Weak convergence of measure-valued processes and r-point functions.

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Abstract: We prove a sufficient set of conditions for a sequence of finite measures on the space of cadlag measure-valued paths to converge to the canonical measure of super-Brownian motion in the sense of convergence of *finite-dimensional distributions*. The conditions are convergence of the Fourier transform of the r-point functions and perhaps convergence of the "survival probabilities". These conditions have recently shown to be satisfied for a variety of statistical mechanical models including critical oriented percolation, the critical contact process, and lattice trees at criticality, all above their respective critical dimensions.

1. Motivation

In the last few years a number of rescaled models from interacting particle systems and statistical physics have been shown to converge to the canonical measure of super-Brownian motion. The models include critical oriented percolation above 4 dimensions [6], critical contact processes above 4 dimensions [5] and critical lattice trees above 8 dimensions [7], all for sufficiently spread-out kernels. In each of these cases what is actually proved is convergence of the Fourier transforms of the moment measures (or r-point functions). Our modest objective here is to translate this result into the more conventional probabilistic language of weak convergence of stochastic processes. To those well-versed in weak convergence arguments we fear this may be one of the proverbial much-needed gaps in the literature, but to others who have complained to us, it is an irritant that should be spelled out once and for all.

The limiting measure is a sigma-finite measure (not a probability) on the space of continuous measure-valued paths which presents some additional minor worries. The full convergence on path space remains open in all of the above settings due to the absence of any tightness result on path space. Even the natural statement of convergence of finite-dimensional distributions requires convergence of the survival probabilities (see Proposition 2.4 below), a result which was only recently discovered for critical oriented percolation [2, 3] and is currently being pursued in the other contexts mentioned above. So in the end we thought someone should advertise this state of affairs and we have acquiesced in the writing of this note. If you are reading this in a journal at least one editor and/or referee has agreed with us. Those who would like to see even more details may find them on the webpage of one of us,

AMS 2000 subject classifications: Primary 60G57,60K35; secondary 60F05

Keywords and phrases: r-point functions, measure-valued processes, super-Brownian motion, canonical measure, critical oriented percolation

http://euridice.tue.nl/~mholmes/fullconvergence.pdf

2. Introduction

Consider a discrete time, critical nearest neighbour branching random walk on $\mathbb{Z}^d$ starting with a single particle at the origin. That is at time $n \in \mathbb{Z}_+$ each individual gives birth to a random number of offspring which immediately take a step to a randomly chosen nearest neighbour of its parent. Assume each parent dies immediately after giving birth, that the offspring distribution has mean one and finite variance $\gamma > 0$, and each of the offspring laws and random walk steps are independently chosen.

Let $\mathcal{B}_n = \{X_n^\alpha : \alpha \in I_n\}$ denote the set of locations of particles in $\mathbb{Z}^d$ alive at time n. We have suppressed the details of the labelling system (see, e.g. section II.3 in [8]) but as multiple occupancies are allowed some labelling scheme is needed here. Extend the branching random walk to all times $t \geq 0$ by making it a right-continuous step function. In order to describe the scaling limit, we represent the model as a *cadlag* measure-valued process by setting

$$X_t^n = \frac{C_1}{n} \sum_{\alpha \in I_{nt}} \delta_{X_{nt}^\alpha/(C_2\sqrt{n})} \equiv \frac{C_1}{n} \sum_{x: C_2\sqrt{n}x \in \mathcal{B}_{nt}} \delta_x. \tag{1}$$

The reader has every right to complain about the last expression as one must sum over sites in $\mathcal{B}_{nt}$ according to their occupancy. This is to make subsequent formulae appear more natural; one can ignore it for now. With probability 1, X_t^n is a finite measure for all $n \in \mathbb{Z}_+$ and $t \geq 0$, so that $\{X_t^n\}_{t\geq 0} \in D(M_F(\mathbb{R}^d))$, where $M_F(E)$ denotes the space of finite measures on E with the topology of weak convergence, and $D(E)$ denotes the space of *cadlag* E-valued paths with the Skorokhod topology.

The *extinction time* $S : D(M_F(\mathbb{R}^d)) \to [0, \infty]$ is defined by

$$S(X) \equiv \inf\{s > 0 : X_s = 0_M\}, \tag{2}$$

where 0_M is the zero measure on $\mathbb{R}^d$. Let

$$D^* \equiv \{X \in D(M_F(\mathbb{R}^d)) : \ S(X) > 0, \ X_t = 0_M \ \forall t \geq S\}, \tag{3}$$

with the topology inherited from $D(M_F(\mathbb{R}^d))$. Next we define a sequence of measures $\mu_n \in M_F(D^*)$ by

$$\mu_n(\bullet) \equiv C_3 n \mathbb{P}(\{X_t^n\}_{t\geq 0} \in \bullet). \tag{4}$$

For the branching random walk we set $C_1 = \gamma^{-1}$, $C_2 = d^{-1/2}$ and $C_3 = 1$.

Let $M_\sigma(D^*)$ denote the σ-finite measures on D^* which assign finite mass to $\{S > \epsilon\}$ for all $\epsilon > 0$, with the topology of weak convergence defined as follows:

Definition 2.1 (Weak convergence)**.** *Let* $\{\nu_n : n \in \mathbb{N} \cup \infty\} \subset M_\sigma(D^*)$. *We write* $\nu_n \stackrel{w}{\Longrightarrow} \nu_\infty$ *if for every* $\epsilon > 0$,

$$\nu_n^\epsilon(\bullet) \equiv \nu_n(\bullet, S > \epsilon) \stackrel{w}{\Longrightarrow} \nu_\infty(\bullet, S > \epsilon) \equiv \nu^\epsilon(\bullet), \qquad \text{as } n \to \infty, \tag{5}$$

where the convergence is in $M_F(D(M_F(\mathbb{R}^d)))$.

It is a standard result in the superprocess literature (see for example [8] Theorem II.7.3.) that there exists $\mathbb{N}_0 \in M_\sigma(D^*)$, supported by the continuous paths in D^*, and called the *canonical measure of super-Brownian motion* (CSBM), such that $\mu_n \overset{w}{\Longrightarrow} \mathbb{N}_0$. In [8] one is working with branching Brownian motion instead of branching random walk but it is trivial to modify the arguments. We have chosen our constants C_i so that the branching and diffusion parameters of our limiting super-Brownian motion are both equal to one.

Let $l \geq 1$, and $\vec{t} = \{t_1, \dots, t_l\} \in [0,\infty)^l$. We use $\pi_{\vec{t}} : D^* \to M_F(\mathbb{R}^d)^l$ to denote the projection map satisfying $\pi_{\vec{t}}(X) = (X_{t_1}, \dots X_{t_l})$. The *finite-dimensional distributions* of $\nu \in M_\sigma(D^*)$ are the measures $\nu^\epsilon \pi_{\vec{t}}^{-1} \in M_F(M_F(\mathbb{R}^d)^l)$ given by

$$\nu^\epsilon \pi_{\vec{t}}^{-1}(H) \equiv \nu^\epsilon\left(\{X : \pi_{\vec{t}}(X) \in H\}\right), \qquad H \in \mathcal{B}(M_F(\mathbb{R}^d)^l). \tag{6}$$

Definition 2.2 (Convergence of f.d.d). *Let* $\{\nu_n : n \in \mathbb{N} \cup \infty\} \subset M_\sigma(D^*)$. *We write* $\nu_n \overset{f.d.d}{\Longrightarrow} \nu_\infty$ *if for every* $\epsilon > 0$, $m \in \mathbb{N}$, *and* $\vec{t} \in [0,\infty)^m$,

$$\nu_n^\epsilon \pi_{\vec{t}}^{-1}(\bullet) \overset{w}{\Longrightarrow} \nu_\infty^\epsilon \pi_{\vec{t}}^{-1}(\bullet), \qquad \text{as } n \to \infty, \tag{7}$$

where the convergence is in $M_F(M_F(\mathbb{R}^d)^m)$.

It is clear that weak convergence (Definition 2.1) implies convergence of finite-dimensional distributions (Definition 2.2) but that an additional tightness condition on ν_n is needed for the converse.

We now work in a more abstract setting (including the above) in which $\{\mu_n\}$ is a sequence of finite measures on D^*. For $k \in \mathbb{R}^d$, let $\phi_k(x) = e^{ik\cdot x}$ and write $E_{\mu_n}[Y]$ for $\int Y d\mu_n$, and $X_t(\phi)$ for $\int \phi X_t(dx)$ respectively. Consider the following convergence condition on the moment measures of μ_n:

$$E_{\mu_n}\left[\prod_{i=1}^{r-1} X_{t_i}(\phi_{k_i})\right] \to E_{\mathbb{N}_0}\left[\prod_{i=1}^{r-1} X_{t_i}(\phi_{k_i})\right], \text{ for } r \geq 2, \vec{t} \in [0,\infty)^{r-1}, \vec{k} \in \mathbb{R}^{d(r-1)}. \tag{8}$$

Of course (8) does hold for the above μ_n but our interest in this condition arises from a number of models, such as oriented percolation, in which $\mathcal{B}_t$ is the (finite) set of occupied sites in $\mathbb{Z}^d$ (at most one particle per site) at time n. For $r \geq 2$ and $\vec{t} \in [0,\infty)^{r-1}$ the r-point functions for this model are $B_{\vec{t}}(\vec{x}) = \mathbb{P}(x_i \in \mathcal{B}_{t_i}, i = 1, \dots, r-1)$, while the $\widehat{r}$*-point functions* are the Fourier transforms of these quantities,

$$\widehat{B}_{\vec{t}}(\vec{k}) = \sum_{\vec{x}} e^{i\vec{k}\cdot\vec{x}} B_{\vec{t}}(\vec{x}), \; \vec{k} \in \mathbb{R}^{d(r-1)}.$$

Now define $X_t^n \in M_F(\mathbb{R}^d)$ by the extreme right-hand side of (1) and assume that μ_n given by (4) defines a finite measure on D^*. An easy calculation then shows that

$$\frac{C_1^{r-1}C_3}{n^{r-2}} \widehat{B}_{n\vec{t}}\left(\frac{\vec{k}}{C_2\sqrt{n}}\right) = E_{\mu_n}\left[\prod_{i=1}^{r-1} X_{t_i}(\phi_{k_i})\right]. \tag{9}$$

Therefore, the asymptotic formulae for the $\widehat{r}$-point functions for sufficiently spread out critical rescaled oriented percolation ($d > 4$), critical rescaled lattice trees ($d > 8$), and critical rescaled contact processes ($d > 4$) derived in [6], [7] and work in progress in [5], respectively, immediately implies (8) in each of these cases. Moreover in each of these models it is known that μ_n is a finite measure supported by D^* as is required above.

In what follows we use $\mathcal{D}_F$ to denote the set of discontinuities of a function F. A function $Q : M_F(\mathbb{R}^d)^m \to \mathbb{R}$ is called a *multinomial* if $Q(\vec{X})$ is a real multinomial in $\{X_1(1), \dots, X_m(1)\}$. A function $F : M_F(\mathbb{R}^d)^m \to \mathbb{C}$ is said to be *bounded by a multinomial* ($|F| \le Q$) if there exists a multinomial Q such that $|F(\vec{X})| \le Q(\vec{X})$ for every $\vec{X} \in M_F(\mathbb{R}^d)^m$. The main results of this paper are the following two propositions. By the above, the first result is applicable in any of the three settings described above.

Proposition 2.3. *Let $\{\mu_n\}_{n\ge 1}$ be a sequence of finite measures on D^* such that (8) holds. Then for every $s > 0$, $\lambda > 0$, $m \ge 1$, $\vec{t} \in [0,\infty)^m$ and every Borel measurable $F : M_F(\mathbb{R}^d)^m \to \mathbb{C}$ bounded by a multinomial and such that $\mathbb{N}_0 \pi_{\vec{t}}^{-1}(\mathcal{D}_F) = 0$,*

1. $$E_{\mu_n}\left[X_s(1)F(\vec{X}_{\vec{t}})\right] \to E_{\mathbb{N}_0}\left[X_s(1)F(\vec{X}_{\vec{t}})\right], \tag{10}$$

   and

2. $$E_{\mu_n}\left[F(\vec{X}_{\vec{t}})I_{\{X_s(1)>\lambda\}}\right] \to E_{\mathbb{N}_0}\left[F(\vec{X}_{\vec{t}})I_{\{X_s(1)>\lambda\}}\right]. \tag{11}$$

For critical oriented percolation above the critical spatial dimension of 4 (and for sufficiently spread out kernels) [2, 3] show that

$$\mu_n(S > \epsilon) \to \mathbb{N}_0(S > \epsilon) \text{ for every } \epsilon > 0. \tag{12}$$

The corresponding results for critical lattice trees and critical contact processes are conjectured to be true above the critical dimension, and are currently work in progress [4, 5]. The next result allows us to strengthen the conclusion of Proposition 2.4 under (12). The latter is clearly necessary for the convergence of f.d.d established below (consider the test function 1).

Proposition 2.4. *Let $\{\mu_n\}_{n\ge 0}$ be a sequence of finite measures on D^* such that (8) and (12) hold. Then $\mu_n \overset{f.d.d}{\Longrightarrow} \mathbb{N}_0$.*

In particular the results of [6, 2, 3] together with Proposition 2.4 imply that above the critical dimension and at the critical occupation probability, the scaling limit (in the sense of finite-dimensional distributions) of sufficiently spread-out oriented percolation is CSBM. Tightness, and hence a full statement of weak convergence, remains an open problem. We show in Section 4 that both Propositions are consequences of standard exponential moment bounds for $\mathbb{N}_0$ and the following theorem in which $\mathcal{F}$ denotes a class of $\mathbb{C}$-valued bounded continuous functions that contains the constant function 1 and that is convergence determining for $M_F(\mathbb{R}^d)$. By convention, an empty product is defined to be 1.

Theorem 2.5. *Let $\mu_n, \mu \in M_F\left(D(M_F(\mathbb{R}^d))\right)$. Suppose that for every $l \in \mathbb{Z}_+$ and every $\vec{t} \in [0,\infty)^l$, $\vec{m} \in \mathbb{Z}_+^l$ we have*

1. *there exists a $\delta = \delta(\vec{t}) > 0$ such that for all $\theta_i < \delta$, $E_{\mu\pi_{\vec{t}}^{-1}}[e^{\sum_{i=1}^l \theta_i X_i(1)}] < \infty$, and*
2. *for every $\phi_{ij} \in \mathcal{F}$,*

$$E_{\mu_n\pi_{\vec{t}}^{-1}}\left[\prod_{i=1}^{l}\prod_{j=1}^{m_i} X_i(\phi_{ij})\right] \to E_{\mu\pi_{\vec{t}}^{-1}}\left[\prod_{i=1}^{l}\prod_{j=1}^{m_i} X_i(\phi_{ij})\right] < \infty. \tag{13}$$

Then for every $m \in \mathbb{N}$ and every $\vec{t} \in [0,\infty)^m$, $\mu_n\pi_{\vec{t}}^{-1} \overset{w}{\Longrightarrow} \mu\pi_{\vec{t}}^{-1}$.

That convergence of the $\widehat{r}$-point functions ($r \geq 2$) is not sufficient for the conclusion of Lemma 3.2 can be illustrated by considering the measures $\mu_n, \mu \in M_F\left(D(M_F(\mathbb{R}^d))\right)$ defined by $\mu_n = n\delta_{n^{-2}\delta_0}$ and $\mu = \delta_{0_M}$. This means that μ_n puts all of its mass (n) on the measure-valued process that is the measure δ_0/n^2 for all time, while μ puts all its mass on the process that is the zero measure for all time. Clearly no subsequence of $\{\mu_n\}_{n\in\mathbb{N}}$ can converge in $M_F\left(D(M_F(\mathbb{R}^d))\right)$. In fact it is easy to show that *except for the $l = 0$ case of the second condition*, for these choices of μ_n and μ, both conditions of Theorem 2.5 hold.

The remainder of this paper is organised as follows. In Section 3 we prove Theorem 2.5. In Section 4 we prove Propositions 2.3 and 2.4.

3. Proof of Theorem 2.5

In this section we prove Theorem 2.5 as a consequence of Lemmas (3.2–3.7). Lemma 3.2 is standard and states that if a sequence of finite measures is tight then it has a convergent subsequence. Lemma 3.3 establishes tightness of the $\{\mu_n\pi_{\vec{t}}^{-1} : n \in \mathbb{N}\}$ for each fixed $\vec{t}$. Thus every subsequence of the $\mu_n\pi_{\vec{t}}^{-1}$ has a further subsequence that converges. Lemma 3.4 states that any limit point of the $\{\mu_n\pi_{\vec{t}}^{-1} : n \in \mathbb{N}\}$ must have the same moments (13) as $\mu\pi_{\vec{t}}^{-1}$. Lemma 3.5 extends equality of the moments on the right-hand side of (13) to all $\phi_{ij} \geq 0$ bounded and continuous. Lemmas 3.6 and 3.7 together imply that under condition 1. of Theorem 2.5, equality of the moments in Lemma 3.5 implies equality of the underlying finite measures on $M_F(\mathbb{R}^d)^m$. Taken together they show that since all subsequential limit points have the same moments (13), the limit points all coincide, and thus the whole sequence converges to that limit point. Thus, Theorem 2.5 follows immediately from the Lemmas proved in this section.

Recall the notion of *tightness* for finite measures.

Definition 3.1. *A set of finite measures $F \subset M_F(E)$ on the Borel σ-algebra of a metric space E is* tight *if $\sup_{\mu\in F}\mu(E) < \infty$ and for every $\eta > 0$ there exists $K \subset E$ compact such that $\sup_{\mu\in F}\mu(K^c) < \eta$.*

Lemma 3.2. *If $F \subset M_F(E)$ is tight, then every sequence in F has a further subsequence which converges in $M_F(E)$ (weak convergence).*

Proof. Let $\{\mu_n\} \subset F$. If there exists a subsequence μ_{n_l} such that $\mu_{n_l}(E) \to 0$ then we have $\mu_{n_l} \to 0_M$ by definition (for every bounded continuous f....) and we are done. So without loss of generality there exists $\eta_0 > 0$ such that $\inf_n \mu_n(E) = \eta_0$. Therefore

$$P_n(\bullet) \equiv \frac{\mu_n(\bullet)}{\mu_n(E)} \tag{14}$$

are probability measures. Let $\eta > 0$. Since the μ_n are tight there exists $K \subset E$ compact such that $\sup_n \mu_n(K^c) < \eta\eta_0$. Therefore

$$\sup_n P_n(K^c) = \sup_n \frac{\mu_n(K^c)}{\mu_n(E)} < \frac{\eta\eta_0}{\eta_0} = \eta, \tag{15}$$

so $\{P_n\}$ is tight as a set of probability measures. Therefore there exists a subsequence $P_{n_k} \to P_\infty$.

Since $\{\mu_{n_k}\}$ is tight, $\{\mu_{n_k}(E)\}$ is a bounded, real-valued sequence, and therefore has a convergent subsequence $\mu_{n_k^*}(E) \to C \geq \eta_0$. So $\mu_{n_k^*}/(\mu_{n_k^*}(E)) \to P_\infty$ and $\mu_{n_k^*}(E) \to C > 0$ and therefore $\mu_{n_k^*} \to CP_\infty \in M_F(E)$ as required. ☐

Lemma 3.3. *Let $\mu_n, \mu \in M_F\left(D(M_F(\mathbb{R}^d))\right)$. Suppose that $E_{\mu_n}[1] \to E_\mu[1] < \infty$ and that for every $t \in [0, \infty)$, and every $\phi \in \mathcal{F}$,*

$$E_{\mu_n\pi_t^{-1}}[X(\phi)] \to E_{\mu\pi_t^{-1}}[X(\phi)] < \infty. \tag{16}$$

Then for each $m \in \mathbb{Z}_+$ and every $\vec{t} \in [0, \infty)^m$, the set of measures $\{\mu_n\pi_{\vec{t}}^{-1} : n \in \mathbb{N}\}$ is tight on $M_F(\mathbb{R}^d)^m$.

Proof. The $m = 0$ case is trivial since $E_{\mu_n}[1] \to E_\mu[1] < \infty$.

We next prove the $m = 1$ case. Let $\epsilon > 0$, $t \geq 0$. Define the mean measures $\nu_n, \nu \in M_F(\mathbb{R}^d)$ by $\nu_n = E_{\mu_n\pi_t^{-1}}[X]$, and $\nu = E_{\mu\pi_t^{-1}}[X]$. Then (16) implies $\nu_n \to \nu$ in $M_F(\mathbb{R}^d)$ and $\sup_n \nu_n(\mathbb{R}^d) \equiv L < \infty$. Choose M such that $L/M < \epsilon/2$. Then by Markov's inequality,

$$\sup_n \mu_n\pi_t^{-1}(X(\mathbb{R}^d) > M) \leq \frac{L}{M} < \frac{\epsilon}{2}. \tag{17}$$

Fix $\eta > 0$. There exists $K_{-1} \subset \mathbb{R}^d$ compact such that $\nu(K_{-1}^c) < \eta/2$. Furthermore there exists $K_0 \subset \mathbb{R}^d$ compact such that $\nu(\overline{K_0^c}) \leq \nu(K_{-1}^c)$ (e.g. the set $K_0 = \{x : d(x, K_{-1}) \leq 1\}$). Since $\nu_n \to \nu$ in $M_F(\mathbb{R}^d)$ and $\overline{K_0^c}$ is closed,

$$\limsup_n \nu_n(\overline{K_0^c}) \leq \nu(\overline{K_0^c}) < \frac{\eta}{2}. \tag{18}$$

It follows easily that there is a compact subset K of $\mathbb{R}^d$ such that

$$\sup_n \nu_n(K^c) < \eta. \tag{19}$$

Another application of Markov's inequality implies that

$$\sup_n \mu_n\pi_t^{-1}\left(X(K^c) > \eta^{\frac{1}{4}}\right) \leq \eta^{-1/4} \sup_n E_{\mu_n\pi_t^{-1}}[X(K^c)] < \eta^{3/4}. \tag{20}$$

Choose $\eta^{\frac{1}{4}} = 2^{-j}$. Then there exists $K_j \subset \mathbb{R}^d$ compact such that

$$\sup_n \mu_n \pi_t^{-1} \left(X(K_j^c) > \frac{1}{2^j} \right) \leq \frac{1}{2^{3j}}. \tag{21}$$

Choose N so that $8^{1-N} < \epsilon/2$ and let

$$\mathbf{K} \equiv \bigcap_{j \geq N} \left\{ X : X(K_j^c) \leq \frac{1}{2^j} \right\} \bigcap \{X : X(\mathbb{R}^d) \leq M\}. \tag{22}$$

Now $\mathbf{K}$ is (sequentially) compact in $M_F(\mathbb{R}^d)$ (see for example in the proof of Theorem II.4.1 of [8]), and

$$\mathbf{K}^c = \bigcup_{j \geq N} \left\{ X : X(K_j^c) > \frac{1}{2^j} \right\} \bigcup \left\{ X : X(\mathbb{R}^d) > M \right\}. \tag{23}$$

Thus, (21) and (17) imply

$$\sup_n \mu_n \pi_t^{-1}(\mathbf{K}^c) \leq \sum_{j=N}^{\infty} \frac{1}{2^{3j}} + \frac{\epsilon}{2} \leq \frac{1}{8^{N-1}} + \frac{\epsilon}{2} < \epsilon, \tag{24}$$

which verifies that the $\mu_n \pi_t^{-1}$ are spatially tight, for $m = 1$.

For $m > 1$ and $\vec{t} \in [0, \infty)^m$, we have from (24) that for each $i \in \{1, \dots, m\}$ there exists $\mathbf{K}_i \subset M_F(\mathbb{R}^d)$ compact such that $\sup_n \mu_n \pi_{t_i}^{-1}(\mathbf{K}_i^c) < \epsilon/m$. Let $\mathbf{K} = \mathbf{K}_1 \times \mathbf{K}_2 \times \cdots \times \mathbf{K}_m$. Then $\mathbf{K} \subset (M_F(\mathbb{R}^d))^m$ is compact and

$$\sup_n \mu_n \pi_{\vec{t}}^{-1} \left(\{ \vec{X} : \vec{X} \in \mathbf{K}^c \} \right) \leq \sup_n \sum_{i=1}^{m} \mu_n \pi_{t_i}^{-1} \left(\mathbf{K}_i^c \right) < \epsilon, \tag{25}$$

which gives the result. □

Lemma 3.4. *Fix $l \geq 0$ and $\vec{t} \in [0, \infty)^l$. Suppose that the second hypothesis of Theorem 2.5 holds for $\mu_n, \mu \in M_F(D(M_F(\mathbb{R}^d)))$, for this $\vec{t}$ and for every $\vec{m} \in \mathbb{Z}_+^l$. If $\mu_{n_k} \pi_{\vec{t}}^{-1} \overset{w}{\Longrightarrow} \nu$ in $M_F\left((M_F(\mathbb{R}^d))^l \right)$, then for each $\vec{m} \in \mathbb{Z}_+^l$ and $\phi_{ij} \in \mathcal{F}$,*

$$E_\nu \left[\prod_{i=1}^{l} \prod_{j=1}^{m_i} X_i(\phi_{ij}) \right] = E_{\mu \pi_{\vec{t}}^{-1}} \left[\prod_{i=1}^{l} \prod_{j=1}^{m_i} X_i(\phi_{ij}) \right]. \tag{26}$$

Proof. The $l = 0$ case is trivial, so we may assume that $l > 0$. Let $\mu_{n_k} \pi_{\vec{t}}^{-1} \overset{w}{\Longrightarrow} \nu$. Then in particular we have $\mu_{n_k} \pi_{\vec{t}}^{-1}(1) \to \nu(1)$. Assume $\nu(1) > 0$. Then there exists k_0 such that for every $k \geq k_0$, $\nu(1)/2 \leq \mu_{n_k} \pi_{\vec{t}}^{-1}(1) \leq 2\nu(1)$ and we define for $k \geq k_0$ the probability measures,

$$P_{n_k}(\bullet) \equiv \frac{\mu_{n_k} \pi_{\vec{t}}^{-1}(\bullet)}{\mu_{n_k} \pi_{\vec{t}}^{-1}(1)}, \quad P(\bullet) \equiv \frac{\nu(\bullet)}{\nu(1)}. \tag{27}$$

Then we have that $P_{n_k} \overset{w}{\Longrightarrow} P$ as probability measures. By Skorokhod's theorem we may construct $\vec{X}^{n_k}$ with laws P_{n_k}, and $\vec{X}$ with law P such that $\vec{X}^{n_k} \to \vec{X}$ a.s. Our hypotheses and a moments thought imply that if $\phi_{ij} \in \mathcal{F}$, then $F_{\vec{\phi}}(\vec{X}^{n_k}) = \prod_{i=1}^{l}\prod_{j=1}^{m_i} X_i^{n_k}(\phi_{ij})$ are L^2 bounded and thus uniformly integrable. Therefore the left-hand side of (26) is $\lim_{k\to\infty} E_{\mu_{n_k}\pi_{\vec{t}}^{-1}(1)}[F_{\vec{\phi}}(\vec{X}^{n_k})]$ and so is the right-hand side by hypothesis.

Consider now the case that $\nu(1) = 0$. By Cauchy-Schwarz,

$$\left(E_{\mu_{n_k}\pi_{\vec{t}}^{-1}}\left[\prod_{i=1}^{l}\prod_{j=1}^{m_i} X_i(\phi_{ij})\right]\right)^2 \leq E_{\mu_{n_k}\pi_{\vec{t}}^{-1}}\left[1^2\right] E_{\mu_{n_k}\pi_{\vec{t}}^{-1}}\left[\left(\prod_{i=1}^{l}\prod_{j=1}^{m_i} X_i(1)\right)^2\right]. \tag{28}$$

Now 1 is a bounded continuous function and $\mu_{n_k}\pi_{\vec{t}}^{-1} \to 0_M$ so the first expectation on the right converges to 0. Since $\sup_k E_{\mu_{n_k}\pi_{\vec{t}}^{-1}}\left[\left(\prod_{i=1}^{l}\prod_{j=1}^{m_i} X_i(1)\right)^2\right] < \infty$ we obtain

$$E_{\mu_{n_k}\pi_{\vec{t}}^{-1}}\left[\prod_{i=1}^{l}\prod_{j=1}^{m_i} X_i(\phi_{ij})\right] \to 0. \tag{29}$$

Since also

$$E_{\mu_{n_k}\pi_{\vec{t}}^{-1}}\left[\prod_{i=1}^{l}\prod_{j=1}^{m_i} X_i(\phi_{ij})\right] \to E_{\mu\pi_{\vec{t}}^{-1}}\left[\prod_{i=1}^{l}\prod_{j=1}^{m_i} X_i(\phi_{ij})\right], \tag{30}$$

we have that $E_{\mu\pi_{\vec{t}}^{-1}}\left[\prod_{i=1}^{l}\prod_{j=1}^{m_i} X_i(\phi_{ij})\right] = 0 = E_\nu\left[\prod_{i=1}^{l}\prod_{j=1}^{m_i} X_i(\phi_{ij})\right]$ which verifies the result. ☐

Lemma 3.5. *Suppose $l \geq 0$, $\mu, \mu' \in M_F\left((M_F(\mathbb{R}^d))^l\right)$. If*

$$E_\mu\left[\prod_{i=1}^{l}\prod_{j=1}^{m_i} X_i(\phi_{ij})\right] = E_{\mu'}\left[\prod_{i=1}^{l}\prod_{j=1}^{m_i} X_i(\phi_{ij})\right] \tag{31}$$

holds (and both quantities are finite) for every $\phi_{ij} \in \mathcal{F}$, then (31) holds for all bounded, continuous $\phi_{ij} \geq 0$.

Proof. If $l = 0$ or $\sum m_i = 0$ then the conclusion is trivial so we may assume that $l > 0$ and $\sum m_i > 0$. Since $1 \in \mathcal{F}$, we have $E_\mu[\prod_{i=1}^{l}\prod_{j=1}^{m_i} X_i(1)] < \infty$. Let $\phi_{ij} \in \mathcal{F}$ and $\varphi((x_{ij})) = \prod_{i=1}^{l}\prod_{j=1}^{m_i} \phi_{ij}(x_{ij})$. Applying Fubini to (31), using the fact that the $\phi_{ij} \in \mathcal{F}$ are bounded we have

$$\int\cdots\int \varphi E_\mu\left[\prod_{i=1}^{l}\prod_{j=1}^{m_i} X_i(dx_{ij})\right] = \int\cdots\int \varphi E_{\mu'}\left[\prod_{i=1}^{l}\prod_{j=1}^{m_i} X_i(dx_{ij})\right]. \tag{32}$$

Since $\mathcal{F}$ is a determining class for $M_F(\mathbb{R}^d)$ one can verify that for any $r \geq 1$, the set of functions $\mathcal{F}_r \equiv \{\prod_{i=1}^r \phi_i(x_i) : \phi_i \in \mathcal{F}\}$ is a determining class for $M_F(\mathbb{R}^{dr})$ (using the fact that this class of functions determines the conditional distribution of the nth coordinate given the first $n-1$ and proceeding by induction). Therefore the products of ϕ_{ij} in (32) uniquely determine the measure ν on $\mathbb{R}^{d\sum m_i}$ defined by $\nu(d\vec{x}) = E_\mu[\prod_{i=1}^l \prod_{j=1}^{m_i} X_i(dx_{ij})]$. Thus (32) holds for all ϕ_{ij} bounded, continuous and applying Fubini again we get the result. □

In the following lemma, $\mathcal{B}_b(\mathbb{R}^d, \mathbb{R}_+)$ denotes the bounded, non-negative real-valued functions on $\mathbb{R}^d$, and $\overline{D_0}^{bp}$ denotes the bounded pointwise closure of $D_0 \subset \mathcal{B}_b(\mathbb{R}^d, \mathbb{R}_+)$, i.e. the smallest set containing D_0 that is closed under bounded pointwise convergence.

Lemma 3.6. *Suppose $\mu, \mu' \in M_F\left((M_F(\mathbb{R}^d))^m\right)$ and assume $D_0 \subset \mathcal{B}_b(\mathbb{R}^d, \mathbb{R}_+)$ satisfies $\overline{D_0}^{bp} = \mathcal{B}_b(\mathbb{R}^d, \mathbb{R}_+)$. If for all $h_j \in D_0$*

$$E_\mu\left[e^{-\sum_{j=1}^m X_i(h_j)}\right] = E_{\mu'}\left[e^{-\sum_{j=1}^m X_i(h_j)}\right], \tag{33}$$

then $\mu = \mu'$.

Proof. If $m = 0$ the conclusion is trivial as both measures are on the single point space with same total mass, so we may assume that $m > 0$. We follow the proof of Lemma II.5.9 of [8].

(a) Equation (33) holds for every $\vec{\phi} \in (C_b(\mathbb{R}^d, \mathbb{R}_+))^m$. We verify the stronger result that the class $\mathcal{L}$ of $\vec{\phi}$ for which (33) holds contains $(\mathcal{B}_b(\mathbb{R}^d, \mathbb{R}_+))^m$.

Let $\vec{\phi}_n \in \mathcal{L}$ be such that $\vec{\phi}_n \xrightarrow{bp} \vec{\phi}$. Now by dominated convergence (using the fact that μ is a finite measure and dominating by $e^0 = 1$),

$$\begin{aligned} E_\mu\left[e^{-\sum_{j=1}^m X_i(\phi_j)}\right] &= E_\mu\left[\lim_{n\to\infty} e^{-\sum_{j=1}^m X_i(\phi_{j,n})}\right] \\ &= \lim_{n\to\infty} E_\mu\left[e^{-\sum_{j=1}^m X_i(\phi_{j,n})}\right] \\ &= \lim_{n\to\infty} E_{\mu'}\left[e^{-\sum_{j=1}^m X_i(\phi_{j,n})}\right] \\ &= E_{\mu'}\left[e^{-\sum_{j=1}^m X_i(\phi_j)}\right]. \end{aligned} \tag{34}$$

Thus $\mathcal{L}$ is closed under bounded pointwise convergence. Since $D_0 \subset \mathcal{L}$ by hypothesis this shows that $(\mathcal{B}_b(\mathbb{R}^d, \mathbb{R}_+))^m \subset \mathcal{L}$ as required.

Before proceeding to the next step we define $e_{\vec{\phi}} : (M_F(\mathbb{R}^d))^m \to \mathbb{R}_+$ by $e_{\vec{\phi}}(\vec{\nu}) = \exp\{-\sum_{j=1}^m \nu_j(\phi_j)\}$, and let

$$\mathcal{H} \equiv \left\{\Phi \in \mathcal{B}_b((M_F(\mathbb{R}^d))^m, \mathbb{R}) : E_\mu[\Phi(\vec{X})] = E_{\mu'}[\Phi(\vec{X})]\right\} \tag{35}$$

and

$$\mathcal{H}_0 = \left\{e_{\vec{\phi}} : \vec{\phi} \in (C_b(\mathbb{R}^d, \mathbb{R}_+))^m\right\}. \tag{36}$$

(b) $\mathcal{H}$ contains all bounded $\sigma(\mathcal{H}_0)$ measurable functions. We show that $\mathcal{H}$ is a linear class containing 1, closed under $\xrightarrow{bp}$, and that $\mathcal{H}_0 \subset \mathcal{H}$ is closed under products. Once we achieve this, we have by Lemma II.5.2 of [8] that $\mathcal{H}$ contains all bounded $\sigma(\mathcal{H}_0)$-measurable functions.

1) that $\mathcal{H}$ is a linear class is immediate by linearity of the integral.
2) $1 \in \mathcal{H}$ by taking $\vec{\phi} = \vec{0}$ and using part (a).
3) Let $\Phi_n \in \mathcal{H}, \Phi_n \xrightarrow{bp} \Phi$. Then $\Phi \in \mathcal{H}$ by dominated convergence since μ, μ' are finite measures.
4) Let $f_1, f_2 \in \mathcal{H}_0$. Then $f_i = e_{\phi_i}$ and
$$f_1 f_2 = e^{-\sum_{j=1}^m X_j(\phi_{j,1})} e^{-\sum_{j=1}^m X_j(\phi_{j,2})} = e^{-\sum_{j=1}^m X_j(\phi_{j,1}+\phi_{j,2})} = e_{\phi_1+\phi_2} \in \mathcal{H}_0. \tag{37}$$
5) $\mathcal{H}_0 \subset \mathcal{H}$ was verified in part (a).

(c) There exists a countable convergence determining set for $(M_F(\mathbb{R}^d))^m$. We use the construction of Proposition 3.4.4 of [1] to obtain a countable set $V \subset (C_b(\mathbb{R}^d, \mathbb{R}_+))^m$ such that $\vec{\nu}_n \to \vec{\nu}$ in $(M_F(\mathbb{R}^d))^m$ if and only if $\vec{\nu}_n(\vec{\phi}) \to \vec{\nu}(\vec{\phi})$ for every $\vec{\phi} \in V$. Let $\{\vec{q}_1, \vec{q}_2, \dots\}$ be an enumeration of $\mathbb{Q}^d$, a dense subset of $\mathbb{R}^d$. For each $(i,j) \in \mathbb{N}^2$ define

$$f_{i,j}(\vec{x}) \equiv 2\,(1 - j|\vec{x} - \vec{q}_i|) \vee 0, \tag{38}$$

and for $A \subset \mathbb{N}^2$ define

$$g_A^m(\vec{x}) \equiv \left(\sum_{\substack{i,j \le m \\ (i,j) \in A}} f_{i,j} \right) \wedge 1. \tag{39}$$

It is an exercise left for the reader to verify that

$$V_0 \equiv \left\{ g_A^m : m \in \mathbb{N}, A \subset \{1, \dots, m\}^2 \right\} \subset C_b(\mathbb{R}^d), \tag{40}$$

is a countable convergence determining set for $M_F(\mathbb{R}^d)$. It follows that $V = \{(\phi_1, \dots, \phi_m) : \phi_i \in V_0 \cup \{0\}\}$ is a countable convergence determining set for $(M_F(\mathbb{R}^d))^m$.

In order to proceed with the next step of the proof we define

$$\mathcal{G} \equiv \sigma(e_{\vec{\phi}} : \vec{\phi} \in V). \tag{41}$$

(d) $\mathcal{B}((M_F(\mathbb{R}^d))^m) \subset \mathcal{G} \subset \sigma(\mathcal{H}_0)$. The second inclusion is trivial since $V \subset (C_b(\mathbb{R}^d, \mathbb{R}))^m$. We claim that $\mathcal{G}$ contains all the open sets in $(M_F(\mathbb{R}^d))^m$ and hence contains $\mathcal{B}((M_F(\mathbb{R}^d))^m)$. Define the metric

$$\varrho'(\vec{\mu}, \vec{\nu}) \equiv \sum_{j=1}^m \sum_{n=1}^\infty \frac{|\int \phi_n d\mu_i - \int \phi_n d\nu_i|}{2^n}, \tag{42}$$

where $\{\phi_1, \phi_2, \dots\}$ is some fixed enumeration of $V_0 \cup \{0\}$. It is a standard result that ϱ' induces the topology of weak convergence.

Let U be an open set in the topology of weak convergence. Then U is also open in $((M_F(\mathbb{R}^d))^m, \varrho')$. Now $M_F(\mathbb{R}^d)$ is separable so every open set is a countable union of balls $B_{\varrho'}(\vec{\nu}, r)$ and therefore to show that $U \in \mathcal{G}$, it is enough to show that $B_{\varrho'}(\vec{\nu}, r) \in \mathcal{G}$. But

$$B_{\varrho'}(\vec{\nu}, r) = \left\{ \vec{\mu} : \sum_{j=1}^{m} \sum_{n=1}^{\infty} \frac{|\int \phi_n d\mu_j - \int \phi_n d\nu_j|}{2^n} < r \right\} \in \mathcal{G} \tag{43}$$

since an infinite series of measurable functions is measurable. Thus $\mathcal{G}$ contains all the open sets of $(M_F(\mathbb{R}^d))^m$ and therefore contains $\mathcal{B}((M_F(\mathbb{R}^d))^m)$.

(e) Conclusion. We have now verified that $\mathcal{B}((M_F(\mathbb{R}^d))^m) \subset \mathcal{G} \subset \sigma(\mathcal{H}_0)$. Therefore every bounded continuous function is measurable with respect to $\sigma(\mathcal{H}_0)$. Furthermore we have that $\mathcal{H}$ contains all $\sigma(\mathcal{H}_0)$-measurable functions (and in particular all the bounded continuous functions). Since $\mu = \mu'$ if and only if $\int f d\mu = \int f d\mu'$ for all bounded continuous $f : (M_F(\mathbb{R}^d))^m \to \mathbb{R}$, we have proved the result. □

Lemma 3.7. *Let $\mu \in M_F\left((M_F(\mathbb{R}^d))^m\right)$. Suppose there exists a $\delta > 0$ such that for all $\theta_i < \delta$,*

$$E_\mu \left[e^{\sum_{i=1}^m \theta_i X_i(1)} \right] < \infty. \tag{44}$$

Then for every bounded continuous $0 \le \psi_i$, the quantity $E_\mu \left[e^{-\sum_{i=1}^m X_i(\psi_i)} \right]$ is uniquely determined by the collection of mixed moments $\{E_\mu \left[\prod_{i=1}^m X_i(h_i)^{n_i}\right] : 0 \le h_i \le 1$ is continuous, $i = 1, \dots, m.\}$

Proof. Without loss of generality we may assume that $m > 0$. Fix one particular choice of $\vec{h} = (h_1, \dots, h_m)$ as above. For $\mathrm{Re} z_i < \delta$, $i = 1, \dots, m$, let

$$f(z_1, \dots, z_m) = E_\mu[e^{\vec{z} \cdot \vec{X}(\vec{h})}].$$

Use (44), the Taylor expansion for the exponential function and Fubini's theorem to see that for $\|\vec{z}\|_\infty < \delta$,

$$f(z_1, \dots, z_m) = \sum_{l=0}^{\infty} \frac{1}{l!} E_\mu \left[\sum_{\substack{\vec{n} \in \mathbb{Z}_+^m : \\ \sum n_i = l}} \frac{l!}{\prod_{i=1}^m n_i!} \prod_{i=1}^m (z_i X_i(h_i))^{n_i} \right].$$

Hence the mixed moments of the form

$$E_\mu \left[\prod_{i=1}^m X_i(h_i)^{n_i} \right], \ n_i \in \mathbb{Z}_+ \tag{45}$$

uniquely determine $f(z)$ for $\|\vec{z}\|_\infty < \delta$.

Now let $\vec{z}$ be such that each $\mathrm{Re} z_i < \delta$ and note that $0 \le X_i(h_i) \le X_i(1)$ for $0 \le h_i \le 1$. Then

$$
\begin{aligned}
&\lim_{\Delta z_i \to 0} \left| \int \frac{e^{(\vec{z}+\Delta z_i)\cdot \vec{X}(\vec{h})} - e^{\vec{z}\cdot \vec{X}(\vec{h})}}{\Delta z_i} d\mu - \int X_i(h_i) e^{\vec{z}\cdot \vec{X}(\vec{h})} d\mu \right| \\
&= \lim_{\Delta z_i \to 0} \left| \int e^{\vec{z}\cdot \vec{X}(\vec{h})} \left[\frac{e^{\Delta z_i \cdot X_i(h_i)} - 1}{\Delta z_i} - X_i(h_i) \right] d\mu \right| \\
&= \lim_{\Delta z_i \to 0} \left| \int e^{\vec{z}\cdot \vec{X}(\vec{h})} \left[\Delta z_i \sum_{l=2}^{\infty} \frac{(\Delta z_i)^{l-2} X_i(h_i)^l}{l!} \right] d\mu \right| \\
&\le \lim_{\Delta z_i \to 0} |\Delta z_i| \int e^{\sum_{j=1}^m \mathrm{Re} z_j X_j(h_j)} X_i(h_i)^2 e^{|\Delta z_i| X_i(h_i)} d\mu \\
&= \lim_{\Delta z_i \to 0} |\Delta z_i| \int e^{\mathrm{Re} z_i + \epsilon + |\Delta z_i|) X_i(h_i) + \sum_{j\neq i}^m \mathrm{Re} z_j X_j(h_j)} X_i(h_i)^2 e^{-\epsilon X_i(h_i)} d\mu.
\end{aligned}
\tag{46}
$$

Now $X_i(\phi_i)^2 e^{-\epsilon X_i(h_i)} \le C_\epsilon$ so this integral converges (uniformly in $|\Delta z_i|$ sufficiently small) for all $\vec{z}$ such that $\mathrm{Re} z_i + \epsilon + |\Delta z_i| < \delta$ and $\mathrm{Re} z_j < \delta$ for all $j \neq i$. This shows that for fixed $z_1, \dots, z_{i-1}, z_{i+1}, \dots, z_m$ satisfying $\mathrm{Re} z_j < \delta$ for $j \neq i$, $f(\vec{z})$ is analytic in $\mathrm{Re} z_i < \delta$. Now use induction on $i \le m+1$ to see that moments of the form (45) uniquely determine $f(z_1, \dots, z_m)$ for $\mathrm{Re} z_1, \dots, \mathrm{Re} z_{i-1} < \delta, |z_i| \vee \dots \vee |z_m| < \delta$. Here one used the aforementioned analyticity in $\mathrm{Re} z_i < \delta$ in the induction step.

Finally if $\psi_i \ge 0$ is bounded and continuous, we apply the above to $h_i = \psi_i / \|\psi_i\|_\infty \in [0,1]$ and $z_i = -\|\psi_i\|_\infty$ (set $h_i = 0$ if $\psi_i \equiv 0$) to complete the proof. □

4. Applications of Theorem 2.5

In this section we prove Propositions 2.3 and 2.4, which relate the asymptotic formulae for the $\hat{r}$-point functions for various spread-out models above their critical dimensions to the convergence to CSBM. Recall that $\phi_k(x) = e^{ik\cdot x}$. In this section we fix our convergence determining class of functions for $M_F(\mathbb{R}^d)$ to be

$$\mathcal{F} = \{\phi_k : k \in \mathbb{R}^d\}. \tag{47}$$

The following Lemma will be used to verify the exponential moment hypothesis of Theorem 2.5. The branching and diffusion parameters for $\mathbb{N}_0$ are taken to be 1.

Lemma 4.1. *For every $b \ge 0$ the following hold.*

1. *For every $\lambda > 0$, $\mathbb{N}_0(X_b(1) = \lambda) = 0$.*
2. *For every $\vec{t} \in [0,\infty)^m$ there exists a $\delta(\vec{t}, b) > 0$ such that for $\theta_i < \delta$*

$$E_{\mathbb{N}_0}\left[X_b(1) e^{\sum_{i=1}^m \theta_i X_{t_i}(1)} \right] < \infty. \tag{48}$$

3. *For every* $\vec{t} \in [0,\infty)^m$ *and every* $\epsilon > 0$ *there exists a* $\delta(\vec{t},\epsilon) > 0$ *such that for* $\theta_i < \delta$
$$E_{\mathbb{N}_0}\left[e^{\sum_{i=1}^m \theta_i X_{t_i}(1)} I_{\{S>\epsilon\}}\right] < \infty. \tag{49}$$

Proof. By Theorem II.7.2(iii) of [8] we have for $b > 0$,
$$\mathbb{N}_0\left(X_b(1) \in A\right) = \left(\frac{2}{b}\right)^2 \int_A e^{-\frac{2}{b}x} dx. \tag{50}$$
Since also $\mathbb{N}_0\left(X_0(1) > 0\right) = 0$, the first assertion is trivial.

The other parts are also well-known but we include a proof for completeness. As above we may assume $b > 0$ in part 2. Use the inequality
$$X_b(1) \le 1(S > b) C_\eta e^{\eta X_b(1)} \text{ for each } \eta > 0$$
to see that part 2 will follow from part 3.

For the last claim of the Lemma we abuse our notation and let E_{X_0} also denote expectation for our standard super-Brownian motion starting at X_0. Let $\mathcal{G}_t$ denote the canonical filtration generated by the coordinates X_s of our super-Brownian motion for $s \le t$. If $H : M_F(\mathbb{R}^d) \to [0,\infty)$ is continuous, then for $t \ge s > 0$,
$$E_{\mathbb{N}_0}(H(X_t)|\mathcal{G}_s) = E_{X_s}(H(X_{t-s})),\ \mathbb{N}_0 - \text{a.e.} \tag{51}$$
This is easily derived, for example, from the convergence of branching random walk to $\mathbb{N}_0$ mentioned in Section 2, the Markov property for branching random walk, and the analogous convergence result for super-Brownian motion (e.g. Theorem II.5.2 of [8]).

We may assume without loss of generality that $0 < \epsilon < t_i < t_{i+1}$ for each i. Observe from (51) that
$$\begin{aligned} E_{\mathbb{N}_0}\left[e^{\sum_{i=1}^m \theta_i X_{t_i}(1)} I_{\{S>\epsilon\}}\right] =& E_{\mathbb{N}_0}\left[E_{X_{t_{m-1}}}\left[e^{\theta_m X_{t_m - t_{m-1}}(1)}\right] e^{\sum_{i=1}^{m-1} \theta_i X_{t_i}(1)} I_{\{S>\epsilon\}}\right] \\ \le& E_{\mathbb{N}_0}\left[e^{2\theta_m X_{t_{m-1}}(1)} e^{\sum_{i=1}^{m-1} \theta_i X_{t_i}(1)} I_{\{S>\epsilon\}}\right], \end{aligned} \tag{52}$$
where the inequality holds for θ_m sufficiently small depending on $t_m - t_{m-1}$ by Lemma III.3.6. of [8]. The last line of (52) has no t_m dependence and proceeding by induction it is enough to show that for sufficiently small $\theta > 0$,
$$E_{\mathbb{N}_0}\left[e^{\theta X_{t_1}(1)} I_{\{S>\epsilon\}}\right] < \infty. \tag{53}$$
However for $\theta > 0$ small enough (depending on ϵ) the left-hand side is
$$\begin{aligned} E_{\mathbb{N}_0}\left[E_{\mathbb{N}_0}\left[e^{\theta X_{t_1}(1)}|\mathcal{G}_\epsilon\right] I_{\{S>\epsilon\}}\right] \le& E_{\mathbb{N}_0}\left[e^{2\theta X_\epsilon(1)} I_{\{S>\epsilon\}}\right] \\ =& E_{\mathbb{N}_0}\left[e^{2\theta X_\epsilon(1)} I_{\{X_\epsilon(1)>0\}}\right] \\ =& \left(\frac{2}{\epsilon}\right)^2 \int_0^\infty e^{2\theta x} e^{-\frac{2x}{\epsilon}} dx, \end{aligned} \tag{54}$$
where the last equality holds by (50). The last line of (54) is finite for sufficiently small $\theta > 0$ (depending on ϵ) and the result follows. □

4.1. Proof of Proposition 2.3.

Define $\mu_{n,s}, \mathbb{N}_{0,s} \in M_F\left(D(M_F(\mathbb{R}^d))\right)$ by

$$\mu_{n,s}(A) = \int_A X_s(1) d\mu_n, \quad \mathbb{N}_{0,s}(A) = \int_A X_s(1) d\mathbb{N}_0. \tag{55}$$

That these measures are finite follows from the fact that for $s > 0$,

$$\mu_{n,s}\left(D(M_F(\mathbb{R}^d))\right) = E_{\mu_n}\left[X_s(1)\right] \to E_{\mathbb{N}_0}\left[X_s(1)\right] < \infty. \tag{56}$$

For all $l \geq 0$, $\vec{m} \in \mathbb{Z}_+^l$,

$$\begin{aligned} E_{\mu_{n,s}\pi_{\vec{t}}^{-1}}\left[\prod_{i=1}^{l}\prod_{j=1}^{m_i} X_i(\phi_{ij})\right] &= E_{\mu_n}\left[X_s(1)\prod_{i=1}^{l}\prod_{j=1}^{m_i} X_{t_i}(\phi_{ij})\right] \\ &\to E_{\mathbb{N}_0}\left[X_s(1)\prod_{i=1}^{l}\prod_{j=1}^{m_i} X_{t_i}(\phi_{ij})\right] \\ &= E_{\mathbb{N}_{0,s}\pi_{\vec{t}}^{-1}}\left[\prod_{i=1}^{l}\prod_{j=1}^{m_i} X_i(\phi_{ij})\right], \end{aligned} \tag{57}$$

where even in the $l = 0$ case, the presence of the factor $X_s(1)$ ensures that the convergence in (57) follows from (8).

By Lemma 4.1 we have that

$$E_{\mathbb{N}_{0,s}\pi_{\vec{t}}^{-1}}\left[e^{\sum_{i=1}^{m}\theta_i X_i(1)}\right] < \infty, \tag{58}$$

for $\theta_i > 0$ sufficiently small depending on $\vec{t}$ and s. In view of (56), (57) and (58) we may apply Theorem 2.5 to the measures $\mu_{n,s}, \mathbb{N}_{0,s}$ to get

$$\mu_{n,s}\pi_{\vec{t}}^{-1} \stackrel{w}{\Longrightarrow} \mathbb{N}_{0,s}\pi_{\vec{t}}^{-1}. \tag{59}$$

Thus (10) holds for every bounded continuous F. The extension to bounded, Borel measurable F satisfying $\mathbb{N}_{0,s}\pi_{\vec{t}}^{-1}(\mathcal{D}_F) = 0$ is standard. For F as in the theorem we may assume that $F \geq 0$. The extension to F dominated by a multinomial Q is obtained by an easy uniform integrability argument since $\lim E_{\mu_{n,s}}[Q(\vec{X}_{\vec{t}})] = E_{\mathbb{N}_{0,s}}[Q(\vec{X}_{\vec{t}})]$.

To prove the second claim we define

$$G \equiv \begin{cases} 0 & , \text{ if } X_s(1) = 0 \\ \frac{I_{\{X_s(1)>\lambda\}}}{X_s(1)} & , \text{ otherwise.} \end{cases} \tag{60}$$

Then G is continuous except when $X_s(1) = \lambda$, and is bounded above by $\frac{1}{\lambda}$. Thus, Lemma 4.1 and (10) show that for F bounded by a multinomial,

$$E_{\mu_n h_{\vec{t}}^{-1}}\left[X_s(1)GF\right] \to E_{\mu_n h_{\vec{t}}^{-1}}\left[X_s(1)GF\right],$$

that is,

$$E_{\mu_n h_{\vec{t}}^{-1}}\left[I_{\{X_s(1)>\lambda\}}F\right] \to E_{\mu_n h_{\vec{t}}^{-1}}\left[I_{\{X_s(1)>\lambda\}}F\right]. \tag{61}$$

□

4.2. Proof of Proposition 2.4.

We apply Theorem 2.5 to the finite measures μ_n^ϵ and $\mathbb{N}_0^\epsilon$ defined by

$$\mu_n^\epsilon(\bullet) = \mu_n(\bullet, S > \epsilon), \quad \mathbb{N}_0^\epsilon(\bullet) = \mathbb{N}_0(\bullet, S > \epsilon). \tag{62}$$

Fix $l \in \mathbb{Z}_+$ and $\vec{t} \in [0,\infty)^l$. By Lemma 4.1, for $\delta(\vec{t},\epsilon) > 0$ sufficiently small and for $\theta_i < \delta$,

$$E_{\mathbb{N}_0^\epsilon}\left[e^{\sum_{i=1}^l \theta_i X_{t_i}(\mathbb{R}^d)}\right] < \infty, \tag{63}$$

so that the first condition of Theorem 2.5 is satisfied. The second condition is trivially true if any $t_i = 0$ so we assume that $t_i > 0$ for each i.

Let $\eta > 0$ and write $F(\vec{X}_{\vec{t}}(\vec{\phi})) \equiv \prod_{i=1}^l \prod_{j=1}^{m_i} X_{t_i}(\phi_{ij})$. By hypothesis we have $E_{\mu_n}\left[F^2(\vec{X}_{\vec{t}}(\vec{1}))\right] \to E_{\mathbb{N}_0}\left[F^2(\vec{X}_{\vec{t}}(\vec{1}))\right] < C$ and so there exists $C_0(\vec{t},\vec{m})$ such that

$$\sup_n E_{\mu_n}\left[F^2(\vec{X}_{\vec{t}}(\vec{1}))\right]^{\frac{1}{2}} \le C_0. \tag{64}$$

Choose $\lambda_0 = \lambda_0(\eta, C_0)$ sufficiently small so that

$$\mathbb{N}_0\left(X_\epsilon(1) \in (0,\lambda_0]\right) < \left(\frac{\eta}{6C_0}\right)^2. \tag{65}$$

By part 2 of Proposition 2.3 with $F \equiv 1$ we have

$$\mu_n\left(X_\epsilon(1) > \lambda_0\right) \to \mathbb{N}_0\left(X_\epsilon(1) > \lambda_0\right). \tag{66}$$

Combining this with (12) we have $\mu_n\left(X_\epsilon(1) \in (0,\lambda_0]\right) \to \mathbb{N}_0\left(X_\epsilon(1) \in (0,\lambda_0]\right)$. It follows from (65) that there exists n_0 such that for all $n \ge n_0$,

$$\mu_n\left(X_\epsilon(1) \in (0,\lambda_0]\right) < \left(\frac{\eta}{3C_0}\right)^2. \tag{67}$$

Using $I_{\{S>\epsilon\}} = I_{\{X_\epsilon(1)>\lambda_0\}} + I_{\{X_\epsilon(1)\in(0,\lambda_0]\}}$ we have

$$\begin{aligned}
&\left|E_{\mu_n}\left[F(\vec{X}_{\vec{t}}(\vec{\phi}))I_{\{S>\epsilon\}}\right] - E_{\mathbb{N}_0}\left[F(\vec{X}_{\vec{t}}(\vec{\phi}))I_{\{S>\epsilon\}}\right]\right| \\
\le &\left|E_{\mu_n}\left[F(\vec{X}_{\vec{t}}(\vec{\phi}))I_{\{X_\epsilon(1)>\lambda_0\}}\right] - E_{\mathbb{N}_0}\left[F(\vec{X}_{\vec{t}}(\vec{\phi}))I_{\{X_\epsilon(1)>\lambda_0\}}\right]\right| \\
&+ \left|E_{\mu_n}\left[F(\vec{X}_{\vec{t}}(\vec{\phi}))I_{\{X_\epsilon(1)\in(0,\lambda_0]\}}\right]\right| + \left|E_{\mathbb{N}_0}\left[F(\vec{X}_{\vec{t}}(\vec{\phi}))I_{\{X_\epsilon(1)\in(0,\lambda_0]\}}\right]\right|.
\end{aligned} \tag{68}$$

We bound the right-hand side of (68) as follows. By part 2 of Proposition 2.3, the first absolute value is less than $\eta/3$ for n sufficiently large. On the second term we use Cauchy-Schwarz to get

$$E_{\mu_n}\left[|F(\vec{X}_{\vec{t}})|I_{\{X_\epsilon(1)\in(0,\lambda_0]\}}\right] \leq E_{\mu_n}\left[F^2(\vec{X}_{\vec{t}}(\vec{1}))\right]^{\frac{1}{2}} E_{\mu_n}\left[I_{\{X_\epsilon(1)\in(0,\lambda_0]\}}\right]^{\frac{1}{2}} \leq \frac{C_0\eta}{3C_0}. \tag{69}$$

The third term is handled similarly. Thus for n sufficiently large,

$$\left|E_{\mu_n^\epsilon}\left[F(\vec{X}_{\vec{t}}(\vec{\phi}))\right] - E_{\mathbb{N}_0^\epsilon}\left[F(\vec{X}_{\vec{t}}(\vec{\phi}))I_{\{S>\epsilon\}}\right]\right| < \eta, \tag{70}$$

which proves the second condition of Theorem 2.5 for $\{\mu_n^\epsilon\}_{n\geq 0}$ and $\mathbb{N}_0^\epsilon$. The result follows by Theorem 2.5. □

Acknowledgements

The work of Holmes was supported in part by a UGF from the University of British Columbia and by the Netherlands Organization for Scientific Research (NWO). The work of Perkins was supported in part by an NSERC research grant. We thank Gordon Slade and Remco van der Hofstad for providing the motivation for studying this problem and for many helpful suggestions.

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