longrightarrow} \frac{1}{1024e} \qquad \text{and} \qquad \beta(a) \stackrel{a\to\infty}{\longrightarrow} 0. \tag{1.7}$$

Thus recurrence is still an open problem.

Set $\Omega = (0,\infty)^E$. For $x \in \Omega$, let $Q_{v_0,x}$ denote the law of a Markovian nearest neighbor random walk on $(\mathbb{Z}^2, E)$ with starting point v_0 in a time-independent environment given by the edge weights x. Given this random walk is at u and given the past of the random walk, the probability it jumps to the neighboring vertex v is proportional to the weight x_e of the edge $e = \{u, v\}$.

Edge-reinforced random walk on $\mathbb{Z}^2$ can be represented as a random walk in a random environment. More precisely:

Theorem 1.2 (ERRW as RWRE, Theorem 2.2 in [MR05d]) *Let $a > 0$. There is a probability measure $\mathbb{Q}_{0,a}$ on Ω, such that for all events $A \subseteq (\mathbb{Z}^2)^{\mathbb{N}_0}$, one has*

$$P_{0,a}[A] = \int_\Omega Q_{0,x}[A]\, \mathbb{Q}_{0,a}(dx). \tag{1.8}$$

It is not known whether $\mathbb{Q}_{0,a}$ is unique up to normalization of the edge weights x. In this article, we prove some decay properties of the random environment. These results hold for at least one choice of $\mathbb{Q}_{0,a}$. For $x \in \Omega$ and $v \in \mathbb{Z}^2$, set

$$x_v = \sum_{e \ni v} x_e. \tag{1.9}$$

The following two theorems show that in some weak probabilistic sense, the ratios of weights x_v/x_0 tend to zero as $|v| \to \infty$. We phrase two formal versions of this statement. The first version only cares about the expected logarithms of ratios x_v/x_0. In this version, we get a fast divergence to $-\infty$ as $|v| \to \infty$ if only $a > 0$ is small enough.

Theorem 1.3 (Decay of the weights – infinite-volume version) *There exist functions $c_2, c_3 : (0,\infty) \to (0,\infty)$ with $-c_2(a) \to -\infty$ as $a \downarrow 0$, such that the following holds: For all $a > 0$ and all $v \in \mathbb{Z}^2 \setminus \{0\}$,*

$$E_{\mathbb{Q}_{0,a}}\left[\log \frac{x_v}{x_0}\right] \le c_3(a) - c_2(a) \log |v|. \tag{1.10}$$

However, Theorem 1.3 does not imply weak convergence of x_v/x_0 to zero. Weak convergence is stated among others in the following theorem, but our bound for the rate of convergence is not as strong as in the preceding theorem.

Theorem 1.4 (Decay of the weights – infinite-volume version) *For all $a > 0$, there are $c_1(a) > 0$ and $\beta(a) > 0$, such that for all $v \in \mathbb{Z}^2 \setminus \{0\}$, the following holds:*

$$E_{\mathbb{Q}_{0,a}}\left[\left(\frac{x_v}{x_0}\right)^{\frac{1}{4}}\right] \le c_1(a)|v|^{-\beta(a)}. \tag{1.11}$$

In particular, x_v/x_0 converges weakly to zero as $|v| \to \infty$.

Theorem 1.5 (Tightness of quotients of neighboring edge weights) *For all $a > 0$ and all $\alpha \in (0, a/2)$, one has*

$$\sup_{\substack{e,f\in E\\ e\cap f\neq\emptyset}} E_{\mathbb{Q}_{0,a}}\left[\left(\frac{x_e}{x_f}\right)^{\alpha}\right] < \infty. \tag{1.12}$$

1.3 Uniform finite-volume results

All our infinite-volume results are derived from uniform finite-volume analogues for edge-reinforced random walk on finite boxes. "Uniform" here means "uniform in the size of the finite box".

We consider a $(2N+1) \times (2N+1)$ box $V^{(N)} = \mathbb{Z}^2/(2N+1)\mathbb{Z}^2$ with periodic boundary conditions. For $v \in \mathbb{Z}^2$, let $v^{(N)} = v + (2N+1)\mathbb{Z}^2$ denote the class of v in $V^{(N)}$. If there is no risk of confusion, we identify $V^{(N)}$ with the subset $\tilde{V}^{(N)} = [-N, N]^2 \cap \mathbb{Z}^2$ of $\mathbb{Z}^2$. Let

$$E^{(N)} = \{\{u^{(N)}, v^{(N)}\} :\ \{u, v\} \in E\}. \tag{1.13}$$

For $u \in V^{(N)}$, set

$$|u| = \min\{|v| :\ v \in u\}. \tag{1.14}$$

This is just the euclidean distance of u to the origin, viewed as an element of $\tilde{V}^{(N)}$.

Just as in the infinite-volume case, for $v_0 \in V^{(N)}$ and $a > 0$, let $P^{(N)}_{v_0,a}$ denote the law of edge-reinforced random walk on $(V^{(N)}, E^{(N)})$ with starting point v_0 and constant initial weights a. Set $\Omega^{(N)} = (0,\infty)^{E^{(N)}}$. For $x \in \Omega^{(N)}$, let $Q^{(N)}_{v_0,x}$ denote the law of a random walk on $(V^{(N)}, E^{(N)})$ with starting point v_0 in a time-independent environment given by weights x. Note that multiplying all components of x by the same (random or deterministic) scaling factor α does not change the law of the corresponding random walk. The following finite-volume analogue of Theorem 1.2 is well-known:

Theorem 1.6 (ERRW as RWRE on finite boxes, Theorem 3.1 in [Rol03]) *Let $a > 0$. There is a probability measure $\mathbb{Q}^{(N)}_{0,a}$ on $\Omega^{(N)}$, such that for all events $A \subseteq (V^{(N)})^{\mathbb{N}_0}$, one has*

$$P^{(N)}_{0,a}[A] = \int_{\Omega^{(N)}} Q^{(N)}_{0,x}[A]\, \mathbb{Q}^{(N)}_{0,a}(dx). \tag{1.15}$$

Up to an arbitrary normalization of the random edge weights $x \in \Omega^{(N)}$, the law $\mathbb{Q}^{(N)}_{0,a}$ of the random environment is unique.

In [Dia88] and [KR00], the distribution $\mathbb{Q}_{0,a}^{(N)}$ is described explicitly; see also Lemma 2.1, below. The weaker statement that the edge-reinforced random walk on $(V^{(N)}, E^{(N)})$ is a mixture of Markov chains follows already from [DF80].

The following two theorems are finite-volume analogues of Theorems 1.3 and 1.4. The bounds are uniform in the size of a finite box.

Theorem 1.7 (Decay of the weights – finite-volume version) *There are functions* $c_2, c_3 : (0,\infty) \to (0,\infty)$ *with* $c_2(a) \to \infty$ *as* $a \downarrow 0$*, such that the following holds: For all* $a > 0$*, all* $N \in \mathbb{N}$*, and all* $v \in V^{(N)} \setminus \{0\}$*,*

$$E_{\mathbb{Q}_{0,a}^{(N)}}\left[\log \frac{x_v}{x_0}\right] \leq c_3(a) - c_2(a) \log |v|. \tag{1.16}$$

Theorem 1.8 (Decay of the weights – finite-volume version) *For all* $a > 0$*, there are* $c_1(a) > 0$ *and* $\beta(a) > 0$*, such that for all* $N \in \mathbb{N}$ *and all* $v \in V^{(N)} \setminus \{0\}$*, the following holds:*

$$E_{\mathbb{Q}_{0,a}^{(N)}}\left[\left(\frac{x_v}{x_0}\right)^{\frac{1}{4}}\right] \leq c_1(a)|v|^{-\beta(a)}. \tag{1.17}$$

Let $e, f \in E^{(N)}$ be two neighboring edges, i.e. two edges containing a common vertex v. The random variables $\log(x_e/x_f)$ with respect to $\mathbb{Q}_{0,a}^{(N)}$ are tight with exponential tails, uniformly in N and uniformly in the choice of the edges e and f. This is stated formally in the following lemma.

Lemma 1.9 *For all* $a > 0$ *and all* $\alpha \in (0, a/2)$*, one has*

$$\sup_{\substack{N\in\mathbb{N}\\N>1}} \max_{\substack{e,f\in E^{(N)}\\e\cap f\neq\emptyset}} E_{\mathbb{Q}_{0,a}^{(N)}}\left[\left(\frac{x_e}{x_f}\right)^{\alpha}\right] < \infty. \tag{1.18}$$

Furthermore,

$$c_4(a) := \sup_{\substack{N\in\mathbb{N}\\N>1}} \max_{\substack{v,w\in V^{(N)}\\|v-w|=1}} E_{\mathbb{Q}_{0,a}^{(N)}}\left[\log \frac{x_v}{x_w}\right] < \infty. \tag{1.19}$$

2 Bounds in finite-volume

2.1 Random environment in finite volume

In this section, we take a fixed box size $N \in \mathbb{N}$ and a fixed $a > 0$, and we examine some random deformation of the random environment.

First, we state a description of the random environment in finite volume. Let $e_0 \in E^{(N)}$ be any reference edge adjacent to the origin. So far, the arbitrary normalization of the random edge weights $x \in E^{(N)}$ has not been specified. However, in this section, it is convenient to choose the normalization

$$x_{e_0} = 1 \quad \mathbb{Q}_{0,a}^{(N)}\text{-a.s.} \tag{2.1}$$

We introduce a reference measure ρ on $\Omega^{(N)}$ to be the following product measure:

$$\rho(dx) = \delta_1(dx_{e_0}) \prod_{e \in E^{(N)} \setminus \{e_0\}} \frac{dx_e}{x_e}. \tag{2.2}$$

Here δ_1 denotes the Dirac measure on $(0, \infty)$ with unit mass at 1.

Let $\mathcal{T}^{(N)}$ denote the set of all spanning trees of $(V^{(N)}, E^{(N)})$, viewed as subsets of the set of edges $E^{(N)}$.

Lemma 2.1 (Random environment for a finite box) *For $v_0 \in V^{(N)}$, the law $\mathbb{Q}_{v_0,a}^{(N)}$ of the random environment is absolutely continuous with respect to the reference measure ρ with density*

$$\frac{d\mathbb{Q}_{v_0,a}^{(N)}}{d\rho}(x) = \frac{1}{z_{v_0,a}^{(N)}} \frac{\prod_{e \in E^{(N)}} x_e^a}{x_{v_0}^{2a} \prod_{v \in V^{(N)} \setminus \{v_0\}} x_v^{2a+1/2}} \sqrt{\sum_{T \in \mathcal{T}^{(N)}} \prod_{e \in T} x_e} \tag{2.3}$$

with some normalizing constant $z_{v_0,a}^{(N)} > 0$.

The claim of the lemma is essentially the formula of Coppersmith and Diaconis [CD86] for the distribution of the random environment, transformed such that one has the normalization $x_{e_0} = 1$. The transformation to this normalization and thus the proof of Lemma 2.1 is given in the appendix.

Now, we consider an interpolation between the random environments $\mathbb{Q}_{0,a}^{(N)}$ and $\mathbb{Q}_{\ell,a}^{(N)}$ associated with two different starting points 0 and ℓ in $V^{(N)}$. We introduce an "external force" $\eta\Sigma_\ell$ with the switching parameter $\eta \in [0,1]$. Turning the external force off ($\eta = 0$) corresponds to $\mathbb{Q}_{0,a}^{(N)}$, while turning the external force completely on ($\eta = 1$) corresponds to $\mathbb{Q}_{\ell,a}^{(N)}$. More formally, we proceed as follows:

Definition 2.2 (Interpolated measures for the environment) *For $\ell \in V^{(N)}$ and $0 \le \eta \le 1$, we define the following probability measure on $\Omega^{(N)}$:*

$$\mathbb{P}_\eta = \mathbb{P}_{\eta,0,\ell,a}^{(N)} := \frac{1}{Z_{\eta,0,\ell,a}^{(N)}} (\mathbb{Q}_{0,a}^{(N)})^{1-\eta} (\mathbb{Q}_{\ell,a}^{(N)})^{\eta} \tag{2.4}$$

with some normalizing constant $Z_{\eta,0,\ell,a}^{(N)}$. This means:

$$\frac{d\mathbb{P}_\eta}{d\rho} = \frac{1}{Z_{\eta,0,\ell,a}^{(N)}} \left(\frac{d\mathbb{Q}_{0,a}^{(N)}}{d\rho} \right)^{1-\eta} \left(\frac{d\mathbb{Q}_{\ell,a}^{(N)}}{d\rho} \right)^{\eta}, \tag{2.5}$$

$$Z_{\eta,0,\ell,a}^{(N)} = \int_{\Omega^{(N)}} \left(\frac{d\mathbb{Q}_{0,a}^{(N)}}{d\rho} \right)^{1-\eta} \left(\frac{d\mathbb{Q}_{\ell,a}^{(N)}}{d\rho} \right)^{\eta} d\rho. \tag{2.6}$$

By Hölder's inequality, $Z^{(N)}_{\eta,0,\ell,a}$ is finite. Note that this definition is independent of the choice of the reference measure ρ, as long as both $\mathbb{Q}^{(N)}_{0,a}$ and $\mathbb{Q}^{(N)}_{\ell,a}$ are absolutely continuous with respect to ρ.

We define the random variable

$$\Sigma_\ell = \Sigma^{(N)}_{0,\ell} := \frac{1}{2}\log\frac{x_\ell}{x_0}. \tag{2.7}$$

Then, the following identity holds:

$$\frac{d\mathbb{Q}^{(N)}_{\ell,a}}{d\mathbb{Q}^{(N)}_{0,a}} = \sqrt{\frac{x_\ell}{x_0}} = \exp\Sigma_\ell. \tag{2.8}$$

In the formula in (2.8), there appears no normalizing constant, because there is a reflection symmetry of the box $V^{(N)}$ which interchanges 0 and ℓ. Recall that the box $V^{(N)}$ has periodic boundary conditions.

Furthermore, using (2.8), note that $\mathbb{P}^{(N)}_{\eta,0,\ell,a}$ is absolutely continuous with respect to $\mathbb{Q}^{(N)}_{0,a}$ with the density

$$\frac{d\mathbb{P}_\eta}{d\mathbb{Q}^{(N)}_{0,a}} = \frac{1}{Z^{(N)}_{\eta,0,\ell,a}}\left(\frac{x_\ell}{x_0}\right)^{\eta/2} = \frac{\exp(\eta\Sigma_\ell)}{Z^{(N)}_{\eta,0,\ell,a}}, \qquad \text{and} \qquad Z^{(N)}_{\eta,0,\ell,a} = E_{\mathbb{Q}^{(N)}_{0,a}}[\exp(\eta\Sigma_\ell)]. \tag{2.9}$$

Together with (2.3) this implies

$$\frac{d\mathbb{P}_\eta}{d\rho}(x) = \frac{1}{z^{(N)}_{0,a}Z^{(N)}_{\eta,0,\ell,a}}\frac{\left(\prod_{e\in E^{(N)}} x_e^a\right)\sqrt{\sum_{T\in\mathcal{T}^{(N)}}\prod_{e\in T}x_e}}{x_0^{2a+\eta/2}x_\ell^{2a+(1-\eta)/2}\prod_{v\in V^{(N)}\setminus\{0,\ell\}}x_v^{2a+1/2}}. \tag{2.10}$$

2.2 Deformation

Recall the notation $\tilde{V}^{(l)} = [-l,l]^2\cap\mathbb{Z}^2$. For $l\in\mathbb{N}_0$, we say that a vertex $\ell\in\mathbb{Z}^2$ is on *level* l and we write $l = \text{level}(\ell)$, if $\ell\in\tilde{V}^{(2l)}\setminus\tilde{V}^{(2(l-1))}$, where we use the convention $\tilde{V}^{(-2)}=\emptyset$. Identifying $V^{(N)}$ with the subset $\tilde{V}^{(N)}\subset\mathbb{Z}^2$, the level of ℓ is also defined for vertices $\ell\in V^{(N)}$. Note that the level sets are defined to have width 2 instead of width 1, see Figure 1.

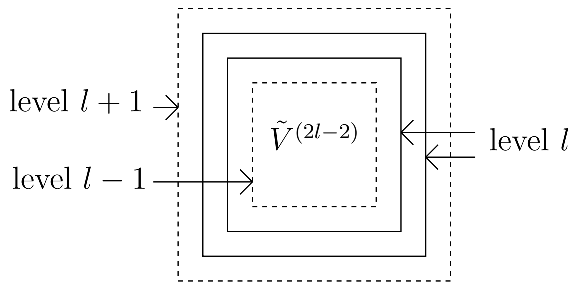
Lemma 2.3 (Deformation of the random environment) *Fix*

(i) $a>0$,

(ii) $n_a = \max\{\lceil 1/\sqrt{a}\rceil, 2\}$,

(iii) $N\in\mathbb{N}$ *with* $N>6n_a$, *and*

Figure 1: Vertices at level l are located on the solid lines



(iv) let $\ell \in V^{(N)}$ *be a vertex on a level* $l > 3n_a$.

Then, for all $\gamma \in \mathbb{R}$*, there is a measurable, measurably invertible map* $\Xi_\gamma = \Xi^{(N)}_{\gamma,0,\ell,a} : \Omega^{(N)} \to \Omega^{(N)}$ *with the following properties:*

(a) One has

$$x_0 \circ \Xi_\gamma = x_0 \quad \textit{and} \quad x_\ell \circ \Xi_\gamma = e^\gamma x_\ell. \tag{2.11}$$

(b) The reference measure ρ *is invariant with respect to* Ξ_γ.

(c) For all $\eta \in [0,1]$*, the image measure*

$$\Pi_{\gamma,\eta} = \Pi^{(N)}_{\gamma,\eta,0,\ell,a} := \Xi^{(N)}_{\gamma,0,\ell,a}\mathbb{P}^{(N)}_{\eta,0,\ell,a} \tag{2.12}$$

of $\mathbb{P}_\eta$ *with respect to* Ξ_γ *is absolutely continuous with respect to* $\mathbb{P}_\eta$.
If furthermore

$$|\gamma| \leq n_a \log \frac{l-1}{n_a} \tag{2.13}$$

holds, one has the entropy bound

$$E_{\Pi_{\gamma,\eta}}\left[\log \frac{d\Pi_{\gamma,\eta}}{d\mathbb{P}_\eta}\right] \leq \frac{c_5(a,\eta)\gamma^2}{n_a \log((l-1)/n_a)} \tag{2.14}$$

with the constant

$$c_5(a,\eta) := 32\left(2a + \frac{1}{2}\right)\left(ec_6(a,\eta)n_a + \frac{1}{\log 2}\right) > 16 \qquad \textit{and} \tag{2.15}$$

$$c_6(a,\eta) := \begin{cases} \min\{\sqrt{a}, 1\} & \textit{if } \eta = 1, \\ 1 & \textit{otherwise.} \end{cases} \tag{2.16}$$

In the special case $\eta = 1$*, one has* $\lim_{a\to 0} c_5(a,1) < \infty$.

Proving this lemma is the main goal of this and the next section.

In the following, we assume that a, n_a, N, ℓ, and l are chosen as in (i)–(iv) of Lemma 2.3.

In the following two definitions, we introduce a modified, truncated version of the Green's function in 2 dimensions. At first, we take the logarithm, appropriately scaled and truncated:

Definition 2.4 (Auxiliary function 1: truncated and scaled logarithm) *We define a function* $\varphi = \varphi_{l,a} : \mathbb{N}_0 \to [0,1]$ *by*

$$\varphi(n) = \begin{cases} 0 & \text{for } 0 \le n < n_a, \\ \dfrac{\log(n/n_a)}{\log((l-1)/n_a)} & \text{for } n_a \le n \le l-1, \\ 1 & \text{for } n \ge l. \end{cases} \tag{2.17}$$

Let $C^{(n)} := \{(-n,-n), (-n,n), (n,-n), (n,n)\}$ denote the set of corner points of the box $\tilde{V}^{(n)}$.

Definition 2.5 (Auxiliary function 2: modified Green's function) *For every* $e = \{u,v\} \in E^{(N)}$*, we define a map* $D_e = D^{(N)}_{e,0,\ell,a} : \Omega^{(N)} \to [0,1]$ *as follows: Let* $x \in \Omega^{(N)}$ *and* $l = \text{level}(\ell)$*.*

- *If for some* $n \in \mathbb{N}_0$ *we have* $\text{level}(u) = \text{level}(v) = n$ *or* $\{u,v\} \cap C^{(2n)} \neq \emptyset$*, then we set*

$$D_e(x) = \varphi(n). \tag{2.18}$$

- *Otherwise, we set*

$$D_e(x) = \begin{cases} \varphi(\text{level}(u)) & \text{if } x_u < x_v, \\ \varphi(\text{level}(v)) & \text{if } x_v < x_u, \\ \frac{1}{2}\{\varphi(\text{level}(u)) + \varphi(\text{level}(v))\} & \text{if } x_u = x_v. \end{cases} \tag{2.19}$$

Thus $e \mapsto D_e(x)$ is an approximation to the Green's function in 2 dimensions, slightly dependent on x for technical reasons. It has the property that for all vertices e not being corner points, at most one neighboring vertex f has a value $D_f(x)$ different from $D_e(x)$. This property is very convenient below, and it is our reason to define the level sets having width 2 instead of width 1.

We write the set $E^{(N)}$ of edges as a disjoint union $E^{(N)} = F \cup F'$, $F \cap F' = \emptyset$, where F denotes the set of all edges $e = \{u,v\} \in E^{(N)}$ where u and v are on the same level, and F' denotes the set of all edges $e = \{u,v\} \in E^{(N)}$ with u and v on different levels. For $e, e' \in E^{(N)}$, we write $e \prec e'$ if and only if $e \in F$ and $e' \in F'$. Let

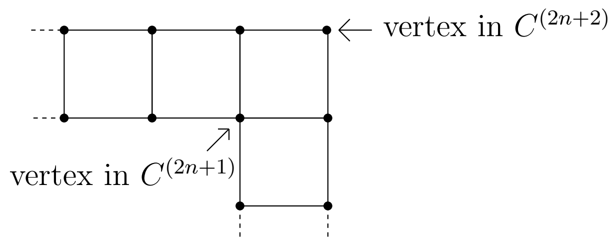
$$\mathcal{F}^{(N)}_e := \sigma(x_{e'} : e' \prec e) \tag{2.20}$$

denote the σ-field on $\Omega^{(N)}$ generated by the canonical projections on the coordinates $e' \in E^{(N)}$ with $e' \prec e$.

Lemma 2.6 *For any $e \in E^{(N)}$, the map D_e is $\mathcal{F}_e^{(N)}$-measurable.*

Proof. Let $e = \{u, v\}$. If level(u) = level(v) or $\{u, v\} \cap C^{(2n)} \neq \emptyset$ for some $n \in \mathbb{N}_0$, then D_e is constant and there is nothing to show. By definition, our levels have thickness 2. Therefore, if $\{u, v\} \cap C^{(2n+1)} \neq \emptyset$ for some $n \in \mathbb{N}_0$, then level(u) = level(v) (see Figure 2), and this case has already been taken care of.

Figure 2: All vertices drawn are on level n.

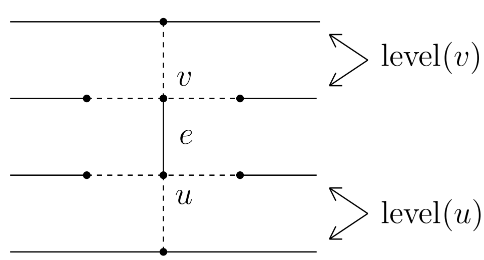


Assume that u and v are on different levels and $\{u, v\} \cap C^{(n)} = \emptyset$ for all $n \in \mathbb{N}_0$. Then, D_e is constant on the sets $\{x_u < x_v\}$, $\{x_v < x_u\}$, and $\{x_u = x_v\}$. Observe that

$$x_u < x_v \quad \text{if and only if} \quad \sum_{\substack{e' \ni u \\ e' \neq e}} x_{e'} < \sum_{\substack{e' \ni v \\ e' \neq e}} x_{e'}. \tag{2.21}$$

Since u and v are on different levels, but not corner points of any box $\tilde{V}^{(n)}$, all edges $e' \neq e$ incident to u have endpoints on the same level, namely on level(u), see Figure 3.

Figure 3: The edge e has endpoints u and v on different levels. The edges $e' \neq e$ with $e' \ni u$ or $e' \ni v$ are drawn with dashed lines.



Similarly, all edges $e' \neq e$ incident to v have endpoints on level(v). In other words, $e' \prec e$ holds for these edges e'. Thus all $x_{e'}$ appearing in the two sums on the right hand side of (2.21) are $\mathcal{F}_e^{(N)}$-measurable; recall the definition (2.20) of $\mathcal{F}_e^{(N)}$. Consequently, because of (2.21), we have $\{x_u < x_v\} \in \mathcal{F}_e^{(N)}$. The same argument shows that $\{x_v < x_u\} \in \mathcal{F}_e^{(N)}$ and $\{x_u = x_v\} \in \mathcal{F}_e^{(N)}$. ■

Definition 2.7 (Deformation) *For any $\gamma \in \mathbb{R}$, we define the deformation map $\Xi_\gamma = \Xi^{(N)}_{\gamma,0,\ell,a} : \Omega^{(N)} \to \Omega^{(N)}$ by*

$$\Xi_\gamma(x) = \left(\exp\left(\gamma D_e(x)\right) \cdot x_e\right)_{e \in E^{(N)}} . \tag{2.22}$$

Proof of parts (a) and (b) of Lemma 2.3. Let $\gamma \in \mathbb{R}$. The measurability of Ξ_γ follows from its definition (2.22) and Lemma 2.6.

(a) Let $x \in \Omega^{(N)}$, and let $e \in E^{(N)}$ with $0 \in e$. Then, $e \cap C^{(0)} \neq \emptyset$, and consequently, $D_e(x) = \varphi(0) = 0$. Hence, $x_e \circ \Xi_\gamma = x_e$, and it follows that $x_0 = \sum_{e' \ni 0} x_{e'} = x_0 \circ \Xi_\gamma$. Let $e \in E^{(N)}$ with $\ell \in e$. Then, $D_e(x)$ takes a value in the set $\{\varphi(l),\ \varphi(l \pm 1), (\varphi(l) + \varphi(l \pm 1))/2\} = \{1\}$. Hence, $x_\ell \circ \Xi_\gamma = e^\gamma x_\ell$. This completes the proof of (2.11).

(b) We list the edges in $E^{(N)}$ in such a way that every $e = \{u, v\} \in E^{(N)}$ with $\mathrm{level}(u) = \mathrm{level}(v)$ gets a smaller index than every $e' = \{u', v'\} \in E^{(N)}$ with $\mathrm{level}(u') \neq \mathrm{level}(v')$. Thus, we get a list $e_0, e_1, \ldots, e_K$ with the property that e has a smaller index than e' whenever $e \prec e'$. We rewrite the definitions (2.22) of Ξ_γ and (2.2) of the reference measure ρ in logarithmic form:

$$\log\left(x_e \circ \Xi_\gamma\right) = \log x_e + \gamma D_e(x), \tag{2.23}$$

$$\rho(dx) = \delta_1(dx_{e_0}) \prod_{e \in E^{(N)} \setminus \{e_0\}} d \log x_e. \tag{2.24}$$

Since D_e is $\mathcal{F}^{(N)}_e$-measurable, we see that the logarithmized component $\log x_{e_j}$ is just translated by a value which depends only on the components x_{e_i} with $i < j$. Since $0 \in e_0$, we have $D_{e_0} = 0$, and the component x_{e_0} remains unchanged. Such translations leave the reference measure ρ invariant. Here we use that every measurable map $f : \mathbb{R}^d \to \mathbb{R}^d$ of the form

$$f(x_1, \ldots, x_d) = (x_i + g_i(x_j; j < i))_{i=1,\ldots,d} \tag{2.25}$$

leaves the Lebesgue measure invariant.

One verifies that the inverse of Ξ_γ is given by

$$x_e \circ [\Xi_\gamma]^{-1} = \exp\left\{-\gamma D_e\left([\Xi_\gamma]^{-1}(x)\right)\right\} x_e, \tag{2.26}$$

$x \in \Omega^{(N)}$, $e \in E^{(N)}$. This is a recursive system for determining the inverse, since $D_e([\Xi_\gamma]^{-1}(x))$ depends only on the components x_f with f earlier in the list $e_0, \ldots, e_K$ than e. It follows that $[\Xi_\gamma]^{-1}$ is measurable. ■

2.3 Entropy bounds

The goal of this section is to prove Lemma 2.3(c). Throughout this section, we assume that a, n_a, N, ℓ, and l are fixed and chosen as in (i)–(iv) of Lemma 2.3. For $\gamma \in \mathbb{R}$ and $\eta \in [0,1]$, we denote by

$$\Pi^-_{\gamma,\eta} = \Pi^{(N)-}_{\gamma,\eta,0,\ell,a} := [\Xi^{(N)}_{\gamma,0,\ell,a}]^{-1}\mathbb{P}^{(N)}_{\eta,0,\ell,a} \tag{2.27}$$

the image measure of $\mathbb{P}_\eta$ under the inverse of Ξ_γ. In the following, we suppress the dependence of D_e on the environment x. Thus, we write D_e instead of $D_e(x)$. We abbreviate for $v \in V^{(N)}$, $\gamma \in \mathbb{R}$, $T \in \mathcal{T}^{(N)}$, and $x \in \Omega^{(N)}$:

$$x_{v,\gamma} := \sum_{e \ni v} \left(e^{\gamma D_e} x_e\right) \qquad \text{and} \qquad Y_{T,\gamma} = Y_{T,\gamma}(x) := \prod_{e \in T} \left(e^{\gamma D_e} x_e\right). \tag{2.28}$$

Lemma 2.8 *(a) For any $\gamma \in \mathbb{R}$ and $\eta \in [0,1]$, we have*

$$\Pi_{\gamma,\eta} \ll \mathbb{P}_\eta \ll \Pi^-_{\gamma,\eta}, \tag{2.29}$$

where "$\ll$" means that the left-hand side is absolutely continuous with respect to the right-hand side.

(b) For any $\gamma \in \mathbb{R}$ and $\eta \in [0,1]$, the Radon-Nikodym derivatives are bounded functions on $\Omega^{(N)}$ and fulfill

$$\begin{aligned}\log \frac{d\Pi_{\gamma,\eta}}{d\mathbb{P}_\eta} \circ \Xi_\gamma &= \log \frac{d\mathbb{P}_\eta}{d\Pi^-_{\gamma,\eta}} \\ &= -\sum_{e \in E^{(N)}} (\gamma D_e a) + \gamma \left(2a + \frac{1-\eta}{2}\right) \\ &\quad + \left(2a + \frac{1}{2}\right) \sum_{v \in V^{(N)} \setminus \{0,\ell\}} \log \frac{x_{v,\gamma}}{x_v} - \frac{1}{2} \log \frac{\sum_{T \in \mathcal{T}^{(N)}} Y_{T,\gamma}}{\sum_{T \in \mathcal{T}^{(N)}} Y_{T,0}}.\end{aligned} \tag{2.30}$$

(c) The following two entropies are finite and coincide:

$$E_{\Pi_{\gamma,\eta}}\left[\log \frac{d\Pi_{\gamma,\eta}}{d\mathbb{P}_\eta}\right] = E_{\mathbb{P}_\eta}\left[\log \frac{d\mathbb{P}_\eta}{d\Pi^-_{\gamma,\eta}}\right]. \tag{2.31}$$

Proof. Let $\gamma \in \mathbb{R}$ and $\eta \in [0,1]$. By (2.10), $\mathbb{P}_\eta$ is absolutely continuous with respect to ρ with a strictly positive Radon-Nikodym-derivative $d\mathbb{P}_\eta/d\rho$. The reference measure ρ is invariant under Ξ_γ by Lemma 2.3 (b). Consequently, we find

$$\frac{d\Pi_{\gamma,\eta}}{d\rho} = \frac{d(\Xi_\gamma \mathbb{P}_\eta)}{d(\Xi_\gamma \rho)} = \frac{d\mathbb{P}_\eta}{d\rho} \circ [\Xi_\gamma]^{-1} \quad \text{and} \quad \frac{d\Pi^-_{\gamma,\eta}}{d\rho} = \frac{d(\Xi_\gamma^{-1}\mathbb{P}_\eta)}{d(\Xi_\gamma^{-1}\rho)} = \frac{d\mathbb{P}_\eta}{d\rho} \circ \Xi_\gamma. \tag{2.32}$$

Taking quotients, this implies claim (a).

The first equality in (2.30) follows from (2.32). To prove the second equality in part (b), recall the explicit form of $d\mathbb{P}_\eta/d\rho$ in formula (2.10). By (2.11), we know that $x_0 \circ \Xi_\gamma = x_0$ and $x_\ell \circ \Xi_\gamma = e^\gamma x_\ell$. Consequently, for any $\gamma \in \mathbb{R}$, we obtain using (2.32),

$$\begin{aligned}\frac{d\Pi^-_{\gamma,\eta}}{d\rho}(x) =& \frac{d\mathbb{P}_\eta}{d\rho}\left(\Xi_\gamma(x)\right) \\ =& \frac{1}{z^{(N)}_{0,a} Z^{(N)}_{\eta,0,\ell,a}} \frac{\prod_{e\in E^{(N)}} \left(e^{\gamma D_e a} x_e^a\right) \sqrt{\sum_{T\in\mathcal{T}^{(N)}} \prod_{e\in T} \left(e^{\gamma D_e} x_e\right)}}{x_0^{2a+\eta/2} (e^\gamma x_\ell)^{2a+(1-\eta)/2} \prod_{v\in V^{(N)}\setminus\{0,\ell\}} \left[\sum_{e\ni v} \left(e^{\gamma D_e} x_e\right)\right]^{2a+1/2}}.\end{aligned} \tag{2.33}$$

Combining (2.10) and (2.33), yields the second equality in the claim (2.30).

Since D_e takes only values in $[0, 1]$, we have $e^{-|\gamma|}x_e \le e^{\gamma D_e} x_e \le e^{|\gamma|} x_e$ for all $e \in E^{(N)}$. Hence,

$$\begin{aligned}& e^{-|\gamma|} x_v \le x_{v,\gamma} \le e^{|\gamma|} x_v \qquad \text{and} \\ & e^{-|E^{(N)}|\cdot|\gamma|} Y_{T,0} \le Y_{T,\gamma} \le e^{|E^{(N)}|\cdot|\gamma|} Y_{T,0}\end{aligned} \tag{2.34}$$

hold. Thus, it follows from (2.30) that $x \mapsto \log(d\mathbb{P}_\eta/d\Pi^-_{\gamma,\eta}(x))$ is a bounded measurable function on $\Omega^{(N)}$. Furthermore, using the first equality in (2.30), we see that $x \mapsto d\Pi_{\gamma,\eta}/d\mathbb{P}_\eta$ is also a bounded measurable function on $\Omega^{(N)}$. Consequently, the entropies in (2.31) are both finite. Using $\Pi_{\gamma,\eta} = \Xi_\gamma \mathbb{P}_\eta$, we obtain

$$E_{\Pi_{\gamma,\eta}}\left[\log \frac{d\Pi_{\gamma,\eta}}{d\mathbb{P}_\eta}\right] = E_{\Xi_\gamma \mathbb{P}_\eta}\left[\log\left(\frac{d\mathbb{P}_\eta}{d\Pi^-_{\gamma,\eta}} \circ \Xi_\gamma^{-1}\right)\right] = E_{\mathbb{P}_\eta}\left[\log \frac{d\mathbb{P}_\eta}{d\Pi^-_{\gamma,\eta}}\right]. \tag{2.35}$$

■

Recall the definitions (2.28). Fix $x \in \Omega^{(N)}$ and take $v \in V^{(N)}$ and $\gamma \in \mathbb{R}$. We define a probability measure $\mu_{v,\gamma} = \mu^{(N)}_{v,x,\gamma}$ on the set $E^{(N)}_v := \{e \in E^{(N)} : e \ni v\}$ by

$$\mu_{v,\gamma} := \sum_{e\in E^{(N)}_v} \frac{e^{\gamma D_e} x_e}{x_{v,\gamma}} \delta_e. \tag{2.36}$$

For our fixed $x \in \Omega^{(N)}$, we view $D_\bullet : E^{(N)}_v \to \mathbb{R}$, $e \mapsto D_e$, as a random variable on the probability space $(E^{(N)}_v, \mathcal{P}(E^{(N)}_v), \mu_{v,\gamma})$, again suppressing the dependence on the parameter x in the notation; here $\mathcal{P}(A)$ denotes the power set of the set A.

Lemma 2.9 *For all $\eta \in [0, 1]$, the function $\mathbb{R} \ni \gamma \mapsto \log(d\mathbb{P}_\eta/d\Pi^-_{\gamma,\eta})$ is twice continuously differentiable. The second derivative satisfies the bound*

$$\frac{\partial^2}{\partial^2\gamma}\left[\log \frac{d\mathbb{P}_\eta}{d\Pi^-_{\gamma,\eta}}\right] \le \left(2a + \frac{1}{2}\right) \sum_{v\in V^{(N)}\setminus\{0,\ell\}} \mathrm{Var}_{\mu_{v,\gamma}}(D_\bullet). \tag{2.37}$$

Proof. Fix $x \in \Omega^{(N)}$. We define a probability measure $\nu_\gamma = \nu_{x,\gamma}^{(N)}$ on the set $\mathcal{T}^{(N)}$ by

$$\nu_\gamma := \sum_{T \in \mathcal{T}^{(N)}} \frac{Y_{T,\gamma}}{\sum_{T' \in \mathcal{T}^{(N)}} Y_{T',\gamma}} \delta_T. \tag{2.38}$$

Note that

$$\frac{\partial}{\partial\gamma} x_{v,\gamma} = \sum_{e \ni v} \left(D_e e^{\gamma D_e} x_e \right), \tag{2.39}$$

$$\frac{\partial}{\partial\gamma} Y_{T,\gamma} = \sum_{e \in T} D_e Y_{T,\gamma} = \Delta_T Y_{T,\gamma}, \tag{2.40}$$

where we set

$$\Delta_T = \Delta_T(x) := \sum_{e \in T} D_e. \tag{2.41}$$

For our fixed $x \in \Omega^{(N)}$, we view $\Delta_\bullet : \mathcal{T}^{(N)} \to \mathbb{R}$, $T \mapsto \Delta_T$, as a random variable on the probability space $(\mathcal{T}^{(N)}, \mathcal{P}(\mathcal{T}^{(N)}), \nu_\gamma)$; again we drop the dependence on x in the notation. We calculate the first and second derivative of $\gamma \mapsto \log(d\mathbb{P}_\eta / d\Pi^-_{\gamma,\eta})$ using the representation (2.30):

$$\begin{aligned}
\frac{\partial}{\partial\gamma}\left[\log \frac{d\mathbb{P}_\eta}{d\Pi^-_{\gamma,\eta}}\right] &= -\sum_{e \in E^{(N)}} (D_e a) + 2a + \frac{1-\eta}{2} \\
&\quad + \left(2a + \frac{1}{2}\right) \sum_{v \in V^{(N)} \setminus \{0,\ell\}} \frac{1}{x_{v,\gamma}} \sum_{e \ni v} \left(D_e e^{\gamma D_e} x_e \right) - \frac{1}{2} \frac{\sum_{T \in \mathcal{T}^{(N)}} \Delta_T Y_{T,\gamma}}{\sum_{T \in \mathcal{T}^{(N)}} Y_{T,\gamma}} \\
&= -\sum_{e \in E^{(N)}} (D_e a) + 2a + \frac{1-\eta}{2} \\
&\quad + \left(2a + \frac{1}{2}\right) \sum_{v \in V^{(N)} \setminus \{0,\ell\}} E_{\mu_{v,\gamma}}[D_\bullet] - \frac{1}{2} E_{\nu_\gamma}[\Delta_\bullet].
\end{aligned} \tag{2.42}$$

We also calculate the second derivative:

$$\frac{\partial^2}{\partial^2\gamma}\left[\log \frac{d\mathbb{P}_\eta}{d\Pi^-_{\gamma,\eta}}\right] = \left(2a + \frac{1}{2}\right) \sum_{v \in V^{(N)} \setminus \{0,\ell\}} \mathrm{Var}_{\mu_{v,\gamma}}(D_\bullet) - \frac{1}{2} \mathrm{Var}_{\nu_\gamma}(\Delta_\bullet). \tag{2.43}$$

Since $\mathrm{Var}_{\nu_\gamma}(\Delta_\bullet) \geq 0$, the claim of the lemma follows. ■

Lemma 2.10 *The function*

$$f : \mathbb{R} \ni \gamma \mapsto f(\gamma) = E_{\mathbb{P}_\eta}\left[\log \frac{d\mathbb{P}_\eta}{d\Pi^-_{\gamma,\eta}}\right] \tag{2.44}$$

is twice continuously differentiable. The derivatives can be obtained by differentiating inside of the expectation, i.e.

$$\frac{\partial^j}{\partial^j \gamma}\left[E_{\mathbb{P}_\eta}\left[\log \frac{d\mathbb{P}_\eta}{d\Pi^-_{\gamma,\eta}}\right]\right] = E_{\mathbb{P}_\eta}\left[\frac{\partial^j}{\partial^j \gamma}\left[\log \frac{d\mathbb{P}_\eta}{d\Pi^-_{\gamma,\eta}}\right]\right] \tag{2.45}$$

for $j = 1, 2$. Furthermore, for any $\gamma \in \mathbb{R}$, one has

$$f(\gamma) = \int_0^\gamma E_{\mathbb{P}_\eta}\left[\frac{\partial^2}{\partial^2 \tilde{\gamma}}\left[\log \frac{d\mathbb{P}_\eta}{d\Pi^-_{\tilde{\gamma},\eta}}\right]\right](\gamma - \tilde{\gamma})\, d\tilde{\gamma}. \tag{2.46}$$

Proof. Note that $0 \le D_e \le 1$ for all e and $0 \le \Delta_T \le |E^{(N)}|$ for all $T \in \mathcal{T}^{(N)}$. These bounds are valid for all $x \in \Omega^{(N)}$. Consequently, it follows from (2.42) and (2.43) that there exists a constant $c_7(a, N) \in (0, \infty)$ such that for $j = 1, 2$ and all $x \in \Omega^{(N)}$, we have

$$\sup_{\gamma \in \mathbb{R}}\left|\frac{\partial^j}{\partial^j \gamma}\left[\log \frac{d\mathbb{P}_\eta}{d\Pi^-_{\gamma,\eta}}\right]\right| \le c_7(a, N). \tag{2.47}$$

By Lemma 2.9, $\gamma \mapsto \log(d\mathbb{P}_\eta/d\Pi^-_{\gamma,\eta})$ is twice continuously differentiable. Thus, by the dominated convergence theorem, the same is true for f, and (2.45) is valid for $j = 1, 2$. We know $f \ge 0$ because entropies are always non-negative. Furthermore, since $\Pi^-_{0,\eta} = \mathbb{P}_\eta$, we have $f(0) = 0$. Consequently, $f'(0) = 0$. A Taylor expansion of f around 0 yields

$$f(\gamma) = \int_0^\gamma f''(\tilde{\gamma})(\gamma - \tilde{\gamma})\, d\tilde{\gamma} = \int_0^\gamma E_{\mathbb{P}_\eta}\left[\frac{\partial^2}{\partial^2 \tilde{\gamma}}\left[\log \frac{d\mathbb{P}_\eta}{d\Pi^-_{\tilde{\gamma},\eta}}\right]\right](\gamma - \tilde{\gamma})\, d\tilde{\gamma}; \tag{2.48}$$

note that the last integral is finite by (2.47). ■

Proof of part (c) of Lemma 2.3. Recall the assumptions (i)–(iv) from Lemma 2.3. By Lemma 2.8(a), $\Pi_{\gamma,\eta}$ is absolutely continuous with respect to $\mathbb{P}_\eta$. To prove the entropy bound (2.14), first combine Lemma 2.8(c) with (2.46), and then, insert the bound (2.37). This yields:

$$E_{\Pi_{\gamma,\eta}}\left[\log \frac{d\Pi_{\gamma,\eta}}{d\mathbb{P}_\eta}\right] \le \left(2a + \frac{1}{2}\right) \sum_{v \in V^{(N)} \setminus \{0,\ell\}} \int_0^\gamma E_{\mathbb{P}_\eta}\left[\mathrm{Var}_{\mu^{(N)}_{v,\tilde{\gamma}}}(D_\bullet)\right](\gamma - \tilde{\gamma}) d\tilde{\gamma}. \tag{2.49}$$

Let $v \in V^{(N)} \setminus \{0, \ell\}$, and abbreviate

$$l_v := \mathrm{level}(v). \tag{2.50}$$

Recall that the measure $\mu_{v,\gamma} = \mu^{(N)}_{v,x,\gamma}$ depends on the environment $x \in \Omega^{(N)}$. In the following, we stress this dependence by writing $\mu^{(N)}_{v,x,\gamma}$ instead of $\mu_{v,\gamma}$. To estimate the variance of $D_\bullet$ with respect to the measure $\mu^{(N)}_{v,x,\gamma}$, we distinguish three cases.

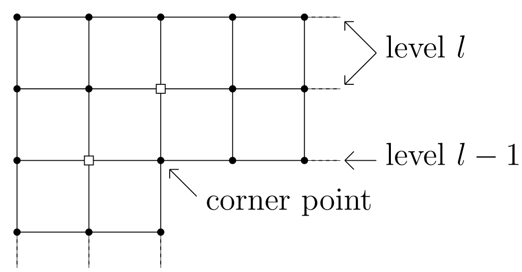
Case 1: Assume that $v \in V^{(N)} \setminus \{0, \ell\}$ is a corner point of a box $[-m, m]^2$, i.e. $v \in C^{(m)}$, for some $m \ge 1$.

Then, there are two possibilities (see Figure 1): If m is odd, all edges incident to v have both endpoints on level l_v. Otherwise, m is even and $m = 2l_v$. In both cases, by Definition 2.5, it follows that $D_{e'} = \varphi(l_v)$ for any e' incident to v. Hence, we have the estimate:

$$\mathrm{Var}_{\mu^{(N)}_{v,x,\gamma}}(D_\bullet) \leq E_{\mu^{(N)}_{v,x,\gamma}}\left[(D_\bullet - \varphi(l_v))^2\right] = 0. \tag{2.51}$$

Case 2: Assume that $v \in V^{(N)} \setminus \{0, \ell\}$ is a neighbor of a corner point u with $\mathrm{level}(u) \neq l_v$ (see Figure 4).

Figure 4: The vertices marked with a square are the neighbors of corner points considered in case 2.



Then, three edges incident to v have both endpoints on level l_v and one edge has one endpoint on the level of the corner point u, namely on level $l_v - 1$. Hence, for any e' incident to v, we have $D_{e'} \in \{\varphi(l_v), \varphi(l_v - 1)\}$, and consequently

$$\begin{aligned}\mathrm{Var}_{\mu^{(N)}_{v,x,\gamma}}(D_\bullet) \leq& E_{\mu^{(N)}_{v,x,\gamma}}\left[(D_\bullet - \varphi(l_v))^2\right] \\ \leq& \left(\varphi(l_v - 1) - \varphi(l_v)\right)^2.\end{aligned} \tag{2.52}$$

Let I denote the set of all vertices $v \in V^{(N)} \setminus \{0, \ell\}$ considered in case 2. Then,

$$\sum_{v \in I} \mathrm{Var}_{\mu^{(N)}_{v,x,\gamma}}(D_\bullet) \leq 8 \sum_{n=1}^{\infty} \left(\varphi(n+1) - \varphi(n)\right)^2. \tag{2.53}$$

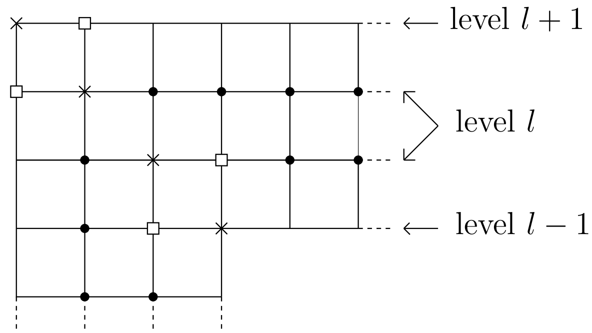
The factor 8 arises since there are 8 edges connecting corner points at level n to vertices at level $n+1$: Each of the 4 relevant corner points at level n is connected to 2 vertices at level $n+1$.

Case 3: Assume that $v \in V^{(N)} \setminus \{0, \ell\}$ is not a corner point of any box and v is not a neighbor of a corner point at a different level (see Figure 5).

Then, there is precisely one vertex u adjacent to v with $l_u := \mathrm{level}(u) \neq l_v$. We set $e(v) := \{u, v\}$. One has $D_{e'} = \varphi(l_v)$ for all $e' \ni v$ with $e' \neq e(v)$, and thus, it follows:

$$\begin{aligned}\mathrm{Var}_{\mu^{(N)}_{v,x,\gamma}}(D_\bullet) \leq& E_{\mu^{(N)}_{v,x,\gamma}}\left[(D_\bullet - \varphi(l_v))^2\right] \\ \leq& \left(D_{e(v)} - \varphi(l_v)\right)^2 \mu^{(N)}_{v,x,\gamma}(e(v)).\end{aligned} \tag{2.54}$$

Figure 5: Corner points are marked with a cross, neighbors of corner points at a different level (as treated in case 2) are marked with a square. The black dots are the vertices at level l covered in case 3.



Furthermore, since we have excluded v to be as in case 2, the definition (2.19) applies to $D_{e(v)}$. In particular, if $x_v < x_u$, then $D_{e(v)} = \varphi(l_v)$, and hence

$$\mathrm{Var}_{\mu^{(N)}_{v,x,\gamma}}(D_\bullet) = 0 \qquad \text{if } x_v < x_u. \tag{2.55}$$

For e' incident to v, we know that the difference $D_{e(v)} - D_{e'}$ takes one of the three values 0, $\varphi(l_u) - \varphi(l_v)$, and $(\varphi(l_u) - \varphi(l_v))/2$. Consequently,

$$\begin{aligned} \mu^{(N)}_{v,x,\gamma}(e(v)) =& \frac{e^{\gamma D_{e(v)}} x_{e(v)}}{x_{v,\gamma}} = \frac{x_{e(v)}}{\sum_{e' \ni v} e^{\gamma(D_{e'} - D_{e(v)})} x_{e'}} \\ \leq& \exp\{|\gamma| \cdot |\varphi(l_u) - \varphi(l_v)|\} \frac{x_{e(v)}}{x_v}. \end{aligned} \tag{2.56}$$

Assume that $x_u \leq x_v$. Then, combining (2.54) and (2.56) and using $\sqrt{x_u x_v} \leq x_v$ yields

$$\mathrm{Var}_{\mu^{(N)}_{v,x,\gamma}}(D_\bullet) \leq (\varphi(l_u) - \varphi(l_v))^2 \exp\{|\gamma| \cdot |\varphi(l_u) - \varphi(l_v)|\} \frac{x_{e(v)}}{\sqrt{x_u x_v}}. \tag{2.57}$$

Because of (2.55), this estimate is also true in the case $x_v < x_u$.

A side remark: At this point, it becomes clear why in Definition 2.5, D_e was introduced in such a tricky, x-dependent way: If we had used a more naive definition of D_e instead, formula (2.57) would have failed to hold.

For $e = \{u, v\} \in E$, we define

$$L_e := \frac{x_e}{\sqrt{x_u x_v}}. \tag{2.58}$$

Recall that $\mathbb{P}_\eta = \mathbb{P}^{(N)}_{\eta,0,\ell,a}$. The following lemma holds for all $a > 0$, N, and ℓ satisfying (i)–(iv) in Lemma 2.3.

Lemma 2.11 *For all $e \in E^{(N)}$ with $0, \ell \notin e$ and all $\eta \in [0,1]$, we have*

$$E_{\mathbb{P}_\eta}[L_e] \leq c_6(a,\eta) \tag{2.59}$$

with $c_6(a,\eta)$ as in (2.16).

Proof. Let $e = \{u, v\}$. Since $x_e \leq x_u$ and $x_e \leq x_v$, we have $L_e \leq 1$ and the claim follows in the case $0 \leq \eta < 1$.

Assume $\eta = 1$. By Definition 2.2, $\mathbb{P}_1 = \mathbb{Q}^{(N)}_{\ell,a}$; the normalizing constant equals $Z^{(N)}_{\eta,0,\ell,a} = 1$ because $\mathbb{Q}^{(N)}_{\ell,a}$ is a probability measure. Hence, we can apply Proposition 4.6 of [DR05] to obtain

$$E_{\mathbb{Q}^{(N)}_{\ell,a}}\left[L_e^2\right] = \frac{a(a+1)}{(4a+1)^2} \leq a. \tag{2.60}$$

Consequently, $E_{\mathbb{Q}^{(N)}_{\ell,a}}\left[L_e\right] \leq \left(E_{\mathbb{Q}^{(N)}_{\ell,a}}\left[L_e^2\right]\right)^{1/2} \leq c_6(a,1)$. ■

Integrating both sides of (2.57) with respect to $\mathbb{P}_\eta$ and applying Lemma 2.11 gives

$$E_{\mathbb{P}_\eta}\left[\mathrm{Var}_{\mu^{(N)}_{v,x,\gamma}}(D_\bullet)\right] \leq c_6(a,\eta)\left(\varphi(l_u) - \varphi(l_v)\right)^2 \exp\left\{|\gamma| \cdot |\varphi(l_u) - \varphi(l_v)|\right\}. \tag{2.61}$$

We sum the preceding inequality over the different vertices v:

$$\begin{aligned}&\sum_{\substack{v \in V^{(N)} \setminus \{0,\ell\} \\ v \notin \bigcup_{m=1}^\infty C^{(m)}, v \notin I}} E_{\mathbb{P}_\eta}\left[\mathrm{Var}_{\mu^{(N)}_{v,x,\gamma}}(D_\bullet)\right] \\ \leq & 8c_6(a,\eta)\sum_{n=1}^\infty (4n+1)\left(\varphi(n+1) - \varphi(n)\right)^2 \exp\left\{|\gamma| \cdot |\varphi(n+1) - \varphi(n)|\right\}.\end{aligned} \tag{2.62}$$

The factor $8(4n+1)$ arises, since there are not more than $4(4n+1)$ edges connecting level n to level $n+1$. Each of these edges is counted at most twice, once for each of its two endpoints.

By the definition of φ, we have

$$\varphi(n+1) - \varphi(n) = 0 \quad \text{if } 0 \leq n \leq n_a - 1 \quad \text{or} \quad n \geq l-1. \tag{2.63}$$

Furthermore, for $n_a \leq n \leq l-2$, we have

$$|\varphi(n+1) - \varphi(n)| \leq \sup_{n \leq x \leq n+1}\left|\frac{\partial}{\partial x}\frac{\log(x/n_a)}{\log((l-1)/n_a)}\right| = \frac{1}{n\log((l-1)/n_a)}. \tag{2.64}$$

Assume that γ satisfies (2.13):

$$|\gamma| \leq n_a \log\frac{l-1}{n_a}. \tag{2.65}$$

Then, for $n_a \le n \le l-2$,

$$\exp\{|\gamma| \cdot |\varphi(n+1) - \varphi(n)|\} \le e. \tag{2.66}$$

In the following step, we use $1 + c_6(a,\eta)e \le 4$, which follows from $c_6(a,\eta) \le 1$. Inserting the bound (2.66) in (2.62) and using (2.51), (2.53), (2.63), and (2.64) yields:

$$\begin{aligned}\sum_{v \in V^{(N)} \setminus \{0,\ell\}} E_{\mathbb{P}_\eta}\left[\mathrm{Var}_{\mu^{(N)}_{v,x,\gamma}}(D_\bullet)\right] &\le 8 \sum_{n=n_a}^{l-2} (1 + c_6(a,\eta)(4n+1)e)\,(\varphi(n+1) - \varphi(n))^2 \\ &\le \frac{8}{(\log((l-1)/n_a))^2} \sum_{n=n_a}^{l-2} \left(\frac{4c_6(a,\eta)e}{n} + \frac{1 + c_6(a,\eta)e}{n^2} \right) \\ &\le \frac{32}{(\log((l-1)/n_a))^2} \sum_{n=n_a}^{l-2} \left(\frac{c_6(a,\eta)e}{n} + \frac{1}{n^2} \right). \end{aligned} \tag{2.67}$$

Observe that

$$\sum_{n=n_a}^{l-2} \frac{1}{n} \le 2 \log \frac{l-1}{n_a} \qquad \text{and} \qquad \sum_{n=n_a}^{l-2} \frac{1}{n^2} \le \frac{2}{n_a}. \tag{2.68}$$

Hence, using that $\log((l-1)/n_a) \ge \log 2$ by (iv) in Lemma 2.3, we obtain

$$\sum_{v \in V^{(N)} \setminus \{0,\ell\}} E_{\mathbb{P}_\eta}\left[\mathrm{Var}_{\mu^{(N)}_{v,x,\gamma}}(D_\bullet)\right] \le \frac{64}{n_a \log((l-1)/n_a)} \left(c_6(a,\eta) e n_a + \frac{1}{\log 2} \right). \tag{2.69}$$

Combining this bound with (2.49) yields

$$E_{\Pi_{\gamma,\eta}}\left[\log \frac{d\Pi_{\gamma,\eta}}{d\mathbb{P}_\eta}\right] \le \frac{c_5(a,\eta)\gamma^2}{n_a \log((l-1)/n_a)} \tag{2.70}$$

with

$$c_5(a,\eta) := 32 \left(2a + \frac{1}{2}\right) \left(e c_6(a,\eta) n_a + \frac{1}{\log 2} \right) \ge \frac{16}{\log 2} > 16. \tag{2.71}$$

In the special case $\eta = 1$, because of (ii) in Lemma 2.3 and (2.16), $c_6(a,1)n_a \to 1$ as $a \to 0$. Hence, $\lim_{a \to 0} c_5(a,1) < \infty$. This completes the proof of part (c) of Lemma 2.3. ■

2.4 Auxiliary finite-volume estimates

The following tail estimates are proved in [MR05d]. We need them below. Recall that the random environment $\mathbb{Q}^{(N)}_{0,a}$ for the finite box $V^{(N)}$ is unique up to a multiplication of the edge weights by a constant.

Theorem 2.12 (Theorems 2.3 and 2.4 in [MR05d]) *For all $a > 0$, there are constants $c_8(a) > 0$ and $c_9(a) > 0$, depending only on a, such that the following estimates hold for all $N \in \mathbb{N}$:*

(a) For all $e, f \in E^{(N)}$ with $e \cap f \neq \emptyset$, and all $M > 0$, one has

$$\mathbb{Q}_{0,a}^{(N)}\left[\frac{x_e}{x_f} \geq M\right] \leq c_8(a) M^{-a/2}. \tag{2.72}$$

(b) For all $v \in V^{(N)}$, all $e \in E^{(N)}$ incident to v, and all $M > 0$, one has

$$\mathbb{Q}_{0,a}^{(N)}\left[\frac{x_e}{x_v} \leq M\right] \leq c_9(a) M^{a/2}. \tag{2.73}$$

Proof of Lemma 1.9. Let $a > 0$, $N \geq 2$, and let $e, f \in E^{(N)}$ satisfy $e \cap f \neq \emptyset$. Using (2.72), we obtain for any $\alpha > 0$:

$$\begin{aligned} E_{\mathbb{Q}_{0,a}^{(N)}}\left[\left(\frac{x_e}{x_f}\right)^\alpha\right] \leq & 1 + \int_1^\infty \mathbb{Q}_{0,a}^{(N)}\left[\left(\frac{x_e}{x_f}\right)^\alpha \geq M\right] dM \\ \leq & 1 + c_8(a) \int_1^\infty M^{-a/(2\alpha)}\, dM. \end{aligned} \tag{2.74}$$

The last integral is finite whenever $\alpha \in (0, a/2)$. Since the upper bound in (2.74) is uniform in e, f, and N, the claim (1.18) follows.

To prove (1.19), let $v, w \in V^{(N)}$ with $|v - w| = 1$. Denote by $e := \{v, w\}$ the edge connecting v and w. Then, using (2.73) and $\log(x_v/x_e) \geq 0$,

$$\begin{aligned} E_{\mathbb{Q}_{0,a}^{(N)}}\left[\log \frac{x_v}{x_w}\right] \leq & E_{\mathbb{Q}_{0,a}^{(N)}}\left[\log \frac{x_v}{x_e}\right] \\ = & \int_0^\infty \mathbb{Q}_{0,a}^{(N)}\left[\log \frac{x_v}{x_e} \geq M\right] dM \\ \leq & 1 + c_9(a) \int_1^\infty e^{-Ma/2}\, dM < \infty; \end{aligned} \tag{2.75}$$

clearly, the upper bound is uniform in v, w, and N. This completes the proof of (1.19). ■

2.5 Proof of Theorems 1.7 and 1.8

Recall the definition (2.7) of Σ_ℓ. We abbreviate

$$\mathbb{Q}_0 := \mathbb{Q}_{0,a}^{(N)}. \tag{2.76}$$

Theorem 2.13 (Key estimate) *Let $a > 0$, n_a, $N \in \mathbb{N}$, and $\ell \in V^{(N)}$ fulfill the assumptions (i)–(iv) from Lemma 2.3, and let $0 \le \eta \le 1$. Then*

$$\log E_{\mathbb{Q}_0}\left[\exp(\eta\Sigma_\ell)\right] - \eta E_{\mathbb{P}_\eta}\left[\Sigma_\ell\right] \le \frac{-\eta^2 n_a \log((l-1)/n_a)}{16 c_5(a,\eta)} \tag{2.77}$$

with $c_5(a,\eta)$ given by (2.15).

Proof. We set

$$\gamma = -\frac{\eta n_a \log((l-1)/n_a)}{4c_5(a,\eta)}. \tag{2.78}$$

The condition (2.13) is satisfied for this choice, since we have $c_5(a,\eta) > 16$ for all $\eta \in [0,1]$ and $a > 0$. Using the entropy bound (2.14) and positivity of entropies, we get

$$\begin{aligned} -\frac{\eta\gamma}{4} =& \frac{c_5(a,\eta)\gamma^2}{n_a \log((l-1)/n_a)} \ge E_{\Pi_{\gamma,\eta}}\left[\log \frac{d\Pi_{\gamma,\eta}}{d\mathbb{P}_\eta}\right] \\ =& E_{\Pi_{\gamma,\eta}}\left[\log \frac{d\mathbb{Q}_0}{d\mathbb{P}_\eta}\right] + E_{\Pi_{\gamma,\eta}}\left[\log \frac{d\Pi_{\gamma,\eta}}{d\mathbb{Q}_0}\right] \\ \ge& E_{\Pi_{\gamma,\eta}}\left[\log \frac{d\mathbb{Q}_0}{d\mathbb{P}_\eta}\right] = E_{\mathbb{P}_\eta}\left[\log\left(\frac{d\mathbb{Q}_0}{d\mathbb{P}_\eta}\circ \Xi_\gamma\right)\right]; \end{aligned} \tag{2.79}$$

here we used the definition (2.12) of the measure $\Pi_{\gamma,\eta}$ in the last step. Note that all expectations occurring in (2.79) are finite. As a consequence of the two equations (2.11), we find

$$\Sigma_\ell \circ \Xi_\gamma = \Sigma_\ell + \frac{\gamma}{2}. \tag{2.80}$$

Thus, using (2.9), we find

$$\log\left(\frac{d\mathbb{Q}_0}{d\mathbb{P}_\eta}\circ\Xi_\gamma\right) = \log Z^{(N)}_{\eta,0,\ell,a} - \eta\Sigma_\ell\circ\Xi_\gamma = \log E_{\mathbb{Q}_0}[\exp(\eta\Sigma_\ell)] - \eta\Sigma_\ell - \frac{\eta\gamma}{2}. \tag{2.81}$$

Consequently, it follows from (2.79):

$$-\frac{\eta\gamma}{4} \ge \log E_{\mathbb{Q}_0}[\exp(\eta\Sigma_\ell)] - \eta E_{\mathbb{P}_\eta}[\Sigma_\ell] - \frac{\eta\gamma}{2}. \tag{2.82}$$

Combining this with our choice (2.78) for γ, we obtain the claim (2.77). ■

Roughly speaking, Theorems 1.7 and 1.8 are just the special cases $\eta = 1$ and $\eta = 1/2$ of Theorem 2.13:

Proof of Theorem 1.7. Let $a > 0$, and let n_a be as in (ii) of Lemma 2.3. We set $c_2(a) = n_a/(8c_5(a,1)) > 0$ with $c_5(a,1)$ as in (2.15) with $\eta = 1$ and

$$c_3(a) = \max\left\{\frac{n_a \log(4n_a)}{8c_5(a,1)}, 12n_a c_4(a) + c_2(a)\log(12n_a)\right\}, \tag{2.83}$$

where the constant $c_4(a)$ is taken from formula (1.19). Note that $c_2(a) \to \infty$ as $a \to 0$ because $n_a \to \infty$ as $a \to 0$ and $\lim_{a\to 0} c_5(a,1) < \infty$. Now let $N \in \mathbb{N}$ and $v \in V^{(N)} \setminus \{0\}$, and set $l := \text{level}(v)$. We distinguish two cases; first finitely many exceptional cases, and then the general case.

Case 1: $N \le 6n_a$ or $l \le 3n_a$. In this case, there is a path $0 = v_0, v_1, \dots, v_k = v$ in $V^{(N)}$ joining the vertices 0 and v of length $k \le 12n_a$; recall that levels have width two. Taking the expectation $E_{\mathbb{Q}^{(N)}_{0,a}}$ of

$$\log \frac{x_v}{x_0} = \sum_{i=1}^{k} \log \frac{x_{v_i}}{x_{v_{i-1}}} \tag{2.84}$$

and using the formula (1.19) from Lemma 1.9 and the fact that $|v| \le 12n_a$, we obtain the bound (1.16):

$$E_{\mathbb{Q}^{(N)}_{0,a}}\left[\log \frac{x_v}{x_0}\right] \le 12n_a c_4(a) \le c_3(a) - c_2(a)\log(12n_a) \le c_3(a) - c_2(a)\log|v|. \tag{2.85}$$

Case 2: $N > 6n_a$ and $l > 3n_a$. In this case, Theorem 2.13 is applicable with $\ell = v$. Using (2.8) and $\mathbb{P}_1 = \mathbb{Q}^{(N)}_{v,a}$, we rewrite (2.77) for the special value $\eta = 1$ in the form

$$\begin{aligned}
&\frac{1}{2}E_{\mathbb{Q}^{(N)}_{v,a}}\left[\log \frac{x_0}{x_v}\right] = -E_{\mathbb{P}_1}\left[\log \frac{d\mathbb{P}_1}{d\mathbb{Q}_0}\right] \\
&= \log E_{\mathbb{Q}_0}\left[\frac{d\mathbb{P}_1}{d\mathbb{Q}_0}\right] - E_{\mathbb{P}_1}\left[\log \frac{d\mathbb{P}_1}{d\mathbb{Q}_0}\right] \le \frac{-n_a\log((l-1)/n_a)}{16c_5(a,1)} \\
&\le \frac{-n_a\log(|v|/(4n_a))}{16c_5(a,1)} \le \frac{1}{2}\left(c_3(a) - c_2(a)\log|v|\right);
\end{aligned} \tag{2.86}$$

note that $l - 1 \ge l/\sqrt{2} \ge |v|/4$ for all $l > 3n_a$. Recall that the box $V^{(N)}$ has periodic boundary conditions. Using reflection symmetry, we interchange 0 and v to obtain the claim (1.16):

$$E_{\mathbb{Q}^{(N)}_{0,a}}\left[\log \frac{x_v}{x_0}\right] = E_{\mathbb{Q}^{(N)}_{v,a}}\left[\log \frac{x_0}{x_v}\right] \le c_3(a) - c_2(a)\log|v|. \tag{2.87}$$

■

Proof of Theorem 1.8. Let $a > 0$, n_a as above, and set

$$\beta(a) = \frac{n_a}{64c_5(a,1/2)} \qquad \text{and} \qquad c_1(a) = 2\cdot(6\sqrt{2}n_a)^{\beta(a)}. \tag{2.88}$$

Let $N \in \mathbb{N}$ and $v \in V^{(N)} \setminus \{0\}$, $l = \text{level}(v)$. We distinguish the same two cases as in the previous proof:

Case 1: $N \le 6n_a$ or $l \le 3n_a$. We observe that $E_{\mathbb{Q}^{(N)}_{0,a}}[(x_v/x_0)^{1/4}]$ is bounded: Using (2.8) and $z^{1/4} \le 1 + z^{1/2}$ for all $z \ge 0$, we get

$$E_{\mathbb{Q}_0}\left[\left(\frac{x_v}{x_0}\right)^{\frac{1}{4}}\right] \le 1 + E_{\mathbb{Q}_0}\left[\sqrt{\frac{x_v}{x_0}}\right] = 1 + E_{\mathbb{Q}_0}\left[\frac{d\mathbb{P}_1}{d\mathbb{Q}_0}\right] = 2 \le c_1(a)|v|^{-\beta(a)}; \tag{2.89}$$

in the last step, we used $|v| \leq 2\sqrt{2}l \leq 6\sqrt{2}n_a$.
Case 2: $N > 6n_a$ and $l > 3n_a$. This time, we apply Theorem 2.13 with $\eta = 1/2$ and $\ell = v$. Note that by (2.4), we have

$$\frac{d\mathbb{P}_{1/2}}{d\mathbb{Q}_0} = \frac{1}{Z^{(N)}_{1/2,0,v,a}} \left(\frac{x_v}{x_0}\right)^{\frac{1}{4}} \tag{2.90}$$

and thus

$$\mathbb{P}_{1/2}(dx) = \frac{1}{Z^{(N)}_{1/2,0,v,a}} \left(\frac{x_v}{x_0}\right)^{\frac{1}{4}} \mathbb{Q}_0(dx) = \frac{1}{Z^{(N)}_{1/2,0,v,a}} \left(\frac{x_0}{x_v}\right)^{\frac{1}{4}} \mathbb{Q}^{(N)}_{v,a}(dx). \tag{2.91}$$

Using reflection symmetry again, we interchange 0 and v in the following computation:

$$\begin{aligned} E_{\mathbb{P}_{1/2}}[\Sigma_v] &= \frac{1}{Z^{(N)}_{1/2,0,v,a}} E_{\mathbb{Q}_0}\left[\left(\frac{x_v}{x_0}\right)^{\frac{1}{4}} \log\sqrt{\frac{x_v}{x_0}}\right] \\ &= \frac{1}{Z^{(N)}_{1/2,0,v,a}} E_{\mathbb{Q}^{(N)}_{v,a}}\left[\left(\frac{x_0}{x_v}\right)^{\frac{1}{4}} \log\sqrt{\frac{x_0}{x_v}}\right] = -E_{\mathbb{P}_{1/2}}[\Sigma_v] \end{aligned} \tag{2.92}$$

and thus

$$E_{\mathbb{P}_{1/2}}[\Sigma_v] = 0. \tag{2.93}$$

Inserting this in the estimate (2.77) of Theorem 2.13 with $\eta = 1/2$, we obtain the claim (1.17) of Theorem 1.8:

$$\begin{aligned} &\log E_{\mathbb{Q}_0}\left[\left(\frac{x_v}{x_0}\right)^{\frac{1}{4}}\right] = \log E_{\mathbb{Q}_0}[\exp(\Sigma_v/2)] \leq \frac{-n_a \log((l-1)/n_a)}{64c_5(a,1/2)} \\ &\leq \frac{-n_a \log(|v|/(4n_a))}{64c_5(a,1/2)} \leq \log[c_1(a)|v|^{-\beta(a)}]. \end{aligned} \tag{2.94}$$

■

3 Bounds in infinite-volume

3.1 Infinite-volume limits

In this section, we deduce the infinite-volume results Theorems 1.3, 1.4, and Theorem 1.5 from their finite volume analogues, namely Theorems 1.7, 1.8, and Lemma 1.9.

Proof of Theorems 1.3, 1.4, and Theorem 1.5. Let $a > 0$, and let $v \in \mathbb{Z}^2$. In the proof of Theorem 2.2 in [MR05d] it is shown that there is a subsequence $(n(k))_{k\in\mathbb{N}}$ such that for any finite subset $F \subset E$, the $\mathbb{Q}^{(n(k))}_{0,a}$-distribution of $(x_e)_{e\in F}$ converges weakly to the $\mathbb{Q}_{0,a}$-distribution of $(x_e)_{e\in F}$. Recall from (2.1) that the weights are normalized such that $x_{e_0} = 1$ holds $\mathbb{Q}_{0,a}^{(n(k))}$-a.s. for a fixed reference edge $e_0 \in E$.

To prove Theorem 1.4, let $v \in \mathbb{Z}^2 \setminus \{0\}$. Since $(x_v/x_0)^{1/4}$ takes only positive values and is a continuous function of the finitely many weights x_e with $e \ni v$ or $e \ni 0$, we conclude

$$E_{\mathbb{Q}_{0,a}}\left[\left(\frac{x_v}{x_0}\right)^{\frac{1}{4}}\right] \leq \liminf_{k\to\infty} E_{\mathbb{Q}_{0,a}^{(n(k))}}\left[\left(\frac{x_v}{x_0}\right)^{\frac{1}{4}}\right] \leq c_1(a)|v|^{-\beta(a)}; \tag{3.1}$$

we used Theorem 1.8 in the last step. This proves Theorem 1.4. Using Lemma 1.9, the same argument yields Theorem 1.5.

To prove Theorem 1.3, we observe that $\log(x_v/x_0)$ is also a continuous function of the finitely many weights x_e with $e \ni v$ or $e \ni 0$.

Let $0 = v_0, v_1, \ldots, v_L = v$ be a path from 0 to v, and let $e_i := \{v_{i-1}, v_i\}$, $1 \leq i \leq L$. Then, for all N large enough that $v_i \in V^{(N)}$ for all $1 \leq i \leq L$ and all $M > 0$, one finds:

$$\begin{aligned}
\mathbb{Q}_{0,a}^{(N)}\left[\left|\log\frac{x_v}{x_0}\right| \geq M\right] \leq & \mathbb{Q}_{0,a}^{(N)}\left[\sum_{i=1}^{L}\left|\log\frac{x_{v_i}}{x_{v_{i-1}}}\right| \geq M\right] \\
\leq & \sum_{i=1}^{L}\mathbb{Q}_{0,a}^{(N)}\left[\left|\log\frac{x_{v_i}}{x_{v_{i-1}}}\right| \geq \frac{M}{L}\right] \\
\leq & \sum_{i=1}^{L}\left\{\mathbb{Q}_{0,a}^{(N)}\left[\log\frac{x_{v_i}}{x_{e_i}} \geq \frac{M}{L}\right] + \mathbb{Q}_{0,a}^{(N)}\left[\log\frac{x_{v_{i-1}}}{x_{e_i}} \geq \frac{M}{L}\right]\right\} \\
\leq & 2Lc_9(a)e^{-Ma/(2L)};
\end{aligned} \tag{3.2}$$

in the last but one step, we used $x_{e_i} \leq \min\{x_{v_i}, x_{v_{i-1}}\}$, and in the last step we used (2.73) from Theorem 2.12. Thus, $\log(x_v/x_0)$ has exponential tails, uniformly in the size N of the box. Consequently, we can take the limit along the subsequence $(n(k))_{k\in\mathbb{N}}$ in (1.16) to obtain (1.10). ■

3.2 Hitting probabilities for ERRW

Finally, we apply our bounds for the random environment to deduce estimates for the hitting probabilities for the edge-reinforced random walk.

Proof of Theorem 1.1.

(a) We claim first that for all $x \in \Omega$ and all $v \in \mathbb{Z}^2 \setminus \{0\}$, the probability $Q_{0,x}[\tau_v < \tau_0]$ for the random walk starting in 0 in the fixed environment x to visit v before returning to 0 and the probability $Q_{v,x}[\tau_0 < \tau_v]$ for the random walk with exchanged roles of 0 and v in the same environment are connected by the following equation:

$$Q_{0,x}[\tau_v < \tau_0] = \frac{x_v}{x_0}Q_{v,x}[\tau_0 < \tau_v]. \tag{3.3}$$

To prove this claim, take two vertices $u \neq w$. Denote by $\Pi_{u,w}$ the set of all admissible finite paths

$$\pi = (u = v_0, v_1, \dots, v_n = w), \tag{3.4}$$

$n \in \mathbb{N}$, joining u and w which do not visit u or w twice. For any such path π, we introduce the event

$$A_\pi := \{X_i = v_i \text{ for } i = 0, 1, \dots, n\} \subseteq (\mathbb{Z}^2)^{\mathbb{N}_0} \tag{3.5}$$

that π is an initial piece of the random path. Note that the events A_π, $\pi \in \Pi_{u,w}$, are pairwise disjoint. Furthermore, let

$$\pi^{\leftrightarrow} := (v_n, \dots, v_1, v_0) \tag{3.6}$$

denote the reversed path. Note that the reversion defines a bijection $\cdot^{\leftrightarrow} : \Pi_{0,v} \to \Pi_{v,0}$. Moreover, for any path π as in (3.4),

$$x_u Q_{u,x}[A_\pi] = x_u \prod_{i=0}^{n-1} \frac{x_{\{v_i,v_{i+1}\}}}{x_{v_i}} = x_w \prod_{i=1}^{n} \frac{x_{\{v_i,v_{i-1}\}}}{x_{v_i}} = x_w Q_{w,x}[A_{\pi^{\leftrightarrow}}]. \tag{3.7}$$

Now, we take $u = 0$ and $w = v$. Summing (3.7) over all $\pi \in \Pi_{0,v}$, we obtain the claim (3.3) as follows:

$$\begin{aligned} x_0 Q_{0,x}[\tau_v < \tau_0] &= x_0 \sum_{\pi \in \Pi_{0,v}} Q_{0,x}[A_\pi] = x_v \sum_{\pi \in \Pi_{0,v}} Q_{v,x}[A_{\pi^{\leftrightarrow}}] \\ &= x_v \sum_{\pi \in \Pi_{v,0}} Q_{v,x}[A_\pi] = x_v Q_{v,x}[\tau_0 < \tau_v]. \end{aligned} \tag{3.8}$$

From this, we conclude

$$Q_{0,x}[\tau_v < \tau_0] \leq \frac{x_v}{x_0}. \tag{3.9}$$

Taking the 1/4-th power and expectations yields

$$\begin{aligned} P_{0,a}[\tau_v < \tau_0] =& E_{\mathbb{Q}_{0,a}}[Q_{0,x}[\tau_v < \tau_0]] \leq E_{\mathbb{Q}_{0,a}}[Q_{0,x}[\tau_v < \tau_0]^{1/4}] \\ \leq& E_{\mathbb{Q}_{0,a}}\left[\left(\frac{x_v}{x_0}\right)^{1/4}\right] \leq c_1(a)|v|^{-\beta(a)}; \end{aligned} \tag{3.10}$$

we used the representation of the edge-reinforced random walk as a random walk in random environment from Theorem 1.2 in the first step and the bound (1.11) from Theorem 1.4 in the last step. This shows part (a) of Theorem 1.1.

(b) To prove part (b), let $\Sigma^n_{u,w}$ denote the set of all admissible paths

$$\pi = (u = v_0, v_1, \dots, v_n = w) \tag{3.11}$$

from u to w of length n. Again, reversion yields a bijection between $\Sigma^n_{u,w}$ and $\Sigma^n_{w,u}$, and the events A_π, $\pi \in \Sigma^n_{u,w}$, are pairwise disjoint. In analogy to (3.8), we obtain

$$x_0 Q_{0,x}[X_n = v] = x_0 \sum_{\pi \in \Sigma^n_{0,v}} Q_{0,x}[A_\pi] = x_v \sum_{\pi \in \Sigma^n_{v,0}} Q_{v,x}[A_\pi] = x_v Q_{v,x}[X_n = 0]. \quad (3.12)$$

Using this, an analogous argument to (3.10) yields the claim (1.5).

■

A Appendix: Proof of Lemma 2.1

In this appendix, we consider a generalization of Lemma 2.1 to arbitrary finite graphs. It essentially states the formula of Coppersmith and Diaconis [CD86] for the law of the random environment, transformed to a special normalization.

Consider edge-reinforced random walk on any finite graph (V,E) with starting point $v_0 \in V$ and initial weights $a = (a_e)_{e\in E} \in (0,\infty)^E$. Recall the definition (1.9) of x_v; we use the similar notation $a_v = \sum_{e \ni v} a_e$. For $x = (x_e)_{e\in E} \in (0,\infty)^E$, we set

$$\phi_{v_0,a}(x) = c_{10}(v_0,a) \frac{\prod_{e\in E} x_e^{a_e-1}}{x_{v_0}^{a_{v_0}/2} \prod_{v\in V\setminus\{v_0\}} x_v^{(a_v+1)/2}} \sqrt{\sum_{T\in\mathcal{T}} \prod_{e\in T} x_e}, \quad (A.1)$$

where the sum is indexed by the set $\mathcal{T}$ of all spanning trees of (V,E), viewed as sets of edges, and the constant $c_{10}(v_0,a)$ is defined by

$$c_{10}(v_0,a) = \frac{\Gamma(a_{v_0}/2) \prod_{v\in V\setminus\{v_0\}} \Gamma((a_v+1)/2)}{\prod_{e\in E} \Gamma(a_e)} \frac{2^{1-|V|+\sum_{e\in E} a_e}}{\pi^{(|V|-1)/2}}. \quad (A.2)$$

Lemma A.1 *The above edge-reinforced random walk on (V,E) has the same distribution as a random walk in a random environment given by random positive weights $\tilde{x} = (\tilde{x}_e)_{e\in E}$ on the edges. Normalizing $\tilde{x}$ such that $\tilde{x}_{e_0} = 1$ for a fixed reference edge e_0, the law of $\tilde{x}$ has the density $\phi_{v_0,a}$ with respect to the Lebesgue measure $\delta_1(d\tilde{x}_{e_0}) \prod_{e\in E\setminus\{e_0\}} d\tilde{x}_e$ on the hyperplane $H = \{(\tilde{x}_e)_{e\in E} \in (0,\infty)^E \mid \tilde{x}_{e_0} = 1\}$.*

Proof. By Theorem 3.1 of [Rol03], the edge-reinforced random walk on (V,E) has the same distribution as a random walk in a random environment given by random positive weights $x = (x_e)_{e\in E}$ on the edges. The law of the random environment $\mathbb{Q}^\Delta_{v_0,a}$, normalized such that $\sum_{e\in E} x_e = 1$, has a density with respect to the normalized surface measure on the simplex $\Delta = \{(x_e)_{e\in E} \in (0,\infty)^E \mid \sum_{e\in E} x_e = 1\}$. The density is provided by Theorem 1 in [KR00]. Combining this theorem with the matrix-tree-theorem ([Mau76], p. 145, theorem 3', see also Theorem 3 in [KR00]), it is given by

$$\frac{d\mathbb{Q}^\Delta_{v_0,a}}{d\sigma}(x) = \frac{\phi_{v_0,a}(x)}{(|E|-1)!}. \quad (A.3)$$

Consider the change of normalization

$$F:\Delta\to H,\qquad F((x_e)_{e\in E})=\left(\frac{x_e}{x_{e_0}}\right)_{e\in E}. \tag{A.4}$$

We factor F as follows: $\Delta\stackrel{\pi}{\longrightarrow}\pi[\Delta]\stackrel{f}{\longrightarrow}(0,\infty)^{E\setminus\{e_0\}}\stackrel{\iota}{\longrightarrow}H$,

$$(x_e)_{e\in E}\stackrel{\pi}{\longmapsto}(x_e)_{e\in E\setminus\{e_0\}}\stackrel{f}{\longmapsto}\left(\tilde{x}_e=\frac{x_e}{1-\sum_{e'\in E\setminus\{e_0\}}x_{e'}}=\frac{x_e}{x_{e_0}}\right)_{e\in E\setminus\{e_0\}}\stackrel{\iota}{\longmapsto}(\tilde{x}_e)_{e\in E} \tag{A.5}$$

where the first map is the canonical projection and the last map ι just includes an extra component $\tilde{x}_{e_0}=1$. Let us calculate the Jacobi determinant of the map f. Using the abbreviation $x_{e_0}=1-\sum_{e'\in E\setminus\{e_0\}}x_{e'}$, we have the Jacobi matrix

$$Df(x)=\left(\frac{\partial\tilde{x}_e}{\partial x_{e'}}\right)_{e,e'\in E\setminus\{e_0\}}=\frac{1}{x_{e_0}}\left(\delta_{ee'}+\frac{x_e}{x_{e_0}}\right)_{e,e'\in E\setminus\{e_0\}}, \tag{A.6}$$

which is $1/x_{e_0}$ times the identity matrix I plus a rank 1 matrix. Since $\det(I+A)=1+\operatorname{tr}A$ holds for rank 1 matrices A, we get the Jacobi determinant

$$\det Df(x)=\frac{1}{x_{e_0}^{|E|-1}}\left(1+\sum_{e\in E\setminus\{e_0\}}\frac{x_e}{x_{e_0}}\right)=\frac{1}{x_{e_0}^{|E|}}. \tag{A.7}$$

Abbreviating

$$\alpha:=\sum_{e\in E}a_e-|E|-\sum_{v\in V}\frac{a_v}{2}-\frac{|V|-1}{2}=-|E|-\frac{|V|-1}{2}=-|E|-\frac{|T|}{2} \tag{A.8}$$

for any spanning tree $T\subseteq E$ in (V,E), we rewrite (A.1) as

$$\phi_{v_0,a}(x)=c_{10}(v_0,a)x_{e_0}^{\alpha}\frac{\prod_{e\in E}\tilde{x}_e^{a_e-1}}{\tilde{x}_{v_0}^{a_{v_0}/2}\prod_{v\in V\setminus\{v_0\}}\tilde{x}_v^{(a_v+1)/2}}\sqrt{\sum_{T\in\mathcal{T}}x_{e_0}^{|T|}\prod_{e\in T}\tilde{x}_e}=x_{e_0}^{-|E|}\phi_{v_0,a}(\tilde{x}) \tag{A.9}$$

and thus

$$\frac{\phi_{v_0,a}(x)}{\det Df(\pi(x))}=\phi_{v_0,a}(\tilde{x}). \tag{A.10}$$

We combine this with (A.3). Using that the projected normalized surface measure $\pi\sigma$ has the density $(|E|-1)!$ with respect to the Lebesgue measure on $\pi[\Delta]$, we get that the transformed distribution $\mathbb{Q}_{v_0,a}=F\mathbb{Q}_{v_0,a}^{\Delta}$ has the density $\phi_{v_0,a}(\tilde{x})$ with respect to the Lebesgue measure $\delta_1(d\tilde{x}_{e_0})\prod_{e\neq e_0}d\tilde{x}_e$ on the hyperplane H. This proves the claim. ■

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