ON THE 2D ISING WULFF CRYSTAL NEAR CRITICALITY

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Abstract. We study the behavior of the two-dimensional Ising model in a finite box at temperatures that are below, but very close to, the critical temperature. In a regime where the temperature approaches the critical point and, simultaneously, the size of the box grows fast enough, we establish a large deviation principle that proves the appearance of a round Wulff crystal.

1. Introduction

The Ising model in two dimensions is the first model where phase transition and non mean-field critical behavior has been established by Onsager [43] in 1944. It is also for that particular model that phase coexistence was rigorously studied and led to the first microscopic justification of the Wulff crystal. The first proof by Dobrushin, Kotecký and Shlosman [23] was valid for temperatures that are much lower than the critical point. Simplifications by Pfister [44], Ioffe and Schonmann [33, 34, 35] improved the result up to the critical point. These results in two dimensions rely on the study of contours to analyze large deviations of surface order. The extension of the Wulff construction to the Ising model in dimensions greater or equal to three required new techniques such as block coarse graining and the use of tools coming from geometric measure theory. This was achieved by Cerf, Pisztora [16] and Bodineau [11]. These results were initially valid up to the slab "percolation" threshold and recently Bodineau [12] proved that this threshold is indeed the usual critical point, thus extending the results of [16, 46] up to the critical temperature T_c. A two-dimensional analogue of the coarse graining developed in [46] is the subject of [19], thereby providing a unified approach to treat the problem for all dimensions. The appearance of the Wulff crystal has been proved in other "percolation" type models as well: [7], [4] in two dimensions and [14], [17] for dimensions greater or equal to three.

In all the works described above, the temperature has been kept fixed away from the critical temperature. Our main goal is to study the impact of the presence of a second order phase transition on the phase coexistence phenomenon. We do this by analyzing phase coexistence in a regime where the temperature approaches the critical point from below while simultaneously taking the thermodynamical limit.

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A priori, one can expect that a second order phase transition has a non trivial effect on phase coexistence. Indeed, when approaching the critical point, the basic quantities describing the model either diverge or stay finite but have divergent derivatives. In the second case, they decay as a power law giving rise to critical exponents. This critical behavior and the existence of these exponents is conjectured for a wide family of two-dimensional statistical mechanics models. The existence of some of these critical exponents in a strong sense is an important ingredient in our analysis. This ingredient is available for the two-dimensional Ising model. For this particular model, the relevant statistical mechanics quantities can be computed explicitly, giving rise to beautiful identities such as Onsager's formula for the magnetization [43] and an explicit formula for the surface tension that describes the geometry of the Wulff droplet in terms of random walks [41, 42]. Such results can be obtained, for example, by using the dimer representation of Kasteleyn [36].

The probabilistic understanding of the critical phenomena is a very active field nowadays. In the case of Bernoulli site percolation on the planar triangular lattice, the existence and the identification of the critical exponents have been rigorously established by Smirnov and Werner [49] after the groundbreaking work of Schramm [47] and Smirnov [48]. In [49], the existence of the critical exponents has been explained in a probabilistic manner. Indeed, this work establishes a rigorous link with the conformal invariance of the scaling limit of critical percolation, described by the Schramm Loewner evolution process. Regarding the above mentioned results, the reader may wonder why we don't investigate the influence of a phase transition on phase coexistence in the a priori simpler and better understood *independent* Bernoulli percolation model instead of the *dependent* spins of the Ising model. The reason is that despite the spectacular progress in the understanding of criticality of independent percolation, essential properties of the critical exponents are still unaccessible by other methods than explicit computation; see in particular the open question 3 at the end of [49]. And since explicit computations work only for the two-dimensional Ising model, our results are confined to this particular model.

2. Statement of the main results

Consider the Ising model at temperature $T < T_c$, defined on a square box $\Lambda(n)$ of side length $n \in \mathbb{N} \setminus \{0\}$ and submitted to $+$-boundary conditions. For every spin configuration $\sigma \in \{-1,1\}^{\Lambda(n)}$, we associate a random signed measure $\sigma_{n,T}$ on the unit square $Q = [-1/2, 1/2]^2 \subset \mathbb{R}^2$, defined by

$$\sigma_{n,T} = \frac{1}{m^*(T)n^2} \sum_{x\in\Lambda(n)} \sigma(x)\delta_{x/n}$$

where $\delta_{x/n}$ is the Dirac mass at x/n and $m^*(T)$ is the spontaneous magnetization at temperature T. The expectation b_n of $\sigma_{n,T}$ is

$$b_n = \frac{1}{m^*(T)n^3} \sum_{x\in\Lambda(n)} \sigma(x)x\,.$$

The main result of our paper is a convergence theorem for $\sigma_{n,T}$ under a conditioned Ising measure:

Theorem 1. *Let $0<\delta<\pi$. Let $B(\delta)$ be the ball of radius $\sqrt{\delta/\pi}$ and w_n be the random measure defined by*

$$w_n(x)\,dx \;=\; \Big(1_Q(x)-2\cdot 1_{B(\delta)}\big(\frac{b_n}{2}+x\big)\Big)\,dx\,.$$

This is the measure having density -1 on $B(\delta)-b_n/2$ and 1 on the complement. Under the conditional probability

$$\mu_{\Lambda(n)}(\cdot)\;=\;\mu^{+,T}_{\Lambda(n)}\Big(\;\cdot\;\Big|\frac{1}{n^2}\sum_{x\in\Lambda(n)}\sigma(x)\le(1-\delta)m^*(T)\Big)$$

the difference between the random measures σ_n and w_n converges weakly in probability towards 0 when $n\uparrow\infty$ and $T\uparrow T_c$ in such a way that $n(T_c-T)^{20}\uparrow\infty$ and $\log(n)/\log(1/(T_c-T))$ stays bounded. That is, for any continuous function $f:Q\mapsto\mathbb{R}$,

$$\forall\varepsilon>0\qquad \lim_{n,T}\,\mu_{\Lambda(n)}\big(|\sigma_n(f)-w_n(f)|\ge\varepsilon\big)\;=\;0\,. \tag{1}$$

The probabilities of the deviations are of order $\exp\left(-\mathrm{constant}(T_c-T)n\right)$.

The last sentence of the theorem means the following. For any continuous function $f:Q\mapsto\mathbb{R}$, any $\varepsilon>0$, there exist positive constants b,c depending on f,ε such that

$$\mu_{\Lambda(n)}\bigg(\bigg|\frac{1}{m^*(T)n^2}\sum_{x\in\Lambda(n)}\sigma(x)\,f\Big(\frac{x}{n}\Big)+2\int_{B(\delta)}f\big(-\frac{b_n}{\delta}+x\big)\,dx-\int_Q f(x)\,dx\bigg|>\delta\bigg)$$
$$\le b\exp(-\mathrm{c}\cdot(T_c-T)n)\,.$$

The main assertion of the theorem is that conditioning on having a defect of magnetization, the random measure $\sigma_{n,T}$ looks like a measure whose density is an indicator of the Wulff crystal which turns out to be an ordinary circle near the critical point. In other words, the defect of magnetization concentrates into a circular region. Note that in our regime, the shape of the Wulff crystal is no more affected by the geometry of the square lattice.

Theorem 1 is a consequence of De Giorgi's isoperimetric inequality [20] and a large deviation principle (LDP) that we prove in this paper. The assumption on $\log(n)/\log(1/(T_c-T))$ is a side hypothesis. Although we believe that a proof in the case where this quantity diverges is possible using the ideas of the current paper, we could not find a proof that includes both cases. Since we do like to study regimes that are as close as possible from criticality, we decided to treat the case where $\log(n)/\log(1/(T_c-T))$ stays bounded. The exponent 20 in the statement of the theorem is not optimal. Indeed, if we introduce the quantity

$$\nu_{\mathcal{W}}=\inf\left\{\begin{array}{l}\gamma>0\text{ such that the convergence (1) is valid when }n\uparrow\infty\\ \text{and }T\uparrow T_c\text{ in such a way that }n(T-T_c)^\gamma\uparrow\infty\end{array}\right\}$$

then our result states that $\nu_{\mathcal{W}} \leq 20$. We believe that $\nu_{\mathcal{W}} = 1$, i.e., the critical exponent for the correlation length. For percolation type models we can introduce a similar exponent that characterizes the maximal regime where a Wulff droplet near criticality appears. We believe that in the case of the two-dimensional Bernoulli percolation $\nu_{\mathcal{W}} = 4/3$.
For the two-dimensional Ising model, several difficulties have to be overcome to obtain the right exponent. In a heuristic manner, what prevents us to go from $\nu_{\mathcal{W}} \leq 20$ to $\nu_{\mathcal{W}} \leq 5$ is the lack of the van den Berg-Kesten inequality for the dependent random cluster model. Then to go from $\nu_{\mathcal{W}} \leq 5$ to $\nu_{\mathcal{W}} \leq 2$, one has to have a better understanding of the influence of the boundary conditions when approaching criticality. More precisely, one has to understand better how weak mixing properties in the sense of [1] behave close to the critical point. The gap $\nu_{\mathcal{W}} \leq 2$ to $\nu_{\mathcal{W}} \leq 1$ is related to the speed of convergence of the empirical magnetization to its thermodynamical limit in a regime where we approach the critical point.
The computation of an explicit bound for $\nu_{\mathcal{W}}$ would have been impossible without the kind advice of Charles Pfister. We thank him warmly for drawing our attention to the paper [13].

2.1. The large deviation principle

Next, we give the notions necessary to state our LDP that describes phase coexistence near the critical point.

2.1.1 Sets of finite perimeter. A fundamental quantity in our study is the perimeter of a set. In order to prove our LDP, we need to define this quantity for Borel subsets of $Q = [-1/2, 1/2]^2$ that may have irregular boundaries. We give next the distributional definition of the perimeter.

Definition 2. *Let A be a Borel subset of Q, its perimeter is defined as*

$$\mathcal{P}(A) = \sup \left\{ \int_A \mathrm{div} f(x) d\mathcal{L}^2(x) : \ f \in C_c^\infty(\mathbb{R}^2, B(0,1)) \right\},$$

where $\mathcal{L}^2$ is the Lebesgue measure on $\mathbb{R}^2$, $C_c^\infty(\mathbb{R}^2, B(0,1))$ is the set of C^∞ vector fields from $\mathbb{R}^2$ to the Euclidean unit ball $B(0,1)$ having a compact support and div *is the usual divergence operator. The set A is said to have finite perimeter if $\mathcal{P}(A)$ is finite.*

If the boundary of A is smooth then an application of the Gauss-Green theorem gives that $\mathcal{P}(A) = \mathcal{H}(\partial A)$, where $\mathcal{H}$ is the one dimensional Hausdorff measure. We denote by $\mathcal{M}(Q)$ the vector space of the finite signed Borel measures on Q. We equip $\mathcal{M}(Q)$ with the weak topology, that is the coarsest topology for which the linear functionals

$$\nu \in \mathcal{M}(Q) \mapsto \int f \, d\nu \,, \qquad f \in C_c(\mathbb{R}^2, \mathbb{R})$$

are continuous, where $C_c(\mathbb{R}^2, \mathbb{R})$ is the set of the continuous functions from $\mathbb{R}^2$ to $\mathbb{R}$ having compact support. The rate function of our LDP is the map

$$\mathcal{J} : \mathcal{M}(Q) \to [0, +\infty]$$

$$\nu \mapsto \mathcal{J}(\nu) = \begin{cases} \tau_c \mathcal{P}(A) \text{ if there exists a Borel set } A \text{ such that } \dfrac{d\nu}{d\mathcal{L}^2} = -1_A + 1_{Q\setminus A} \\ \infty \text{ otherwise.} \end{cases}$$

The positive constant τ_c will be defined later, it plays the role of the unit length by which we measure the perimeter.

Theorem 3. *If $T \uparrow T_c$ and $n \uparrow \infty$ in such a way that $n(T_c - T)^{20} \uparrow \infty$ and such that $\log(n)/\log(1/(p-p_c))$ stays bounded then the law of the random measure $\sigma_{n,T}$ under $\mu^{+,T}_{\Lambda(n)}$ satisfies a large deviation principle in $\mathcal{M}(Q)$ with respect to the weak topology. The speed of the LDP is $(T_c - T)n$ and the good rate function is $\mathcal{J}$, i.e., for any Borel subset $\mathbb{M}$ of $\mathcal{M}(Q)$,*

$$-\inf\{\mathcal{J}(\nu) : \nu \in \mathring{\mathbb{M}}\} \le \liminf_{n,T} \frac{1}{(T_c - T)n} \log \mu^{+,T}_{\Lambda(n)}[\sigma_{n,T} \in \mathbb{M}]$$
$$\le \limsup_{n,T} \frac{1}{(T_c - T)n} \log \mu^{+,T}_{\Lambda(n)}[\sigma_{n,T} \in \mathbb{M}] \le -\inf\{\mathcal{J}(\nu) : \nu \in \overline{\mathbb{M}}\}.$$

2.2. Structure of the paper

In section 3 we define the Ising model and introduce the basic notions that we will use in the rest of the paper. In this section we also give the Ising specific properties on which we rely. This way we isolate and minimize the use of the specificities of the Ising model. The other techniques are more robust and take their roots in the study of the Wulff crystal in dimensions greater or equal to three. In section 4 we start by establishing weak mixing properties that permit then the implementation of an adequate block coarse graining, and we also establish the block estimates required for the subsequent proofs. Section 5 contains the proof of the upper bound for our LDP. Finally section 6 finishes the proof of the LDP by establishing the corresponding lower bound.

3. Preliminaries

In this section we define the Ising model and its representation in terms of the random cluster model. In a second part, we isolate the Ising specific properties that are required for our study.

3.1. The Ising model

Let $Q = [-1/2, 1/2]^2$ be the centered planar unit box. For a positive integer n, we define the discrete set of sites $\Lambda(n) = nQ \cap \mathbb{Z}^2$ that we turn into a graph by considering the following set of edges : $\mathbb{E}(\Lambda(n)) = \{\{x, y\} \subset \Lambda(n) : |x - y| = 1\}$, where $|\cdot|$ is the usual Euclidean norm. We also define the boundary $\partial\Lambda(n)$ of the graph $\Lambda(n)$ by

$\partial\Lambda(n) = \{x \in \Lambda(n) : \exists\, y \in \Lambda(n)^c : |x-y| = 1\}$. For every value $\beta = 1/T > 0$ of the inverse temperature, the Ising model in $\Lambda(n)$ with $+1$ boundary conditions is the probability measure on the spin configurations $\Omega_{\Lambda(n)} = \{-1,+1\}^{\Lambda(n)}$ defined by

$$\forall \sigma \in \{+1,-1\}^{\Lambda(n)} \quad \mu^{+,\beta}_{\Lambda(n)}[\sigma] = \frac{1}{Z^{+,\beta}_n} \exp\left(-\beta \mathcal{H}^+_n(\sigma)\right),$$

where

$$\mathcal{H}^+_n(\sigma) = -\frac{1}{2} \sum_{\substack{x,y \in \Lambda(n)\setminus\partial\Lambda(n) \\ |x-y|_1 = 1}} \sigma(x)\,\sigma(y) - \sum_{x \in \Lambda(n)\setminus\partial\Lambda(n)} \sum_{\substack{y \in \partial\Lambda(n) \\ |x-y|_1=1}} \sigma(x)$$

and $Z^{+,\beta}_n$ is the adequate normalization constant.

3.2. The FK-representation

There exists a useful and well known coupling between the Ising model at inverse temperature β and the random cluster model with parameter $q=2$ and $p = 1-\exp(-2\beta)$, see [24, 26]. We will use this coupling in order to derive several probabilistic estimates from the corresponding FK-percolation model. The coupling is a probability measure $\mathbb{P}^+_n$ on the edge-spin configuration space $\{0,1\}^{\mathbb{E}(\Lambda(n))} \times \{-1,+1\}^{\Lambda(n)}$. To construct $\mathbb{P}^+_n$ we first consider Bernoulli percolation of parameter p on the edge space $\{0,1\}^{\mathbb{E}(\Lambda(n))}$, then we choose the spins of the sites in $\Lambda(n)$ independently with the uniform distribution on $\{-1,+1\}$ and finally we condition the edge-spin configuration on the event that there is no open edge in $\Lambda(n)$ between two sites with different spin values. The construction can be summed up with a formula, we have

$$\begin{aligned} &\forall (\sigma,\omega) \in \{0,1\}^{\mathbb{E}(\Lambda(n))} \times \{-1,+1\}^{\Lambda(n)} \\ &\qquad \mathbb{P}^+_n(\sigma,\omega) = \frac{1}{Z} \prod_{e \in \mathbb{E}(\Lambda(n))} p^{\omega(e)}(1-p)^{1-\omega(e)} 1_{(\sigma(x)-\sigma(y))\omega(e)=0}, \end{aligned}$$

where Z is the appropriate normalization factor. It can be verified that the marginal of $\mathbb{P}^+_n$ on the spin configurations is the Ising model with parameter β given by the formula $p = 1-\exp(-2\beta)$ and the marginal on the edge configurations is the random cluster measure with parameters p, $q=2$ and subject to wired boundary conditions, i.e., the probability measure on $\Omega_{\Lambda(n)} = \{0,1\}^{\mathbb{E}(\Lambda(n))}$ defined by

$$\forall \omega \in \Omega_{\Lambda(n)} \qquad \Phi^{p,w}_{\Lambda(n)}[\omega] = \frac{1}{Z} q^{\mathrm{cl}^w(\omega)} \prod_{e \in \mathbb{E}(\Lambda(n))} p^{\omega(e)}(1-p)^{1-\omega(e)}, \tag{2}$$

where $\mathrm{cl}^w(\omega)$ is the number of connected components with the convention that two clusters that touch the boundary $\partial\Lambda(n)$ are identified. This coupling says that one may obtain an Ising configuration by first drawing a FK-percolation configuration with the measure $\Phi^{w,p}_{\Lambda(n)}$, then coloring all the sites in the clusters that touch the boundary $\partial\Lambda(n)$ in $+1$ and finally coloring the remaining clusters independently in $+1$ and -1

with probability 1/2 each. Also, the coupling permits to obtain a $\Phi^{w,p}_{\Lambda(n)}$ percolation configuration by first drawing a spin configuration with $\mu^{+,\beta}_{\Lambda(n)}$, then declaring that all the edges between two sites with different spins are closed, while the other edges are independently declared open with probability p and closed with probability $1-p$.
Let $\Lambda \subset \mathbb{Z}^2$ and $0 \le p \le 1$. In addition to the wired boundary conditions we will also work with *partially wired boundary conditions*. In order to define them, we consider a partition π of $\partial\Lambda = \{x \in \Lambda : \exists y \in \mathbb{Z}^2 \setminus \Lambda, |x-y|_1 = 1\}$. Let us say that π consists of $\{B_1, \cdots, B_k\}$, where the B_i are non-empty disjoint subsets of $\partial\Lambda$ and such that $\cup_i B_i = \partial\Lambda$. For every configuration $\omega \in \Omega_\Lambda$, we define $\mathrm{cl}^\pi(\omega)$ as the number of open connected clusters in Λ computed by identifying two clusters that are connected to the same set B_i. The π-wired FK-measure $\Phi^{p,\pi}_\Lambda$ is defined by substituting $\mathrm{cl}^w(\omega)$ for $\mathrm{cl}^\pi(\omega)$ in (2). We will denote the set of all partially wired FK-measures in Λ by $\mathcal{FK}(p,\Lambda)$. Note that $\Phi^{p,w}_\Lambda$ corresponds to $\pi = \{\partial\Lambda\}$. We define the FK-measure with free boundary conditions $\Phi^{p,f}_\Lambda$ as the partially wired measure corresponding to $\pi = \emptyset$.
Let $U \subseteq V \subseteq \mathbb{Z}^2$. For every configuration $\omega \in \{0,1\}^{\mathbb{E}(\mathbb{Z}^2)}$, we denote by ω_V the restriction of ω to $\Omega_V = \{0,1\}^{\mathbb{E}(V)}$. More generally we will denote by ω_V^U the restriction of ω to $\Omega_V^U = \{0,1\}^{\mathbb{E}(V)\setminus\mathbb{E}(U)}$. If $V = \mathbb{Z}^2$ or $U = \emptyset$ then we drop them from the notation. We will denote by $\mathcal{F}_V^U$ the σ-algebra generated by the finite dimensional cylinders of Ω_V^U.
Note that every configuration $\eta \in \Omega_V$ induces a partially wired boundary condition $\pi(\eta)$ on the set U. The partition $\pi(\eta)$ is obtained by identifying the sites of ∂U that are connected through an open path of η^U. We will denote by $\Phi_U^{p,\pi(\eta)}$ the corresponding FK measure.

3.3. Planar Duality

The duality of the FK-measures in dimension two is well known. In this paper we will use the notation of [19] that we summarize next. Let $0 \le p \le 1$ and Λ be a connected subset of $\mathbb{Z}^2$. To construct the dual process of $\Phi^{p,w}_\Lambda$, we introduce the set of sites $\widehat{\Lambda} \subset \mathbb{Z}^2 + (1/2,1/2)$ consisting of the centers of the squares formed by the edges in $\mathbb{E}(\Lambda)$. To each edge $e \in \mathbb{E}(\Lambda)$ we associate the edge $\widehat{e} \in \mathbb{E}(\widehat{\Lambda})$ that crosses e. Next, to each configuration $\omega \in \Omega_\Lambda$ we associate the dual configuration $\widehat{\omega} \in \Omega_{\widehat{\Lambda}}$ defined by

$$\forall\, e \in \mathbb{E}(\Lambda) \quad \widehat{\omega}(\widehat{e}) = 1 - \omega(e).$$

Similarly, for every event $A \subset \Omega_\Lambda$ we define the *dual event*

$$\widehat{A} = \{\eta \in \Omega_{\widehat{\Lambda}} : \exists \omega \in A, \widehat{\omega} = \eta\}.$$

The duality property asserts that

$$\Phi^{p,w}_\Lambda[A] = \Phi^{\widehat{p},f}_{\widehat{\Lambda}}[\widehat{A}],$$

where $\widehat{p} = 2(1-p)/(2-p)$.

3.4. The critical point

It is known that the critical point of the Ising model on $\mathbb{Z}^2$ is given by the fixed point of a duality relation (see [29]). For the random cluster model with $q=2$, the dual point $\widehat{p}$ is related to p through the relation

$$\frac{p}{1-p}\,\frac{\widehat{p}}{1-\widehat{p}}=2,\ \text{and the fixed point is } p_c=\frac{\sqrt{2}}{1+\sqrt{2}}. \tag{3}$$

The duality relation (3) is translated in the Ising setting into

$$\sinh(2\beta)\sinh(2\widehat{\beta})=1,\ \text{and the critical point is } \beta_c=\frac{\operatorname{arcsinh}(1)}{2}.$$

As we will work in the limit $p\to p_c$, it is worth noting that the derivative of the function $p\mapsto\widehat{p}$ is non-zero at p_c. Thus $p-p_c$ is of the same order as $p_c-\widehat{p}$ when $p\to p_c$. Also $\beta\mapsto p=1-\exp(-2\beta)$ has a non-zero derivative at β_c and thus $p-p_c$ is of the same order as $\beta-\beta_c$ when $\beta\to\beta_c$.
For the general q-Potts model, the identification of the critical point and the self-dual point, i.e., $p_c=\sqrt{q}/(1+\sqrt{q})$, is still an open problem for the values $2<q<25$. When $q>25.72$, this identity has been established and in this situation the Potts model exhibits a first order phase transition [28,38]. Thus the 2d-Ising model is the only two dimensional Potts model exhibiting a second order phase transition for which the critical point has been rigorously identified to be the self-dual point.
We end this section by setting the following convention concerning the use of the word *dual* in the rest of the paper: we always consider that the original model is the super-critical one, i.e., $p>p_c$, which is defined on the edges of $\mathbb{Z}^2$. The *dual* model is always the dual of the super-critical model. That is, it is a sub-critical model defined on the edges of $\mathbb{Z}^2+(1/2,1/2)$ and at percolation parameter $\widehat{p}=2(1-p)/(2-p)\le p_c$. A dual path, circuit or site will always denote a path, circuit or site in $\mathbb{Z}^2+(1/2,1/2)$. The term *open dual* will always designate edges $\widehat{e}$ of $\mathbb{Z}^2+(1/2,1/2)$ that are open with respect to the dual configuration, i.e., $\widehat{\omega}(\widehat{e})=1$. The law of the dual edges $\widehat{e}$ will always be the dual measure $\Phi^{\widehat{p}}$ which is sub-critical, i.e., $\widehat{p}<p_c$.

3.5. The surface tension and critical exponents

3.5.1 The surface tension. By duality, the surface tension τ_β of the two-dimensional Ising model at inverse temperature $\beta>\beta_c$ is given by the directional dependence of the exponential decay of the correlations at dual inverse temperature $\widehat{\beta}<\beta_c$:

$$\forall x\in\mathbb{Z}^2\quad \tau_\beta(x)=-\lim_{n\to\infty}\frac{1}{n}\log\mu_\infty^{\widehat{\beta}}[\sigma(0)\sigma(nx)]=-\lim_{n\to\infty}\frac{1}{n}\log\Phi_\infty^{\widehat{p}}[0\leftrightarrow nx],$$

with $\widehat{p}=1-\exp(-2\widehat{\beta})$ and where we have used the FK-representation to derive the second equality. We will also consider the unique continuous extension of τ_β into a norm on $\mathbb{R}^2$.

In this paper, we are interested in the situation where the spatial scale n goes to infinity and simultaneously β goes to β_c. To study phase coexistence in such a context, we first need to define the joint limit surface tension.

Proposition 4. *If $n \uparrow \infty$ and $\widehat{\beta} \uparrow \beta_c$ in such a way that $n(\beta_c - \widehat{\beta})/\log n \to \infty$, then uniformly on $x \in \mathbb{Z}^2$, we have that*

$$\lim_{(n,\widehat{\beta})\to(\infty,\beta_c)} -\frac{1}{(\beta_c - \widehat{\beta})n} \log \mu_\infty^{\widehat{\beta}}[\sigma(0)\sigma(nx)] = \tau_c |x|_2, \tag{4}$$

where τ_c is a positive constant and $|\cdot|_2$ is the Euclidean norm on $\mathbb{R}^2$.

Note that τ_c does not depend on x and that the appearance of the Euclidean norm means that the surface tension is isotropic near the critical point. The presence of the factor $(\beta - \beta_c)$ in the denominator of the limit is directly related to the critical exponent for the correlation length. In the planar Ising model this exponent is equal to 1.
The proof of the last proposition is an extension of the computation [18] of the *classical* surface tension at fixed temperature. Using subadditivity, it follows that

$$\forall x \in \mathbb{Z}^2 \quad \log \mu_\infty^{\widehat{\beta}}[\sigma(0)\sigma(x)] \leq \tau_\beta(x).$$

Thus the upper bound of (4) follows directly from the formula obtained in [18] describing τ_β. On the other hand, the corresponding lower bound is harder to obtain. It follows by extending the asymptotic computations of $\mu_\infty(\sigma(0)\sigma(nx))$ from a regime where $\widehat{\beta} < \beta_c$ is kept fixed and $n \uparrow \infty$ to a situation where $\widehat{\beta} \uparrow \beta_c$ and simultaneously $n \uparrow \infty$. This can be done by using, as in [18], Kasteleyn's dimer representation of the 2d Ising model [36]. These computations are long and rely on quite different mathematical tools and thus, we consider here (4) as a starting point and we will present the derivation of proposition 4 in a separate paper [42]. In words, the extension [42] relies on a random walk interpretation of the computation [18]. This interpretation allows the derivation of proposition 4 from classical moderate deviation results for the random walk. In fact, the isotropy of the right hand side of (4) follows from the isotropy of the rate function for the moderate deviation principle of the simple random walk on $\mathbb{Z}^2$.
The regime in proposition 4 is nearly optimal, since from results of [10] it appears that if $(n, \widehat{\beta}) \uparrow (\infty, \beta_c)$ in a regime where $n(\beta_c - \widehat{\beta})$ stays constant then

$$\mu_\infty^{\widehat{\beta}}[\sigma(0)\sigma(ne_1)] \sim c\frac{1}{n^{1/4}},$$

where $e_1 = (1, 0) \in \mathbb{Z}^2$ and c is a positive constant. Thus below the regime of proposition 4 we do not expect phase coexistence to happen. In this situation the critical phenomena take over and hence we expect that $\nu_{\mathcal{W}} \geq 1$.

3.5.2 The magnetization. We will need to know how fast the magnetization is going to zero near the critical point. In our case, from Onsager's famous formula we know that

$$\mu_\infty^{+;\beta}[\sigma(0)] \sim (\beta - \beta_c)^{1/8} \quad \text{when} \quad \beta \downarrow \beta_c. \tag{5}$$

The Ising model is the only Potts model for which this exponent has been computed. For independent site percolation on the planar triangular lattice, this exponent has been derived in [49], along with several other exponents. For the values $q = 1, 3, 4$, the existence of the exponent as well as its values are conjectured by physicists but currently not proved.

In addition to that, we shall need to estimate the speed of convergence of the empirical magnetization near the critical point. In order to do so, we rely on correlation inequalities that are specific to the Ising model. Furthermore, we rely on explicit computations to control the behavior of the relevant quantities near the critical point. The next result is inspired by [13], it shows how the speed of convergence of the empirical magnetization is affected when we approach the critical point.

Proposition 5. *Let $\xi > 0$ and $a > \xi + 1$. There exist two positive constants $c = c(\xi, a)$ and ρ such that*

$$\forall \beta \neq \beta_c,\ n > c|\beta - \beta_c|^{-a} \qquad \frac{1}{n^2} \sum_{x \in \Lambda(n)} \mu^{+,\beta}_{\Lambda(n)}[\sigma(x)] - m^*(\beta) \leq \rho |\beta - \beta_c|^{\xi}.$$

Proof. Let n, k, l be three integers larger than one. For $h > 0$, we note $\mu^{+,\beta,h}_{n+k+l}$ the Ising measure on the box $\Lambda(n+k+l)$ with boundary conditions $+$, at inverse temperature β and where every spin in $\Lambda(n+k+l) \setminus \Lambda(n+k)$ is submitted to a positive field h/β. Let $x \in \Lambda(n)$. The measure $\mu^{+,\beta,h}_{n+k+l}$ has the property that

$$\lim_{h \uparrow \infty} \mu^{+,\beta,h}_{n+k+l}[\sigma(x)] = \mu^{+,\beta}_{\Lambda(n+k)}[\sigma(x)].$$

It is thus a sort of interpolation between the measures $\mu^{+,\beta}_{\Lambda(n+k+l)}$ and $\mu^{+,\beta,h}_{\Lambda(n+k)}$. Furthermore, it is easy to check that

$$\frac{\partial \mu^{+,\beta,h}_{n+k+l}[\sigma(x)]}{\partial h} = \sum_{y \in \Lambda(n+k+l) \setminus \Lambda(n+k)} \mu^{+,\beta,h}_{n+k+l}[\sigma(x)\sigma(y)] - \mu^{+,\beta,h}_{n+k+l}[\sigma(x)] \mu^{+,\beta,h}_{n+k+l}[\sigma(y)].$$

Therefore, we have

$$\begin{gathered} 0 \leq \mu^{+,\beta}_{\Lambda(n+k)}[\sigma(x)] - \mu^{+,\beta}_{\Lambda(n+k+l)}[\sigma(x)] = \\ \sum_{y \in \Lambda(n+k+l) \setminus \Lambda(n+k)} \int_0^\infty \mu^{+,\beta,h}_{n+k+l}[\sigma(x)\sigma(y)] - \mu^{+,\beta,h}_{n+k+l}[\sigma(x)] \mu^{+,\beta,h}_{n+k+l}[\sigma(y)]\, dh, \end{gathered}$$

Next, applying the Ising specific G.H.S inequality [25], we get that

$$\mu^{+,\beta,h}_{n+k+l}[\sigma(x)\sigma(y)] - \mu^{+,\beta,h}_{n+k+l}[\sigma(x)] \mu^{+,\beta,h}_{n+k+l}[\sigma(y)] \leq \mu^{+,\beta}_{\infty}[\sigma(x)\sigma(y)] - \mu^{+,\beta}_{\infty}[\sigma(x)] \mu^{+,\beta}_{\infty}[\sigma(y)].$$

Note that the right hand side depends only on the infinite volume measure. On the other hand, by using Griffith's inequalities [25], we may estimate

$$\mu^{+,\beta,h}_{n+k+l}[\sigma(x)\sigma(y)] - \mu^{+,\beta,h}_{n+k+l}[\sigma(x)] \mu^{+,\beta,h}_{n+k+l}[\sigma(y)] \leq \exp(-\lambda_1 h),$$

uniformly in $n+k+l$, x, y and in β, where λ_1 is a positive constant.
Combining the two last inequalities with the magnetic field representation of the boundary conditions, we finally obtain

$$\mu^{+,\beta}_{\Lambda(n+k)}[\sigma(x)] - \mu^{+,\beta}_{\Lambda(n+k+l)}[\sigma(x)] \leq \\ \int_0^\infty dh \sum_{y\in\Lambda(n+k+l)\setminus\Lambda(n+k)} \left\{(\mu^{+,\beta}_\infty[\sigma(x)\sigma(y)] - \mu^{+,\beta}_\infty[\sigma(x)]\,\mu^{+,\beta}_\infty[\sigma(y)]) \wedge \exp(-\lambda_1 h)\right\}.$$

First, let us consider the case where $0 < \beta < \beta_c$. In this situation, the explicit computation (see [42]) yields that the correlation is bounded above as follows:

$$\mu^{+,\beta}_\infty[\sigma(x)\sigma(y)] - \mu^{+,\beta}_\infty[\sigma(x)]\,\mu^{+,\beta}_\infty[\sigma(y)] \leq \exp(-\lambda_2(\beta-\beta_c)|x-y|). \tag{6}$$

Thus

$$\begin{aligned}\mu^{+,\beta}_{\Lambda(n+k)}[\sigma(x)] - \mu^{+,\beta}_{\Lambda(n+k+l)}[\sigma(x)] &\leq \int_0^\infty dh \sum_{y\in\Lambda(n+k+l)\setminus\Lambda(n+k)} \exp(-(\lambda_1 h \vee \lambda_2(\beta_c-\beta)|x-y|)) \\ &\leq \int_0^\infty dh \exp(-\frac{\lambda_1}{2}h) \sum_{y\in\Lambda(n+k+l)\setminus\Lambda(n+k)} \exp(-\frac{\lambda_2}{2}(\beta_c-\beta)|x-y|) \\ &\leq \frac{2}{\lambda_1} \sum_{y\in\Lambda(n+k+l)\setminus\Lambda(n+k)} \exp(-\frac{\lambda_2}{2}(\beta_c-\beta)|x-y|) \\ &\leq \frac{8}{\lambda_1}\exp(-\frac{\lambda_2}{4}(\beta_c-\beta)k)\sum_{r=0}^{l}(n+k+r)\exp(-\frac{\lambda_2}{4}(\beta_c-\beta)r) \\ &\leq c_1\frac{8}{\lambda_1}\frac{n+k}{\beta_c-\beta}\exp(-\frac{\lambda_2}{4}(\beta_c-\beta)k)\sum_{r=0}^{l}\exp(-\frac{\lambda_2}{8}(\beta_c-\beta)r).\end{aligned}$$

The last inequality has been obtained by bounding $n+k+r$ by $(n+k)(r+1)$ and by choosing $c_1 > 0$ in such a way that

$$\forall r \geq 0 \qquad r+1 \leq \frac{c_1}{\beta_c-\beta}\exp(\frac{\lambda_2}{8}(\beta_c-\beta)(r+1)).$$

Sending l to infinity yields

$$\mu^{+,\beta}_{\Lambda(n+k)}[\sigma(x)] \leq \frac{8c_1}{\lambda_1\lambda_2}\frac{n+k}{(\beta_c-\beta)^2}(8+\lambda_2(\beta_c-\beta))\exp(-\frac{\lambda_2}{4}(\beta_c-\beta)k).$$

Thus, there exists a positive constant c_2 such that

$$\mu^{+,\beta}_{\Lambda(n+k)}[\sigma(x)] \leq c_2\frac{n+k}{(\beta_c-\beta)^2}\exp(-\frac{\lambda_2}{4}(\beta_c-\beta)k).$$

Applying this inequality to the box $\Lambda(n)$ and the sites in $\Lambda(n-k)$, we deduce that for all $k<n$

$$\frac{1}{n^2}\sum_{x\in\Lambda(n)}\mu^{+,\beta}_{\Lambda(n)}[\sigma(x)]\le 2\frac{k}{n}+c_2\frac{n}{(\beta_c-\beta)^2}\exp(-\frac{\lambda_2}{4}(\beta_c-\beta)k).$$

We fix $\xi>0$ and $a>\xi+1$. For all $0<\beta<\beta_c$ we restrict ourself to $n>(\beta_c-\beta)^{-\xi}$ so that we can choose $k=(\beta_c-\beta)^{\xi}n$ and obtain

(7)
$$\frac{1}{n^2}\sum_{x\in\Lambda(n)}\mu^{+,\beta}_{\Lambda(n)}[\sigma(x)]\le$$
$$2(\beta_c-\beta)^{\xi}+\frac{c_2}{(n(\beta_c-\beta)^a)^{\frac{2}{a}}}\exp\left(\Big(-\frac{\lambda_2}{4}(\beta_c-\beta)^{1+\xi}\frac{n}{\log n}+1+\frac{2}{a}\Big)\log n\right).$$

The last expression suggests to impose a regime on $(\beta_c-\beta)^a n$. Indeed, there exists a positive $n_0=n_0(\xi,a)$ such that $n>n_0$ implies that $n/\log n>n^{\frac{\xi+1}{a}}$, hence by imposing

$$(\beta_c-\beta)^a n>\Big(\frac{8}{\lambda_2}(1+\frac{2}{a})\Big)^{a/(\xi+1)}\vee 1\vee\beta_c^a n_0$$

we obtain that

$$\begin{aligned}\frac{1}{n^2}\sum_{x\in\Lambda(n)}\mu^{+,\beta}_{\Lambda(n)}[\sigma(x)]\le&2(\beta_c-\beta)^{\xi}+c_2\exp\left(-\frac{\lambda_2}{8}(\beta_c-\beta)^{\xi+1}n\right)\\ \le&(\beta_c-\beta)^{\xi}\left[2+c_2\exp\left(\log\frac{1}{\beta_c-\beta}\Big(\xi-\frac{\lambda_2}{8}\frac{(\beta_c-\beta)^{\xi+1}}{\log\frac{1}{\beta_c-\beta}}n\Big)\right)\right],\end{aligned}$$

Finally, note that there exists a positive $\varepsilon=\varepsilon(\xi,a)$ such that for all $\beta_c-\varepsilon\le\beta\le\beta_c$

$$\frac{(\beta_c-\beta)^{\xi+1}}{\log\frac{1}{\beta_c-\beta}}>(\beta_c-\beta)^a.$$

This implies that

$$\begin{aligned}\frac{1}{n^2}\sum_{x\in\Lambda(n)}\mu^{+,\beta}_{\Lambda(n)}[\sigma(x)]\le&(\beta_c-\beta)^{\xi}\left[2+c_2\exp\left(\log\frac{1}{\beta_c-\beta}\Big(\xi-\frac{\lambda_2}{8}\frac{(\beta_c-\beta)^{\xi+1}}{\log\frac{1}{\beta_c-\beta}}n\Big)\right)\right]\\ \le&(\beta_c-\beta)^{\xi}\left[2+c_2(\beta_c-\beta)^{-\xi+\frac{\lambda_2}{8}(\beta_c-\beta)^a n}\right],\end{aligned}$$

Thus, if we impose that

$$n>\frac{16\xi}{\lambda_2}(\beta_c-\beta)^{-a},$$

we get that

$$\frac{1}{n^2}\sum_{x\in\Lambda(n)}\mu^{+,\beta}_{\Lambda(n)}[\sigma(x)]\le(\beta_c-\beta)^{\xi}[2+c_2(\beta_c-\beta)^{\xi}].$$

Thus, there exists a positive $c=c(\xi,a)$ such that for all $\beta_c-\varepsilon\le\beta\le\beta_c$ and for all $n>c(\beta_c-\beta)^{-a}$ we have that

$$\frac{1}{n^2}\sum_{x\in\Lambda(n)}\mu^{+,\beta}_{\Lambda(n)}[\sigma(x)]\le\rho'(\beta_c-\beta)^{\xi}, \tag{8}$$

where ρ' is a positive constant. When $\beta<\beta_c-\varepsilon$, (8) also holds, provided ρ' is replaced by $\rho=\rho'\vee\varepsilon^{-\xi}$.
In order to treat the case where $\beta>\beta_c$, one proceeds in the same way. In this situation (6) is replaced by the following bound that can be obtained from the results of [41]: there exist positive constants λ_3, c_3 and δ such that for all $x,y\in\mathbb{Z}^2$ satisfying $|x-y|(\beta-\beta_c)>1/\delta$ we have that

$$\mu^{+,\beta}_{\infty}[\sigma(x)\sigma(y)]-m^*(\beta)^2\le c_3\exp(-\lambda_3(\beta-\beta_c)|x-y|)\,.$$

□

4. Block arguments

Besides the Ising specific properties that we stated in the last section, our analysis is based on rather robust techniques that have been developed by Cerf and Pisztora [14, 15, 16, 17] to study phase coexistence in dimensions greater or equal to three. The probabilistic estimates are obtained by translating the relevant Ising events into the random cluster model via the FK representation. In this paper, an essential tool in analyzing the random cluster model is an adaptation of block coarse graining techniques [46] to the situation where $p\downarrow p_c$.

4.1. Notation and preparatory lemmas

In this section, we introduce the notation used in coarse graining arguments and state useful preliminary estimates that we will use repeatedly in the rest of the paper.

4.1.1 The rescaled lattice. First we fix a positive integer K, that will typically depend on p later on and will diverge when $p\downarrow p_c$. For each $\underline{x}\in\mathbb{Z}^2$, we define the block indexed by $\underline{x}$ as $B(\underline{x})=\Lambda(K)+K\underline{x}$. Let A be a region in $\mathbb{R}^2$. We define the rescaled region $\underline{A}$:

$$\underline{A}=\{\underline{x}\in\mathbb{Z}^2\ :\ B(\underline{x})\cap A\ne\emptyset\}.$$

From now on, underlining means that we are dealing with rescaled objects. For instance, $\underline{\Lambda}(n)$ means the rescaled box $\Lambda(n)$. Note that $|\underline{\Lambda}(n)|$ is now of order n^2/K^2, which is the order of the number of boxes necessary to cover $\Lambda(n)$.

4.1.2 The lattice $\mathbb{L}^\infty$*.* When dealing with block arguments it will be convenient to equip $\mathbb{Z}^2$ with another graph structure. We denote by d_∞ the distance associated to the norm $|\cdot|_\infty$ defined by

$$\forall (x_1,\, x_2) \in \mathbb{R}^2 \qquad |(x_1,\, x_2)|_\infty = \max(|x_1|,\, |x_2|).$$

Thus $d_\infty(x,y) = |x-y|_\infty$. For every set $E \subseteq \mathbb{R}^2$ and positive real number r, we define the r-neighborhood of E as

$$\mathcal{V}_\infty(E,r) = \{x \in \mathbb{R}^2,\, d_\infty(x,E) < r\}.$$

We will also use the associated diameter given by

$$\operatorname{diam}_\infty(E) = \sup\{|x-y|_\infty \,:\, x,y \in E\}.$$

The new graph structure on $\mathbb{Z}^2$ is induced by the set of edges

$$\mathbb{E}^{2,\infty} = \big\{\{x,y\} \subset \mathbb{Z}^2,\, d_\infty(x,y) = 1\big\}\,.$$

Then the lattice $\mathbb{L}^\infty$ is the graph $(\mathbb{Z}^2, \mathbb{E}^{2,\infty})$. This lattice has the useful property that the exterior boundary of any connected finite set A in $\mathbb{Z}^2$ is itself connected when regarded as a subgraph of $\mathbb{L}^\infty$; for a proof we refer the reader to [21].

4.1.3 Block events. For the renormalization to be useful it is almost always required to use block events on a set of blocks that are overlapping. Thus, in addition to the partitioning blocks $B(\underline{x})$ we define the event blocks $B'(\underline{x})$, by setting

$$B'(\underline{x}) = \bigcup_{\substack{\underline{y} \in \mathbb{Z}^2 \\ |\underline{y}-\underline{x}|_\infty \le 1}} B(\underline{y}).$$

4.1.4 Rough estimates on the block process. Given the events that describe a good block, we define the block process $(X(\underline{x}), \underline{x} \in \underline{\Lambda}(n))$ as the dependent site percolation process on $\underline{\Lambda}(n)$ that indicates if a block is good or not. We cite several rough estimates on the block process from [15].

Lemma 6. *Suppose that there exists $\delta > 0$ such that*

$$\forall \underline{x} \in \mathbb{Z}^2 \qquad P[X(\underline{x}) = 0 | X(\underline{z}), |\underline{x}-\underline{z}|_\infty \ge 3] \le \delta,$$

then for any $\underline{A} \subset \mathbb{Z}^2$, we have

$$P[\forall \underline{x} \in \underline{A} : X(\underline{x}) = 0] \le \exp\left(\frac{1}{9}|\underline{A}|\log\delta\right).$$

The block process can be viewed as a dependent site percolation process where a site $\underline{x}$ is occupied if and only if $X(\underline{x}) = 0$. The occupied $\mathbb{L}^\infty$ cluster of the site $\underline{x}$, i.e., the $\mathbb{L}^\infty$ connected component of the occupied sites containing $\underline{x}$, is then denoted by $\underline{C}(\underline{x})$. The next lemma is a standard counting Peierls argument:

Lemma 7. *Suppose that there exists $\delta > 0$ such that*

$$\forall \underline{x} \in \mathbb{Z}^2 \qquad P[X(\underline{x}) = 0 | X(\underline{z}), |\underline{x} - \underline{z}|_\infty \geq 3] \leq \delta.$$

There exists a constant b such that, for any bounded open subset O of $\mathbb{R}^2$, any $s, t > 0$, any $K, n \in \mathbb{N}$ with $n \geq K$,

$$P[|\{\underline{x} \in \mathbb{Z}^2 : B(\underline{x}) \cap O \neq \emptyset,\ |\underline{C}(\underline{x})| \geq t\}| \geq s] \leq \\ 2 \sum_{j \geq s} \exp j \left(\frac{1}{t} \log \mathcal{L}^2(\mathcal{V}(O,2)) + \log b + \frac{1}{9} \log \delta \right),$$

where $\mathcal{V}(O,2) = \{x \in \mathbb{R}^2 : d(x,O) \leq 2\}$.

Here is the last rough estimate:

Lemma 8. *We consider the box $\Lambda(n)$ rescaled by a factor K:*

$$\underline{\Lambda}(n) = \{\underline{x} : B(\underline{x}) \cap \Lambda(n) \neq \emptyset\}.$$

If there exists $\delta > 0$ such that

$$\forall \underline{x} \in \mathbb{Z}^2 \qquad P[X(\underline{x}) = 0 | X(\underline{z}), |\underline{x} - \underline{z}|_\infty \geq 3] \leq \delta,$$

then for any n, K, ε satisfying $n \geq 6K, \delta < \varepsilon$, we have

$$P\left[\frac{1}{|\underline{\Lambda}(n)|} \sum_{\underline{x} \in \underline{\Lambda}(n)} 1_{X(\underline{x})=0} \geq \varepsilon \right] \leq 9 \exp\left(-\Lambda^*(\varepsilon, \delta) \left\lfloor \frac{n}{6K} \right\rfloor^2 \right),$$

where

$$\Lambda^*(\varepsilon,\delta) = \varepsilon \log \frac{\varepsilon}{\delta} + (1-\varepsilon) \log \frac{1-\varepsilon}{1-\delta}$$

is the Cramér transform of a Bernoulli variable with parameter δ.

We finish with Hoeffding's inequality that will be useful:

Lemma 9. *(Theorem 1 of [32]) If $(X_i)_{1 \leq i \leq n}$ are independent random variables with values in $[-1,1]$ and with mean m, then*

$$\forall t \in]0, 1-m[\qquad P\Big[\sum_{i=1}^n (X_i - m) \geq n\ t \Big] \leq \exp(-nt^2).$$

4.2. Weak mixing near criticality

In this part we establish weak mixing properties in the situation where $p \downarrow p_c$. These results are crucial in order to bound the influence of the boundary conditions and also in order to decouple spatially separated events. As it appears from [19], in order to implement a useful coarse graining in dimension two, it is necessary to have a control of the boundary conditions. When p is fixed, this control can be obtained by using the weak mixing properties proved in [1, 2]. To handle the situation where $p \downarrow p_c$, we give an alternative way to establish weak mixing and generalize the results of [1, 2] to a situation where the exponential decay of connectivities becomes degenerate.

4.2.1 Control of the number of boundary connected sites. Let $p < p_c, n \geq 1$. In this paragraph, we are interested in the control of the number of boundary connected sites

$$M_{\Lambda(n)} = |\,\{x \in \Lambda(n) : x \leftrightarrow \partial\Lambda(n)\}\,|. \tag{9}$$

The coming results depend on the speed of convergence of the mean of M_n near the critical point. We characterize this speed by introducing the following quantity:

$$\forall p < p_c,\ \delta > 0 \quad m_{\mathrm{sub}}(\delta, p) = \inf\left\{ m \geq 1 : \forall n > m \quad \frac{1}{|\Lambda(n)|}\Phi^{w,p}_{\Lambda(n)}[M_{\Lambda(n)}] \leq \delta \right\}. \tag{10}$$

Lemma 10. *Let $\delta > 0$, $p \leq p_c$. If $n \geq 16 m_{\mathrm{sub}}(\delta/2, p)/\delta$, then*

$$\log \Phi^{w,p}_{\Lambda(n)}\left[\frac{M_{\Lambda(n)}}{|\Lambda(n)|} \geq \delta\right] \leq -\left(\frac{\delta n}{6 m_{\mathrm{sub}}(\delta/2, p)}\right)^2.$$

Proof. First we partition $\Lambda(n)$ into translates of the square $\Lambda(m)$ where

$$m = m_{\mathrm{sub}}(\delta/2, p). \tag{11}$$

Next, we take

$$n > 16m/\delta, \tag{12}$$

and consider the set

$$\Lambda'(n) = \bigcup_{\underline{x} \in \mathbb{Z}^2 : B(\underline{x}) \subset \Lambda(n)} B(\underline{x}),$$

where $B(\underline{x}) = m\underline{x} + \Lambda(m)$. Note that $|\Lambda(n) \setminus \Lambda'(n)| \leq 4mn$. The number of partitioning blocks satisfies

$$\frac{n^2}{2m^2} \leq |\underline{\Lambda}'(n)| \leq \frac{n^2}{m^2}. \tag{13}$$

Since M_Λ is subadditive, by (13) and (12), we obtain

$$\begin{aligned}\frac{M_{\Lambda(n)}}{|\Lambda(n)|} &\leq \frac{1}{n^2}\sum_{\underline{x}\in\underline{\Lambda}'(n)} |\{v\in B(\underline{x}) : v\leftrightarrow \partial\Lambda(n)\}| + \frac{4m}{n}\\ &\leq \frac{1}{|\underline{\Lambda}'(n)|}\sum_{\underline{x}\in\underline{\Lambda}'(n)} \frac{M_{B(\underline{x})}}{|B(\underline{x})|} + \frac{\delta}{4}.\end{aligned}$$

By the FKG inequality, we get

$$\Phi^{w,p}_{\Lambda(n)}\left[\frac{M_{\Lambda(n)}}{|\Lambda(n)|}\geq\delta\right] \leq \Phi^{w,p}_{\Lambda(n)}\left[\frac{1}{|\underline{\Lambda}'(n)|}\sum_{\underline{x}\in\underline{\Lambda}'(n)}\frac{M_{B(\underline{x})}}{|B(\underline{x})|}\geq\frac{3\delta}{4}\,\middle|\, E\right] \tag{14}$$

where E is the increasing event $\big\{\forall\underline{x}\in\underline{\Lambda}'(n),\ \text{all the edges of } \partial B(\underline{x}) \text{ are open}\big\}$. The random variables $M_{B(\underline{x})}/|B(\underline{x})|$, $\underline{x}\in\underline{\Lambda}'(n)$, take their values in $[0,1]$ and they are independent under $\Phi^{w,p}_{\Lambda(n)}[\cdot\,|E]$. By (11), their mean satisfies

$$\forall\ \underline{x}\in\underline{\Lambda}'(n)\qquad \Phi^{w,p}_{\Lambda(n)}\left[\frac{M_{B(\underline{x})}}{|B(\underline{x})|}\,\middle|\, E\right] = \Phi^{w,p}_{B(\underline{x})}\left[\frac{M_{B(\underline{x})}}{|B(\underline{x})|}\right]\leq\frac{\delta}{2}. \tag{15}$$

Finally, by lemma 9 and by the inequalities (13), (14) and (15) we get

$$\Phi^{w,p}_{\Lambda(n)}\left[\frac{M_{\Lambda(n)}}{|\Lambda(n)|}\geq\delta\right]\leq\exp\left(-\frac{\delta^2n^2}{32m^2}\right).$$

□

4.3. Control of the boundary conditions

In this section, we determine a regime where we can still control the influence of the boundary conditions when $p\to p_c$. The regime will be characterized by the speed by which the quantity m_{sub} defined in (10) diverges near the critical point. We thus need to give an upper bound for the speed of this divergence.

Lemma 11. *Let $\kappa>0,\xi>0$. For every $a>\xi+1$ there exists a positive constant $c=c(a,\kappa)$ such that*

$$\forall p<p_c\qquad m_{\text{sub}}(\kappa(p_c-p)^\xi,p)\leq c(p_c-p)^{-a}.$$

Proof. From the Ising-FK coupling it follows that

$$\frac{1}{|\Lambda(n)|}\Phi^{w,p}_{\Lambda(n)}[M_{\Lambda(n)}] = \frac{1}{|\Lambda(n)|}\sum_{x\in\Lambda(n)}\mu^{+,\beta}_{\Lambda(n)}[\sigma(x)].$$

Let $a > 1$ and $\xi \in (0, a-1)$. From proposition 5 we know that for every $\eta \in (\xi, \xi+1)$ there exist two positive constants ρ and c_1 such that

$$\forall p < p_c \quad \forall\, n > c_1(p_c - p)^{-a} \qquad \frac{1}{|\Lambda(n)|}\Phi^{w,p}_{\Lambda(n)}[M_{\Lambda(n)}] \le \rho(p_c - p)^{\eta}.$$

Furthermore, since $\eta > \xi$, there exists a positive constant $\varepsilon = \varepsilon(\rho, \xi, \kappa, \eta)$ such that

$$\forall\, p \in (p_c - \varepsilon, p_c) \qquad \rho(p_c - p)^{\eta} \le \kappa(p_c - p)^{\xi}.$$

Note also that if $p \le p_c - \varepsilon$ then $\kappa(p - p_c)^{\xi} \ge \kappa\varepsilon^{\xi}$ and there exists $n_0(\varepsilon^{\xi})$ such that $n > n_0$ implies

$$\forall\, p < p_c - \varepsilon \qquad m_{\mathrm{sub}}(\kappa(p - p_c)^{\xi}, p) \le n_0.$$

Hence the result follows by choosing $c = \max(c_1, \varepsilon^a n_0)$. $\square$

Proposition 12. *Let $p < p_c$ and $a > 5$. There exist two positive constants $c = c(a)$ and λ such that if $n > c(p_c - p)^{-a}$ then*

$$\log \Phi^{w,p}_{\Lambda(n)}[0 \leftrightarrow \partial\Lambda(n)] \le -\lambda(p_c - p)n\,.$$

Proof. Let $A = \{0 \leftrightarrow \partial\Lambda(n/2)\}$. In order to control the influence of the boundary conditions imposed on $\Lambda(n)$ we first write

$$\begin{aligned}\Phi^{w,p}_{\Lambda(n)}[A] \le &\Phi^{w,p}_{\Lambda(n)}[A \cap \{M_{\Lambda(n)} \le |\Lambda(n)|\delta\}] \\ &+ \Phi^{w,p}_{\Lambda(n)}[M_{\Lambda(n)} > |\Lambda(n)|\delta\ ],\end{aligned} \tag{16}$$

where $M_{\Lambda(n)}$ is defined in (9). On the event $A' = A \cap \{M_{\Lambda(n)} \le |\Lambda(n)|\delta\}$ of the first term we can bound the influence of the boundary conditions in an adequate way by using a judicious trick due to David Barbato [9], while the second term will be made negligible thanks to lemma 10.

Barbato's trick: This trick has initially been introduced in order to simplify the proof of the so called interface lemma in the case of dimensions higher or equal to three. Here we will use this trick in a different context. From the definition of the FK-measures it is clear that the influence of the boundary conditions comes from the connected components that connect $\partial\Lambda(n/2)$ to $\partial\Lambda(n)$. Thus if one can cut all these connections without altering too much the probability of the event A then one gets a control over the influence of the boundary conditions. To do this we first define $M'_{\Lambda(n)}$ as

$$M'_{\Lambda(n)} = \Big|\{x \in \Lambda(n) : x \leftrightarrow \partial\Lambda(n) \text{ in } \Lambda(n) \setminus \Lambda(2|x|_\infty)\}\Big|.$$

This is the same quantity as $M_{\Lambda(n)}$ with the difference that we count only the sites x that are connected to the boundary with a *direct* path that does not use the edges in

$\mathbb{E}(\Lambda(2|x|_\infty))$. Now suppose that $A' = A \cap \{M_{\Lambda(n)} \leq |\Lambda(n)|\delta\}$ occurs. Since $M'_{\Lambda(n)} \leq M_{\Lambda(n)}$ we also have $M'_{\Lambda(n)} \leq \delta|\Lambda(n)|$. Next, for $0 < h < 1/4$, we define the set

$$\mathfrak{b}(h) = \partial[-n(1-h)/2, n(1-h)/2]^2.$$

Note that for $0 < h < 1/4$, we always have

$$\mathfrak{b}(h) \cap \Lambda(n/2) = \emptyset.$$

Next, we concentrate on the finite set of values $0 < h_1 < \cdots < h_K$ that satisfy

$$\mathfrak{b}(h_k) \cap \Lambda(n) \neq \emptyset.$$

We notice that the number K of such values h_k satisfies

$$\frac{n}{8} - 1 < K < \frac{n}{8} + 1.$$

Until here, the construction does not depend on the configuration. Next, we scan the configuration in $\Lambda(n)$ from outside inwards and define for each h_k the set of *bad* sites intersected by $\mathfrak{b}(h_k)$:

$$V(h_k) \;=\; M'_{\Lambda(n)} \cap \mathfrak{b}(h_k).$$

On A' we have that $\sum_{k=1}^{K} |V(h_k)| \leq M'_{\Lambda(n)} \leq \delta|\Lambda(n)|$ whence, for n large enough,

$$\min_k |V(h_k)| \leq \frac{\delta|\Lambda(n)|}{K} \leq \frac{\delta|\Lambda(n)|}{\frac{n}{8} - 1} \;\leq\; 16\delta n\,.$$

Thus there exists at least one $k \in \{1, \ldots, K\}$ such that

$$|V(h_k)| \leq 16\delta n. \tag{17}$$

We define h^* as the first (smallest) value h_k that satisfies (17). Notice that h^* is a sort of stopping time, in the sense that

$$\forall 0 < h < 1/4 \qquad \{h^* = h\} \in \mathcal{F}_{\Lambda(n)\setminus\Lambda((1-h)n)}. \tag{18}$$

Then we define the set of *bad* edges as the set of edges that have one extremity in $\Lambda((1-h^*)n)$ and the other in $V(h^*)$:

$$I_n \;=\; \big\{\, e = \{v, u\} \in \mathbb{E}^2 :\; v \in \Lambda((1-h^*)n),\; u \in V(h^*) \,\big\}.$$

Even though

$$I_n \cap \mathbb{E}(\Lambda(n) \setminus \Lambda((1-h^*)n)) = \emptyset, \tag{19}$$

we obtain from (18) and from the definition of $V(h^*)$ that

$$\forall I \subseteq \mathbb{E}(\Lambda(n)) \quad \{I_n = I\} \in \mathcal{F}_{\Lambda(n)\setminus\Lambda((1-h^*)n)}. \tag{20}$$

It is also important to notice that

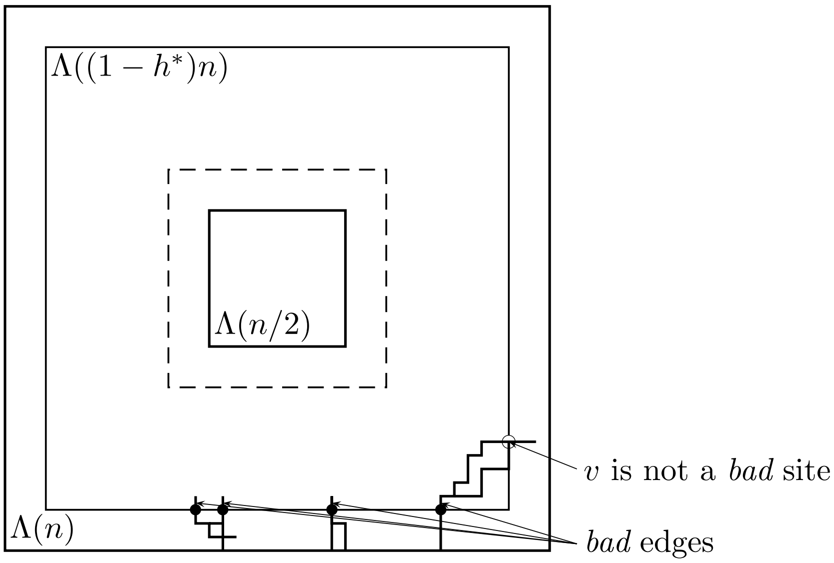
$$I_n \cap \mathbb{E}(\Lambda(n/2)) = \emptyset. \tag{21}$$

Now, for each site $v \in V(h^*)$ there is at most one edge e in I_n with extremity v thus we get from (17) that

$$|I_n| \leq 16\delta n. \tag{22}$$

Let $\Psi : A' \to \Omega$ be the map defined by:

$$\forall \omega \in A' \quad \forall e \in \Lambda(n) \quad \Psi(\omega)(e) = \begin{cases} 0 & \text{if } e \in I_n(\omega) \\ \omega(e) & \text{otherwise} \end{cases}$$



The configurations in $\Psi(A')$ have the following three crucial properties:

i) We claim that

$$\max_{\omega' \in \Psi(A')} |\Psi^{-1}(\omega')| \leq 2^{16\delta n}. \tag{23}$$

To prove (23), we first write for each $\widetilde{\omega} \in \Psi(A')$

$$|\Psi^{-1}(\widetilde{\omega})| \leq \sum_{I \subset \mathbb{E}(\Lambda(n))} \left|\{\omega \in \Omega_{\Lambda(n)} : I_n(\omega) = I,\, \omega^I = \widetilde{\omega}^I\}\right|.$$

By (19) and (20), the above sum contains only one term corresponding to $I = I(\widetilde{\omega})$. Hence

$$|\Psi^{-1}(\widetilde{\omega})| \leq \left|\{\omega \in \Omega_{\Lambda(n)} : I_n(\omega) = I(\widetilde{\omega}),\, \omega^I = \widetilde{\omega}^I\}\right| \leq 2^{|I_n(\widetilde{\omega})|},$$

and the claim follows from (22). Finally, using the finite energy property and (23) we get

$$\tag{24} \begin{aligned} \Phi^{w,p}_{\Lambda(n)}[A'] &\leq \max_{\omega' \in \Psi(A')} \left|\Psi^{-1}(\omega')\right| \left(1 \vee \frac{p}{1-p}\right)^{16\delta n} \Phi^{w,p}_{\Lambda(n)}[\Psi(A')] \\ &\leq \exp(c_1 \delta n) \Phi^{w,p}_{\Lambda(n)}[\Psi(A')], \end{aligned}$$

where $0 < c_1 < \infty$ is a constant.

ii) By (21), the map Ψ does not modify the configuration inside $\Lambda(n/2)$, thus

$$\Psi(A') \subset A.$$

iii) By our cutting procedure we disconnect $\Lambda((1-h^*)n)$ from $\partial\Lambda(n)$ hence

$$\Psi(A') \subset \left\{ \Lambda(3n/4) \nleftrightarrow \partial\Lambda(n) \right\}.$$

By the property *iii)* and by duality, if the event $\Psi(A')$ occurs, there exists an outermost open dual circuit Γ in $\Lambda(n)$ that surrounds $\Lambda(3n/4)$. Let Ξ be the set of such dual circuits surrounding $\Lambda(3n/4)$. For every $\widehat{\gamma} \in \Xi$, we define $\mathrm{Int}(\widehat{\gamma})$ as the set of all the sites of $\Lambda(n)$ that are surrounded by $\widehat{\gamma}$ and $\mathrm{Ext}(\widehat{\gamma})$, the set of the sites of $\Lambda(n)$ that are not surrounded by $\widehat{\gamma}$. Note that

$$\tag{25} \{\Gamma = \widehat{\gamma}\} = \mathrm{Open}(\widehat{\gamma}) \cap G_{\widehat{\gamma}}\,,$$

where $\mathrm{Open}(\widehat{\gamma}) = \{\forall \widehat{e} \in \widehat{\gamma} :\ \widehat{\omega}(\widehat{e}) = 1\}$ and where $G_{\widehat{\gamma}}$ is a $\mathcal{F}_{\mathrm{Ext}(\widehat{\gamma})}$-measurable event. By using properties ii) and iii) and by (25) we can write

$$\tag{26} \begin{aligned} \Phi^{w,p}_{\Lambda(n)}[\Psi(A')] &\leq \Phi^{w,p}_{\Lambda(n)}[A \cap \bigcup_{\widehat{\gamma}\in\Xi} \{\Gamma = \widehat{\gamma}\}] \\ &= \sum_{\widehat{\gamma}\in\Xi} \Phi^{w,p}_{\Lambda(n)}[A \cap G_{\widehat{\gamma}} | \mathrm{Open}(\widehat{\gamma})]\ \Phi^{w,p}_{\Lambda(n)}[\mathrm{Open}(\widehat{\gamma})]. \end{aligned}$$

Since A is $\mathcal{F}_{\mathrm{Int}\widehat{\gamma}}$-measurable, $G_{\widehat{\gamma}}$ is $\mathcal{F}_{\mathrm{Ext}\widehat{\gamma}}$-measurable, we can use the independence of the σ-algebras $\mathcal{F}_{\mathrm{Int}\widehat{\gamma}}$ and $\mathcal{F}_{\mathrm{Ext}\widehat{\gamma}}$ under $\Phi^{w,p}_{\Lambda(n)}[\,\cdot\,|\mathrm{Open}(\widehat{\gamma})]$ and the spatial Markov property to get

$$\tag{27} \begin{aligned} \Phi^{w,p}_{\Lambda(n)}[A \cap G_{\widehat{\gamma}} | \mathrm{Open}(\widehat{\gamma})] &= \Phi^{w,p}_{\Lambda(n)}[A|\mathrm{Open}(\widehat{\gamma})]\ \Phi^{w,p}_{\Lambda(n)}[G_{\widehat{\gamma}}|\mathrm{Open}(\widehat{\gamma})] \\ &= \Phi^{f,p}_{\mathrm{Int}(\widehat{\gamma})}[A]\ \Phi^{w,p}_{\Lambda(n)}[G_{\widehat{\gamma}}|\mathrm{Open}(\widehat{\gamma})]. \end{aligned}$$

Also A is an increasing event, so using (27), we get

$$\tag{28} \forall \widehat{\gamma} \in \Xi \quad \Phi^{w,p}_{\Lambda(n)}[A \cap G_{\widehat{\gamma}}|\mathrm{Open}(\widehat{\gamma})] \leq \Phi^{f,p}_{\Lambda(n)}[A]\ \Phi^{w,p}_{\Lambda(n)}[G_{\widehat{\gamma}}|\mathrm{Open}(\widehat{\gamma})].$$

Using (26) and (28) we obtain

$$
\begin{aligned}
\Phi^{w,p}_{\Lambda(n)}[\Psi(A')] \leq &\Phi^{f,p}_{\Lambda(n)}[A] \sum_{\widehat{\gamma}\in\Xi} \Phi^{w,p}_{\Lambda(n)}[G_{\widehat{\gamma}}|\mathrm{Open}(\widehat{\gamma})]\ \Phi^{w,p}_{\Lambda(n)}[\mathrm{Open}(\widehat{\gamma})] \\
=&\Phi^{f,p}_{\Lambda(n)}[A]\ \Phi^{w,p}_{\Lambda(n)}[\exists\widehat{\gamma}\in\Xi:\ \Gamma=\widehat{\gamma}] \leq \Phi^{p}_{\infty}[A].
\end{aligned}
\tag{29}
$$

Combining (29) with (24) gives us

$$
\Phi^{w,p}_{\Lambda(n)}[A'] \leq \exp(c_1\delta n)\Phi^{p}_{\infty}[A]. \tag{30}
$$

Now we turn to the second term of (16), namely $\Phi^{w,p}_{\Lambda(n)}[M_{\Lambda(n)} > |\Lambda(n)|\delta\ ]$. Assuming that n is bigger than $16m_{\mathrm{sub}}(\delta/2,p)/\delta$, we can apply lemma 10 to get

$$
\Phi^{w,p}_{\Lambda(n)}[M_{\Lambda(n)} > |\Lambda(n)|\delta\ ] \leq \exp\left[-\left(\frac{\delta n}{6m_{\mathrm{sub}}(\delta/2,p)}\right)^2\right]. \tag{31}
$$

Substituting (30) and (31) into (16) one has

$$
\Phi^{w,p}_{\Lambda(n)}[A] \leq \exp(c_1\delta n)\ \Phi^{p}_{\infty}[A] + \exp\left[-\left(\frac{\delta n}{6m_{\mathrm{sub}}(\delta/2,p)}\right)^2\right]. \tag{32}
$$

It follows from the comments after proposition 4 that there exists a positive τ_c such that for all $p < p_c$ and $n > 1$,

$$
\Phi^{p}_{\infty}[A]\ \leq\ |\partial\Lambda(n/2)| \sup_{x\in\partial\Lambda(n/2)} \Phi^{p}_{\infty}[0\leftrightarrow x]\ \leq\ 2n\exp(-\tau_c(p_c-p)n/4).
$$

So that (32) becomes

$$
\Phi^{w,p}_{\Lambda(n)}[A] \leq 2n\exp(-\tau_c(p_c-p)n/4 + c_1\delta n) + \exp\left[-\left(\frac{\delta n}{6m_{\mathrm{sub}}(\delta/2,p)}\right)^2\right]. \tag{33}
$$

From (33), it is clear that the only way not to destroy our estimates is to take δ at most of order (p_c-p). So let us choose $\delta = \frac{\tau_c}{8c_1}(p_c-p)$. Let $a > 2$. By lemma 11 we know that there exists a positive constant c_2 such that $m_{\mathrm{sub}}(\tau_c(p_c-p)/(16c_1),p) < c_2(p_c-p)^{-a}$. Thus there exists a positive c_3 such that for all $n > c_3(p_c-p)^{-1-a}$, (33) becomes

$$
\Phi^{w,p}_{\Lambda(n)}[A] \leq \exp(-(\tau_c/16)(p_c-p)n) + \exp(-c_4(p_c-p)^{2+2a}n^2), \tag{34}
$$

where $c_4 > 0$. Furthermore, we require that the first term is the main contribution, we do this by imposing that $n > \tau_c(p_c-p)^{-1-2a}/(16c_4)$. We conclude the proof by choosing $c = (c_3p_c^a) \vee (\tau_c/(16c_4))$ and $\lambda = \tau_c/16$. $\square$

Now we are ready to state an analogue of weak mixing in our regime.

Proposition 13. *Let $a > 5$ and $p \neq p_c$. Let $\Delta \subset \Gamma$ be two subsets of $\mathbb{Z}^2$. There exist two positive constants $c = c(a)$ and λ such that if $d(\Delta, \partial\Gamma) > c|p_c - p|^{-a}$ then, uniformly over all $\Phi_1, \Phi_2 \in \mathcal{FK}(p, \Gamma)$, we have*

$$\sup_{A \in \mathcal{F}_\Delta} |\Phi_1[A] - \Phi_2[A]| \leq 2 \sum_{x \in \partial^{\text{in}} \Delta} \exp\big(- \lambda |p_c - p| d(x, \partial\Gamma)\big).$$

Proof. Once we have proposition 12, weak mixing follows from arguments contained in theorem 3.1 of [1]. We include the proof for the sake of completeness. Let $\rho_1, \rho_2 \in \Omega_{\partial\Gamma}$. It is sufficient to prove the claim for the FK-measures $\Phi_\Gamma^{\pi(\rho_1)}$ and $\Phi_\Gamma^{\pi(\rho_2)}$.
It is known, see lemma 3.2 of [1], that there exists a coupling $P_\Gamma^{\rho_1,\rho_2}$ on $\Omega_\Gamma \times \Omega_\Gamma$ with marginals $\Phi_\Gamma^{\pi(\rho_1)}$ and $\Phi_\Gamma^{\pi(\rho_2)}$ such that

$$P_\Gamma^{\rho_1,\rho_2}\left[\left\{ (\omega_1, \omega_2) : \begin{array}{l} \text{Each } e = \{x, y\} \in \mathbb{E}(\Gamma) \text{ such that } x \nleftrightarrow \partial\Gamma \\ \text{in } \omega_1 \text{ and } \omega_2 \text{ satisfies } \omega_1(e) = \omega_2(e) \end{array} \right\}\right] = 1.$$

This means that ω_1 and ω_2 can be made identical on the edges that are not in the union of the boundary connected clusters of the configurations ω_1 and ω_2. Using this coupling, we write

$$\begin{aligned} |\Phi_\Gamma^{\pi(\rho_1)}[A] - \Phi_\Gamma^{\pi(\rho_2)}[A]| \leq & \sum_{\substack{\omega_1 \in A \\ \omega_2 \in A^c}} 1_{P[(\omega_1,\omega_2)] > P[(\omega_2,\omega_1)]} P[(\omega_1, \omega_2)] \left(1 - \frac{P[(\omega_2, \omega_1)]}{P[(\omega_1, \omega_2)]}\right) \\ & + \sum_{\substack{\omega_1 \in A \\ \omega_2 \in A^c}} 1_{P[(\omega_2,\omega_1)] > P[(\omega_1,\omega_2)]} P[(\omega_2, \omega_1)] \left(1 - \frac{P[(\omega_1, \omega_2)]}{P[(\omega_2, \omega_1)]}\right) \\ \leq & \sum_{\substack{\omega_1 \in A \\ \omega_2 \in A^c}} \max(P[(\omega_1, \omega_2)], P[(\omega_2, \omega_1)]), \end{aligned}$$

where we omitted the subscripts and superscripts in the notation of the coupling. Observe that if $\omega_1 \in A$ and $\omega_2 \in A^c$, then there exists $e \in \mathbb{E}(\Delta)$ such that $\omega_1(e) \neq \omega_2(e)$. Hence, using the property of the coupling, we obtain

$$\begin{aligned} |\Phi_\Gamma^{\pi(\rho_1)}[A] - \Phi_\Gamma^{\pi(\rho_2)}[A]| = & |P[A, A^c] - P[A^c, A]| \\ \leq & P[(\omega_1, \omega_2) : \exists e \in \Delta : \omega_1(e) \neq \omega_2(e)] \\ \leq & 2\, \Phi_\Gamma^w[\exists\, x \in \Delta :\ x \leftrightarrow \partial\Gamma] \\ \leq & 2 \sum_{x \in \partial^{\text{in}} \Delta} \Phi^w_{\Lambda(r(x))}[0 \leftrightarrow \partial\Lambda(r(x))], \end{aligned}$$

where $r(x) = 2d(x, \partial\Gamma) \geq 2d(\Delta, \partial\Gamma)$.
Let $a > 5$. By proposition 12, there exist two positive constants $c = c(a)$ and λ such that $d(\Delta, \partial\Gamma) \geq c|p - p_c|^{-a}$ implies

$$|\Phi_\Gamma^{\pi(\rho_1)}[A] - \Phi_\Gamma^{\pi(\rho_2)}[A]| \leq \sum_{x \in \partial^{\text{in}} \Delta} \exp(-\lambda |p - p_c| d(x, \partial\Gamma)).$$

□

The just proved weak mixing property permits to establish the following decoupling lemma:

Lemma 14. *Let $p \neq p_c$, $a > 5$, Λ a box and $\Phi \in \mathcal{FK}(p, \Lambda)$. There exist two positive constants λ and $c = c(a)$ such that for every two sets $\Gamma, \Delta \subset \Lambda$ satisfying*

$$d(\Gamma, \Delta) > c\left(|p - p_c|^{-a} \vee \frac{\log|\Gamma|}{|p - p_c|} \vee \frac{\log|\Delta|}{|p - p_c|}\right),$$

and for every two events $A \in \mathcal{F}_\Gamma$ and $B \in \mathcal{F}_\Delta$, we have

$$|\Phi[A \cap B] - \Phi[A]\ \Phi[B]| \leq \exp[-\lambda(p - p_c)d(\Gamma, \Delta)]\Phi[A]\ \Phi[B].$$

Proof. An adaptation of the arguments in lemma 3.2 of [4] establishes the result. □

We conclude this section with a decoupling lemma.

Lemma 15. *Let $p \neq p_c$, $a > 5$ and $\delta > 0$. There exist two positive constants $c = c(a, \delta)$ and λ such that uniformly over the events $A \in \mathcal{F}_{\Lambda(n)}$ and uniformly over two measures Φ_1, Φ_2 in $\mathcal{FK}(\Lambda(n(1+\delta)), p)$ we have*

$$n > c|p - p_c|^{-a} \quad \Rightarrow \quad (1 - e^{-\delta\lambda|p - p_c|n/2})^2\Phi_1[A] \leq \Phi_2[A] \leq (1 + e^{-\delta\lambda|p - p_c|n/2})^2\Phi_1[A].$$

Proof. Consider $A \in \mathcal{F}_{\Lambda(n)}$ and two partially wired boundary conditions π_1 and π_2 on the boundary $\partial\Lambda((1+\delta)n)$. It is sufficient to prove the statement for the measures $\Phi_1 = \Phi^{\pi_1,p}_{\Lambda((1+\delta)n)}$ and $\Phi_2 = \Phi^{\pi_2,p}_{\Lambda((1+\delta)n)}$. Let $m > (1+2\delta)n$ and define the following $\mathcal{F}^{\Lambda((1+\delta)n)}_{\Lambda(m)}$-measurable events, for $i = 1, 2$:

$$W_i = \left\{ \omega \in \Omega_{\Lambda(m)} : \begin{array}{l} \text{with wired boundary conditions on } \Lambda(m) \\ \text{and the configuration } \omega \text{ on } \Lambda(m) \setminus \Lambda((1+\delta)n), \\ \text{the boundary conditions induced on } \Lambda((1+\delta)n) \text{ are } \pi_i \end{array} \right\}$$

Since π_1 and π_2 are partially wired boundary conditions, it is possible to find a large enough finite m such that $\Phi^{w,p}_{\Lambda(m)}[W_i] > 0, i = 1, 2$. We fix such an m and write $\Phi_i[A] = \Phi^{w,p}_{\Lambda(m)}[A|W_i], i = 1, 2$. Since $d(\Lambda(m) \setminus \Lambda((1+\delta)n), \Lambda(n)) > \delta n/2$, it is possible to find a positive $c = c(a, \delta)$ such that $n > c|p - p_c|^{-a}$ verifies the hypothesis of lemma 14 and we obtain that

$$n > c|p - p_c|^{-a} \quad \Rightarrow \quad |\Phi^{w,p}_{\Lambda(m)}[A|W_i] - \Phi^{w,p}_{\Lambda(m)}[A]| \leq e^{-\delta\lambda|p - p_c|n/2}\Phi^{w,p}_{\Lambda(m)}[A] \quad i = 1, 2.$$

Using the last inequality, we finally get

$$\Phi_2[A] \geq (1 - e^{-\delta\lambda|p - p_c|n/2})\Phi^{w,p}_{\Lambda(m)}[A] \geq (1 - e^{-\delta\lambda|p - p_c|n/2})^2\Phi_1[A],$$

and

$$\Phi_2[A] \leq (1 + e^{-\delta\lambda|p - p_c|n/2})\Phi^{w,p}_{\Lambda(m)}[A] \leq (1 - e^{-\delta\lambda|p - p_c|n/2})^2\Phi_1[A].$$

□

4.4. Block estimates near criticality

In this section we establish the necessary block estimates to implement coarse graining techniques in a regime where p approaches p_c from above. In the following we will use the boxes

$$\widetilde{\Lambda}(n) \;=\; \{x\in\mathbb{Z}^2 : d(x,\Lambda(n))\le n/10\}\,.$$

We take this bigger boxes in order to give estimates on events that occur in $\Lambda(n)$ uniformly over the boundary conditions on $\widetilde{\Lambda}(n)$. In fact, an adaptation of [2] to our regime would spare us this precaution. But this would also increase the size of the paper.
We will say that a FK-cluster C of a box Λ is a *crossing* cluster or that C *crosses* the box Λ if C connects all the sides of Λ. Note that in dimension two if there exists a crossing cluster in a box then it is necessarily unique. We will give estimates on the following block events:

$$U(\Lambda) = \{\exists \text{ an open crossing cluster } C^* \text{ in } \Lambda\}\,.$$

For $M>0$, we define

$$\begin{aligned} R(\Lambda,M) = U(\Lambda)\cap\{&\text{ every open path } \gamma\subset\Lambda \text{ with } \mathrm{diam}(\gamma)\ge M \text{ is in } C^*\}\\ &\cap\{C^* \text{ crosses every sub-box of } \Lambda \text{ with diameter } \ge M\}\,. \end{aligned}$$

For $\delta>0$, we define

$$\begin{aligned} &V(\Lambda,\delta) = U(\Lambda)\cap\{|C^*|\ge(1-\delta)\theta|\Lambda|\}\,,\\ &F(\Lambda,\delta) = \left\{\begin{array}{l} \exists \text{ an open circuit } \gamma \text{ enclosing a volume } \ge(1-\delta)|\Lambda| \\ \text{and such that } \sup\limits_{x\in\gamma} d(x,\partial\Lambda)\le\delta|\partial\Lambda| \end{array}\right\},\\ &W(\Lambda,\delta) = \{|\{x\in\Lambda : x\leftrightarrow\partial\Lambda\}|\le(1+\delta)\theta|\Lambda|\}\,,\\ &T(\Lambda,\delta) = \left\{\Big|\sum_{x\in\Lambda:x\not\leftrightarrow\partial\Lambda}\sigma(x)\Big|\le\delta\theta|\Lambda|\right\}\,. \end{aligned}$$

Notice that the last event involves the FK-Ising coupling. Let us begin with the first two events:

Lemma 16. *Let $a>5$. There exist two positive constants $\lambda, c=c(a)$ such that if $p>p_c$ and $n>c(p-p_c)^{-a}$ then*

$$\forall\Phi\in\mathcal{FK}(\widetilde{\Lambda}(n),p)\qquad \log\Phi[U(\Lambda(n))^c]\le-\lambda(p-p_c)n.$$

Moreover, if M is such that

$$\frac{\log n}{\kappa(p-p_c)}<M\le n, \tag{35}$$

with $\kappa>0$ small enough, then

$$\forall\Phi\in\mathcal{FK}(\widetilde{\Lambda}(n),p)\qquad \log\Phi[R(\Lambda(n),M)^c]\le-\lambda(p-p_c)M.$$

Proof. Let $\Phi \in \mathcal{FK}(\widetilde{\Lambda}(n),p)$. Since $U(\Lambda(n))$ is increasing, we have that $\Phi[U(\Lambda(n))^c] \leq \Phi^{f,p}_{\Lambda(n)}[U(\Lambda(n))^c]$. By duality we get that

$$\Phi[U(\Lambda(n))^c] \leq 2\ \Phi^{f,p}_{\Lambda(n)}[\exists \text{ an open dual path in } \widehat{\Lambda}(n) \text{ of diameter } \geq n],$$

Let $a > 5$. By lemma 15 and proposition 4 there exist two positive constants $c = c(a)$ and λ_1 such that for all $p > p_c$ and for all $n > c(p-p_c)^{-a}$ we have

$$\begin{aligned}
&\Phi^{f,p}_{\Lambda(n)}[\exists \text{ an open dual path in } \widehat{\Lambda}(n) \text{ of diameter} \geq n] \\
&\qquad \leq \Phi^{f,p}_{\Lambda(n)}[\exists \text{ an open dual path in } \widehat{\Lambda}(n) \text{ of diameter} \geq n] \\
&\qquad \leq \Phi^{p}_{\infty}[\exists \text{ an open dual path in } \widehat{\Lambda}(n) \text{ of diameter} \geq n] \\
&\qquad \leq 2n^4 \exp(-\lambda_1(p-p_c)n) \\
&\qquad \leq 2\exp((p_c-p)n(\lambda_1 - 4\frac{\log n}{n(p-p_c)})),
\end{aligned}$$

Note that there exists n_0 independent of everything such that

$$\forall n > \max(n_0, c(p-p_c)^{-a}) \qquad \frac{\log n}{n(p-p_c)} \leq \frac{n^{-1/2}}{p-p_c} \leq \frac{1}{c}(p-p_c)^{3/2}.$$

Thus, the result follows by choosing $\lambda = \lambda_1/2$ and c big enough. To estimate the event R, notice that

$$\Phi[R(\Lambda(n),M)^c] \leq \Phi[U(\Lambda(n))^c] + \Phi^{f,p}_{\Lambda(n)}[\exists \text{ an open dual path of diameter } \geq M].$$

Then, as before, we use lemma 15 and proposition 4 to get

$$\begin{aligned}
\Phi[R(\Lambda(n),M)^c] &\leq \exp(-\lambda(p-p_c)n) + n^4\exp(-\lambda(p-p_c)M) \\
&\leq (1+n^4)\exp(-\lambda(p-p_c)M).
\end{aligned}$$

Finally, condition (35) ensures that the prefactor does not destroy our estimates and this concludes the proof. $\square$

Now we turn to the estimation of the crossing cluster's size:

Lemma 17. *Let $p > p_c$ and $\delta > 0$. Let $a > 5$ and $\alpha \in]0,(1+\frac{1}{8a})^{-1}[$. There exists a positive constant $c = c(a,\alpha)$ such that, if $n \uparrow \infty$ and $p \downarrow p_c$ in such a way that $n^\alpha(p-p_c)^a > c$ then*

$$\sup_{\Phi\in\mathcal{FK}(\widetilde{\Lambda}(n),p)} \Phi[V(\Lambda(n),\delta)^c] \leq \exp(-\lambda\delta(p-p_c)n^\alpha) + \exp(-\frac{\delta^2\theta^2(p)}{4}n^{2-2\alpha}),$$

where λ is a positive constant. In particular

$$\lim_{n,p} \inf_{\Phi\in\mathcal{FK}(\widetilde{\Lambda}(n),p)} \Phi[V(\Lambda(n),\delta)] = 1.$$

Proof. Let $\Phi \in \mathcal{FK}(\widetilde{\Lambda}(n),p)$. We renormalize $\Lambda(n)$ into $\underline{\Lambda}(n)$ by partitioning it into blocks $B(\underline{x})$ of size $N \leq n$ to get the renormalized box

$$\underline{\Lambda}(n) = \{\underline{x} \in \mathbb{Z}^2 : \ (-N/2, N/2]^2 + N\underline{x} \subset (-n/2, n/2]^2\}.$$

Next, we define the following events:

- For $\{\underline{x},\underline{y}\} \in \mathbb{E}(\underline{\Lambda}(n))$, we denote by $m(\underline{x},\underline{y})$ the middle point of the face between $B(\underline{x})$ and $B(\underline{y})$. We also introduce the box $D_{\underline{x},\underline{y}} = m(\underline{x},\underline{y}) + \Lambda(\lfloor N/4 \rfloor)$ of width $\lfloor N/4 \rfloor$ and centered at $m(\underline{x},\underline{y})$. Then, we define

$$K_{\underline{x},\underline{y}} = \{\exists \text{ crossing in } D_{\underline{x},\underline{y}}\}, \qquad K_{\underline{x}} = \bigcap_{\underline{z} \in \underline{\Lambda}(n)\,:\,|\underline{x}-\underline{z}|=1} K_{\underline{x},\underline{z}}.$$

- For $\underline{x} \in \underline{\Lambda}(n)$ and $M > 0$, we define

$$\begin{aligned} R(\underline{x}) = \{&\exists! \text{ crossing cluster } C^*_{\underline{x}} \text{ in } B(\underline{x})\} \cap \\ &\big\{\text{every open path } \gamma \subset B(\underline{x}) \text{ with } \operatorname{diam}(\gamma) \geq M \text{ is included in } C^*_{\underline{x}}\big\}. \end{aligned} \tag{36}$$

On $\underline{\Lambda}(n)$, we define the $0-1$ renormalized process $(X(\underline{x}), \underline{x} \in \underline{\Lambda}(n))$ as the indicator of the occurrence of the above mentioned events:

$$\forall \underline{x} \in \underline{\Lambda}(n) \quad X(\underline{x}) = \begin{cases} 1 \text{ on } R(\underline{x}) \cap K(\underline{x}) \\ 0 \text{ otherwise} \end{cases}$$

By lemma 16, we get the following estimate on the probability that a specific box is *bad*. There exist $\kappa, \lambda > 0$ such that if

$$n > N > 4M > \frac{\log N}{\kappa(p - p_c)} \tag{37}$$

then

$$\forall \underline{x} \in \underline{\Lambda}(n) \qquad \Phi[X(\underline{x}) = 0] \leq \exp\left(-\lambda(p - p_c)M\right). \tag{38}$$

As M will grow, we can restrict ourselves to the case where there is no bad block at all and where the event $R(\Lambda(n), N)$ is satisfied, namely for all $\Phi \in \mathcal{FK}(\widetilde{\Lambda}(n),p)$, we write

$$\begin{aligned} \Phi[V(\Lambda(n),\delta)^c] \leq &\Phi[\exists \text{ a bad block }] + \Phi[R(\Lambda(n),N)^c] \\ &+ \Phi[\nexists \text{ a bad block } \cap R(\Lambda(n),N) \cap V(\Lambda(n),\delta)^c]. \end{aligned} \tag{39}$$

By (38), we get

$$\Phi[\exists \text{ a bad block }] \leq \frac{n^2}{N^2} \exp(-\lambda_1(p - p_c)M). \tag{40}$$

For the second term of (39),we apply lemma 16 to get

$$\Phi[R(\Lambda(n),N)^c] \le \exp(-\lambda_2(p-p_c)N), \tag{41}$$

For the third term of (39), we observe that if there is no bad block then there is one single cluster in the renormalized process that consists of all the blocks of $\underline{\Lambda}(n)$. By the definition of the events associated to $(X(\underline{x}),\, \underline{x}\in\underline{\Lambda}(n))$, this induces one crossing cluster $\widetilde{C}^*$ of $\cup_{\underline{x}\in\underline{\Lambda}(n)} B(\underline{x})$ that contains all the crossing clusters $C^*_{\underline{x}}$, $\underline{x}\in\underline{\Lambda}(n)$. On the other hand, since $R(\Lambda(n),N)$ is satisfied, we have that $\widetilde{C}^*\subset C^*$, where C^* is the crossing cluster of $\Lambda(n)$, which is guaranteed to exists thanks to the event $U(\Lambda(n))$. Now, we define for every $\underline{x}\in\underline{\Lambda}(n)$ the random variables

$$Y(\underline{x}) = N^{-2}|\{v\in B(\underline{x}) : \mathrm{diam}(C_v)\ge M\}|$$

and observe that

$$|C^*| < (1-\delta)\theta n^2 \quad\Rightarrow\quad \sum_{\underline{x}\in\underline{\Lambda}(n)} |C^*_{\underline{x}}| < (1-\delta)\theta n^2\,. \tag{42}$$

Yet if $B(\underline{x})$ is a good box then every cluster of $B(\underline{x})$ that is of diameter larger than M is included in $C^*_{\underline{x}}$, thus using (39), (40),(41), (42) and by the FKG inequality we get

$$\Phi[V(\Lambda(n),\delta)^c] \le 2\frac{n^2}{N^2}\exp(-\lambda_3(p-p_c)M) + \Phi\Big[\Big(\frac{N}{n}\Big)^2 \sum_{\underline{x}\in\underline{\Lambda}(n)} Y(\underline{x}) \le (1-\delta)\theta|E\Big], \tag{43}$$

where E is the event that all the edges that touch the boundary of the boxes $B(\underline{x})$ are *closed* and $\lambda_3=\min(\lambda_1,\lambda_2)$. Now we choose N and M such that the mean of the random variables $Y(\underline{x})$ is big enough: by using lemma 15 we have for $\underline{x}\in\underline{\Lambda}(n)$

$$\begin{aligned}
\Phi^{f,p}_{\Lambda(N)}[Y(\underline{x})] &\ge N^{-2}\Phi^{f,p}_{\Lambda(N)}[|\{x\in\Lambda(N-4M) : x\leftrightarrow\partial\Lambda(2M)+x\}|]\\
&\ge N^{-2}\sum_{x\in\Lambda(N-4M)} \Phi^{f,p}_{\Lambda(N)}[x\leftrightarrow\partial\Lambda(2M)+x]\\
&\ge N^{-2}\sum_{x\in\Lambda(N-4M)} \Phi^{f,p}_{x+\Lambda(4M)}[x\leftrightarrow\partial\Lambda(2M)+x]\\
&\ge (1-e^{-(p-p_c)M/2})^2\frac{(N-4M)^2}{N^2}\Phi^{w,p}_{\Lambda(4M)}[0\leftrightarrow\partial\Lambda(2M)]\\
&\ge (1-e^{-(p-p_c)M/2})^2\frac{(N-4M)^2}{N^2}\Phi^{p}_{\infty}[0\leftrightarrow\partial\Lambda(2M)]\\
&\ge (1-2e^{-(p-p_c)M/2})(1-\frac{8M}{N})\theta\,.
\end{aligned}$$

By Onsager's formula, we have

$$\theta \;=\; (p-p_c)^{1/8} + o((p-p_c)^{1/8})\,,\qquad p\downarrow p_c\,.$$

Thus if we choose

$$M = \frac{\delta}{32} N \quad \text{and} \quad M(p - p_c) \geq c, \tag{44}$$

where $c > 0$ is a large enough constant we get

$$\forall \underline{x} \in \underline{\Lambda}(n) \qquad \Phi^{f,p}_{\Lambda(N)}[Y(\underline{x})] \geq \theta(1 - \frac{\delta}{2}). \tag{45}$$

The random variables $Y(\underline{x})$, $\underline{x} \in \underline{\Lambda}(n)$, take their values in $[0,1]$ and they are independent under $\Phi[\cdot\ |E]$, thus we can use lemma 9 with (45) to bound (43) by

$$\Phi[V(\Lambda(n), \delta)^c] \leq 2\frac{n^2}{N^2} \exp(-\lambda(p - p_c)M) + \exp\left(-\frac{\delta^2\theta^2(p)}{4}\frac{n^2}{N^2}\right). \tag{46}$$

Let $a > 3$ and $0 < \alpha < 1$. If $n^\alpha > (p - p_c)^{-a}$ and letting $N = n^\alpha$, one gets

$$\Phi[V(\Lambda(n), \delta)^c] \leq 2\exp(-\frac{\lambda}{32}\delta(p - p_c)n^\alpha + 2(1 - \alpha)\log n) + \exp\left(-\frac{\delta^2\theta^2(p)}{4} n^{2-2\alpha}\right).$$

Also, under the above regime we have that $(p - p_c)n^\alpha / \log n \to \infty$. Thus, by choosing n, N, M such that (37) and (44) are satisfied and using Onsager's formula we obtain the desired result. □

Next, we consider the deviations from above for the size of the crossing cluster.

Lemma 18. *Let $p > p_c$ and $\delta > 0$. If $n > 8m_{\sup}(\delta, p)/\delta$ where*

$$m_{\sup}(\delta, p) = \inf\left\{ m \geq 1 : \forall n \geq m \quad \frac{1}{|\Lambda(n)|}\Phi^{w,p}_{\Lambda(n)}[M_{\Lambda(n)}] \leq (1 + \delta/2)\theta \right\},$$

then

$$\log \Phi^{w,p}_{\Lambda(n)}[W(\Lambda(n), \delta)^c] \leq -\left(\frac{\delta\theta n}{4m_{\sup}(\delta, p)}\right)^2. \tag{47}$$

In particular, for every $a > 5/4$, there exists a positive constant $c = c(a, \delta)$ such that whenever $n \uparrow \infty$ and $p \downarrow p_c$ in such a way that $n > c(p - p_c)^{-a}$ then

$$\lim_{(n,p)} \frac{1}{(p - p_c)^{2a+1/4} n^2} \log \Phi^{w,p}_{\Lambda(n)}[W(\Lambda(n), \delta)^c] < 0.$$

Proof. To get (47), one proceeds as in lemma 10. For the second statement, one proceeds as in lemma 11 to prove that for every $a > 9/8$, there exists a positive constant $c = c(a, \delta)$ such that $m_{\sup}(\delta, p) \leq C(p - p_c)^{-a}$. The desired result follows then from (47). □

Lemma 19. *Let $a > 5$ and $\delta > 0$. There exist two positive constants $\lambda, c = c(a,\delta)$ such that for all $p > p_c$ and $n > 4$ such that $n > c(p-p_c)^{-a}$, we have*

$$\log \Phi[F(\Lambda(n),\delta)^c] \leq -\lambda\delta n(p-p_c),$$

uniformly in $\Phi \in \mathcal{FK}(\widetilde{\Lambda}(n),p)$.

Proof. It suffices to note that

$$\Phi[F(\Lambda(n),\delta)^c] \leq \Phi[\partial\Lambda(n(1-2\delta)) \leftrightarrow \partial\Lambda(n) \text{ by an open dual path}]$$

Thus, by proposition 4 and lemma 15, there exist two positive constants λ and $c = c(a)$ such that

$$n > c(p-p_c)^{-a} \quad \Rightarrow \quad \Phi[F(\Lambda(n),\delta)^c] \leq 16n^2 \exp(-\lambda\delta n(p-p_c)).$$

By making c a bit larger and replacing λ by $\lambda/2$, we get the desired result. □

Finally, we consider the event T where the edge and the spin configuration of the FK-Ising coupling $\mathbb{P}_n^+$, defined in section 3.2, is involved.

Lemma 20. *Let $\delta > 0$ and $a > 5$. If $p \downarrow p_c$ and $n \uparrow \infty$ in such a way that $n > (p-p_c)^{-a}$, then*

$$\lim_{n,p} \mathbb{P}_n^+[T(\Lambda(n),\delta)] = 1.$$

Proof. Let $\mathcal{C}$ be the collection of the open clusters which do not touch the boundary $\partial\Lambda(n)$. Let $p, n, M = (\log n)/\kappa(p-p_c)$ where κ is like in lemma 16. Let $\omega \in R(\Lambda(n),M)$. Using Chebyshev's inequality,

$$\begin{aligned}\mathbb{P}_n^+[T(\Lambda(n),\delta)^c|\,\omega] \leq & \mathbb{P}_n^+\left[\frac{1}{|\Lambda(n)|}\left|\sum_{C\in\mathcal{C}}\sigma(C)|C|\right| \geq \delta\theta \,\middle|\, \omega\right] \\ \leq & \frac{1}{\delta^2\theta^2|\Lambda(n)|^2}\sum_{C\in\mathcal{C}}|C|^2 \leq \frac{M^4}{\delta^2\theta^2 n^2}.\end{aligned}$$

Imposing $\log n/((p-p_c)\sqrt{\theta n}) \to 0$, using lemma 16 and the previous inequalities we get the desired result. □

5. The upper bound

Let x be a point of $\mathbb{R}^2$. The closed ball of center x and Euclidean radius $r > 0$ is denoted by $B(x,r)$. For w in the unit sphere S^1, we define the half balls

$$B_-(x,r,w) = B(x,r) \cap \{y \in \mathbb{R}^2 : (y-x)\cdot w \leq 0\},$$
$$B_+(x,r,w) = B(x,r) \cap \{y \in \mathbb{R}^2 : (y-x)\cdot w \geq 0\}.$$

To prove the local upper bound we need to estimate a FK percolation event which occurs when the locally averaged magnetization exhibits a jump. We will do this by showing that this event implies the existence of an interface. The relevant event is that there exists a collection $\mathcal{G}$ of open clusters in $B(nx, nr)$ such that

$$\sum_{C \in \mathcal{G}} |C \cap B_-(nx, nr, w)| \geq (1 - \theta\delta)\mathcal{L}^2(B_-(nx, nr, w)),$$
$$\sum_{C \in \mathcal{G}} |C \cap B_+(nx, nr, w)| \leq \delta\theta\mathcal{L}^2(B_+(nx, nr, w)).$$

We will denote this event by $\mathrm{Sep}(n, x, r, w, \delta)$. Next, we state the so-called interface lemma:

Lemma 21. *Let $x \in Q$ and $0 < r \leq 1$ such that $B(x, r) \subset Q$. Let $\delta > 0$ and $w \in S^1$. If $p \downarrow p_c$ and $n \uparrow \infty$ in such a way that $n(p-p_c)^{20} \to \infty$ and such that $\log(n)/\log(1/(p-p_c))$ stays bounded, then*

$$\limsup_{(n,p)} \frac{1}{(p - p_c)n} \log \Phi^{w,p}_{\Lambda(n)} [\mathrm{Sep}(n, x, r, w, \delta)] \leq -2r\tau_c(1 - c'\delta^{1/2}),$$

where c' is a positive constant.

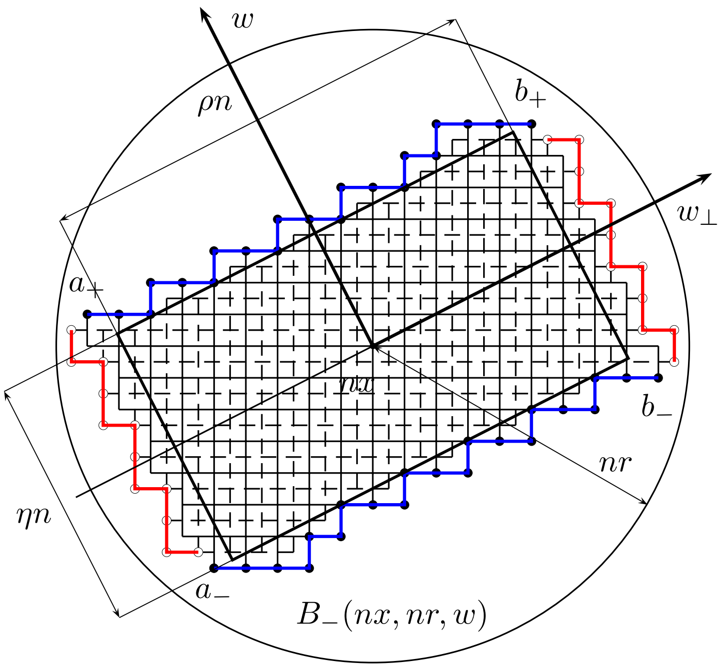
In [14, 15, 16, 17] a cutting procedure has been used to create an interface from the event Sep without altering too much the probability. In our context such an approach does not work. This stems from the fact that the monotone perturbation lemma (lemma 6.3 of [16]) is not appropriate when $p \downarrow p_c$. We thus have to proceed differently. We start by showing that the event Sep is well approximated by a similar event involving filled clusters instead of clusters with a lot of small holes, then instead of cutting some edges in order to create the interface we will detect a piecewise interface. Let us fix a small positive η that will be determined later and ρ such that

$$0 < \eta < \rho < r, \quad 0 < 2\eta < \sqrt{r^2 - \rho^2},$$

and we restrict our attention to the rectangle

$$R = \{y \in B(nx, nr) : -\eta n \leq (y - nx) \cdot w \leq \eta n, \quad -\rho n \leq (y - nx) \cdot w_\perp \leq \rho n\},$$

where $w_\perp$ is the vector perpendicular to w such that $(w_\perp, w)$ is a direct basis.



We define the *right* to be the direction at which $w_\perp$ points and the *top* the direction at which w points and accordingly we define the *left* and the *bottom*. We consider the graph $D \subset \Lambda(n)$ whose edges are the boundaries of the squares centered in $\mathbb{Z}^2+(1/2,1/2)$ that intersect R. In this way, the set ∂D is a simple closed circuit. We denote by a_+ the upper left site of the square that contains the upper left point of ∂D. Going clockwise we define successively and in a similar way b_+, b_- and a_-, see the figure above. We define the top boundary $\partial^+ D$ as the path of ∂D that joins clockwise a_+ to b_+. Similarly, the bottom boundary $\partial^- D$ is the path of ∂D that joins counterclockwise a_- to b_-. We define also $D_+ = B_+(nx, nr, w) \cap D$ and $D_- = B_-(nx, nr, w) \cap D$. Since the interface is an open *dual* path that goes from the left to the right, we need also to consider the dual graph $\widehat{D}$ of D which is depicted in the previous figure. This permits us to define the right boundary $\partial^R \widehat{D}$ as the piece of the boundary $\partial\widehat{D}$ that joins clockwise the center of the square containing b_+ to the center of the square containing b_-. Similarly we define the left boundary $\partial^L \widehat{D}$. The definitions of $\partial^+ D, \partial^- D, \partial^L \widehat{D}$ and $\partial^R \widehat{D}$ guarantee that, if a configuration $\omega \in \Omega_D$ does not contain any open cluster that connects $\partial^+ D$ to $\partial^- D$ then, in the dual configuration $\widehat{\omega} \in \Omega_{\widehat{D}}$, there exists an open cluster that connects $\partial^L \widehat{D}$ to $\partial^R \widehat{D}$.

In order to prove the upper bound, it is sufficient to consider the consequence of $\mathrm{Sep}(\delta)$ on the configuration restricted to D which is a convenient set for duality arguments. To depict our restriction to D, we denote by $\mathcal{C}$ the clusters in D that connect $\partial^+ D$ to $\partial^- D$ and suppose that there exists a collection $\mathcal{G}$ of open clusters in $B(nx, nr, w)$ that realizes $\mathrm{Sep}(\delta)$. In this situation, we can make the decomposition $\mathcal{C} = \mathcal{C}_- \cup \mathcal{C}_+$, where $\mathcal{C}_-$ is the collection of the open clusters in $\mathcal{C}$ that are contained in a cluster of the collection $\mathcal{G}$ and $\mathcal{C}_+$ is the collection of the open clusters in $\mathcal{C}$ that are not contained in any cluster of the collection $\mathcal{G}$. By the definition of the event Sep, the cardinality of the intersection of D_+ with the clusters in $\mathcal{C}_-$ is less than $\theta\delta\mathcal{L}^2(B_+(nx,nr,w))$ and the cardinality of the intersection of D_- with the clusters in $\mathcal{C}_+$ is also less than $\theta\delta\mathcal{L}^2(B_-(nx,nr,w))$. Thus the event Sep implies the following event involving only the clusters in D:

$$\mathrm{Sep}_D(\delta) = \Big\{ \text{ there exists a decomposition } \mathcal{C} = \mathcal{C}_- \cup \mathcal{C}_+ \text{ such that}$$

$$\sum_{C\in\mathcal{C}_-} |C \cap D_+(nx,nr,w)| \le \pi\delta\theta(nr)^2, \tag{48}$$

$$\sum_{C\in\mathcal{C}_+} |C \cap D_-(nx,nr,w)| \le \pi\delta\theta(nr)^2 \Big\}.$$

5.1. Elimination of the small holes

In our situation it is necessary to fill the small holes of the clusters that enter in the definition of Sep in order to give an adequate upper bound. Namely, we will replace the event Sep with an event Sep$'$ that uses only filled clusters and we will show that the probability of Sep is well approximated by the probability of the auxiliary event Sep$'$. In order to construct our filling procedure we need some definitions. Let $C \in \mathcal{C}$. We introduce the notion of *holes* of C. For this we consider the dual $\widehat{\mathbb{E}(D)\setminus\mathbb{E}(C)}$ of the complement of C. Each maximally connected set $\widehat{F}$ of $\widehat{\mathbb{E}(D)\setminus\mathbb{E}(C)}$ that is isolated from the other clusters of $\mathcal{C}$ by C will be designated as a *hole* of C. For each hole we also define the following notion of boundary

$$\Delta\widehat{F} = \{\widehat{e} \in \widehat{F} : d(\widehat{e}, \mathbb{E}(C)) = \frac{1}{2}\}.$$

Note that by the definition of a hole, the edges of $\Delta\widehat{F}$ are all open dual edges.
Next, we fix $M < n$ and say that a hole $\widehat{F}$ is small (respectively big) if $\mathrm{diam}(\widehat{F}) < M$ (respectively if $\mathrm{diam}(\widehat{F}) \ge M$). For each $C \in \mathcal{C}$ we define its filling fill C as

$$\text{fill } C = C \cup \bigcup_{\widehat{F}} F,$$

where the union runs over all the small holes of C and where F is the set of edges in D whose dual is $\widehat{F}$. Note that if $C_1 \neq C_2$ then fill $C_1 \neq$ fill C_2.
For $* = -, +$, let

$$\mathcal{C}_*^{\text{fill}} = \{\text{fill } C : C \in \mathcal{C}_*\}.$$

We define then a modified Sep_D event that involves only the filled clusters:

$$\mathrm{Sep}'_D(n,x,r,w,\delta) = \Big\{ \text{ there exists a decomposition } \mathcal{C} = \mathcal{C}_- \cup \mathcal{C}_+ \text{ such that}$$

$$\sum_{S\in\mathcal{C}_-^{\text{filled}}} |S \cap D_+| \le \delta\pi(nr)^2, \qquad \sum_{S\in\mathcal{C}_+^{\text{filled}}} |S \cap D_-| \le \delta\pi(nr)^2 \Big\}.$$

Note that the event Sep'_D involves only the filled clusters of D and even if Sep has been defined originally in $B(nx,nr)$, we will only use with its consequence (48). Now we show that the event Sep_D is well approximated by the event Sep'_D:

Lemma 22. *Let $\delta > 0$, $a > 5$ and $\alpha \in]0, (1+\frac{1}{8a})^{-1}[$. There exists a positive constant $c = c(a, \alpha, \delta) > 0$ such that if*

$$\eta r n > M > c(p-p_c)^{-a/\alpha},$$

then uniformly in n, M, x, r, w we have that

$$\log \Phi^{w,p}_{\Lambda(n)}[\mathrm{Sep}_D(n,x,r,w,\delta) \setminus \mathrm{Sep}'_D(n,x,r,w,4\delta)] \leq -\lambda \frac{n^2}{M^2} \delta (p-p_c)^{1/8} \log(\frac{1}{p-p_c}),$$

where λ is a positive constant.

Proof. First we renormalize D into $\underline{D}$ by partitioning it with blocks $B(\underline{x})$ of size $M/2$. We say that a block $B(\underline{x})$ is good if and only if the event

$$V(B(\underline{x}),\delta) \cap W(B(\underline{x}),\delta) \cap F(B(\underline{x}),\delta) \cap R(B'(\underline{x}), M/4)$$

occurs. Recalling the definitions of V, W, F, R the above event is

$$\begin{aligned}
&\left\{ \begin{array}{l} \exists \text{ crossing cluster } C^* \text{ in } B(\underline{x}) \\ \text{and } |C^*| \geq (1-\delta)\theta |B(\underline{x})| \end{array} \right\} \\
\cap &\left\{ \big| \{ x \in B(\underline{x}) : x \leftrightarrow \partial B(\underline{x}) \} \big| \leq (1+\delta)\theta |B(\underline{x})| \right\} \\
\cap &\left\{ \begin{array}{l} \exists \text{ an open circuit } \gamma \text{ in } B(\underline{x}) \text{ enclosing a volume } \geq (1-\delta)|B(\underline{x})| \\ \text{ and such that } \sup_{x\in\gamma} d(x, \partial B(\underline{x})) \leq \delta |\partial B(\underline{x})| \end{array} \right\} \\
\cap &\left\{ \exists \text{ crossing cluster } \widetilde{C}^* \text{ in } B'(\underline{x}) \right\} \\
\cap &\left\{ \begin{array}{l} \text{Every open path } \gamma' \subset B'(\underline{x}) \text{ with} \\ \mathrm{diam}\, \gamma' \geq M/4 \text{ is included in } \widetilde{C}^* \end{array} \right\} \\
\cap &\left\{ \widetilde{C}^* \text{ crosses every sub-box of } B'(\underline{x}) \text{ of diam} \geq M/4 \right\}.
\end{aligned}$$

We define the block process $(X(\underline{x}), \underline{x} \in \underline{D})$ by $X(\underline{x}) = 1_{B(\underline{x}) \text{ is good}}$ for $\underline{x} \in \underline{D}$. We have

$$\begin{aligned}
\sum_{C \in \mathcal{C}_-} |C \cap D_+| &\leq \sum_{\underline{x} \in \underline{D}_+} \sum_{C \in \mathcal{C}_-} |C \cap B(\underline{x})| \\
&= \sum_{\substack{\underline{x} \in \underline{D}_+ \\ B(\underline{x}) \text{ is good}}} \sum_{C \in \mathcal{C}_-} |C \cap B(\underline{x})| + \sum_{\substack{\underline{x} \in \underline{D}_+ \\ B(\underline{x}) \text{ is bad}}} \sum_{C \in \mathcal{C}_-} |C \cap B(\underline{x})|.
\end{aligned}$$

When $B(\underline{x})$ is good and fill $C \cap B(\underline{x}) \neq \emptyset$ then $C \cap B(\underline{x}) \neq \emptyset$. We also have that

$$|C \cap B(\underline{x})| \geq |C^*| \geq (1-\delta)\theta |B(\underline{x})|. \tag{49}$$

The first inequality holds because when $B(\underline{x})$ is good, then every cluster $C \in \mathcal{C}_-$ that intersects $B(\underline{x})$ will contain a path of diameter at least $M/2$ in the box $B'(\underline{x})$, this path is included in $\widetilde{C}^*$, thus the cluster C contains $\widetilde{C}^*$ which also contains C^*. Next

$$|C \cap B(\underline{x})| \leq |\{x \in B(\underline{x}) : x \leftrightarrow \partial B(\underline{x})\}| \leq (1+\delta)\theta|B(\underline{x})|\,. \tag{50}$$

This inequality is true because for reasons of diameter no cluster $C \in \mathcal{C}_-$ fits into a box $B(\underline{x})$. Thus all the connected components of $C \cap B(\underline{x})$ have to be connected to $\partial B(\underline{x})$. Next, as before, when $B(\underline{x})$ is good then $\widetilde{C}^* \cap B(\underline{x}) \subset C \cap B(\underline{x})$. And since $\operatorname{diam}\gamma > M/4$, we have that $\gamma \subset \widetilde{C}^* \cap B(\underline{x}) \subset C \cap B(\underline{x})$. But the diameter of γ is less than M and thus the interior of γ is included in fill C, hence

$$(1-\delta)\theta|B(\underline{x})| \leq \theta|\text{fill } C \cap B(\underline{x})| \leq \theta|B(\underline{x})|. \tag{51}$$

By (49), (50), (51) we get

$$-2\delta\theta|B(\underline{x})| \leq |\text{fill } C \cap B(\underline{x})|\theta - |C \cap B(\underline{x})| \leq 2\delta\theta|B(\underline{x})|.$$

Since two different clusters of $\mathcal{C}$ can not intersect a same good block, we obtain

$$\begin{aligned}\sum_{C \in \mathcal{C}_-} \Big| |C \cap D_+| - \theta|\text{fill } C \cap D_+| \Big| &\leq \sum_{\underline{x} \text{ good}} 2\delta\theta|B(\underline{x})| + \sum_{\underline{x} \text{ bad}} |B(\underline{x})| \\ &\leq 2\delta\theta\pi(nr)^2 + \frac{|\{\underline{x} \in \underline{D} :\ X(\underline{x}) = 0\}|}{|\underline{D}|} 2\pi(nr)^2.\end{aligned}$$

Doing the same reasoning for D_- with $\mathcal{C}_+$, we get

$$\sum_{C \in \mathcal{C}_+} \Big| |C \cap D_-| - \theta|\text{fill } C \cap D_-| \Big| \leq 2\delta\theta\pi(nr)^2 + \frac{|\{\underline{x} \in \underline{D} :\ X(\underline{x}) = 0\}|}{|\underline{D}|} 2\pi(nr)^2.$$

From this, we conclude that

$$\mathrm{Sep}_D(\delta) \cap \left\{ \frac{|\{\underline{x} \in \underline{D} :\ X(\underline{x}) = 0\}|}{|\underline{D}|} \leq \frac{\delta\theta}{2} \right\}$$

is included in $\mathrm{Sep}'_D(4\delta)$. Thus

$$\begin{aligned}&\Phi^{w,p}_{\Lambda(n)}[\mathrm{Sep}_D(n,x,r,w,\delta) \setminus \mathrm{Sep}'_D(n,x,r,w,4\delta)] \leq \\ &\qquad \Phi^{w,p}_{\Lambda(n)}\left[\frac{|\{\underline{x} \in \underline{D} :\ X(\underline{x}) = 0\}|}{|\underline{D}|} \geq \frac{\delta\theta}{2} \right].\end{aligned}$$

Finally, we show that it is possible to tune our regime so that with probability very close to one, the fraction of bad boxes in D remains negligible. Fix $a > 5$ and $\alpha \in]0, (1+\frac{a}{8})^{-1}[$. There exists a positive $c = c(\alpha, a, \delta)$ such that by lemmas 16, 17, 18, 19

$$\sup_{\Phi \in \mathcal{FK}(p,D)} \Phi[X(\underline{x}) = 0 | X(\underline{y}), |\underline{x} - \underline{y}|_\infty \geq 3] = \rho(M,p) \downarrow 0,$$

when $M \uparrow \infty$ and $p \downarrow p_c$ in such a way that $M^\alpha > c(p - p_c)^{-a}$. Thus, by lemma 8 we get that

$$\log \Phi^{w,p}_{\Lambda(n)} [\mathrm{Sep}_D(n,x,r,w,\delta) \setminus \mathrm{Sep}'_D(n,x,r,w,4\delta)] \leq -\delta\theta \log \frac{\delta\theta}{\rho(M,p)} \left\lfloor \frac{nr}{3M} \right\rfloor^2.$$

By using Onsager's formula we get that $\theta \sim (p - p_c)^{1/8}$ when $p \downarrow p_c$. The conclusion follows from the speed of the convergence $\rho(M,p) \downarrow 0$ provided by lemmas 16, 17, 18, 19. $\square$

5.2. The piecewise interface

In this section we will detect a piecewise interface from the occurrence of the event Sep'_D. We suppose that the event Sep'_D occurs and let $\mathcal{C}_-$, $\mathcal{C}_+$ be a decomposition of $\mathcal{C}$ realizing it. We define

$$\mathcal{C}^{\text{filled}} = \mathcal{C}_-^{\text{filled}} \cup \mathcal{C}_+^{\text{filled}},$$

where $\mathcal{C}_*^{\text{filled}}$ has been defined in (48). Note that there is a natural order in $\mathcal{C}^{\text{filled}}$ and thus it is possible to enumerate the elements of $\mathcal{C}^{\text{filled}}$ from the left to the right from S_1 to $S_{|\mathcal{C}^{\text{filled}}|}$. Next, for each $h \in \mathbb{R}$, we define the line

$$\pi(h) = \{y \in \mathbb{R}^2 : (y - nx) \cdot w = h\}.$$

Let

$$E_+ = \bigcup_{S \in \mathcal{C}_-^{\text{filled}}} \mathbb{E}(S \cap D_+)$$

be the set of the edges in D_+ that belong to a filled cluster of $\mathcal{C}_-^{\text{filled}}$. We define similarly E_- as the set of the edges in D_- that belong to a filled cluster of $\mathcal{C}_+^{\text{filled}}$. Then

$$\int_{\eta n/3}^{2\eta n/3} |\{e \in E_+ : e \cap \pi(h) \neq \emptyset\}| dh \leq |E_+|,$$

where η has been defined in the paragraph after lemma 21. Since $\mathcal{C}_-$, $\mathcal{C}_+$ realize the event $\mathrm{Sep}'_D(n, x, r, w, \delta)$, then we have that $|E_+| \leq \delta\pi(nr)^2$ and by the previous inequality, there exists $h \in [\eta n/3, 2\eta n/3]$ such that

$$|\{e \in E_+ : e \cap \pi(h) \neq \emptyset\}| \leq \frac{3\delta}{\eta} n\pi r^2. \tag{52}$$

Let h^* be the infimum in $[\eta n/3, 2\eta n/3]$ of the real numbers h satisfying this inequality. If we increase the value of h by a small $\varepsilon > 0$ then the inequality (52) still holds and $\pi(h^* + \varepsilon) \cap \mathbb{Z}^2 \cap D = \emptyset$. We choose one such $h^* + \varepsilon$ and we call it h_+. Moreover any edge of E_+ which intersects $\pi(h_+)$ has an endpoint in each of the two half spaces delimited by $\pi(h_+)$. In a symmetric way we get from E_- a value h_- in $[-2\eta n/3, -\eta n/3]$. The edges in $\{\, e \in E_- : e \cap \pi(h_-) \neq \emptyset \,\} \cup \{\, e \in E_+ : e \cap \pi(h_+) \neq \emptyset \,\}$ will be designated as *bad edges*. We end with an horizontal segment $\pi(h_+)$ in D_+ that crosses at most $3\pi\delta n r^2/\eta$ edges belonging to a cluster $C \in \mathcal{C}_-$ and an horizontal segment $\pi(h_-)$ in D_- that crosses at most $3\pi\delta n r^2/\eta$ edges belonging to a cluster $C \in \mathcal{C}_+$. Note that if these $6\pi\delta n r^2/\eta$ bad edges were closed then by duality, there would exist an open dual path connecting $\partial^L \widehat{D}$ to $\partial^R \widehat{D}$. For $* = -, +$, we introduce the following sets of edges:

$$\Pi_* = \{\, e \in \mathbb{E}(D) : e \cap \pi(h_*) \neq \emptyset \,\}, \qquad \widehat{\Pi}_* = \{\, \widehat{e} \in \mathbb{E}(\widehat{D}) : e \in \Pi_* \,\}.$$

The set Π_* is the set of all the edges that intersect $\pi(h_*)$ and $\widehat{\Pi}_*$ is its dual set. Note that $\widehat{\Pi}_*$ is always a simple dual path connecting $\partial^L \widehat{D}$ to $\partial^R \widehat{D}$.

In order to capture the relevant dual connections, we introduce for each dual path $\gamma \subset \widehat{D}$ its w-diameter:

$$\mathrm{diam}_w(\gamma) = \max_{x,y \in \gamma} (y - x) \cdot w_\perp.$$

Lemma 23. *If the event* $\mathrm{Sep}'_D(n,x,r,w,\delta)$ *occurs, then there exists a family of open dual paths* $(\widehat{\xi}_1,\widehat{\xi}_2,\cdots,\widehat{\xi}_K)$ *such that*

$$\mathrm{diam}_w(\widehat{\xi}_1)+\cdots+\mathrm{diam}_w(\widehat{\xi}_K)\geq 2n\rho-\frac{6\delta}{\eta}n\pi r^2,$$

where ρ *has been defined just after lemma* 21*. Moreover the number* $K-1$ *is bounded above by the number of open dual clusters of diameter* $\geq M$ *that intersect* $\pi(h_+)\cup\pi(h_-)$*.*

Proof. First choose $\omega\in\mathrm{Sep}'(n,x,r,w,\delta)$. If there is no top-bottom crossing cluster in D then by duality, there exists an open dual path connecting $\partial^L\widehat{D}$ to $\partial^R\widehat{D}$ and we are done. If there are crossing clusters, then we construct an algorithm that detects in every dual configuration $\widehat{\omega}$ of $\omega\in\mathrm{Sep}'(n,x,r,w,\delta)$, a way to move from $\partial^L\widehat{D}$ to $\partial^R\widehat{D}$ by using either open dual paths or paths of bad edges, i.e., edges of $(\widehat{\Pi}_+\cap E_+)\cup(\widehat{\Pi}_-\cap E_-)$. Note that the paths of bad edges are not necessarely open dual paths and we will denote them by *tunnels*. Using tunnels will mean following a path of bad edges along $\widehat{\Pi}_+\cup\widehat{\Pi}_-$ from the left to the right until we reach an edge that is not bad. Note already that the total length of the tunnels is bounded by

$$|(\widehat{\Pi}_+\cap\widehat{E}_+)\cup(\widehat{\Pi}_-\cap\widehat{E}_-)|\leq 6\delta\pi r^2n/\eta.$$

Let us first sketch the idea behind the algorithm: we want to move from the left to the right along open dual paths. The only obstacles preventing us from doing that is the existence of top-bottom crossing clusters. To overcome the problem, when we meet such a filled crossing cluster S, we check if S is in $\mathcal{C}_-^{\mathrm{fill}}$ or in $\mathcal{C}_+^{\mathrm{fill}}$. Accordingly, we move to $\widehat{\Pi}_+$ or $\widehat{\Pi}_-$ and traverse the obstacle using a tunnel. After such a tunnel, we meet holes $\widehat{F}$ of S that we traverse using open dual paths included in $\Delta\widehat{F}$. We continue like this until we reach the right side of S. After this, we find an open dual path that reaches the next top-bottom crossing cluster and so on. At the end, the total number of closed dual edges that we have used is negligeabe. But this is not enough as the number of segments of open dual paths may be very large and this may prevent us from decoupling properly the probability of these segments. It is at this point that the filling of the small holes is important. Indeed, with our filling, we are guaranteed that at each time we produce a new open dual path, we will meet a large open dual cluster that intersects $\widehat{\Pi}_+\cap\widehat{\Pi}_-$. The number of such clusters can be controlled in order to decouple the relevant dual connections.

Next, we give the precise description of our algorithm:

Initialization

First we check the leftest edge e_{1+} of Π_+ and the leftest edge e_{1-} of Π_-.

1) If e_{1+} is in $\mathcal{C}_-^{\mathrm{fill}}$, then we use the tunnel included in $\widehat{\Pi}_+$ that starts in $\widehat{e}_{1+}$ and ends at an edge of $\widehat{\Pi}_+$ that is not bad.
2) if e_{1+} is in $\mathcal{C}_+^{\mathrm{fill}}$, then two subcases arise according to e_{1-}:

2a) If $e_{1-}\in S_1$, where S_1 is the first, from the left, top-bottom filled crossing cluster. We use the tunnel included in $\widehat{\Pi}_-$ that starts at $\widehat{e}_{1-}$ and that ends at an edge which is not bad.

2b) If $e_{1-} \notin S_1$, then e_{1-} is isolated from $\partial^R \widehat{D}$ by S_1. So, there exists an open dual path from $\partial^L \widehat{D}$ to a site in $\widehat{\Pi}_-$. Let $\widehat{\xi}_1$ be such a path whose endpoint on $\widehat{\Pi}_-$ is rightmost. By doing so, the right successor edge of $\widehat{\xi}_1$ on $\widehat{\Pi}_-$ must be a bad edge and thus the entrance of a tunnel. We use this tunnel until we reach an edge that is not bad.

Intermediate steps
Next, we suppose that we have reached an edge $\widehat{e}_j \in \widehat{\Pi}_+ \cup \widehat{\Pi}_-$ that is not a bad edge and describe how to proceed with the algorithm in order to reach an other edge $\widehat{e}_{j+1} \in \widehat{\Pi}_+ \cup \widehat{\Pi}_-$ that is not bad. Whether $\widehat{e}_j$ is in $\widehat{\Pi}_+$ or $\widehat{\Pi}_-$ is completly symmetric. We thus suppose that $\widehat{e}_j \in \widehat{\Pi}_+$ and the other case can be deduced by symmetry. If $\widehat{e}_j$ is described as above, then it is either included in a hole of a cluster in $\mathcal{C}^{\text{fill}}$ or it is at the right of the rightmost top-bottom crossing path of a cluster in $\mathcal{C}^{\text{fill}}$.

1) If $\widehat{e}_j$ is in a hole $\widehat{F}$ of a filled cluster $S \in \mathcal{C}^{\text{fill}}$, then we choose a path from $\Delta \widehat{F}$ that takes us to the rightmost edge of $\widehat{\Pi}_+$ and then we are again at the entrance of a tunnel that we cross. We denote by $\widehat{e}_{j+1}$ the successor on $\widehat{\Pi}_+$ of the exit of the just traversed tunnel. By definition $\widehat{e}_{j+1}$ is not a bad edge.
2) $\widehat{e}_j$ is just at the right of the rightmost top-bottom crossing path of a filled cluster $S \in \mathcal{C}^{\text{fill}}$. Let S' be the next filled crossing cluster of $\mathcal{C}^{\text{fill}}$. If $S' \in \mathcal{C}^{\text{fill}}_+$, then we go along an open dual path that joins $\widehat{e}_j$ to the rightmost dual intersection $\widehat{e}'_{j+1}$ of $\widehat{\Pi}_-$ with the top-bottom crossing open dual path that is just on the left of the leftmost top-bottom crossing path of S'. The edge $\widehat{e}'_{j+1}$ is the entrance of a tunnel that we take until we reach an edge in $\widehat{\Pi}_-$ that is not a bad edge. We call this edge $\widehat{e}_{j+1}$.

The final step is reached when an edge of $\partial^R \widehat{D}$ has been seen. This must happen in a finite number of steps since we explore partially without repetition the edges of $\widehat{\Pi}_- \cup \widehat{\Pi}_+$ from the left to the right. The number of the edges in the tunnels is bounded by $6\delta\pi r^2 n/\eta$, thus the created open dual paths $\widehat{\xi}_1, \dots, \widehat{\xi}_K$ satisfy

$$\operatorname{diam}_w(\widehat{\xi}_1) + \cdots + \operatorname{diam}_w(\widehat{\xi}_K) \geq 2n\rho - \frac{6\delta}{\eta} n\pi r^2,$$

In addition to that, the just described algorithm has the property that the creation of a new open dual path corresponds to an additional open dual cluster of diameter larger than M that intersects $\widehat{\Pi}_+ \cup \widehat{\Pi}_-$. Thus we can bound $K-1$ as stated in the lemma. □

5.3. Separating the pieces of the interface

In order to get the right probabilistic upper bound from the existence of the piecewise interface, we have to factorize the probability of the dual connections obtained in lemma 23 without altering too much our estimates. If we were working in independent Bernoulli percolation then we would simply apply the van den Berg-Kesten inequality. Unfortunately this inequality does not hold in dependent FK-percolation models. To decouple our events, we start by constructing a new family of paths from $(\widehat{\xi}_1, \cdots, \widehat{\xi}_K)$. The new paths will be well separated from each other by a distance of at least $0 < \ell < \delta n$. In order to simplify the notation, we consider without loss of generality that our domain $\widehat{D}$ is centered at the origin. For $-\rho n < h < \rho n$ we define the line $v(h)$ parallel to w and at a relative distance $hn\rho$ from the origin:

$$v(h) = \{x \in \mathbb{R}^2 : x \cdot w_\perp = h\}.$$

In addition to that we will need vertical strips that separate the events, so we define for every site $x \in \mathbb{R}^2$ the strip of width ℓ on the right of x:

$$H_\ell(x) = \{y \in \mathbb{R}^2 : 0 < (y - x) \cdot w_\perp < \ell\}.$$

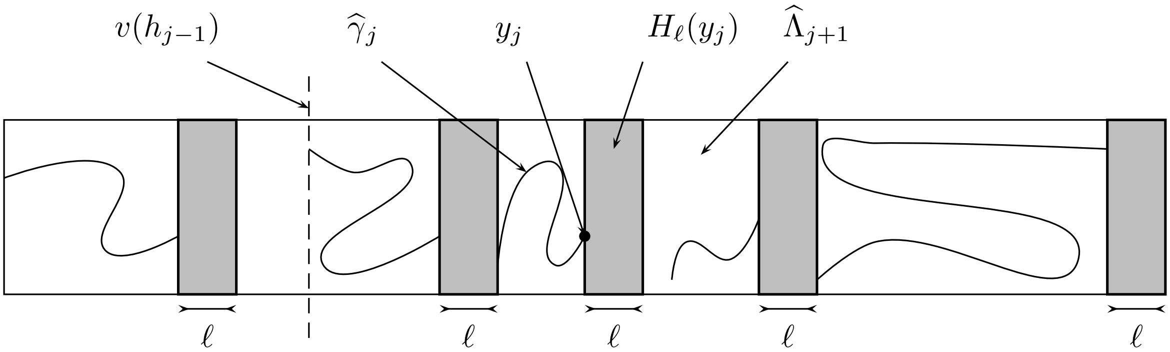
Now we give the construction of our new well separated dual open paths. First we start with the value $h = -n$ and we increase h until the first time we find at least one dual open path $\widehat{\gamma}$ that satisfies:

i) $\widehat{\gamma}$ is part of one of the paths of the piecewise interface $(\widehat{\xi}_1, \cdots, \widehat{\xi}_K)$.
ii) $\widehat{\gamma}$ starts at a site on $v(h)$ and does not intersect the left half plane defined by $v(h)$.
iii) $\mathrm{diam}_w(\widehat{\gamma}) \geq \ell$.

Let us call h_1 the first value of h where we stopped. Since $\ell < \delta n$, it is clear that $h_1 < n$ as soon as Sep'_D occurs. Let us pick, among the above mentioned paths, a path $\widehat{\gamma}$ of maximal w-diameter. On $\widehat{\gamma}$ we choose two sites $\widehat{x}_1, \widehat{y}_1 \in \widehat{\gamma}$ that satisfy

$$(\widehat{y}_1 - \widehat{x}_1) \cdot w_\perp = \mathrm{diam}_w(\widehat{\gamma})$$

and we define $\widehat{\gamma}_1$ as a dual open path that joins $\widehat{x}_1$ to $\widehat{y}_1$. Right after this path we put the strip $H_\ell(\widehat{y}_1)$.
Now suppose that $\widehat{\gamma}_1, \cdots, \widehat{\gamma}_j$ and $H_\ell(\widehat{y}_1), \cdots, H_\ell(\widehat{y}_j)$ have been constructed. Then we start with the value $h = h_j + \ell = y_j \cdot w_\perp + \ell$, we increase h until we find a path $\widehat{\gamma}$ that satisfies the above three criteria i)-iii) and we define $\widehat{\gamma}_{j+1}$ in the same way than the way we defined $\widehat{\gamma}_1$.
We continue this process until we reach the boundary $\partial^R \widehat{D}$.
After this construction, we end with a sequence of strips separating a family of dual open paths $(\widehat{\gamma}_1, \cdots, \widehat{\gamma}_{K'})$ (see the figure below).



The constructed paths verify:
- For every $1 \leq j \leq K'$ we have that $\mathrm{diam}_w(\widehat{\gamma}_j) \geq \ell$.
- The number K' of the new paths is bounded above by the number of paths K in the original piecewise interface. Indeed, two different paths $\widehat{\gamma}_i, \widehat{\gamma}_j$ cannot be part of the same path $\widehat{\xi}$ of the original interface because when defining the paths $\widehat{\gamma}_j$ we always choose one with maximal w-diameter.

- The total w-diameter of the new family of paths satisfies

$$\sum_{k=1}^{K'} \mathrm{diam}_w(\widehat{\gamma}_k) \geq 2n\rho - \frac{6\delta}{\eta} n\pi r^2 - 2K\ell. \tag{53}$$

  Indeed, we lost from the original total w-diameter only for two reasons. The first reason is the fact that we have chosen paths of w-diameter larger than ℓ, this gives a maximal loss of ℓK. The second reason is the fact that we have put strips. These strips are of width ℓ, this gives in the worst case another loss of ℓK.

For each j, we denote by $\widehat{\Lambda}_j$ the region of $\widehat{D}$ between $H_\ell(y_j)$ and $H_\ell(y_{j+1})$ and for each $k > 0$, we define $\Xi(k)$ as the set of families $(\widehat{\Lambda}_1, \cdots, \widehat{\Lambda}_k)$ that partition the set $\widehat{D}$ as above. Also we define $\Upsilon(k)$ as the set of the families $(s_1, \cdots, s_k) \subset \mathbb{R}^k$ such that

$$\forall j \in \{1, \cdots, k\} \quad s_j \geq \ell, \qquad \sum_{j=1}^{k} s_j \;\geq\; 2n\rho - \frac{6\delta}{\eta} n\pi r^2 - 2k\ell\,. \tag{54}$$

From lemma 23 and from the last construction, we get the following result.

Lemma 24. *Suppose that the event* $\mathrm{Sep}'_D(n, x, r, w, \delta)$ *occurs and let* K *be the number of open dual clusters of diameter larger than* M *that cross* $\pi(h_+) \cup \pi(h_-)$.

Then there exist $k \leq K$, $(\widehat{\Lambda}_1, \cdots, \widehat{\Lambda}_k) \in \Xi(k)$ *and* $(s_1, \cdots, s_k) \in \Upsilon(k)$ *such that the event*

$$A(\widehat{\Lambda}_1, \cdots, \widehat{\Lambda}_k; s_1, \cdots, s_k) \;=\; \bigcap_{j=1}^{k} \left\{ \exists \text{ an open dual path } \widehat{\gamma}_j \subset \widehat{\Lambda}_j \text{ with } \mathrm{diam}_w \widehat{\gamma}_j = s_j \right\}$$

occurs.

5.3.1 Control of the big dual clusters.

Let $h \in (-2r\eta n/3, -r\eta n/3) \cup (r\eta n/3, 2r\eta n/3)$. In what follows, we estimate the number of big open dual clusters that intersect the set $\widehat{\Pi}(h) = \{\widehat{e} \in \mathbb{E}(\widehat{D}) : e \cap \pi(h) \neq \emptyset\}$. Here *big cluster* means a cluster whose diameter exceeds a certain threshold $M > 0$. This estimate is crucial in order to decouple the different pieces of our spatially separated piecewise interface.

Lemma 25. *Let* $p > p_c$ *and fix* $h \in (-2r\eta n/3, -r\eta n/3) \cup (r\eta n/3, 2r\eta n/3)$. *Let* $K(h)$ *be the number of big open dual clusters of* $\widehat{D}$ *intersecting* $\widehat{\Pi}(h)$. *If* $\xi > 0$ *and* $a > 2\xi + 1$ *then there exist two positive constants* $c = c(a, \xi)$ *and* $\lambda = \lambda(a, \xi)$ *such that*

$$c(p - p_c)^{-a} < M < r\eta n/3 \quad \Rightarrow \quad \log \Phi[K(h) \geq (p - p_c)^{\xi} n] \leq -\lambda (p - p_c)^{4\xi + 2} n M,$$

uniformly over $\Phi \in \mathcal{FK}(p, D)$.

Proof. For a given $h \in (-2r\eta n/3, -r\eta n/3) \cup (r\eta n/3, 2r\eta n/3)$, let

$$\begin{aligned}
\widehat{\Pi}_+(M) \;&=\; \left\{ y \in \widehat{D} : 0 \leq (y - nx) \cdot w - h \leq M,\ |(y - nx) \cdot w_\perp| \leq n + M \right\},\\
\widehat{\Pi}_-(M) \;&=\; \left\{ y \in \widehat{D} : -M \leq (y - nx) \cdot w - h \leq 0,\ |(y - nx) \cdot w_\perp| \leq n + M \right\}.
\end{aligned}$$

Let $\widehat{C}$ be an open dual cluster of diameter $\geq M$ which intersects $\widehat{\Pi}(h)$. Since $|\widehat{C}| \geq M$, either $\widehat{\Pi}_-(M) \cap \widehat{C}$ contains at least $M/2$ sites connected inside $\widehat{\Pi}_-(M)$ to $\partial\widehat{\Pi}_-(M) \cap v(h)$ or $\widehat{\Pi}_+(M) \cap \widehat{C}$ contains at least $M/2$ sites connected inside $\widehat{\Pi}_+(M)$ to $\partial\widehat{\Pi}_+(M) \cap v(h)$. Thus for $\xi > 0$, we have that

$$\Phi[K \geq (p-p_c)^\xi n] \leq \Phi[\widehat{M}_{\widehat{\Pi}_-(M)} \geq (p-p_c)^\xi \frac{Mn}{4}] + \Phi[\widehat{M}_{\widehat{\Pi}_+(M)} \geq (p-p_c)^\xi \frac{Mn}{4}],$$

where $\widehat{M}_{\widehat{\Lambda}} = |\{\widehat{x} \in \widehat{\Lambda} : \widehat{x} \leftrightarrow \partial\widehat{\Lambda}\}|$. We proceed as in lemma 10. We choose $a > 2\xi + 1$ and partition $\widehat{\Pi}_-(M)$ and $\widehat{\Pi}_+(M)$ into blocks $B(\underline{x})$ of size $m = c(p-p_c)^{\xi - a}/4$. We assume that

$$\min(\frac{1}{6}, \frac{\eta r}{3})n > M > 10c(p-p_c)^{-a}, \tag{55}$$

where c is a positive constant that will be determined later. Next, we define for $* = +, -$,

$$\widehat{\Pi}'_*(M) = \bigcup_{\underline{x} \in \mathbb{Z}^2 : B(\underline{x}) \cap \widehat{\Pi}_*(M) \neq \emptyset} B(\underline{x}).$$

Also, the The number of partitioning blocks $|\underline{\widehat{\Pi}}'_*(M)|$ satisfies

$$\frac{nM}{m^2} \leq |\underline{\widehat{\Pi}}'_*(M)| \leq \frac{2(M+m)(n+M+m)}{m^2} \leq 7\frac{nM}{m^2}.$$

By subadditivity, one gets

$$\frac{\widehat{M}_{\widehat{\Pi}_*(M)}}{Mn} \leq \frac{7}{|\underline{\widehat{\Pi}}'_*(M)|} \sum_{\underline{x} \in \underline{\widehat{\Pi}}'_*(M)} \frac{\widehat{M}_{B(\underline{x})}}{|B(\underline{x})|}.$$

Thus, by using the FKG inequality we have that

$$\Phi[\widehat{M}_{\widehat{\Pi}_*(M)} \geq \frac{(p-p_c)^\xi}{4} Mn] \leq \Phi\left[\frac{1}{|\underline{\widehat{\Pi}}'_*(M)|} \sum_{\underline{x} \in \underline{\widehat{\Pi}}'_*(M)} \frac{\widehat{M}_{B(\underline{x})}}{|B(\underline{x})|} \geq \frac{(p-p_c)^\xi}{28} \,\middle|\, E\right],$$

where E is the event that all the dual edges of the boundaries of the blocks $B(\underline{x})$ are *open*. Furthermore, from lemma 11, we have that for every $a' \in (\xi+1, a-\xi)$, there exists a positive constant $c' = c'(a')$ such that for every $\underline{x} \in \underline{\widehat{\Pi}}'_-(M) \cup \underline{\widehat{\Pi}}'_+(M)$ we have

$$m > c'(p-p_c)^{-b'} \quad \Rightarrow \quad \Phi\Big[\frac{\widehat{M}_{B(\underline{x})}}{|B(\underline{x})|}\Big|E\Big] \leq \Phi^{\widehat{p},w}_{B(\underline{x})}\Big[\frac{\widehat{M}_{B(\underline{x})}}{|B(\underline{x})|}\Big] \leq \frac{(p-p_c)^\xi}{56}.$$

Observe that the random variables $(\widehat{M}_{B(\underline{x})}/|B(\underline{x})|, \underline{x} \in \underline{\widehat{\Pi}}'_-(M) \cup \underline{\widehat{\Pi}}'_+(M))$ take their values in $[0,1]$, are independent and identically distributed under $\Phi[\cdot|E]$. Also, by choosing $c = c'/10$ in (55), we obtain that their mean is bounded above by $(p-p_c)^\xi/56$. Therefore, we can apply lemma 9 to get

$$\log \Phi[\widehat{M}_{\widehat{\Pi}_*(M)} \geq (p-p_c)^\xi \frac{Mn}{4}] \leq -\lambda(p-p_c)^{2\xi}\frac{nM}{m^2},$$

where $* = -, +$ and $\lambda > 0$ is a positive constant. $\square$

5.3.2 Proof of the interface lemma. Now we have all the ingredients to give an upper bound on the probability of Sep that captures the existence of an interface.

Proof of lemma 21. First we approximate Sep with Sep′ and we have

$$\Phi[\mathrm{Sep}(\delta)] \le \Phi[\mathrm{Sep}_D(\delta) \setminus \mathrm{Sep}'_D(4\delta)] + \Phi[\mathrm{Sep}'_D(4\delta)]. \tag{56}$$

Let $a > 5, \alpha \in (0, (1 + \frac{1}{8a})^{-1})$. Lemma 22 ensures the existence of a positive constant c_1 such that for every M satisfying

$$c_1(p - p_c)^{-a/\alpha} < M < \eta r n/3, \tag{57}$$

we can bound the first term of (56) by

$$\log \Phi[\mathrm{Sep}_D(\delta) \setminus \mathrm{Sep}'_D(4\delta)] \le -c_1 \frac{n^2}{M^2} \delta (p - p_c)^{1/8} \log(\frac{1}{p - p_c}). \tag{58}$$

Next, we turn to the estimation of the second term of (56). We fix $\xi > 2$ and decompose the event under consideration as follows

(59)

$$\Phi[\mathrm{Sep}'_D(4\delta)] \le \Phi[\exists\, h \quad K(h) \ge (p - p_c)^\xi n] + \Phi[\{\forall\, h \quad K(h) < (p - p_c)^\xi n\} \cap \mathrm{Sep}'_D(4\delta)],$$

where $K(h)$ is the number of big open dual clusters that intersect $\widehat{\Pi}(h)$ ($\widehat{\Pi}(h)$ is defined before lemma 25) and h takes its values in $(-2r\eta n/3, -r\eta n/3) \cup (r\eta n/3, 2r\eta n/3)$. Next, we impose further to the exponent a to be larger than $2\xi + 1$ so that by (57) and by lemma 25, there exists a positive $\lambda = \lambda(a, \xi)$ such that for p close enough to p_c, we have

$$\Phi[\exists\, h \quad K(h) \ge (p - p_c)^\xi n] \le n \exp(-\lambda (p - p_c)^{4\xi+2} n M). \tag{60}$$

Now we turn to the second term of (59). By lemma 24 we can bound from above $\Phi[\{\forall\, h \quad K(h) < (p - p_c)^\xi n\} \cap \mathrm{Sep}'_D(4\delta)]$ by

$$\sum_{k=1}^{\lfloor (p-p_c)^\xi n \rfloor} \sum_{\substack{(\widehat{\Lambda}_1, \cdots, \widehat{\Lambda}_k) \in \Xi(k) \\ (s_1, \cdots, s_k) \in \Upsilon(k)}} \Phi[A(\widehat{\Lambda}_1, \cdots, \widehat{\Lambda}_k; s_1, \cdots, s_k)], \tag{61}$$

where $\Xi(k)$ and $\Upsilon(k)$ have been defined just before lemma 24. By Stirling's formula, for all $1 \le k \le \lfloor (p - p_c)^\xi n \rfloor$, the cardinality of the sets $\Xi(k)$ and $\Upsilon(k)$ are uniformly bounded from above by

$$\begin{aligned} |\Xi(k)| &\le \binom{2n}{k} \le \exp(4(p - p_c)^\xi n \log n), \\ |\Upsilon(k)| &\le \binom{2n + k}{k} \le \exp(4(p - p_c)^\xi n \log n). \end{aligned} \tag{62}$$

Next, we fix $k \in [1, \lfloor (p-p_c)^\xi n \rfloor]$, $(\widehat{\Lambda}_1, \cdots, \widehat{\Lambda}_k) \in \Xi(k)$ and $(s_1, \cdots, s_k) \in \Upsilon(k)$ and we use lemma 14 to decouple the occurrence of the k separated dual open connections appearing in the event $A(\cdot)$. To do so, we must require that the distance ℓ separating the regions $\widehat{\Lambda}_j$'s is large enough. More precisely, there exists a positive constant c_2 such that if

$$\delta n \geq \ell = c_2(p-p_c)^{-a}, \tag{63}$$

then we can apply lemma 14 k times and use (54) to obtain

(64)

$$\begin{aligned} \Phi[A(\widehat{\Lambda}_1, \cdots, \widehat{\Lambda}_k; s_1, \cdots, s_k)] \leq & 2^k \prod_{j=1}^{k} \Phi[\exists \text{ an open dual path } \widehat{\gamma}_j \subset \widehat{\Lambda}_j \text{ of } \operatorname{diam}_w \widehat{\gamma}_j = s_j] \\ \leq & (2n)^k \exp\Big[-\tau_p(w) n (2\rho - \frac{6\delta\pi}{\eta} r^2 - \frac{2k\ell}{n}) \Big]. \end{aligned}$$

Combining (64) and (62) we can bound (61) from above by

(65)

$$\exp\Big(-\tau_p(w) n (2\rho - \frac{6\delta\pi}{\eta} r^2) + 8(p-p_c)^\xi n \log n \Big) \sum_{k=1}^{\lfloor n(p-p_c)^\xi \rfloor} \exp\big(k \big(\log(2n) + 2\tau_p(w)\ell \big) \big).$$

In order to satisfy condition (63), we are limited to regimes where $n \geq c_2 \delta^{-1}(p-p_c)^{-a}$. By making c_2 a bit bigger and by using proposition 4, we can find $n_0 > 1$ such that for every $n > n_0$, we have

(66)

$$\begin{aligned} & \Phi[\{\forall h \quad K(h) < (p-p_c)^\xi n\} \cap \mathrm{Sep}'_D(4\delta)] \\ & \leq \exp\Big(10(p-p_c)^\xi n \log n + \tau_c c_2 (p-p_c)^{-a+\xi+1} n \Big) \exp\Big(-\tau_c (p-p_c) n (2\rho - \frac{14\delta\pi}{\eta} r^2) \Big) \end{aligned}$$

By choosing $a > \xi + 1$ and $\log n / \log(1/(p-p_c))$ bounded from above, the first exponential becomes negligible. It remains to specify a regime satisfying (57) and (63) such that the bounds (58) and (60) are smaller than (66). That is, we have to choose $\xi > 2, a > 2\xi + 1, \alpha \in (0, (1 + \frac{1}{8a})^{-1})$ and $n \uparrow \infty, p \downarrow p_c, M$ such that

$$\begin{aligned} \eta r n/3 \geq & M \geq \max\left(c_1 (p-p_c)^{-a/\alpha}, \Big(2\rho\tau_c (p-p_c) + \frac{\log n}{n} \Big) \Big/ \Big(\lambda (p-p_c)^{4\xi+2} \Big) \right), \\ & n \geq \max\left(\frac{c_2}{\delta}(p-p_c)^{-a}, \frac{2\rho}{\delta c_1} \tau_c M^2 \frac{(p-p_c)^{7/8}}{-\log(p-p_c)} \right). \end{aligned} \tag{67}$$

For the choice $\xi > 2, a = 2\xi + 2, \alpha = (1 + \frac{1}{8a})^{-1}, M = (p-p_c)^{4\xi+1}, n = (p-p_c)^{-8\xi - 9/8}$, it is easy to check that there exists $n_0 > 1$ such that for all $n > n_0$, the conditions (67) are satisfied. Since ξ has to be larger than 2, we obtain that for every $\gamma > 20$, if $n \uparrow \infty, p \downarrow p_c$ in such a way that $n(p-p_c)^\gamma \to \infty$ then it is possible to find ξ, a, α, M such that asymptotically the conditions (67) are satisfied. Finally, we obtain

$$\limsup_{(n,p)\to(\infty,p_c)} \frac{1}{(p-p_c)n} \log \Phi\left[\mathrm{Sep}(\delta)\right] \leq -2\rho\tau_c + \frac{14\delta}{\eta} r^2 \tau_c.$$

By choosing $\eta = r\sqrt{2\delta}/14$ and $\rho = r\sqrt{1-\rho}$, we get the desired result. $\square$

5.4. The local upper bound

In order to prove the upper bound we will need the following approximation result.

Lemma 26. *Let A be a subset of Q having finite perimeter. For any positive ε, δ, there exists a finite collection of disjoint balls $B(x_i, r_i), i \in I \cup I^{\mathrm{bd}}$, such that:*

- $\forall i \in I \qquad x_i \in \partial^* A \cap \mathring{Q}$ *and* $B(x_i, r_i) \subset \mathring{Q}$.
- $\forall i \in I^{\mathrm{bd}} \qquad x_i \in \partial^* A \cap \partial Q$ *and* $B_-(x_i, r_i, \nu_Q(x_i)) \subset Q$.
- $\forall i \in I \cup I^{\mathrm{bd}} \qquad \mathcal{L}^2((A \cap B(x_i, r_i)) \Delta B_-(x_i, r_i, \nu_A(x_i))) \leq \delta r^2$.
- *Finally*

$$\Big| \mathcal{P}(A) - \sum_{i \in I \cup I^{\mathrm{bd}}} 2r_i \Big| \leq \varepsilon.$$

Proof. The proof can be found in lemma 82 of [15]. □

Lemma 27. *Let $\nu \in \mathcal{M}(Q)$ be such that $\mathcal{J}(\nu) < +\infty$. If $\beta \downarrow \beta_c$ and $n \uparrow \infty$ in such a way that $n(\beta - \beta_c)^{20} \to \infty$, and such that $\log(n)/\log(1/(\beta - \beta_c))$ stays bounded then for every $\varepsilon > 0$ there exists a weak neighborhood $\mathcal{U}$ of ν in $\mathcal{M}(Q)$ such that*

$$\limsup_{(n,\beta)} \frac{1}{n(\beta - \beta_c)} \log \mu^{+,\beta}_{\tilde{\Lambda}(n)}(\sigma_n \in \mathcal{U}) \;\leq\; -(1 - \varepsilon)\mathcal{J}(\nu)\,.$$

Proof. By definition of $\mathcal{J}$, since $\mathcal{J}(\nu) < +\infty$, there exists a Borel subset A of Q such that ν is the measure with density $-1_A + 1_{Q \setminus A}$ with respect to the Lebesgue measure, and

$$\mathcal{J}(\nu) \;=\; \tau_c \mathcal{P}(A) \;=\; \tau_c \mathcal{P}(Q \setminus A)\,.$$

If $\mathcal{P}(A) = 0$, there is nothing to prove. Suppose that $\mathcal{P}(A) > 0$. Let $\delta_0, \varepsilon' \in]0,1[$ that we will tune later. Let $B(x_i, r_i), i \in I \cup I^{bd}$, be a finite collection of disjoint balls associated with A, ε' and $\delta_0/3$, as given in lemma 26. For i in $I \cup I^{bd}$, let f_i, g_i be two continuous functions having compact support and taking values in $[0,1]$ such that

$$\begin{gathered}
\forall x \in \mathbb{R}^2 \setminus \mathring{B}_-(x_i, r_i, \nu_A(x_i)) \qquad f_i(x) = 0\,,\\
\forall x \in \overline{B}_+(x_i, r_i, \nu_A(x_i)) \qquad g_i(x) = 1\,,\\
\Big(\frac{1}{2} - \frac{\delta_0}{4}\Big)\pi r_i^2 \;\leq\; \int f_i \, d\mathcal{L}^2\,, \qquad \int g_i \, d\mathcal{L}^2 \;\leq\; \Big(\frac{1}{2} + \frac{\delta_0}{4}\Big)\pi r_i^2\,,
\end{gathered}$$

where $\nu_A(x_i)$ is the exterior normal vector of A at x_i and where $\mathring{B}_-$ and $\overline{B}_+$ denote the interior and the closure of the half balls. Also, we require that there exists $s_i > 0$ such that, if we set

$$\begin{aligned}
D^i_- \;&=\; \big\{\, y \in B_-(x_i, r_i, \nu_A(x_i)) : \mathrm{d}(y, \mathbb{R}^2 \setminus B_-(x_i, r_i, \nu_A(x_i))) \leq s_i \,\big\}\,,\\
D^i_+ \;&=\; \big\{\, y \notin B_+(x_i, r_i, \nu_A(x_i)) : \mathrm{d}(y, B_+(x_i, r_i, \nu_A(x_i))) \leq s_i \,\big\}
\end{aligned}$$

then we have

$$\mathcal{L}^2\big(D_-^i\big) \le \frac{\delta_0}{8}\pi r_i^2\,, \qquad \forall x\in B_-(x_i,r_i,\nu_A(x_i)))\setminus D_-^i \qquad f_i(x)=1\,,$$
$$\mathcal{L}^2\big(D_+^i\big) \le \frac{\delta_0}{8}\pi r_i^2\,, \qquad \forall x\in \mathbb{R}^d\setminus B_+(x_i,r_i,\nu_A(x_i))\setminus D_+^i \qquad g_i(x)=0\,.$$

These conditions imply that

$$\begin{aligned}\nu(f_i) &= -\int_A f_i\,d\mathcal{L}^2+\int_{Q\setminus A} f_i\,d\mathcal{L}^2 \le -\int f_i\,d\mathcal{L}^2+2\mathcal{L}^2\big(B_-(x_i,r_i,\nu_A(x_i))\setminus A\big)\\ &\le (-1+\delta_0)\frac{1}{2}\pi r_i^2\,,\\ \nu(g_i) &= -\int_A g_i\,d\mathcal{L}^2+\int_{Q\setminus A} g_i\,d\mathcal{L}^2 \ge \int g_i\,d\mathcal{L}^2-2\mathcal{L}^2\big(A\cap B_+(x_i,r_i,\nu_A(x_i))\big)-2\mathcal{L}^2(D_+^i)\\ &\ge (1-\delta_0)\frac{1}{2}\pi r_i^2\,.\end{aligned}$$

Let $\mathcal{U}$ be the weak neighbourhood of ν in $\mathcal{M}(Q)$ defined by

$$\mathcal{U} = \Big\{\,\varrho\in\mathcal{M}(Q):\forall i\in I\quad \rho(f_i)<\nu(f_i)+\frac{\delta_0}{2}\pi r_i^2\,,\rho(g_i)>\nu(g_i)-\frac{\delta_0}{2}\pi r_i^2\,\Big\}.$$

Next, we choose $\gamma>20$ and consider $n\uparrow\infty, p\downarrow p_c$ in such a way that $n(p-p_c)^\gamma\to\infty$. We rescale the lattice by a factor $M=\sqrt{n}$ and choose $L=\sqrt{M}$. Let $\delta>0$, for $\underline{x}\in\mathbb{Z}^2$, the block variable $X(\underline{x})$ is defined as the indicator function of the event

$$R(B'(\underline{x}),L)\cap V(B(\underline{x}),\delta)\cap W(B(\underline{x}),\delta)\cap T(B(\underline{x}),\delta).$$

Let us fix $i\in I$. Let $*$ be a symbol representing either $-$ or $+$. We define

$$\begin{aligned}B_*(n,i) &= B_*(nx_i,nr_i,\nu_A(x_i)),\\ \underline{B}_*(n,i) &= \big\{\,\underline{x}\in\mathbb{Z}^2 : B(\underline{x})\subset \mathring{B}_*(n,i)\,\big\}\,.\end{aligned}$$

By the above choice of n,p and M, for n large enough we have

$$\mathcal{L}^2\Bigg(B_*(n,i)\setminus\bigcup_{\underline{x}\in\underline{B}_*(n,i)} B(\underline{x})\Bigg)\le\theta\delta\mathcal{L}^2(B_*(n,i)).$$

We define $\mathcal{S}_*$ as the collection of the clusters which are included in one of the boxes $B(\underline{x}),\underline{x}\in\underline{B}_*(n,i)$, but which do not intersect their boundaries:

$$\mathcal{S}_*=\bigcup_{\underline{x}\in\underline{B}_*(n,i)}\big\{C\text{ cluster in }B(\underline{x})\text{ such that }C\cap\partial B(\underline{x})=\emptyset\,\big\}.$$

Let $\underline{x} \in \underline{B}_*(n,i)$ be such that $X(\underline{x}) = 1$. Then

$$\sum_{C\in\mathcal{S}_*} |C \cap B(\underline{x})| \geq |B(\underline{x})| - |\{x \in B(\underline{x}) : x \leftrightarrow \partial B(\underline{x})\}| \geq (1-\theta-\delta\theta)|B(\underline{x})|.$$

Summing these inequalities, we get:

$$\sum_{C\in\mathcal{S}_*} |C \cap B_*(n,i)| \geq (1-\theta-2\delta\theta)\mathcal{L}^2(B_*(n,i)). \tag{68}$$

We define also $\mathcal{C}$ as the collection of the $B(n,i)$ clusters which do not belong to $\mathcal{S}_+ \cup \mathcal{S}_-$. For a cluster C, we denote by $\sigma(C)$ its color. For $\underline{x} \in \underline{B}_-(n,i) \cup \underline{B}_+(n,i)$, we have

$$\theta n^2 \sigma_n(B(\underline{x})) = \sum_{C\in\mathcal{C}} \sigma(C)|C \cap B(\underline{x})| + \sum_{\substack{x\in B(\underline{x})\\ x\not\leftrightarrow\partial B}} \sigma(x).$$

Whenever $X(\underline{x}) = 1$, the event T occurs and the modulus of the last sum is less than $\delta\theta|B(\underline{x})|$. Suppose that, for $* = -$ and $* = +$

$$M^2 \sum_{\underline{x}\in\underline{B}_*(n,i)} 1_{X(\underline{x})=0} \;\leq\; \delta\theta\mathcal{L}^2(B_*(n,i))\,.$$

Summing the previous inequalities, we get

$$\theta n^2 \sigma_n(\overset{\circ}{B}_+(n,i)) \leq \sum_{C\in\mathcal{C}} \sigma(C)|C \cap B_+(n,i)| + 5\delta\theta\mathcal{L}^2(B_+(n,i))\,,$$
$$\theta n^2 \sigma_n(\overline{B}_-(n,i)) \geq \sum_{C\in\mathcal{C}} \sigma(C)|C \cap B_-(n,i)| - 5\delta\theta\mathcal{L}^2(B_-(n,i))\,.$$

Let us denote by $\mathcal{C}_-$ (respectively $\mathcal{C}_+$) the collection of the negatively (respectively positively) colored clusters of $\mathcal{C}$. Notice that the collections $\mathcal{S}_+ \cup \mathcal{S}_-, \mathcal{C}_-, \mathcal{C}_+$ are disjoint. Suppose in addition that $\sigma_n \in \mathcal{U}$. The very definition of the neighborhood $\mathcal{U}$, the two previous inequalities and (68) yield that

$$\sum_{C\in\mathcal{C}_-\cup\mathcal{S}_-} |C \cap B_+(n,i)| \leq \theta(8\delta+\delta_0)\mathcal{L}^2(B_+(n,i)),$$
$$\sum_{C\in\mathcal{C}_-\cup\mathcal{S}_-} |C \cap B_-(n,i)| \geq (1-\theta(8\delta+\delta_0))\mathcal{L}^2(B_-(n,i)).$$

Thus the collection $\mathcal{C}_- \cup \mathcal{S}_-$ realizes the event $\mathrm{Sep}(n, x_i, r_i, w_i, 8\delta+\delta_0)$. In fact, some care is needed on the boundary of Q and one needs to define a variant of the event Sep for the balls that intersect the boundary. As the same reasoning holds in this situation, we omit the details. Choosing $\delta < \delta_0/8$, we conclude that

$$\mu^{+,\beta}_{\Lambda(n)}[\sigma_n \in \mathcal{U}] \leq \sum_{*=-,+} \sum_{i \in I \cup I^{bd}} \mathbb{P}^+_n\Bigg(M^2 \sum_{\underline{x} \in \underline{B}_*(n,i)} 1_{X(\underline{x})=0} > \theta\delta\mathcal{L}^2(B_*(n,i)) \Bigg)$$
$$+ \Phi^{w,p}_{\Lambda(n)}\Bigg[\bigcap_{i \in I \cup I^{bd}} \mathrm{Sep}(n, x_i, r_i, \nu_A(x_i), 2\delta_0) \Bigg].$$

Since we chose $\gamma > 20$, $p \downarrow p_c$, $n \uparrow \infty$ in such a way that $n(p-p_c)^\gamma \to \infty$, $M = \sqrt{n}, L = \sqrt{M}$, we can apply lemmas $16, 17, 18, 19$ to conclude that the block process $(X(\underline{x}))$ satisfies the hypothesis of lemma 8 and that for all $i \in I \cup I^{bd}$ and for $* = +,-$ the mean of the normalized sum below converges faster to zero than θ. Thus

$$\limsup_{(n,p)} \frac{1}{n(p-p_c)} \log \mathbb{P}^+_n\Bigg[\frac{M^2}{\mathcal{L}^2(B_*(n,i))} \sum_{\underline{x} \in \underline{B}_*(n,i)} 1_{X(\underline{x})=0} > \delta\theta \Bigg] = -\infty.$$

Next, the sets $B(x_i, r_i), i \in I \cup I^{bd}$, are compact and disjoint. Also $I \cup I^{bd}$ is finite and fixed, thus applying lemma 21 with lemma 14 we get

$$\limsup_{(n,\beta)} \frac{1}{(\beta-\beta_c)n} \log \mu^{+,\beta}_{\Lambda(n)}[\sigma_n \in \mathcal{U}] \leq -\sum_{i \in I \cup I^{bd}} 2r_i\tau_c(1 - c'\sqrt{2\delta_0})$$
$$\leq -\tau_c\big(\mathcal{P}(A) - \varepsilon'\big)(1 - c'\sqrt{2\delta_0}),$$

where c' is the constant appearing in lemma 21. Let $\varepsilon > 0$. By choosing ε' such that $\varepsilon = \varepsilon'(1 + 1/\mathcal{P}(A))$ and δ_0 such that $c'\sqrt{2\delta_0} < \varepsilon'$, we finally get the desired upper bound. $\square$

5.5. Exponential tightness

In order to prove the exponential tightness, we proceed as in [15]. The same approach works in our context with the exception that one has to be careful with the scales of renormalization and some extra care is needed because θ converges to 0 when $p \downarrow p_c$. As in [15], we will first define a roughening $\widetilde{\sigma}_n$ of the random measure σ_n. This auxiliary measure will be regular enough to produce suitable surface energy estimates and the proof will be completed by proving that the two random measures σ_n and $\widetilde{\sigma}_n$ are exponentially contiguous.

5.5.1 The rough measure $\widetilde{\sigma}_n$ and surface energy estimates. In order to construct $\widetilde{\sigma}_n$ we will work with the box $\Lambda(n)$ rescaled by a factor K that will depend on p in a way to be made precise in the course of our analysis. The renormalized box will be denoted by

$$\underline{\Lambda}(n) = \{\, \underline{x} \in \mathbb{Z}^2 : B(\underline{x}) \cap \Lambda(n) \neq \emptyset \,\}.$$

On $\underline{\Lambda}(n)$ we define the 0-1 valued random field $X(\underline{x})$, $\underline{x} \in \underline{\Lambda}(n)$, by

$$\forall \underline{x} \in \underline{\Lambda}(n) \qquad X(\underline{x}) = 1_{R(B'(\underline{x}),K)},$$

where $R(B'(\underline{x}), K)$ has been defined just before lemma 16. When $X(\underline{x}) = 1$ we will say that the block $B(\underline{x})$ is *good* and if $X(\underline{x}) = 0$ the block will be said to be *bad*. We also need a very similar filling procedure as before but in the renormalized lattice. Namely, for every connected subset $\underline{A}$ of $\underline{\Lambda}(n)$ we define

$$\mathrm{fill}\,\underline{A} \;=\; \underline{A} \cup \bigcup \underline{R}\,,$$

where the union runs over the residual $\mathbb{L}^\infty$-connected components $\underline{R}$ of $\underline{A}$ such that $\mathrm{diam}_\infty \underline{R} < \log n$ and $\underline{R} \cap \partial\Lambda(n) = \emptyset$. For each cluster C, we define its coarse graining $\underline{C} = \{\underline{x} \in \underline{\Lambda}(n) : B(\underline{x}) \cap C \neq \emptyset\}$. We say that a cluster C is large if $\mathrm{diam}\, C \geq K \log n$, in which case we have $\mathrm{diam}\, \underline{C} \geq \log n$. For a large cluster C we define

$$\widehat{\underline{C}} = \bigcup \mathrm{fill}\ \underline{A},$$

where the union runs over all the connected components $\underline{A}$ of good blocks such that $\underline{C} \cap \underline{A} \neq \emptyset$. Lemma 18.1 of [15] ensures that if C_1, C_2 are two distinct large clusters then $\widehat{\underline{C}}_1 \cap \widehat{\underline{C}}_2 = \emptyset$. We define $\widetilde{\sigma}_n$ as the random measure on Q whose density with respect to the Lebesgue measure is $1_{P_n} - 1_{M_n} + 1_{(Q \setminus P_n) \setminus M_n}$, where

$$P_n \;=\; \bigcup_{\substack{C \text{ large cluster}\\ \sigma(C)=+}} \;\bigcup_{\underline{x} \in \widehat{\underline{C}}} \;\frac{1}{n} B(\underline{x}) \cup \big(Q \setminus \Lambda(1 - \frac{6K}{n})\big)\,,$$

$$M_n \;=\; \bigcup_{\substack{C \text{ large cluster}\\ \sigma(C)=-}} \;\bigcup_{\underline{x} \in \widehat{\underline{C}}} \;\frac{1}{n} B(\underline{x}) \setminus \big(Q \setminus \Lambda(1 - \frac{6K}{n})\big)\,.$$

This measure $\widetilde{\sigma}_n$ is regular enough to establish the required exponential tightness. To do this, we consider the set $\widehat{\underline{F}}$ of all the $\mathbb{L}^\infty$-connected components of bad blocks in $\underline{\Lambda}(n)$ that intersect $\underline{\Lambda}(n - 6K \log n)$ and one of the sets

$$\partial_\infty^{\mathrm{out}} \widehat{\underline{C}} = \big\{ \underline{x} \notin \widehat{\underline{C}} :\ \exists \underline{y} \in \widehat{\underline{C}},\, |\underline{x} - \underline{y}|_\infty = 1 \big\},$$

where C is a large cluster.

Lemma 28. *Let $a > 5$. There exist two positive constants λ and $c = c(a)$ such that if $K > c(p - p_c)^{-a}$ and n is such that $n > K \log n$ then, for $u > 0$,*

$$\Phi^w_{\Lambda(n)}[|\widehat{\underline{F}}| \geq u] \leq \exp(-\lambda(p - p_c)Ku),$$

and

$$\mathbb{P}_n^+[\mathcal{P}(P_n) + \mathcal{P}(M_n) \geq u] \leq \exp\left(-\lambda \frac{u - 16}{8}(p - p_c)n\right).$$

Proof. Let C be a large cluster. By definition, the $\mathbb{L}^\infty$ outer boundary of $\widehat{\underline{C}}$ consists of bad blocks whenever $\widehat{\underline{C}} \neq \emptyset$. In the case $\widehat{\underline{C}} = \emptyset$, we define $\partial_\infty^{\mathrm{out}} \widehat{\underline{C}}$ as $\underline{C}$ which again consists only of bad blocks. Let $\underline{F}$ be an $\mathbb{L}^\infty$-connected component of bad blocks intersecting $\partial_\infty^{\mathrm{out}}\underline{\widehat{C}}$ and the rescaled box $\underline{\Lambda}(n-6K\log n)$. As proved in lemma 18.2 of [15], we have $|\underline{F}| \geq \log n$. Thus

$$\Phi^w_{\Lambda(n)}[|\underline{\widehat{F}}| \geq u] \;\leq\; \Phi^w_{\Lambda(n)}\Big[\big|\{\underline{x} \in \underline{\Lambda}(n-6K\log n) : |\underline{C}(\underline{x})| \geq \log n\}\big| \geq u\Big]\,,$$

where $\underline{C}(\underline{x})$ denotes the $\mathbb{L}^\infty$-connected component of occupied sites containing $\underline{x}$ (a site $\underline{y}$ is said to be occupied if $X(\underline{y})=0$). Let $a>5$, lemma 16 ensures the existence of two positive constants λ and $c=c(a)$ such that if $K>c(p-p_c)^{-a}$, then

$$\sup_{\underline{x}\in\underline{\Lambda}(n)} \log \Phi[X(\underline{x})=0|X(\underline{z}), |\underline{x}-\underline{z}|_\infty \geq 3] \leq -\lambda(p-p_c)K.$$

Combining this with lemma 7, we get that

$$\Phi^w_{\Lambda(n)}[|\underline{\widehat{F}}| \geq u] \leq 2\sum_{j\geq u} \exp\left(j\left(c-\frac{1}{9}\lambda(p-p_c)K\right)\right),$$

which yields the first claim.

For the second claim, note that the boundaries of P_n and M_n are located either on $\partial Q \cup \partial\Lambda(1-6K\log n/n)$ or on the faces of the blocks of $\underline{\widehat{F}}$. Thus $\mathcal{P}(P_n)+\mathcal{P}(M_n) \leq 16+8\frac{K}{n}|\underline{\widehat{F}}|$, so that

$$\begin{aligned}\mathbb{P}^+_n[\mathcal{P}(P_n)+\mathcal{P}(M_n)\geq u] &\leq \Phi^w_{\Lambda(n)}\Big[|\underline{\widehat{F}}| \geq (u-16)\frac{n}{8K}\Big]\\ &\leq \exp\Big(-(p-p_c)\lambda K(u-16)\frac{n}{8K}\Big)\end{aligned}$$

which yields the desired result. $\square$

5.5.2 Exponential contiguity.

Here we show that the rough measure $\widetilde{\sigma}_n$ is a good approximation of the original random measure σ_n. Let $f:\mathbb{R}^2\to\mathbb{R}$ be a continuous function having compact support. In order to estimate $|\sigma_n(f)-\widetilde{\sigma}_n(f)|$, we use another block coarse graining with scale $L=K\log n$. We fix $\varepsilon>0$ and define for $y\in\mathbb{Z}^2$ the block variable $Y(\underline{y})$ as the indicator function of the event

$$R(B'(\underline{y}),\sqrt{L})\cap V(B(\underline{y}),\varepsilon)\cap W(B(\underline{y}),\varepsilon).$$

Let

$$\underline{A}=\{\,\underline{y}\in\mathbb{Z}^2 : B(\underline{y})\cap\Lambda(n-6L)\neq\emptyset\}.$$

Note that $|\underline{A}|L^2\leq n^2$. We further introduce

$$\forall \underline{x}\in\underline{\Lambda}(n)\qquad B_n(\underline{x})=\frac{1}{n}B(\underline{x}),$$

$$A_0 = \bigcup_{\underline{y}\in\underline{A}:Y(\underline{y})=0} B_n(\underline{y}) \quad , \quad A_1 = \bigcup_{\underline{y}\in\underline{A}:Y(\underline{y})=1} B_n(\underline{y}),$$

and make the following decomposition

$$\begin{aligned}|\sigma_n(f)-\widetilde{\sigma}_n(f)| \leq & \Big|\int_{A_0} f(x)(d\sigma_n(x)-d\widetilde{\sigma}_n(x))\Big| + \Big|\int_{(Q\setminus A_0)\setminus A_1} f(x)(d\sigma_n(x)-d\widetilde{\sigma}_n(x))\Big| \\ & + \Big|\int_{A_1} f(x)(d\sigma_n(x)-d\widetilde{\sigma}_n(x)).\end{aligned} \tag{69}$$

We bound the first term of (69) as follows:

$$\Big|\int_{A_0} f(x)(d\sigma_n(x)-d\widetilde{\sigma}_n(x))\Big| \leq 2||f||_\infty \frac{1}{\theta|\underline{A}|}\sum_{\underline{y}\in\underline{A}} 1_{Y(\underline{y})=0}.$$

In a similar way we bound the second term:

$$\Big|\int_{(Q\setminus A_0)\setminus A_1} f(x)(d\sigma_n(x)-d\widetilde{\sigma}_n(x))\Big| \leq \frac{\mathcal{L}^2(\Lambda(n)\setminus\Lambda(n-6L))}{n^2}\Big(\frac{1}{\theta}+1\Big)||f||_\infty \leq \frac{24L}{\theta n}||f||_\infty\,.$$

For the third term of (69), we further decompose it into

$$\begin{aligned}\Big|\int_{A_1} f(x)(d\sigma_n(x)-d\widetilde{\sigma}_n(x))\Big| \leq & \frac{1}{\theta n^2}\Big|\sum_{\underline{y}\in\underline{A}:Y(\underline{y})=1}\ \sum_{\substack{C\subset B(\underline{y})\\ C\cap\partial^{\mathrm{in}}B(\underline{y})=\emptyset}}\sigma(C)\sum_{x\in C} f(x/n)\Big| \\ & + \frac{1}{\theta n^2}\Big|\sum_{\underline{y}\in\underline{A}:Y(\underline{y})=1}\ \sum_{\substack{x\in B(\underline{y})\setminus C(\underline{y})\\ x\leftrightarrow\partial^{\mathrm{in}}B(\underline{y})}}\sigma(x)f(x/n)\Big| \\ & + \Big|\frac{1}{\theta n^2}\sum_{\substack{\underline{y}\in\underline{A}\\ Y(\underline{y})=1}}\sigma(C(\underline{y}))\sum_{x\in C(\underline{y})} f(x/n) - \int_{A_1} f(x)d\widetilde{\sigma}_n(x)\Big|,\end{aligned} \tag{70}$$

where $C(\underline{y})$ denotes the unique crossing cluster of $B(\underline{y})$ whenever $Y(\underline{y})=1$. Using the definition of the good blocks, we further bound the second term of the right hand side in (70):

$$\frac{1}{\theta n^2}\Big|\sum_{\underline{y}\in\underline{A}:Y(\underline{y})=1}\ \sum_{\substack{x\in B(\underline{y})\setminus C(\underline{y})\\ x\leftrightarrow\partial^{\mathrm{in}}B(\underline{y})}}\sigma(x)f(x/n)\Big| \leq 2\varepsilon||f||_\infty.$$

Since f is continuous and has compact support, we have for $K\log n/n$ small enough that for all $y\in\mathbb{Z}^2$ $\sup_{x,z\in B_n(\underline{y})}|f(x)-f(z)| \leq \varepsilon||f||_\infty$. Using this observation and the properties of the good blocks we can bound the third term of (70) by

$$
\begin{aligned}
&\Big|\frac{1}{\theta n^2}\sum_{\substack{\underline{y}\in\underline{A}\\ Y(\underline{y})=1}}\sigma(C(\underline{y}))\sum_{x\in C(\underline{y})}f(x/n)-\int_{A_1}f(x)d\widetilde{\sigma}_n(x)\Big|\\
\le&\Big|\Big[\sum_{\substack{\underline{y}\in\underline{A}\\ Y(\underline{y})=1}}\left(\frac{1}{\theta n^2}\sigma(C(\underline{y}))|C(\underline{y})|-\widetilde{\sigma}_n(B_n(\underline{y}))\right)\max_{x\in B_n(\underline{y})}f(x)\Big|\\
&+\frac{1}{\theta n^2}\Big|\sum_{\substack{\underline{y}\in\underline{A}\\ Y(\underline{y})=1}}\sigma(C(\underline{y}))\sum_{x\in C(\underline{y})}\Big(f(\frac{x}{n})-\max_{z\in B_n(\underline{y})}f(z)\Big)\Big|\\
&+\Big|\sum_{\substack{\underline{y}\in\underline{A}\\ Y(\underline{y})=1}}\int_{B_n(\underline{y})}f(x)-\max_{z\in B_n(\underline{y})}f(z)\,d\widetilde{\sigma}_n(x)\Big|\\
\le&||f||_\infty\sum_{\substack{\underline{y}\in\underline{A}\\ Y(\underline{y})=1}}\Big|\frac{1}{\theta n^2}\sigma(C(\underline{y}))|C(\underline{y})|-\widetilde{\sigma}_n(B_n(\underline{y}))\Big|+\varepsilon(\varepsilon+2)||f||_\infty.
\end{aligned}
\tag{71}
$$

Next, we study the sum in the above inequality. Let $\underline{y}\in\underline{A}$ such that $Y(\underline{y})=1$. Several cases arise:

- $B_n(\underline{y})\subset M_n$ and $\sigma(C(\underline{y}))=-1$. We have then

$$\Big|\frac{1}{\theta n^2}\sigma(C(\underline{y}))|C(\underline{y})|-\widetilde{\sigma}_n(B_n(\underline{y}))\Big|=\frac{1}{n^2}\Big|-\frac{|C(\underline{y})|}{\theta}+|B(\underline{y})|\Big|\le\varepsilon\frac{|B(\underline{y})|}{n^2}.$$

- $B_n(\underline{y})\subset P_n$ and $\sigma(C(\underline{y}))=1$: This case is symmetric to the previous one and

$$\Big|\frac{1}{\theta n^2}\sigma(C(\underline{y}))|C(\underline{y})|-\widetilde{\sigma}_n(B_n(\underline{y}))\Big|\le\varepsilon\frac{|B(\underline{y})|}{n^2}.$$

- $B_n(\underline{y})\cap M_n\neq\emptyset$ and $B_n(\underline{y})\not\subset M_n$, then $B_n(\underline{y})$ meets the boundary of M_n.
- $B_n(\underline{y})\cap P_n\neq\emptyset$ and $B_n(\underline{y})\not\subset P_n$, then $B_n(\underline{y})$ meets the boundary of P_n.

In the two last cases, we bound crudely as follows:

$$\Big|\frac{1}{\theta n^2}\sigma(C(\underline{y}))|C(\underline{y})|-\widetilde{\sigma}_n(B_n(\underline{y}))\Big|\le(2+\varepsilon)\frac{|B(\underline{y})|}{n^2}.$$

This will suffice because in this case

$$B_n(\underline{y})\subset\left\{x\in\mathbb{R}^2:d_\infty(x,(\partial P_n\cup\partial M_n)\cap\Lambda(1-3L/n))\le\frac{L}{n}\right\}.$$

Moreover

$$
\begin{aligned}
&\mathcal{L}^2\left(\left\{x\in\mathbb{R}^2:d_\infty(x,(\partial P_n\cup\partial M_n)\cap\Lambda(1-3L/n))\le\frac{L}{n}\right\}\right)\\
&\quad\le(|\partial^{in}nP_n|+|\partial^{in}nM_n|)\big(\frac{2L+2}{n}\big)^2\le(\mathcal{P}(M_n)+\mathcal{P}(P_n))\frac{9L^2}{n}.
\end{aligned}
$$

- $\big(B_n(\underline{y}) \subset M_n$ and $\sigma_n(B_n(\underline{y})) > 0\big)$ or $\big(B_n(\underline{y}) \subset P_n$ and $\sigma_n(B_n(\underline{y})) < 0\big)$. These conditions imply that the whole block $B_n(\underline{y})$ has been added to M_n or P_n by the filling operation. Yet the regions which are added by the filling operation have a diameter at most $K(\log n - 1)$, so this case cannot occur.
- $B_n(\underline{y}) \cap M_n = \emptyset$ and $B_n(\underline{y}) \cap P_n = \emptyset$. In this case, $B(\underline{y}) \subset \widehat{F}$.

Summing the previous inequalities, we get

$$\begin{aligned}|\sigma_n(f) - \widetilde{\sigma}_n(f)| \leq & \frac{1}{\theta n^2}\Big|\sum_{C\in\mathcal{S}} \sigma(C) \sum_{x\in C} f(x/n)\Big| \\ & + ||f||_\infty\Big(\frac{2}{\theta|\underline{A}|}\sum_{\underline{y}\in\underline{A}} 1_{Y(\underline{y})=0} + \frac{L^2}{\theta n^2}|\widehat{F}| + 9(2+\varepsilon)\frac{L^2}{n}(\mathcal{P}(M_n)+\mathcal{P}(P_n))\Big) \\ & + ||f||_\infty\Big(\frac{24L}{\theta n} + \varepsilon(\varepsilon+6)\Big).\end{aligned}$$

where

$$\mathcal{S} = \bigcup_{\underline{y}\in\underline{A}:Y(\underline{y})=1} \left\{C \in B(\underline{y}) : C \cap \partial^{in} B(\underline{y}) = \emptyset\right\}\ .$$

Note that by the definition of a good block, any cluster C of a good block $B(\underline{y})$ that has a diameter larger than $\sqrt{L} = \sqrt{K\log n}$ is connected to the crossing cluster of $B'(\underline{y})$ and thus, such a cluster C is connected to $\partial^{in} B(\underline{y})$. Therefore, any cluster of $\mathcal{S}$ has a diameter that is smaller than $\sqrt{K\log n}$. Next, we analyze the deviations of the first term in the last inequality.
Since $|\mathcal{S}| \leq n^2$, we have

$$\begin{aligned}&\mathbb{P}_n^+\Big[\frac{1}{\theta n^2}\Big|\sum_{C\in\mathcal{S}}\sigma(C)\sum_{x\in C} f(\frac{x}{n})\Big| > \varepsilon||f||_\infty\Big] \\ &\qquad \leq \sum_{\omega\in\Omega_{\Lambda(n)}} \mathbb{P}_n^+\Big[\frac{1}{|\mathcal{S}|}\Big|\sum_{C\in\mathcal{S}} Y_C\Big| > \frac{\varepsilon\theta}{K\log n}\Big|\omega\Big]\Phi^{w,p}_{\Lambda(n)}[\omega],\end{aligned}$$

where $Y_C = \sigma(C)\sum_{x\in C} f(x/n)/\left(||f||_\infty K\log n\right)$.
Under $\mathbb{P}_n^+[\,\cdot\,|\omega]$, the sequence $(Y_C, C \in \mathcal{S}(\omega))$ is independent and takes its values in $[-1,1]$ (recall that the diameters of the clusters of $\mathcal{S}$ are bounded by $\sqrt{K\log n}$). Therefore we can apply theorem 9 to get

$$\begin{aligned}&\mathbb{P}_n^+\Big[\frac{1}{\theta n^2}\Big|\sum_{C\in\mathcal{S}}\sigma(C)\sum_{x\in C} f(\frac{x}{n})\Big| > \varepsilon||f||_\infty\Big] \leq 2\sum_{\omega\in\Omega_{\Lambda(n)}} \exp\left[-\left(\frac{\varepsilon\theta}{K\log n}\right)^2|\mathcal{S}|\right]\Phi^{w,p}_{\Lambda(n)}[\omega] \\ &\leq 2\Phi^{w,p}_{\Lambda(n)}\Big[\frac{1}{|\underline{A}|}\sum_{\underline{y}\in\underline{A}} 1_{Y(\underline{y})=0} \geq \varepsilon\Big] + 2\exp\left(-\frac{(1-\varepsilon)(1-(1+\varepsilon)\theta)\varepsilon^2\theta^2 n^2}{(K\log n)^3}\right),\end{aligned}$$

where we used the fact that

$$|\mathcal{S}| \geq \frac{|\{\underline{y}\in\underline{A} : Y(\underline{y}) = 1\}|L^2(1-(1+\varepsilon)\theta)}{\sup_{C\in\mathcal{S}}(\mathrm{diam}^2(C))}.$$

Therefore, by imposing

$$n \geq \frac{24K}{\varepsilon\theta} \log n, \tag{72}$$

we have

$$\begin{aligned} &\mu_n^{+,\beta}\Big[|\sigma_n(f) - \widetilde{\sigma}_n(f)| > ||f||_\infty(28+10\varepsilon)\Big] \leq \\ &\leq \Phi^{w,p}_{\Lambda(n)}\Big[|\widehat{F}| > \varepsilon\theta\frac{n^2}{L^2}\Big] + \mathbb{P}^+_n\Big[\mathcal{P}(M_n) + \mathcal{P}(P_n) > \varepsilon\frac{n}{L^2}\Big] \\ &\quad + 2\exp\left(-\frac{(1-\varepsilon)(1-(1+\varepsilon)\theta)\varepsilon^2\theta^2 n^2}{(K\log n)^3}\right) + 3\Phi^{w,p}_{\Lambda(n)}\Big[\frac{1}{|\underline{A}|}\sum_{\underline{y}\in\underline{A}} 1_{X(\underline{y})=0} \geq \frac{\varepsilon\theta}{2}\Big]. \end{aligned} \tag{73}$$

By lemma 28, for any $a > 5$, there exists a positive constant c such that if we impose that

$$K > c(p-p_c)^{-a} \qquad \text{and} \qquad n > K\log n, \tag{74}$$

then

$$\Phi^{w,p}_{\Lambda(n)}\Big[|\widehat{F}| > \varepsilon\theta\frac{n^2}{L^2}\Big] \leq \exp\left(-\lambda(p-p_c)\frac{\varepsilon\theta}{K}\frac{n^2}{\log^2 n}\right). \tag{75}$$

If we further impose that

$$\frac{\varepsilon n}{(K\log n)^2} > 32 \tag{76}$$

then lemma 28 gives

$$\mathbb{P}^+_n\Big[(\mathcal{P}(M_n) + \mathcal{P}(P_n)) > \varepsilon\frac{n}{L^2}\Big] \leq \exp\left(-\frac{\lambda\varepsilon}{16}\frac{p-p_c}{K^2}\frac{n^2}{\log^2 n}\right). \tag{77}$$

Furthermore, under the condition (74), lemmas 16, 17, 18 imply that the block process $(Y(\underline{y}),\, \underline{y}\in\underline{A})$ satisfies the hypothesis of lemma 8 and that the mean of the normalized sum below converges to zero faster than θ. Thus

$$\limsup_{n\uparrow\infty, p\downarrow p_c} \frac{1}{(p-p_c)} \log \Phi^{w,p}_{\Lambda(n)}\Big[\frac{1}{|\underline{A}|}\sum_{\underline{y}\in\underline{A}} 1_{Y(\underline{y})=0} \geq \frac{\varepsilon\theta}{2}\Big] = -\infty. \tag{78}$$

Finally, we verify that if we choose $a > 5$ and c large enough and impose

$$K > c(p-p_c)^{-a} \qquad \text{and } n > K^3\log^3 n, \tag{79}$$

then for p close enough to p_c the conditions (72), (74) and (76) are satisfied. Moreover, if $n \uparrow \infty$ and $p \downarrow p_c$ in a regime where (79) is satisfied, then (75), (77) and (78) imply that all the terms of (73) decay exponentially fast with a speed larger than $(p-p_c)n$. Thus, when $n\uparrow\infty$ and $p\downarrow p_c$ in such a way that $n(p-p_c)^{20}\uparrow\infty$, we have for every positive ε

$$\limsup_{(n,p)} \frac{1}{n(p-p_c)} \log \mathbb{P}^+_n\big(|\sigma_n(f) - \widetilde{\sigma}_n(f)| > \varepsilon\big) = -\infty\,.$$

Together with lemma 28, this concludes the proof of the exponential tightness.

6. The Lower bound

We are only left with the lower bound in order to finish our large deviation principle.

6.1. Preparatory lemmas

To prove the lower bound we will consider an event whose probability is of the correct order. For this we will use the following lemma to approximate sets of finite perimeter with polyhedral sets.

Lemma 29. *Let A be a subset of $Q = [-1,1]^2$ having finite perimeter. For any $\varepsilon > 0$, there exists a finite union of polyhedral sets D such that*

$$\overline{D} \subset \overset{\circ}{Q}, \qquad \mathcal{L}^2(A \Delta D) \leq \varepsilon\,, \qquad \mathcal{P}(D) \leq \mathcal{P}(A) + \varepsilon.$$

Proof. For the proof, we refer the reader to [15].

In addition to that, in order to create pieces of the interface that are well localized we will need a lower bound for the existence of an interface inside a certain region. Let x be a site in $\mathbb{Z}^2 + (1/2, 1/2)$ and let

$$H(0, nx, \sqrt{n}) \;=\; \big\{\, y \in \mathbb{R}^2 : \; d_\infty(y, [(1/2,1/2), nx]) \leq \sqrt{n}\,\big\}$$

be the region formed by the points of $\mathbb{R}^2$ that are at a ∞-distance less than $\sqrt{n}$ from the segment $[(1/2,1/2), nx]$. We will need a lower bound on the following event:

$$\mathrm{Wall}(0, nx, \sqrt{n}) = \{(1/2,1/2) \leftrightarrow nx \text{ by an open dual path in } H(0, nx, 2\sqrt{n})\}.$$

Lemma 30. *Let $x \in Q + (1/2,1/2)$ be a dual site and let $a > 5$. If $p \downarrow p_c$ and $n \uparrow \infty$ in such a way that $n > (p - p_c)^{-a}$ then*

$$\liminf_{(n,p)} \frac{1}{(p-p_c)n} \log \Phi^{w,p}_{\Lambda(n)}[\mathrm{Wall}(0, nx, \sqrt{n})] \geq -\tau_c |x|_2.$$

Proof. We consider the case $x = (1,0) + (1/2,1/2)$, the proof for a general x is similar. Let $M > 0$ be an integer. We denote by $n = Mq + r$ the Euclidean division of n by M. Using translation invariance and the FKG inequality, we get that

$$\log \Phi^p_\infty[\mathrm{Wall}(0, nx, \sqrt{n})] \geq q \log \Phi^p_\infty[\mathrm{Wall}(0, Mx, \sqrt{M})] + \log \Phi^p_\infty[\mathrm{Wall}(0, rx, \sqrt{r})].$$

Now, note that the event $\mathrm{Wall}(0, Mx, \sqrt{M})$ is realized as soon as $0 \leftrightarrow Mx$ by an open dual path and there exists no open dual path from $H(0, nx, \sqrt{n}/2)$ to $H(0, nx, \sqrt{n})^c$. Thus, by proposition 4

$$\Phi^p_\infty[\mathrm{Wall}(0, Mx, \sqrt{M})] \geq \Phi^p_\infty[0 \leftrightarrow Mx \text{ by an open dual path }] - 4n^{3/2} e^{-\lambda(p-p_c)\sqrt{n}},$$

where λ is a positive constant. This gives us the following lower bound:
(80)
$$\frac{1}{(p-p_c)n}\log\Phi^p_\infty[\mathrm{Wall}(0,nx,\sqrt{n})]\geq$$
$$\frac{q}{(p-p_c)n}\log\Phi^p_\infty[0\leftrightarrow Mx \text{ by an open dual path }]$$
$$+\frac{q}{(p-p_c)n}\log\left[1-\frac{4n^{3/2}\exp(-\lambda(p-p_c)\sqrt{n})}{\Phi^p_\infty[0\leftrightarrow Mx \text{ by an open dual path }]}\right]$$
$$+\frac{1}{(p-p_c)n}\log\Phi^p_\infty[\mathrm{Wall}(0,rx,\sqrt{r})].$$

Now we consider a double sequence $M\uparrow\infty, p\downarrow p_c$ such that $M(p-p_c)/\log M\uparrow\infty$ and we suppose also that $(p-p_c)\sqrt{n}/M\uparrow\infty$. Then, by proposition 4 we have that

- $\dfrac{q}{(p-p_c)n}\sim\dfrac{1}{(p-p_c)M}$
- $\displaystyle\lim_{M,p}\frac{1}{(p-p_c)M}\log\Phi^p_\infty[0\leftrightarrow Mx \text{ by an open dual path }]=-\tau_c$
- $\displaystyle\lim_{n,M,p}\log\left[1-\frac{4n^{3/2}\exp(-\lambda(p-p_c)\sqrt{n})}{\Phi^p_\infty[0\leftrightarrow Mx \text{ by an open dual path }]}\right]=0.$

Since $r<M$ we have by the finite energy property that there exist two constants $c,\lambda>0$ such that $\Phi^p_\infty[\mathrm{Wall}(0,rx,\sqrt{r})]\geq c\exp(-\lambda M)$ thus, in the regime specified above we have that
$$\lim_{n,M,p}\frac{1}{(p-p_c)n}\log\Phi^p_\infty[\mathrm{Wall}(0,rx,\sqrt{r})]=0.$$

Thus, from (80) we get that for every sequence $(n,p)\to(\infty,p_c)$ such that there exists M satisfying $(p-p_c)\sqrt{n}/M\uparrow\infty$ and $M(p-p_c)/\log M\uparrow\infty$ then
$$\liminf_{(n,p)}-\frac{1}{(p-p_c)n}\log\Phi^p_\infty[\mathrm{Wall}(0,nx,\sqrt{n})]\geq\tau_c.$$

The result for the finite volume measure $\Phi^{w,p}_{\Lambda(n)}$ follows from lemma 14. □

6.2. Proof of the lower bound

Now we have all the ingredients to complete the last part of the large deviation principle, namely the proof of the lower bound.

Proposition 31. *Let $a>5$ and $\nu\in\mathcal{M}(Q)$. If $n\uparrow\infty$ and $\beta\downarrow\beta_c$ in such a way that $n(\beta-\beta_c)^a\uparrow\infty$ then for any weak neighborhood $\mathcal{U}$ of ν*
$$\liminf_{(n,p)}\frac{1}{n(\beta-\beta_c)}\log\mu^+_{\Lambda(n)}[\sigma_n\in\mathcal{U}]\geq-\mathcal{J}(\nu).$$

Proof. Let $\nu \in \mathcal{M}(Q)$. The statement is not trivial only if $\mathcal{J}(\nu) < +\infty$. In this case, by the definition of the rate function $\mathcal{J}$, there exists a Borel set A of Q such that ν is the measure with density $-1_A + 1_{Q\setminus A}$ with respect to the Lebesgue measure and $\mathcal{J}(\nu) = \tau_c \mathcal{P}(A)$. Let $\mathcal{U}$ be a weak neighborhood of ν and let $\varepsilon > 0$. By lemma 29, there exists a polyhedral set D such that $\overline{D} \subset \overset{\circ}{Q}$, the measure $\widetilde{\nu}$ with density $-1_D + 1_{Q\setminus D}$ with respect to the Lebesgue measure belongs to $\mathcal{U}$ and $\mathcal{P}(D) < \mathcal{P}(A) + \varepsilon$. By definition, the boundary ∂D is a union of s segments included in $\overset{\circ}{Q}$, which we denote by $[a_i, a_{i+1}]$, $1 \leq i \leq s$. Thus

$$\sum_{1\leq i\leq s} |a_i - a_{i+1}|_2 \leq \mathcal{P}(A) + \varepsilon\,.$$

Now we will give a lower bound for the probability that σ_n stays in a neighborhood of $\widetilde{\nu}$. Let f be a continuous function on Q. For reasons that will be clear a bit later, we choose

$$\delta = \frac{\theta}{6(1+\theta)(\mathcal{P}(Q) + \mathcal{P}(D))}. \tag{81}$$

In order to evaluate the probability that $|\sigma_n(f) - \widetilde{\nu}(f)| \leq \varepsilon$, we assume that we are in a regime where $n \uparrow \infty$ and $p \downarrow p_c$ in such a way that $n\delta \uparrow \infty$ and rescale the lattice by a factor $L = \lfloor \delta n \rfloor$. Next, we define the sets

$$\underline{A} = \{\underline{y} \in \mathbb{Z}^2 : B_n(\underline{y}) \cap Q \neq \emptyset\},$$

and

$$\underline{E} = \{\underline{y} \in \underline{A} : B_n(\underline{y}) \cap D^c = \emptyset\}\,, \qquad \underline{F} = \{\underline{y} \in \underline{A} : B_n(\underline{y}) \cap D = \emptyset\}.$$

By choosing p close enough to p_c, we can assume that

$$\mathcal{L}^2\left(Q \setminus \Big(\bigcup_{\underline{y}\in\underline{E}\cup\underline{F}} B_n(\underline{y})\Big)\right) \leq 6(\mathcal{P}(Q) + \mathcal{P}(D))\delta. \tag{82}$$

Moreover $|\underline{E}| + |\underline{F}| \leq 1/\delta^2$. The set $\underline{E}$ (respectively $\underline{F}$) can be regarded as a coarse graining of the region inside D (respectively outside D). To evaluate $\sigma_n(f) - \widetilde{\nu}(f)$ we will restrict ourselves to an event $\mathcal{E}$ that gives nice properties to the blocks and such that $\mathcal{E}$ has the right probability of decay. For this let us define the block process $(Y(\underline{x}), \underline{x} \in \mathbb{Z}^2)$ as the indicator functions of the events

$$R(B'(\underline{x}), L^{1/8}) \cap V(B(\underline{x}), \varepsilon) \cap W(B(\underline{x}), \varepsilon)\,, \qquad \underline{x} \in \mathbb{Z}^2\,.$$

We choose $\mathcal{E}$ to be the intersection of the events

$$\{Y(\underline{x}) = 1\}\,, \quad \underline{x} \in \underline{E} \cup \underline{F}\,, \qquad \mathrm{Wall}(A_i, A_{i+1}, n, \sqrt{n})\,, \quad 1 \leq i \leq s,$$

where A_i is a site on the dual of $\Lambda(n)$ that is closest to na_i. Let us evaluate the probability of $\mathcal{E}$:

(83)

$$\mathbb{P}_n^+(\mathcal{E}) \geq$$
$$\mathbb{P}_n^+\Big[\bigcap_{\underline{x}\in\underline{E}\cup\underline{F}}\{Y(\underline{x})=1\}\big|\bigcap_{1\leq i\leq s}\text{Wall}(A_i,A_{i+1},n,\sqrt{n})\Big]\ \Phi^{w,p}_{\Lambda(n)}\Big[\bigcap_{1\leq i\leq s}\text{Wall}(A_i,A_{i+1},n,\sqrt{n})\Big].$$

First observe that $\bigcap_{1\leq i\leq s}\text{Wall}(A_i,A_{i+1},n,\sqrt{n})$ occurs outside the set $\bigcup_{\underline{x}\in\underline{E}\cup\underline{F}}\widetilde{B'}(\underline{x})$, where

$$\widetilde{B'}(\underline{x}) \;=\; \big\{\, y\in\mathbb{Z}^2 : d(x,\widetilde{B}(\underline{x}))\leq L/10\,\big\}\,.$$

Let $a>5$. The estimates of lemmas 16, 17, 18 ensure that, uniformly over the boundary conditions on $B'(\underline{x})$, the probability $\mathbb{P}_n^+\big[Y(\underline{x})=1\big]$ goes to one when $n\uparrow\infty$ and $p\downarrow p_c$ in such a way that

$$n(p-p_c)^a\delta(p)\uparrow\infty. \tag{84}$$

Thus by [40] the first factor of (83) goes to 1. On the other hand, by the FKG inequality and by lemma 30, if $n>(p-p_c)^{-a}$ then

$$\liminf_{n,p}\frac{1}{n(p-p_c)}\log\Phi^{w,p}_{\Lambda(n)}\left[\bigcap_{1\leq i\leq s}\text{Wall}(A_i,A_{i+1},n,\sqrt{n})\right]\geq-\sum_{1\leq i\leq s}|A_{i+1}-A_i|_2\tau_c.$$

Combining the previous inequalities we get

$$\liminf_{n,p}\frac{1}{n(p-p_c)}\log\mathbb{P}_n^+[\mathcal{E}]\;\geq\;-\tau_c\mathcal{P}(A)-\tau_c\varepsilon. \tag{85}$$

Now we are left with the evaluation of $|\sigma_n(f)-\widetilde{\nu}(f)|$ when $\mathcal{E}$ occurs. Suppose that $\mathcal{E}$ occurs and let $\underline{E}_i$, $i\in I$ (respectively $\underline{F}_j, j\in J$) be the connected components of $\underline{E}$ (respectively $\underline{F}$). For $i\in I$ (respectively $j\in J$), all the crossing clusters of the good blocks $B(\underline{x}),\underline{x}\in\underline{E}_i$ (respectively $\underline{x}\in\underline{F}_j$) are connected and belong to one big cluster that we denote by C^i_- (respectively C^j_+). The events $\text{Wall}(A_k,A_{k+1},n,\sqrt{n})$, $1\leq k\leq s$ isolate completely the set $\underline{E}$ from $\underline{F}$, thus for every $i\in I$ and $j\in J$, the two clusters C^i_- and C^j_+ are disjoint and moreover C^i_- cannot be connected to $\partial\Lambda(n)$. Now suppose for a while that all the clusters $C^i_-, i\in I$, are colored negatively and that all the clusters $C^j_+, j\in J$, are colored positively. We will see later that this restriction does not decrease the probability too much.

Next, we define

$$\mathcal{S}=\bigcup_{\underline{y}\in\underline{E}\cup\underline{F}}\big\{C\subset B(\underline{y})\,:\,C\cap\partial^{in}B(\underline{y})=\emptyset\big\}\,,$$

and keeping in mind the suppositions made in the last paragraph, we do the following decomposition

$$
\begin{aligned}
|\sigma_n(f)-\widetilde{\nu}(f)| \leq & \frac{1}{\theta n^2}\Big|\sum_{C\in\mathcal{S}}\sigma(C)\sum_{x\in C}f(x/n)\Big| + \frac{1}{\theta n^2}\Big|\sum_{\underline{y}\in\underline{E}\cup\underline{F}}\ \sum_{\substack{x\in B(\underline{y})\setminus C(\underline{y})\\ x\leftrightarrow\partial^{in}B(\underline{y})}}\sigma(x)f(x/n)\Big| \\
& + \Big|\sum_{\underline{y}\in\underline{E}\cup\underline{F}}\Big(\frac{\sigma(C(\underline{y}))}{\theta n^2}\sum_{x\in C(\underline{y})}f(x/n)-\int_{B_n(\underline{y})}f(x)\,d\widetilde{\nu}(x)\Big)\Big| \\
& + \Big|\int_{Q\setminus\cup_{\underline{y}\in\underline{E}\cup\underline{F}}B_n(\underline{y})}f(x)(d\sigma_n(x)-d\widetilde{\nu}(x))\Big|.
\end{aligned}
$$

The second and the third term can be bounded as in the proof of the exponential contiguity by using the properties of the good blocks and the imposed coloring of the clusters $(C_-^i, i\in I)$ and $(C_+^j, j\in J)$. To deal with the fourth term of the last inequality, we use (81) and (82) to get

$$
|\sigma_n(f)-\widetilde{\nu}(f)| \leq \frac{1}{\theta n^2}\Big|\sum_{C\in\mathcal{S}}\sigma(C)\sum_{x\in C}f(\frac{x}{n})\Big| + 2\varepsilon||f||_\infty + \varepsilon(\varepsilon+4)||f||_\infty.
$$

Thus, the estimate burns down to the analysis of the deviations of the first term in the last inequality. Let $\mathcal{E}'$ be the event $\mathcal{E}$ intersected with the color constraint made above. By the same sort of computations than in the proof of the exponential contiguity, we obtain

$$
\mathbb{P}_n^+\left[\frac{1}{\theta n^2}\Big|\sum_{C\in\mathcal{S}}\sigma(C)\sum_{x\in C}f(\frac{x}{n})\Big| > \varepsilon||f||_\infty\Big|\mathcal{E}'\right] \leq \mathbb{P}_n^+\left[\frac{1}{|\mathcal{S}|}\Big|\sum_{C\in\mathcal{S}}Y_C\Big| > \frac{\varepsilon\theta}{(\delta n)^{1/4}}\Big|\mathcal{E}'\right],
$$

where $Y_C=\sigma(C)\sum_{x\in C}f(x/n)/(||f||_\infty(\delta n)^{1/4})$. Fix $\omega\in\mathcal{E}'$. Observe that under the measure $\mathbb{P}_n^+[\cdot|\omega]$, the random variables $(Y_C, C\in\mathcal{S})$ are independent and take their values in $[-1,1]$. So, we can apply theorem 9 to control this deviation and we get

$$
\mu_n^{+,\beta}\left(|\sigma_n(f)-\widetilde{\nu}(f)|\geq||f||_\infty(7\varepsilon+\varepsilon^2)\right) \geq \frac{1}{2^{|\underline{E}|+|\underline{F}|}}\Big(1-\exp\Big(-c\varepsilon^2\theta^{3/2}n^{3/2}\Big)\Big)\mathbb{P}_n^+[\mathcal{E}],
$$

where c is a positive constant.
We can do the same reasoning with any finite number of continuous functions $f_1,\dots,f_k$ to get

$$
\mu_n^+[\forall l\in\{1,\dots,k\}\quad |\sigma_n(f_l)-\widetilde{\nu}(f_l)|\geq\varepsilon] \geq \frac{1}{2^{|\underline{E}|+|\underline{F}|}}\Big(1-\exp\Big(-c\varepsilon^2\theta^{3/2}n^{3/2}\Big)\Big)\mathbb{P}_n^+[\mathcal{E}].
$$

Finally, since $|\underline{E}|+|\underline{F}|\leq 1/\delta^2(p)$ and by (85) we get that if $n\uparrow\infty$ and $p\downarrow p_c$ in such a way that (84) is satisfied, then for every weak neighborhood $\mathcal{U}$ of ν,

$$
\forall\varepsilon>0\qquad \liminf_{n,\beta}\frac{1}{n(\beta-\beta_c)}\log\mu_n^{+,\beta}[\sigma_n\in\mathcal{U}] \geq -\mathcal{J}(\nu)-\varepsilon\tau_c.
$$

Sending ε to 0 yields the desired lower bound. $\square$

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