Senile reinforced random walks

Holmes, M. and Sakai, A.

August 21, 2006

Abstract

We consider random walks with transition probabilities depending only on the number of consecutive traversals n of the edge most recently traversed. Such walks may get stuck on a single edge, or have every vertex recurrent or every vertex transient, depending on the reinforcement function $f(n)$ that characterizes the model. We prove recurrence/transience results when the walk does not get stuck on a single edge. We also show that the diffusion constant need not be monotone in the reinforcement.

1 Introduction

Random walks with edge reinforcement were introduced by Coppersmith and Diaconis [1]. Many problems that are simple to state remain unsolved for edge-reinforced random walks on $\mathbb{Z}^d$, however there are also many interesting existing results in the general theory of reinforced random walks. There are strong results for example in 1 dimension [2], on finite graphs [6] and on trees [3]. There is also an interesting connection with random walk in a random environment (see for example [11]). The most recent survey that we know of is [12].

A nearest-neighbour *senile reinforced random walk* on $\mathbb{Z}^d$, $\{S_n\}_{n\geq 0}$ is defined in terms of a function $f:\mathbb{N}\mapsto[-1,\infty)$ such that if the current undirected edge (S_{n-1},S_n) has been traversed m consecutive times in the immediate past, then the probability of traversing that edge in the next step is $\frac{1+f(m)}{2d+f(m)}$ with the rest of the possible $2d-1$ choices being equally likely. The reinforcement of the current edge continues until a new edge is traversed, at which point the reinforcement of the previous edge is forgotten (i.e. the weight of that edge returns to its initial value).

At the completion of our work we were made aware of two papers [5, 10] in which a different model with a similar flavour was studied. Their model has the property that the walk prefers (as defined by the reinforcement function) to continue in the same direction, rather than traverse the same edge, and as such we might call their model *senile persistent random walk*.

Let $\mathcal{S}$ be a finite subset of $\mathbb{Z}^d$ such that $o\notin\mathcal{S}$, $\{y\in\mathbb{Z}^d:|y|=1\}\subseteq\mathcal{S}$ and $x\in\mathcal{S}\Rightarrow -x\in\mathcal{S}$. We say that there is an edge between $x\in\mathbb{Z}^d$ and $y\in\mathbb{Z}^d$ and write $x\sim y$ if $x-y\in\mathcal{S}$. Formally, a senile random walk ($SeRW_f$) is a sequence $\{S_n\}_{n\geq 0}$ of $\mathbb{Z}^d$-valued random variables on a probability space $(\Omega,\mathcal{F},\mathbb{P}_f)$ (with corresponding filtration $\{\mathcal{F}_n=\sigma(S_0,\ldots,S_n)\}_{n\geq 0}$) defined by:

- The walk begins at the origin of $\mathbb{Z}^d$, i.e. $S_0=o$, $\mathbb{P}_f$-almost surely,
- $\mathbb{P}_f(S_1=x)=D(x)$, where $D(x)=\frac{1}{|\mathcal{S}|}\mathbb{1}_{\{x\in\mathcal{S}\}}$.

- For $n \in \mathbb{N}$, $e_n = (S_{n-1}, S_n)$ is a random *undirected* edge ($\mathcal{F}_n$-measurable) and

$$m_n = \max\{k \geq 1 : e_{n-l+1} = e_n \text{ for all } 1 \leq l \leq k\} \tag{1.1}$$

is an $\mathbb{N}$-valued ($\mathcal{F}_n$-measurable) random variable.

- For $n \in \mathbb{N}$ and $x \in \mathcal{S}$,

$$\mathbb{P}_f(S_{n+1} = S_n + x) = \begin{cases} \dfrac{1 + f(m_n)}{|\mathcal{S}| + f(m_n)}, & \text{if } (S_n, S_n + x) = e_n, \\ \dfrac{1}{|\mathcal{S}| + f(m_n)}, & \text{if } (S_n, S_n + x) \neq e_n. \end{cases} \tag{1.2}$$

Examples of D satisfying the above definition include the usual nearest-neighbour model, where $\mathcal{S}$ is the set of unit vectors in $\mathbb{Z}^d$ and the spread-out model where $\mathcal{S}$ is the closed ball in $\mathbb{Z}^d$ of radius L for some $L \geq 1$. Many of our results remain valid for more general classes of D, however we at least require that the distribution of the number of times in succession that the walk traverses the first edge traversed is the same for each edge incident to the origin. This is ensured by the uniformity and symmetry conditions. The additional assumptions on $\mathcal{S}$ enable us to avoid reducible cases such as where some vertices or edges of $\mathbb{Z}^d$ may not be reachable by the walk. For notational convenience we often write $\mathbb{P}$ for $\mathbb{P}_f$ when there is no ambiguity.

If $f \equiv 0$ then the model is nothing but random walk on $\mathbb{Z}^d$ with transition kernel given by D. If in addition $\mathcal{S}$ is the set of unit vectors in $\mathbb{Z}^d$ we arrive at nearest-neighbour simple random walk. Let N_x denote the number of times the walk S_n visits x.

If $\mathbb{P}(N_x = \infty) = 1$ for all x we say that the walk is recurrent(I). If $\mathbb{P}(N_x = \infty) = 0$ for all x we say that the walk is transient(I). If $\mathbb{E}[N_x] = \infty$ for every x then we say that the walk is recurrent(II), and if $\mathbb{E}[N_x] < \infty$ for every x then we say that the walk is transient(II). For simple random walk (equivalently senile random walk with $f \equiv 0$) the two characterisations of recurrence/transience are equivalent and it is standard that simple random walk is recurrent for $d \leq 2$ and transient otherwise. For senile reinforced random walks the two notions of recurrence need not be the same.

Let $\tau = \sup\{n \geq 1 : S_m \in \{0, S_1\} \ \forall \ m \leq n\}$ denote the (random) number of times that the walk traverses the first edge before leaving that edge for the first time. Note that τ is not a stopping time (however $\tau + 1 = \inf\{n \geq 2 : S_n \neq S_{n-2}\}$ is a stopping time). Intuitively if the overall effect of the function f is one of *positive reinforcement* but such that the probability it gets stuck on the first edge it traverses is 0, then the walk should in some sense be more recurrent than simple random walk. Similar intuition suggests that if the overall effect of the function f is one of *negative reinforcement*, the senile random walk should in some sense be more transient than simple random walk.

By definition of τ we have for all $n \geq 1$,

$$\mathbb{P}(\tau = n) = \prod_{l=1}^{n-1} \frac{1 + f(l)}{|\mathcal{S}| + f(l)} \frac{|\mathcal{S}| - 1}{|\mathcal{S}| + f(n)}, \qquad \mathbb{P}(\tau \geq n) = \prod_{l=1}^{n-1} \frac{1 + f(l)}{|\mathcal{S}| + f(l)}, \tag{1.3}$$

where an empty product is defined to be 1. Moreover the probability that the senile random walk gets stuck on the first edge it traverses without ever traversing another edge is

$$\mathbb{P}(\tau = \infty) = \prod_{l=1}^{\infty} \frac{1 + f(l)}{|\mathcal{S}| + f(l)}. \tag{1.4}$$

When $f(l) = -1$ for some l, the walk cannot traverse the same edge more than l times in succession (so does not get stuck), and the definition of the function on integer values greater than l is irrelevant. If $f(1) = -1$ then the walk never traverses the same edge on two consecutive steps, a model that is sometimes called *memory-2 self-avoiding walk*. In particular for the nearest-neighbour model when $d = 1$, there are only two possible paths for the walk, and the path is determined by the first step.

Obviously if $f \geq g$ then $\mathbb{P}_f(\tau \geq n) \geq \mathbb{P}_g(\tau \geq n)$, and similarly the probability of being stuck on an edge is monotone in the reinforcement function f.

2 Results

In this section we state the main results of this paper and briefly discuss some interesting open problems. As a first step towards recurrence/transience type results, the following proposition immediately implies that the senile random walk visits either 0,2, or all vertices infinitely often.

Proposition 2.1. *Let A_i be the event that the senile random walk traverses exactly i edges infinitely often and let $A_{\mathbb{Z}^d}$ be the event that every edge in the edge set of $\mathbb{Z}^d$ generated by $\mathcal{S}$ is traversed infinitely often. Then $\mathbb{P}_f(A_0) + \mathbb{P}_f(A_1) + \mathbb{P}_f(A_{\mathbb{Z}^d}) = 1$ and each is a 0-1 event. Furthermore, $\mathbb{P}_f(A_1) = 1$ if and only if $(1 + f(l))^{-1}$ is summable.*

The proof of Proposition 2.1 is easily adapted to show that for any edge-reinforced random walk (or any senile random walk) on $\mathbb{Z}^d$ such that the weight attached to any edge is bounded i.e. $\sup_m f(m) < \infty$, one must have that every site is recurrent almost surely or no site is recurrent almost surely. The last statement of Proposition 2.1 is consistent with the results of [7, 8, 13] for the edge reinforced random walk.

The following theorem is one of the two main results of this paper.

Theorem 2.2. *Let $SeRW_f$ denote senile random walk in $\mathbb{Z}^d$, with f satsfying $\mathbb{P}_f(\tau = \infty) = 0$. Excluding the degenerate case $|\mathcal{S}| = 2$, $f(1) = -1$, we have the following:*

(1) $SeRW_f$ is recurrent(I)/transient(I) if and only if $SeRW_0$ is recurrent(I)/transient(I).

(2) When $\mathbb{E}_f[\tau] < \infty$, $SeRW_f$ is recurrent(II)/transient(II) if and only if $SeRW_0$ is recurrent(II)/ transient(II).

(3) When $\mathbb{E}_f[\tau] = \infty$, $SeRW_f$ is recurrent(II).

Our proof of Theorem 2.2 is via a time change of the process and ultimately by comparison of the Green's function for $SeRW_f$ to that for $SeRW_0$. We will complete the proof in the beginning of Section 4.

The following Corollary is a simple consequence of Theorem 2.2 applied to senile linearly reinforced random walk.

Corollary 2.3. *The senile random walk with linear reinforcement of the form $f(m) = Cm$ is recurrent(I),(II) when $d = 1, 2$ and transient(I) when $d > 2$. It is transient(II) for $d > 2$ if and only if $C < |\mathcal{S}| - 1$.*

Definition 2.4. *The diffusion constant $v = v_f \geq 0$ is defined as*

$$v = \lim_{n\to\infty} \frac{1}{n}\mathbb{E}[|S_n|^2] = \lim_{n\to\infty} \frac{1}{n} \sum_{x\in\mathbb{Z}^d} |x|^2 \mathbb{P}(S_n = x). \tag{2.1}$$

Note that when $f \equiv 0$ (simple random walk), S_n is a sum of independent random variables with mean squared displacement $\sigma^2 = \sum_x |x|^2 D(x)$, and thus $v_0 = \sigma^2$ (=1 for the nearest-neighbour model).

The second main result of the paper is the following Theorem.

Theorem 2.5. *Suppose that $\mathbb{E}[\tau^{1+\epsilon}] < \infty$. Then*

$$v = \frac{\mathbb{P}(\tau \text{ odd})}{1 - \frac{2}{|\mathcal{S}|}\mathbb{P}(\tau \text{ odd})} \frac{\sigma^2}{\mathbb{E}[\tau]}. \tag{2.2}$$

In the degenerate case where $|\mathcal{S}| = 2$ and $f(1) = -1$ we have $|S_n|^2 = \sigma^2 n^2$, $\mathbb{P}_f$-almost surely, and (2.2) should be interpreted as $\infty = 1/0$. It is easy to show that for nearest-neighbour models and for any fixed reinforcement function f, if one has $\mathbb{E}[\tau^{1+\epsilon}] < \infty$ for some ϵ for sufficiently large d, then $v = v(d) \to 1$ as $d \to \infty$. This holds for example when $f(m) = Cm$ for any fixed $C > 0$.

Our proof of Theorem 2.5 is based on the formula for the Green's function, and a standard Tauberian theorem, whose application requires the $(1+\epsilon)$th moment of τ to be finite. We expect that (2.2) holds for all f by a time-change argument of similar flavour to what appears in Section 3.3. When $\mathbb{E}[\tau] = \infty$, the right-hand side of (2.2) is zero, which suggests that the walk is subdiffusive. When $\mathbb{P}(\tau = \infty) > 0$, $\mathbb{E}[|S_n|^2]$ is bounded uniformly in n.

The following corollary follows easily from Theorem 2.5 and implies that the diffusion constant is not monotone in the reinforcement function f.

Corollary 2.6. *For f such that $\mathbb{E}_f[\tau^{1+\epsilon}] < \infty$, the diffusion constant is a decreasing function of $x = f(j)$ for each odd j. However for each even j there exist f, g with $f(m) = g(m)$ for $m \neq j$ and $f(j) < g(j)$ but $v_g > v_f$.*

Indeed for each even j there are examples where f is strictly positive and increasing yet an increase in $f(j)$ results in a decrease of the relevant diffusion constant.

Interestingly, when $f(l) = l$, special hypergeometric functions become relevant and various well known properties of these functions enable a proof of the following proposition.

Proposition 2.7. *The diffusion constant v of the senile random walk with reinforcement $f(l) = l$ satisfies $0 < v < \sigma^2$ when $|\mathcal{S}| > 2$. For the 1-dimensional nearest-neighbour model,*

$$\lim_{n\to\infty} \frac{\log n}{n} \mathbb{E}[|S_n|^2] = \frac{1 - \log 2}{2\log 2 - 1}. \tag{2.3}$$

We expect that (2.3) holds with a different constant whenever $f(l) = (|\mathcal{S}| - 1)l$, and that some other scaling is appropriate when the reinforcement becomes stronger, depending on which moments of τ are finite. Note that when $f(l) = (|\mathcal{S}|-1)l$, it is easy to show that $\mathbb{E}[\tau(\log(\tau+1))^{-2}] < \infty$ but $\mathbb{E}[\tau(\log(\tau+1))^{-1}] = \infty$.

3 Proofs of qualitative results

In this section we prove Proposition 2.1 which in particular shows that when $\mathbb{P}(\tau = \infty) = 0$, *SeRW* is almost surely recurrent(I) or almost surely transient(I). Two further lemmas, introduced in this section will be used in the proof of Theorem 2.2. Lemma 3.1 shows that recurrence(I)/transience(I) and recurrence(II)/transience(II) are equivalent when $\mathbb{E}[\tau] < \infty$. Lemma 3.2 shows that provided $\mathbb{P}(\tau = \infty) = 0$, the quantity $\mathbb{E}[\tau]$ is irrelevant in determining recurrence(I)/transience(I) of a senile random walk.

Lemma 3.1. *If $\mathbb{E}[\tau] < \infty$ then $\mathbb{P}(N_o = \infty) = 1$ if and only if $\mathbb{E}[N_o] = \infty$.*

Lemma 3.2. *For every $f : \mathbb{N} \mapsto [-1, \infty)$ such that $\mathbb{P}_f(\tau = \infty) = 0$, there exists $g : \mathbb{N} \mapsto [-1, \infty)$ satisfying $\mathbb{E}_g[\tau] < \infty$ such that SeRW$_f$ is recurrent(I)/transient(I) if and only if SeRW$_g$ is recurrent(I)/transient(I).*

3.1 Proof of Proposition 2.1

The number of edges that the walk leaves before getting stuck is Geometric with parameter $\mathbb{P}(\tau = \infty)$, and thus $\mathbb{P}(A_1) \in \{0, 1\}$ and is equal to 1 if and only if $\mathbb{P}(\tau = \infty) > 0$.

If $|\mathcal{S}| = 2$ and $f(1) = -1$ then trivially $\mathbb{P}(A_0) = 1$.

Therefore we may assume that $\mathbb{P}(\tau < \infty) = 1$ and $|\mathcal{S}| + f(1) > 1$. Suppose that a fixed edge $(y, y') \in \mathbb{Z}^d$ is traversed infinitely often and fix $(x, x') \in \mathbb{Z}^d$, $x \neq y, y'$. Since $\mathcal{S}$ contains the unit vectors, there is a finite set of edges connecting x and y. Since the walk does not get stuck on any edge $\mathbb{P}$-almost surely, it leaves the edge (y, y') infinitely often and returns infinitely often, $\mathbb{P}$-almost surely. In particular, there are infinitely many times (not necessarily every time, e.g. consider the one dimensional nearest-neighbour case) at which the walk leaves (y, y') from (without loss of generality) y with probability at least $q > 0$ (depending on $|x - y|$, $|\mathcal{S}|$, $f(1)$) of traversing the edge (x, x') before returning. Note that in the nearest-neighbour case in 1 dimension this does hold for exactly one of y or y' (the one nearest x) using the fact that $f(1) > -1$. Each time the walk leaves the edge (y, y') at y, the event that the walk traverses (x, x') before the next traversal of (y, y') is independent of previous departures from (y, y') at y. Thus (x, x') is traversed infinitely often if (y, y') is. Since there are countably many edges we have that

$$\mathbb{P}(A_{\mathbb{Z}^d} | (y, y') \text{ i.o.}) = 1, \tag{3.1}$$

whenever $\mathbb{P}((y, y') \text{ i.o.}) > 0$.

Now the number of times that the walk leaves the first edge traversed is Geometric with parameter $p \in [0, 1]$, and therefore $\mathbb{P}((o, S_1) \text{ i.o.}) \in \{0, 1\}$. If $\mathbb{P}((o, S_1) \text{ i.o.}) = 1$ then by (3.1) we must have $\mathbb{P}(A_{\mathbb{Z}^d}) = 1$. Similarly if $\mathbb{P}((o, S_1) \text{ i.o.}) = 0$ then $\mathbb{P}(A_{\mathbb{Z}^d}) = 0$ and (3.1) implies that $\mathbb{P}((y, y') \text{ i.o.}) = 0$ for each (y, y') and therefore that $\mathbb{P}(A_0) = 1$.

For the last claim of the Proposition, if any $f(l) = -1$ then $\mathbb{P}(A_1) = 0$. Otherwise we may assume that $f > -1$. The product (1.4) converges to a *non-zero* constant if and only if

$$\sum_{l=1}^{\infty} \log\left(\frac{|\mathcal{S}| + f(l)}{1 + f(l)}\right) < \infty. \tag{3.2}$$

Now

$$\frac{|\mathcal{S}| - 1}{|\mathcal{S}| + f(i)} \leq \log\left(\frac{|\mathcal{S}| + f(l)}{1 + f(l)}\right) \leq \frac{|\mathcal{S}| - 1}{1 + f(i)}. \tag{3.3}$$

Since $f > -1$, the lower bound is summable if and only if the upper bound is summable, so that (3.2) is finite if and only if $1/(1 + f(i))$ is summable. □

3.2 Proof of Lemma 3.1

Fix f such that $\mathbb{E}[\tau] < \infty$ (whence $\mathbb{P}(\tau < \infty) = 1$) and recall that N_o is the number of times that the walk visits the origin. From Proposition 2.1 we have that $\mathbb{P}(N_o = \infty) \in \{0, 1\}$.

If $\mathbb{P}(N_o = \infty) = 1$ then $\mathbb{E}[N_o] = \infty$ holds trivially. Now suppose that $\mathbb{E}[N_o] = \infty$, and let $\mathcal{T}_1 = \inf\{n > 0 : S_n \neq o, S_{n-1} \neq o\}$ denote the first time that the walk traverses an edge not incident to the origin. Let τ_i ($\overset{\text{i.i.d}}{\sim} \tau$) denote the random number of consecutive traversals of the ith edge traversed. Since

$$\mathcal{T}_1 - 1 \equiv \sum_{n=1}^{\infty} \sum_{i=1}^{n} \tau_i \prod_{j=1}^{n-1} \mathbb{1}_{\{\tau_j \text{ even}\}} \mathbb{1}_{\{\tau_n \text{ odd}\}} \leq \tau_1 + \sum_{n=2}^{\infty} \sum_{i=1}^{n} \tau_i \prod_{\substack{j=1 \\ (j \neq i)}}^{n-1} \mathbb{1}_{\{\tau_j \text{ even}\}}, \tag{3.4}$$

we have

$$\mathbb{E}[\mathcal{T}_1] - 1 \leq \mathbb{E}[\tau] + \sum_{n=2}^{\infty} \sum_{i=1}^{n} \mathbb{E}[\tau_i] \prod_{\substack{j=1 \\ (j \neq i)}}^{n-1} \mathbb{P}(\tau_j \text{ even}) = \mathbb{E}[\tau] + \mathbb{E}[\tau] \sum_{n=2}^{\infty} n\, \mathbb{P}(\tau \text{ even})^{n-2}, \tag{3.5}$$

which is finite since $\mathbb{E}[\tau] < \infty$ and $f(1) < \infty$ (so that $\mathbb{P}(\tau \text{ even}) < 1$). In particular $\mathcal{T}_1$ is almost surely finite.

Let $\sigma_0 = 0$, and $\sigma_1 = \inf\{n > \mathcal{T}_1 : S_n = o\}$ denote the first time after $\mathcal{T}_1$ that the walk returns to the origin. We then define for $i \geq 2$,

$$\mathcal{T}_i = \inf\{n > \sigma_{i-1} : S_n \neq o,\ S_{n-1} \neq o\}, \qquad\qquad \sigma_i = \inf\{n > \mathcal{T}_i : S_n = o\}. \tag{3.6}$$

As explained above, $\mathcal{T}_1$ is almost surely finite. This is also true of $\mathcal{T}_i - \sigma_{i-1}$, conditionally on $\{\sigma_{i-1} < \infty\}$. In the degenerate case $f(1) = -1$ and $|\mathcal{S}| = 2$ the claim of the Lemma holds trivially since $N_0 = 1$, $\mathbb{P}_f$-almost surely. Otherwise $\mathbb{P}(\sigma_i < \infty) > 0$ for every i, and an easy exercise in conditioning shows that $\mathbb{P}(\sigma_i < \infty) = \mathbb{P}(\sigma_1 - \mathcal{T}_1 < \infty)^i$.

With probability one,

$$N_o \leq \mathcal{T}_1 + \sum_{i=2}^{\infty} (\mathcal{T}_i - \sigma_{i-1}) \mathbb{1}_{\{\sigma_{i-1} < \infty\}} \tag{3.7}$$

Therefore

$$\begin{aligned} \mathbb{E}[N_o] \leq \mathbb{E}[\mathcal{T}_1] + \sum_{i=2}^{\infty} \mathbb{E}\big[(\mathcal{T}_i - \sigma_{i-1}) \mathbb{1}_{\{\sigma_{i-1} < \infty\}}\big] &= \mathbb{E}[\mathcal{T}_1] + \sum_{i=2}^{\infty} \mathbb{E}\big[\mathcal{T}_i - \sigma_{i-1} \big| \{\sigma_{i-1} < \infty\}\big]\, \mathbb{P}(\sigma_{i-1} < \infty) \\ &= \mathbb{E}[\mathcal{T}_1] + \sum_{i=2}^{\infty} \mathbb{E}\big[\mathcal{T}_2 - \sigma_1 \big| \{\sigma_1 < \infty\}\big]\, \mathbb{P}(\sigma_1 - \mathcal{T}_1 < \infty)^{i-1}. \end{aligned} \tag{3.8}$$

It follows as in (3.5) (with minor modifications) that $\mathbb{E}[\mathcal{T}_2 - \sigma_1 | \{\sigma_1 < \infty\}] < \infty$. The left-hand side of (3.8) is infinite by assumption, which implies that $\mathbb{P}(\sigma_1 - \mathcal{T}_1 < \infty) = 1$. Since this is true if and only if $\mathbb{P}_f(N_o = \infty) = 1$, we have the result. □

3.3 Proof of Lemma 3.2

First observe that, since f is such that $\mathbb{P}_f(\tau < \infty) = 1$, the sequence of edges $\{e_n\}_{n \geq 1} \equiv \{(S_{n-1}, S_n)\}_{n \geq 1}$ has the property that, almost surely, for every n_0 there exists $n_1 \geq n_0$ such that $e_{n_1+1} \neq e_{n_1}$. Define a subsequence of edges $\{e_{n_k}\}_{k\geq 0} \subseteq \{e_n\}_{n\geq 0}$ by the following algorithm: Whenever an edge e appears exactly $2m$ (resp., $2m-1$) times in succession in $\{e_n\}_{n\geq 1}$, for any $m \geq 1$, only the first 2 (resp., 1) successive occurrences of this edge are listed in the subsequence $\{e_{n_k}\}_{k\geq 1}$. This sequence is almost surely well defined by the previous observation, and depends only on the parity of the number of consecutive traversals of each edge by the walk $\{S_n\}$.

The sequence $\{e_{n_k}\}_{k\geq 1}$ defines a random walk $\{R_k^f\}_{k\geq 0}$ by setting $R_0^f = 0$ and $(R_{k-1}^f, R_k^f) = e_{n_k}$. Since $\{S_n\}$ spends an (i.i.d.) almost surely finite amount of time traversing each edge before moving on, the walk $\{R_k^f\}$ visits a vertex infinitely often if and only if $\{S_n\}$ does, and therefore by Proposition 2.1, $\{R_k^f\}$ is recurrent(I)/transient(I) if and only if $\{S_n\}$ is (0-1 events). Now by construction, and the fact that $\mathbb{P}_f(\tau < \infty) = 1$, the law of the walk $\{R_k^f\}$ is completely determined by $\rho = \mathbb{P}_f(\tau \text{ odd}) > 0$.

If $\rho = 1$, which is possible only when $f(1) = -1$, then $\mathbb{E}_f[\tau] = 1$ so that $g \equiv f$ satisfies the statement of the Lemma. Otherwise $\rho < 1$, and let $g(1) = \frac{1}{\rho}(|\mathcal{S}|-1) - |\mathcal{S}| > -1$ and $g(2) = -1$. Then $\mathbb{P}_g(\tau \text{ odd}) = \mathbb{P}_g(\tau = 1) = \frac{|\mathcal{S}|-1}{|\mathcal{S}|+g(1)} = \rho$ and $\mathbb{E}_g[\tau] \leq 2 < \infty$. The walk $\{R_k^g\}$ is recurrent(I)/transient(I) if and only if $SeRW_g$ is. However $\{R_k^g\}$ has the same law as $\{R_k^f\}$ since it depends only on ρ, and hence $\{R_k^g\}$ is recurrent(I)/transient(I) if and only if $SeRW_f$ is. $\square$

3.4 Proof of Corollary 2.3

From Theorem 2.2 we know that the senile random walk with $f(m) = Cm$ is recurrent(I) (and therefore also recurrent (II)) in dimensions $d = 1, 2$ and transient(I) in dimensions $d > 2$. Moreover it is transient(II) for $d > 2$ unless $\mathbb{E}[\tau] = \infty$ in which case it is recurrent(II). Since $\mathbb{E}[\tau]$ is monotone increasing in the reinforcement, to complete the proof it is enough to show that $\mathbb{E}[\tau] < \infty$ when $C < |\mathcal{S}| - 1$ and $\mathbb{E}[\tau] = \infty$ when $C = |\mathcal{S}| - 1$.

For the latter, observe that when $C = |\mathcal{S}| - 1$,

$$\begin{aligned}\mathbb{E}[\tau] &= \sum_{n=1}^{\infty} n \left(\prod_{i=1}^{n-1} \frac{1 + (|\mathcal{S}|-1)i}{|\mathcal{S}| + (|\mathcal{S}|-1)i} \right) \frac{|\mathcal{S}|-1}{|\mathcal{S}| + (|\mathcal{S}|-1)n} \\ &= \sum_{n=1}^{\infty} \frac{|\mathcal{S}|n}{|\mathcal{S}| + (|\mathcal{S}|-1)(n-1)} \, \frac{|\mathcal{S}|-1}{|\mathcal{S}| + (|\mathcal{S}|-1)n} = \infty, \end{aligned} \tag{3.9}$$

where we have used the fact that $|\mathcal{S}| + (|\mathcal{S}|-1)i = 1 + (|\mathcal{S}|-1)(i+1)$ to cancel terms in the numerator and denominator of the product to obtain the second equality.

When $(|\mathcal{S}|-1)/C = 1 + 2\alpha > 1$, observe that

$$\begin{aligned}\frac{\mathbb{E}[\tau^{1+\alpha}]}{1+2\alpha} &= \sum_{n=1}^{\infty} \frac{n^{1+\alpha}}{1+2\alpha} \left(\prod_{i=1}^{n-1} \frac{1+Ci}{|\mathcal{S}|+Ci} \right) \frac{|\mathcal{S}|-1}{|\mathcal{S}|+Cn} \leq \sum_{n=1}^{\infty} n^{\alpha} \prod_{i=1}^{n-1} \left(1 - \frac{|\mathcal{S}|-1}{|\mathcal{S}|+Ci} \right) \\ &\leq \sum_{n=1}^{\infty} n^{\alpha} \exp\left(- \sum_{i=1}^{n-1} \frac{1+2\alpha}{i + |\mathcal{S}|/C} \right) \leq \sum_{n=1}^{\infty} n^{\alpha} \left(\frac{1+|\mathcal{S}|/C}{n+|\mathcal{S}|/C} \right)^{1+2\alpha} < \infty, \end{aligned} \tag{3.10}$$

where we compared the sum in the exponential with an integral. Thus $\mathbb{E}[\tau]$ is finite as soon as $C < |\mathcal{S}| - 1$. $\square$

4 Generating function analysis: the proof of Theorem 2.2

For $z \in [0,1]$ we define

$$G_z(x) = \sum_{n=0}^{\infty} z^n \, \mathbb{P}(S_n = x). \tag{4.1}$$

This is the Green's function for senile random walk, and is obviously convergent for $z < 1$. Note that $G_1(x) = \mathbb{E}[N_x]$, so that the behaviour of G_z near $z = 1$ has implications for the recurrence and transience properties of the walk. For an absolutely summable function $F : \mathbb{Z}^d \mapsto \mathbb{R}$ and $k \in [-\pi, \pi]^d$ we write

$$\hat{F}(k) = \sum_{x \in \mathbb{Z}^d} e^{ik\cdot x} F(x) \tag{4.2}$$

for the Fourier transform of F. Note in particular that for any $|z| < 1$, G_z is absolutely summable and $\hat{G}_z(0) = (1-z)^{-1}$. Our analysis of the generating function will essentially involve the parity of τ.

In this section we analyse the Green's function and prove Theorem 2.2. We first expand $G_z(x)$ in terms of two other quantities: $u_z(x)$ and $v_z(x)$ (Section 4.1). We use inclusion-exclusion on those quantities (Section 4.2). After taking the Fourier transform we are left with three equations in three unknowns (i.e., $\hat{G}_z(k)$, $\hat{u}_z(k)$ and $\hat{v}_z(k)$) and that we can solve for $\hat{G}_z(k)$ (Section 4.3), the result of which appears in Proposition 4.1. From this formula together with Lemmas 3.1–3.2 we easily obtain Theorem 2.2.

Before stating Proposition 4.1, we introduce a number of quantities. For any $z \in [0,1]$ we define

$$\begin{cases} a_z = \displaystyle\sum_{n=2}^{\infty} z^n \, \mathbb{P}(\tau \geq n) \, \mathbb{1}_{\{n \text{ even}\}}, \\ b_z = \displaystyle\sum_{n=2}^{\infty} z^n \, \mathbb{P}(\tau \geq n) \, \mathbb{1}_{\{n \text{ odd}\}}, \end{cases} \qquad \begin{cases} p_z = \displaystyle\sum_{n=1}^{\infty} \frac{z^n \, \mathbb{P}(\tau = n) \, \mathbb{1}_{\{n \text{ even}\}}}{|\mathcal{S}| - 1}, \\ q_z = \displaystyle\sum_{n=1}^{\infty} \frac{z^n \, \mathbb{P}(\tau = n) \, \mathbb{1}_{\{n \text{ odd}\}}}{|\mathcal{S}| - 1}. \end{cases} \tag{4.3}$$

The quantities a_z and b_z converge for $|z| < 1$ trivially, and p_z and q_z converge for $|z| \leq 1$. It is easy to show that

$$z + a_z + b_z = \mathbb{E}\left[\frac{z(1 - z^{\tau})}{1 - z} \, \mathbb{1}_{\{\tau < \infty\}}\right], \qquad \begin{cases} b_z \leq z a_z, \\ a_z \leq z^2 + z b_z, \end{cases} \tag{4.4}$$

and that $a_1 = \mathbb{E}[\lfloor \frac{\tau}{2} \rfloor]$, $b_1 = \mathbb{E}[\lfloor \frac{\tau - 1}{2} \rfloor]$ and

$$1 + a_1 + b_1 = \mathbb{E}[\tau]. \tag{4.5}$$

Moreover, by definition we have $p_1 = \frac{1}{|\mathcal{S}|-1} \mathbb{P}(\tau \text{ even})$, $q_1 = \frac{1}{|\mathcal{S}|-1} \mathbb{P}(\tau \text{ odd})$ and

$$(|\mathcal{S}| - 1)(p_1 + q_1) = \mathbb{P}(\tau < \infty). \tag{4.6}$$

Next we define

$$U_z = 1 + p_z - (|\mathcal{S}| - 1)(p_z(1 + p_z) - q_z^2), \qquad \begin{cases} X_z = a_z(1 + p_z) - (z + b_z) q_z + U_z, \\ Y_z = (z + b_z)(1 + p_z) - a_z q_z - |\mathcal{S}| q_z. \end{cases} \tag{4.7}$$

Note that all of these quantities converge at $z = 1$ if $\mathbb{E}[\tau] < \infty$, and that U_1 converges for any f. Moreover,

$$U_z - |\mathcal{S}|q_z = (1 + p_z - q_z)\Big(1 - (|\mathcal{S}| - 1)(p_z + q_z)\Big) \geq 0, \quad \text{with equality iff } z = 1,\ \mathbb{P}(\tau < \infty) = 1, \tag{4.8}$$

and

$$X_z + Y_z = (1 + p_z - q_z)(z + a_z + b_z) + U_z - |\mathcal{S}|q_z \geq 0, \quad \text{with equality iff } z = 1,\ |\mathcal{S}| = 2,\ f(1) = -1. \tag{4.9}$$

The importance of the quantities discussed thus far in this section is given by the form of the generating function presented in the following Proposition.

Proposition 4.1. *For $z < 1$,*

$$\hat{G}_z(k) = \frac{X_z + Y_z\hat{D}(k)}{U_z - |\mathcal{S}|q_z\hat{D}(k)}. \tag{4.10}$$

When $f \equiv 0$ we easily get that $zU_z = |\mathcal{S}|q_z$, $X_z = U_z$ and $Y_z = 0$ which yields the standard result for $\hat{G}_z(k)$ for simple random walk. It follows from (4.10) and the definition of $\hat{G}_z(0)$ that

$$\frac{X_z + Y_z}{U_z - |\mathcal{S}|q_z} = \frac{1}{1 - z} \qquad (z < 1). \tag{4.11}$$

Proof of Theorem 2.2 using Lemmas 3.1–3.2 and Proposition 4.1. Let f be such that $\mathbb{P}(\tau = \infty) = 0$, and that $f(1) \neq -1$ if $|\mathcal{S}| = 2$.

We first prove Theorem 2.2(2)–(3) using Proposition 4.1. For $z \in (0, 1)$, U_z is strictly positive, due to (4.8) and $q_z > 0$. By rearranging (4.10), it follows that for $z \in (0, 1)$

$$\hat{G}_z(k) = \frac{X_z}{U_z} + \frac{\big(\frac{X_z}{U_z} + \frac{Y_z}{|\mathcal{S}|q_z}\big)\frac{|\mathcal{S}|q_z}{U_z}\hat{D}(k)}{1 - \frac{|\mathcal{S}|q_z}{U_z}\hat{D}(k)} = \frac{X_z}{U_z} + \left(\frac{X_z}{U_z} + \frac{Y_z}{|\mathcal{S}|q_z}\right)\sum_{m=1}^{\infty}\left(\frac{|\mathcal{S}|q_z}{U_z}\right)^m \hat{D}(k)^m. \tag{4.12}$$

By Fourier inversion we obtain

$$G_z(x) = \frac{X_z}{U_z}\,\delta_{0,x} + \left(\frac{X_z}{U_z} + \frac{Y_z}{|\mathcal{S}|q_z}\right)\sum_{m=1}^{\infty}\left(\frac{|\mathcal{S}|q_z}{U_z}\right)^m D^{*m}(x), \tag{4.13}$$

where D^{*m} denotes the m-fold convolution of D, and $|\mathcal{S}|q_z/U_z$ tends to 1 as $z \to 1$ by (4.8).

If $\mathbb{E}[\tau] < \infty$, then a_1 and b_1 are both finite, and so are X_1 and Y_1. Therefore $G_1(x)$ is finite if and only if $\sum_{m=1}^{\infty} D^{*m}(x)$ is finite. This completes the proof of Theorem 2.2(2).

If $\mathbb{E}[\tau] = \infty$, then a_1 and b_1 are both $+\infty$. Moreover, by the inequalities in (4.4), a_z and b_z both diverge in the same manner. However, p_z and q_z are both bounded for all $z \leq 1$, and so is U_z. Therefore, X_z and Y_z both diverge to $+\infty$ as $z \uparrow 1$, and $G_1(x)$ is infinite. This completes the proof of Theorem 2.2(3).

It remains to prove Theorem 2.2(1). First we note that, by Lemma 3.2, there is a reinforcement function g with $\mathbb{E}_g[\tau] < \infty$ such that $SeRW_f$ and $SeRW_g$ are type-I equivalent. By Lemma 3.1, the two types of recurrence/transience are equivalent for $SeRW_g$. Then, by Theorem 2.2(2) proved above, $SeRW_g$ is recurrent(II)/transient(II) if and only if $SeRW_0$ is recurrent(II)/transient(II). The proof is completed by collecting these statements. $\square$

We prove Proposition 4.1 in the remainder of this section.

4.1 First stage of the expansion

In this subsection we explain the first stage of the expansion, in which we use the notation

$$\mathbb{P}^y(\cdots) = \mathbb{P}(\cdots \,|\, S_0 = y). \tag{4.14}$$

By definition, we have

$$\begin{aligned} G_z(x) &= \delta_{o,x} + zD(x) + \sum_{n\geq 2} z^n \Big(\mathbb{P}^o(S_n = x,\, \tau \geq n) + \mathbb{P}^o(S_n = x,\, \tau < n)\Big) \\ &= \delta_{o,x} + zD(x) + \sum_{n\geq 2} z^n\, \mathbb{P}^o(S_n = x,\, \tau \geq n) + \sum_{l\geq 1} \sum_{n\geq l+1} z^n\, \mathbb{P}^o(S_n = x,\, \tau = l). \end{aligned} \tag{4.15}$$

First we consider the third term on the right-hand side. If n is even, then we have

$$\mathbb{P}^o(S_n = x,\, \tau \geq n) = \mathbb{P}^o(\tau \geq n)\, \delta_{o,x}. \tag{4.16}$$

If n is odd, then since D is uniform over $\mathcal{S}$, we obtain

$$\mathbb{P}^o(S_n = x,\, \tau \geq n) = \sum_{y\sim o} \mathbb{P}^o(\tau \geq n,\, S_1 = y)\, \delta_{y,x} = \frac{\mathbb{P}^o(\tau \geq n)}{|\mathcal{S}|} \sum_{y\sim o} \delta_{y,x} = \mathbb{P}^o(\tau \geq n)\, D(x). \tag{4.17}$$

Therefore, we have

$$\sum_{n\geq 2} z^n\, \mathbb{P}^o(S_n = x,\, \tau \geq n) = a_z \delta_{o,x} + b_z D(x), \tag{4.18}$$

where a_z and b_z are given by (4.3).

Next we consider the last term on the right-hand side of (4.15). First we note that

$$\mathbb{P}^o(S_n = x,\, \tau = l) = \sum_{y\sim o} \mathbb{P}^o(S_n = x,\, \tau = l,\, S_1 = y). \tag{4.19}$$

If l is even, then the right-hand side is

$$\begin{aligned} &\sum_{\substack{y,u\sim o\\ u\neq y}} \mathbb{P}^o(S_n = x,\, S_{l+1} = u,\, \tau = l,\, S_l = o,\, S_1 = y) \\ &= \sum_{\substack{y,u\sim o\\ u\neq y}} \underbrace{\mathbb{P}^o(S_n = x,\, S_{l+1} = u \,|\, \tau = l,\, S_l = o,\, S_1 = y)}_{= \frac{|\mathcal{S}|}{|\mathcal{S}|-1}\, \mathbb{P}^o(S_{n-l} = x,\, S_1 = u)} \;\underbrace{\mathbb{P}^o(\tau = l,\, S_l = o,\, S_1 = y)}_{= \frac{1}{|\mathcal{S}|}\, \mathbb{P}^o(\tau = l)} \\ &= \frac{\mathbb{P}^o(\tau = l)}{|\mathcal{S}| - 1} \sum_{y\sim o} \mathbb{P}^o(S_{n-l} = x,\, S_1 \neq y). \end{aligned} \tag{4.20}$$

While if l is odd, then the right-hand side of (4.19) is

$$\begin{aligned}
&\sum_{y\sim o}\sum_{\substack{u\sim y\\ u\neq o}}\mathbb{P}^o(S_n=x,\,S_{l+1}=u,\,\tau=l,\,S_l=S_1=y)\\
&=\sum_{y\sim o}\sum_{\substack{u\sim y\\ u\neq o}}\underbrace{\mathbb{P}^o(S_n=x,\,S_{l+1}=u\,|\,\tau=l,\,S_l=S_1=y)}_{=\frac{|\mathcal{S}|}{|\mathcal{S}|-1}\,\mathbb{P}^y(S_{n-l}=x,\,S_1=u)}\ \underbrace{\mathbb{P}^o(\tau=l,\,S_l=S_1=y)}_{=\frac{1}{|\mathcal{S}|}\,\mathbb{P}^o(\tau=l)}\\
&=\frac{\mathbb{P}^o(\tau=l)}{|\mathcal{S}|-1}\sum_{y\sim o}\mathbb{P}^y(S_{n-l}=x,\,S_1\neq o).
\end{aligned}\tag{4.21}$$

Define

$$u_z(x)=\sum_{l\geq 1}z^l\sum_{y\sim o}\mathbb{P}^o(S_l=x,\,S_1\neq y),\qquad\qquad v_z(x)=\sum_{l\geq 1}z^l\sum_{y\sim o}\mathbb{P}^y(S_l=x,\,S_1\neq o).\tag{4.22}$$

Then, by the above computation, the last term on the right-hand side of (4.15) is rewritten as

$$\begin{aligned}
\sum_{l\geq 1}\sum_{n\geq l+1}z^n\mathbb{P}^o(S_n=x,\,\tau=l)&=\sum_{l\geq 1}z^l\frac{\mathbb{P}^o(\tau=l)}{|\mathcal{S}|-1}\Big(u_z(x)\,\mathbb{1}_{\{l\text{ even}\}}+v_z(x)\,\mathbb{1}_{\{l\text{ odd}\}}\Big)\\
&=p_zu_z(x)+q_zv_z(x),
\end{aligned}\tag{4.23}$$

where p_z and q_z are given by (4.3). Together with (4.15) and (4.18), we arrive at

$$G_z(x)=(1+a_z)\,\delta_{o,x}+(z+b_z)\,D(x)+p_zu_z(x)+q_zv_z(x).\tag{4.24}$$

4.2 Second stage of the expansion

In this subsection we derive equations for u_z and v_z in terms of G_z. By definition, it is easy to see that

$$\begin{aligned}
u_z(x)&=\sum_{l\geq 1}z^l\sum_{y\sim o}\Big(\mathbb{P}^o(S_l=x)-\mathbb{P}^o(S_l=x,\,S_1=y)\Big)\\
&=|\mathcal{S}|\big(G_z(x)-\delta_{o,x}\big)-\sum_{l\geq 1}z^l\sum_{y\sim o}\mathbb{P}^o(S_l=x,\,S_1=y)=(|\mathcal{S}|-1)\big(G_z(x)-\delta_{o,x}\big).
\end{aligned}\tag{4.25}$$

Similarly, we can write v_z as

$$\begin{aligned}
v_z(x)&=\sum_{l\geq 1}z^l\sum_{y\sim o}\Big(\mathbb{P}^y(S_l=x)-\mathbb{P}^y(S_l=x,\,S_1=o)\Big)\\
&=\sum_{y\sim o}\big(G_z(x-y)-\delta_{y,x}\big)-\bigg(z\delta_{o,x}+\sum_{l\geq 2}z^l\sum_{y\sim o}\mathbb{P}^y(S_l=x,\,S_1=o)\bigg),
\end{aligned}\tag{4.26}$$

where $\mathbb{P}^y(S_l=x,\,S_1=o)$ can be written as

$$\mathbb{P}^y(S_l=x,\,S_1=o)=\mathbb{P}^y(S_l=x,\,\tau\geq l,\,S_1=o)+\sum_{m=1}^{l-1}\mathbb{P}^y(S_l=x,\,\tau=m,\,S_1=o).\tag{4.27}$$

By the uniformity of D and translation invariance, the contribution from the first term to (4.26) equals

$$\sum_{l\ge2} z^l \sum_{y\sim o} \mathbb{P}^y(S_l=x,\,\tau\ge l,\,S_1=o) = \sum_{l\ge2} z^l \sum_{y\sim o} \underbrace{\mathbb{P}^y(\tau\ge l,\,S_1=o)}_{=\frac{1}{|\mathcal{S}|}\mathbb{P}^y(\tau\ge l)} \Big(\delta_{y,x}\,\mathbb{1}_{\{l\text{ even}\}}+\delta_{o,x}\,\mathbb{1}_{\{l\text{ odd}\}}\Big)$$

$$= \sum_{l\ge2} z^l \mathbb{P}^o(\tau\ge l)\,\frac{1}{|\mathcal{S}|}\sum_{y\sim o}\Big(\delta_{y,x}\,\mathbb{1}_{\{l\text{ even}\}}+\delta_{o,x}\,\mathbb{1}_{\{l\text{ odd}\}}\Big) = a_z D(x) + b_z\delta_{o,x}. \tag{4.28}$$

For the sum over m in (4.27), we follow a similar course to the proof of (4.23). If m is even, then by translation invariance we have

$$\begin{aligned}
&\mathbb{P}^y(S_l=x,\,\tau=m,\,S_1=o) = \mathbb{P}^y(S_l=x,\,\tau=m,\,S_m=y,\,S_1=o)\\
&=\sum_{\substack{u\sim y\\ u\ne o}} \underbrace{\mathbb{P}^y(S_l=x,\,S_{m+1}=u\,|\,\tau=m,\,S_m=y,\,S_1=o)}_{=\frac{|\mathcal{S}|}{|\mathcal{S}|-1}\mathbb{P}^y(S_{l-m}=x,\,S_1=u)}\ \underbrace{\mathbb{P}^y(\tau=m,\,S_m=y,\,S_1=o)}_{=\frac{1}{|\mathcal{S}|}\mathbb{P}^y(\tau=m)}\\
&=\frac{\mathbb{P}^o(\tau=m)}{|\mathcal{S}|-1}\,\mathbb{P}^y(S_{l-m}=x,\,S_1\ne o).
\end{aligned} \tag{4.29}$$

Similarly, if m is odd, then we have

$$\begin{aligned}
&\mathbb{P}^y(S_l=x,\,\tau=m,\,S_1=o) = \mathbb{P}^y(S_l=x,\,\tau=m,\,S_m=S_1=o)\\
&=\sum_{\substack{u\sim o\\ u\ne y}} \underbrace{\mathbb{P}^y(S_l=x,\,S_{m+1}=u\,|\,\tau=m,\,S_m=S_1=o)}_{=\frac{|\mathcal{S}|}{|\mathcal{S}|-1}\mathbb{P}^o(S_{l-m}=x,\,S_1=u)}\ \underbrace{\mathbb{P}^y(\tau=m,\,S_m=S_1=o)}_{=\frac{1}{|\mathcal{S}|}\mathbb{P}^y(\tau=m)}\\
&=\frac{\mathbb{P}^o(\tau=m)}{|\mathcal{S}|-1}\,\mathbb{P}^o(S_{l-m}=x,\,S_1\ne y).
\end{aligned} \tag{4.30}$$

Therefore, the contribution to (4.26) from the sum over m in (4.27) equals

$$\begin{aligned}
&\sum_{m\ge1} z^m\frac{\mathbb{P}^o(\tau=m)}{|\mathcal{S}|-1}\sum_{l\ge m+1} z^{l-m}\sum_{y\sim o}\Big(\mathbb{P}^y(S_{l-m}=x,\,S_1\ne o)\,\mathbb{1}_{\{m\text{ even}\}}+\mathbb{P}^o(S_{l-m}=x,\,S_1\ne y)\,\mathbb{1}_{\{m\text{ odd}\}}\Big)\\
&=p_z v_z(x)+q_z u_z(x).
\end{aligned} \tag{4.31}$$

Summarizing the above and using (4.25), we obtain

$$v_z(x)=\sum_{y\sim o}\big(G_z(x-y)-\delta_{y,x}\big)-z\delta_{o,x}-\big(a_zD(x)+b_z\delta_{o,x}\big)-\big(p_zv_z(x)+q_zu_z(x)\big). \tag{4.32}$$

4.3 Completion of the expansion

Now we solve (4.24), (4.25) and (4.32) in terms of G_z. Taking the Fourier transform of these expressions, we have

$$\hat G-1=a+(z+b)\hat D+p\hat u+q\hat v, \tag{4.33}$$

$$\hat u=(|\mathcal{S}|-1)(\hat G-1), \tag{4.34}$$

$$q\hat u+(1+p)\hat v=|\mathcal{S}|\hat D(\hat G-1)-(z+b+a\hat D), \tag{4.35}$$

where we have abbreviated z and k (e.g. $\hat{G}$ for $\hat{G}_z(k)$). Let

$$\mathcal{M} = \begin{pmatrix} 1 & 0 \\ q & 1+p \end{pmatrix}, \tag{4.36}$$

so that (4.34) and (4.35) are combined as

$$\mathcal{M} \begin{pmatrix} \hat{u} \\ \hat{v} \end{pmatrix} = (\hat{G} - 1) \begin{pmatrix} |\mathcal{S}| - 1 \\ |\mathcal{S}|\hat{D} \end{pmatrix} - \begin{pmatrix} 0 \\ z + b + a\hat{D} \end{pmatrix}. \tag{4.37}$$

Since $1 + p > 0$, the inverse $\mathcal{M}^{-1}$ exists and hence

$$\begin{pmatrix} \hat{u} \\ \hat{v} \end{pmatrix} = (\hat{G} - 1)\mathcal{M}^{-1} \begin{pmatrix} |\mathcal{S}| - 1 \\ |\mathcal{S}|\hat{D} \end{pmatrix} - \mathcal{M}^{-1} \begin{pmatrix} 0 \\ z + b + a\hat{D} \end{pmatrix}. \tag{4.38}$$

Substituting this to (4.33), we obtain

$$\begin{aligned}
\hat{G} - 1 &= a + (z+b)\hat{D} + \begin{pmatrix} p \\ q \end{pmatrix} \cdot \begin{pmatrix} \hat{u} \\ \hat{v} \end{pmatrix} \\
&= a + (z+b)\hat{D} + (\hat{G} - 1) \begin{pmatrix} p \\ q \end{pmatrix} \cdot \mathcal{M}^{-1} \begin{pmatrix} |\mathcal{S}| - 1 \\ |\mathcal{S}|\hat{D} \end{pmatrix} - \begin{pmatrix} p \\ q \end{pmatrix} \cdot \mathcal{M}^{-1} \begin{pmatrix} 0 \\ z + b + a\hat{D} \end{pmatrix} \\
&= a + (z+b)\hat{D} + (\hat{G} - 1)\bigg((|\mathcal{S}| - 1)p - \frac{|\mathcal{S}| - 1}{1+p} q^2 + \frac{|\mathcal{S}|q}{1+p}\hat{D} \bigg) - \bigg(\frac{z+b}{1+p} q + \frac{aq}{1+p}\hat{D} \bigg),
\end{aligned} \tag{4.39}$$

which is equivalent to

$$(\hat{G} - 1)\bigg(1 - (|\mathcal{S}| - 1)p + \frac{|\mathcal{S}| - 1}{1+p} q^2 - \frac{|\mathcal{S}|q}{1+p}\hat{D} \bigg) = a - \frac{z+b}{1+p} q + \bigg(z + b - \frac{aq}{1+p} \bigg)\hat{D}, \tag{4.40}$$

or, by multiplying both sides by $1 + p > 0$ and using U, X, Y in (4.7),

$$\hat{G}(U - |\mathcal{S}|q\hat{D}) = X + Y\hat{D}. \tag{4.41}$$

Since $U - |\mathcal{S}|q > 0$ for $z < 1$ (cf., (4.8)), this completes the proof of Proposition 4.1. $\square$

5 Diffusion constant: the proof of Theorem 2.5

In this section we discuss the diffusion constant assuming $\mathbb{E}[\tau^{1+\epsilon}] < \infty$ for some $\epsilon > 0$ and excluding the degenerate case where $|\mathcal{S}| = 2$ and $f(1) = -1$ (see below Theorem 2.5 for the degenerate case).

First we note that, since $G_z(x) = \delta_{0,x} + \sum_{n=1}^{\infty} z^n \, \mathbb{P}(S_n = x)$, it is easy to see that $-\nabla^2 \hat{G}_z(0)$ is the generating function of $\sum_x |x|^2 \mathbb{P}(S_n = x)$:

$$-\nabla^2 \hat{G}_z(0) = \sum_{n=1}^{\infty} z^n \sum_{x \in \mathbb{Z}^d} |x|^2 \mathbb{P}(S_n = x) \qquad (z < 1). \tag{5.1}$$

However, by differentiating (4.10) using the symmetry of D, we have

$$-\nabla^2 \hat{G}_z(0) = \frac{-\nabla^2 \hat{D}(0)\, Y_z}{U_z - |\mathcal{S}| q_z \hat{D}(0)} + \frac{-\nabla^2 \hat{D}(0)\big(X_z + Y_z \hat{D}(0)\big)|\mathcal{S}| q_z}{\big(U_z - |\mathcal{S}| q_z \hat{D}(0)\big)^2} = \frac{|\mathcal{S}| q_z X_z + U_z Y_z}{(U_z - |\mathcal{S}| q_z)^2} \sigma^2, \tag{5.2}$$

where $\sigma^2 = \sum_x |x|^2 D(x)$. Using (4.11), we obtain

$$-\nabla^2 \hat{G}_z(0) = \frac{c_z \sigma^2}{(1-z)^2}, \qquad \text{where} \;\; c_z = \frac{|\mathcal{S}| q_z X_z + U_z Y_z}{(X_z + Y_z)^2}, \tag{5.3}$$

with the denominator $(X_z + Y_z)^2$ strictly positive, as explained in (4.9).

Now we investigate (5.3) using a Tauberian theorem to derive the formula (2.2) for the diffusion constant. First we rewrite c_z, by simple algebra and using (4.11), as

$$c_z = \frac{|\mathcal{S}| q_1}{X_1 + Y_1} + \frac{|\mathcal{S}| q_1((X_1 + Y_1) - (X_z + Y_z))}{(X_1 + Y_1)(X_z + Y_z)} + \frac{|\mathcal{S}|(q_z - q_1)}{X_z + Y_z} + \frac{Y_z(1-z)}{X_z + Y_z}. \tag{5.4}$$

Note that $a_1 - a_z$ and $b_1 - b_z$ are $\mathcal{O}((1-z)^\epsilon)$, while $p_1 - p_z$ and $q_1 - q_z$ are $\mathcal{O}(1-z)$, since for example,

$$0 \le a_1 - a_z = \sum_{n \ge 2} (1 - z^n)\, \mathbb{P}(\tau \ge n)\, \mathbb{1}_{\{n \text{ even}\}} \le (1-z)^\epsilon \sum_{n \ge 2} n^\epsilon\, \mathbb{P}(\tau \ge n) \le (1-z)^\epsilon \mathbb{E}[\tau^{1+\epsilon}], \tag{5.5}$$

where we have used the inequality $1 - z^n \le (1-z)^\epsilon n^\epsilon$, which holds for all $n \ge 1$, $z \in [0,1]$ and $\epsilon \in [0,1]$ as follows. For $z = 1$ the inequality is trivial so we may assume that $z < 1$. Let $h(x) = ((1-z)n)^x - (1 - z^n)$. Then $h(0) \ge 0$ and $h(1) \ge 0$, and $h'(x) = ((1-z)n)^x \log((1-z)n)$ is nonnegative if and only if $(1-z)n \ge 1$. Thus for each fixed n, z, the function $h(x)$ is either nonincreasing or nondecreasing, and thus is never negative. The term $b_1 - b_z$ can be handled in the same way, and $p_1 - p_z$ and $q_1 - q_z$ are handled similarly using $\epsilon = 1$. Therefore, the last two terms in (5.4) are $\mathcal{O}(1-z)$. Also, the second term of (5.4) is $\mathcal{O}((1-z)^\epsilon)$ because $X_1 - X_z$ and $Y_1 - Y_z$ are sums of factors of $a_1 - a_z$ and $b_1 - b_z$, as well as factors of $p_1 - p_z$ and $q_1 - q_z$.

We have proved that

$$-\nabla^2 \hat{G}_z(0) = \frac{c_1 \sigma^2}{(1-z)^2} + \mathcal{O}((1-z)^{-2+\epsilon}), \qquad \text{where} \;\; c_1 = \frac{|\mathcal{S}| q_1}{X_1 + Y_1}. \tag{5.6}$$

The error term has radius of convergence at least 1 and it follows from [9, Lemma 6.3.3] that its coefficients in z^n satisfy $|a_n| \le \mathcal{O}(n^{1-\epsilon/2})$. Since $(1-z)^{-2} = \sum_{n \ge 0} (n+1) z^n$, we obtain

$$v = \lim_{n \to \infty} \frac{1}{n} \sum_{x \in \mathbb{Z}^d} |x|^2 \mathbb{P}(S_n = x) = c_1 \sigma^2 = \frac{|\mathcal{S}| q_1}{X_1 + Y_1} \sigma^2. \tag{5.7}$$

Now use (4.5), (4.9) and the expressions for p_1, q_1 (stated above (4.6)) to complete the result. □

Proof of Corollary 2.6. For the first claim, observe that for j odd, $\mathbb{P}(\tau \text{ even})$ is an increasing function of $x = f(j)$, and therefore $\mathbb{P}(\tau \text{ odd})$ is a decreasing function of $x = f(j)$. Since $\mathbb{E}[\tau]$ is an increasing function of $x = f(j)$, the first claim follows from (2.2).

For the second claim, for a fixed even j and reinforcement function F such that $\mathbb{E}_F[\tau^{1+\epsilon}] < \infty$ for some $\epsilon > 0$, let F_x denote the reinforcement function with $F_x(i) = F(i)$ when $i \neq j$ and $F_x(j) = x$. Let $v(x) \equiv v_{F_x}$, and $\mathbb{P}_x \equiv \mathbb{P}_{F_x}$. By (2.2) for every x,

$$v(x) = \frac{\mathbb{P}_x(\tau \text{ odd})}{1 - \frac{2}{|\mathcal{S}|}\mathbb{P}_x(\tau \text{ odd})} \frac{\sigma^2}{\mathbb{E}_x[\tau]}, \tag{5.8}$$

and elementary differentiation rules show that $v'(x) \lesseqqgtr 0$ is equivalent to

$$\frac{|\mathcal{S}|}{2}\,\mathbb{E}_x[\tau]\,\mathbb{P}_x(\tau \text{ odd}, \tau \geq j+1) \;\lesseqqgtr\; \mathbb{P}_x(\tau \text{ odd})\left(\frac{|\mathcal{S}|}{2} - \mathbb{P}_x(\tau \text{ odd})\right)\mathbb{E}_x[(\tau - j)\mathbb{1}_{\{\tau \geq j+1\}}]. \tag{5.9}$$

Fix $0 < \eta \leq \frac{1}{2}$. Let $F(j+1) = -1$ so that $\mathbb{P}_F(\tau \leq j+1) = 1$, and choose $F(1), \dots, F(j-1)$ and $F(j) = x_0$ sufficiently large such that $\mathbb{P}_F(\tau = j+1) > 1 - \eta$. Since j is even, $1 - \eta < \mathbb{P}_x(\tau \text{ odd}) < 1$ whenever $x \geq x_0$, so that the "$>$" inequality in (5.9) holds for all $x \geq x_0$ if

$$\frac{|\mathcal{S}|}{2}(j+1)(1-\eta)^2 > \frac{|\mathcal{S}|}{2} - (1-\eta), \tag{5.10}$$

which holds by our choice of η. Thus for this choice of F we have shown that $v(x)$ is increasing for $x \geq x_0$ and therefore $f = F_{x_0}$ and $g = F_{x_0+1}$ are reinforcement functions satisfying the second claim of the corollary. $\square$

6 Critical Senile linearly reinforced random walk in 1 dimension

In this section we fix the reinforcement function to be $f(m) = m$. As long as $|\mathcal{S}| > 2$ it follows from (3.10) that this senile random walk is such that $\mathbb{E}[\tau^{1+\alpha}] < \infty$ for some $\alpha > 0$. In particular by Theorem 2.5 the diffusion constant for this walk is given by (2.2). The critical case for $f(m) = m$ is when $|\mathcal{S}| = 2$ which corresponds to the nearest-neighbour model in 1 dimension. In analyzing this model we will use the fact that for all $|\mathcal{S}| \geq 2$, when $f(m) = m$ special hypergeometric functions enter the analysis. Special properties of the hypergeometric functions will be used to prove Proposition 2.7 in the following subsections.

6.1 Special hypergeometric functions and analytic continuation

We begin by investigating the quantities in (4.3) and (4.7) using the hypergeometric function ${}_2F_1(c, c', c''; z)$ defined as

$${}_2F_1(c, c', c''; z) = \sum_{n=0}^{\infty} \frac{z^n}{n!} \frac{(c)_n\,(c')_n}{(c'')_n}, \qquad \text{where } (c)_n \equiv \begin{cases} 1, & (n=0), \\ (c+n-1)\cdot(c)_{n-1}, & (n \geq 1). \end{cases} \tag{6.1}$$

When $f(m) = m$, we can rewrite a_z, b_z, p_z and q_z for $|\mathcal{S}| \geq 2$ and $z \in (0,1)$ as

$$\begin{cases} a_z = |\mathcal{S}| \displaystyle\sum_{n \geq 2,\text{ even}} \frac{z^n}{n!} \frac{(1)_n(1)_n}{(|\mathcal{S}|)_n} = a_{-z}, \\ b_z = |\mathcal{S}| \displaystyle\sum_{n \geq 2,\text{ odd}} \frac{z^n}{n!} \frac{(1)_n(1)_n}{(|\mathcal{S}|)_n} = -b_{-z}, \end{cases} \qquad \begin{cases} p_z = \displaystyle\sum_{n \geq 1,\text{ even}} \frac{z^n}{n!} \frac{(1)_n(1)_n}{(|\mathcal{S}|+1)_n} = p_{-z}, \\ q_z = \displaystyle\sum_{n \geq 1,\text{ odd}} \frac{z^n}{n!} \frac{(1)_n(1)_n}{(|\mathcal{S}|+1)_n} = -q_{-z}, \end{cases} \tag{6.2}$$

where, as in the proof of Corollary 2.3 a_1 and b_1 do not converge in the critical case $|\mathcal{S}| = 2$. Let

$$F_z = {}_2F_1(1, 1, |\mathcal{S}| + 1; z), \qquad\qquad F^*_z = {}_2F_1(1, 1, |\mathcal{S}|; z). \tag{6.3}$$

Represented in terms of these hypergeometric series, a_z and p_z are given respectively by

$$\begin{aligned} a_z = |\mathcal{S}| \sum_{n\geq 2} \frac{z^n}{n!} \frac{(1)_n\,(1)_n}{(|\mathcal{S}|)_n} \frac{1 + (-1)^n}{2} &= \frac{|\mathcal{S}|}{2}\left(\left(F^*_z - 1 - \frac{z}{|\mathcal{S}|}\right) + \left(F^*_{-z} - 1 + \frac{z}{|\mathcal{S}|}\right)\right) \\ &= \frac{|\mathcal{S}|}{2}(F^*_z + F^*_{-z}) - |\mathcal{S}|, \end{aligned} \tag{6.4}$$

$$p_z = \sum_{n\geq 1} \frac{z^n}{n!} \frac{(1)_n\,(1)_n}{(|\mathcal{S}|+1)_n} \frac{1 + (-1)^n}{2} = \frac{1}{2}\Big((F_z - 1) + (F_{-z} - 1)\Big) = \frac{1}{2}(F_z + F_{-z}) - 1. \tag{6.5}$$

Similarly we have

$$b_z = \frac{|\mathcal{S}|}{2}(F^*_z - F^*_{-z}) - z, \qquad\qquad q_z = \frac{1}{2}(F_z - F_{-z}). \tag{6.6}$$

Further arithmetic shows that

$$\begin{cases} U_z = U_{-z} = -(|\mathcal{S}| - 1)F_zF_{-z} + \dfrac{|\mathcal{S}|}{2}(F_z + F_{-z}), \\ X_z = X_{-z} = -(|\mathcal{S}| - 1)F_zF_{-z} + \dfrac{|\mathcal{S}|}{2}(F^*_zF_{-z} + F^*_{-z}F_z), \\ Y_z = -Y_{-z} = \dfrac{|\mathcal{S}|}{2}(F^*_zF_{-z} - F^*_{-z}F_z). \end{cases} \tag{6.7}$$

Of course, the parity of these $a_z, b_z, p_z, q_z, U_z, X_z$ and Y_z are invariant for any reinforcement function.

Euler's formula for the hypergeometric function shows that

$${}_2F_1(1, 1, c; z) = (c - 1)\int_0^1 \frac{(1 - t)^{c-2}}{1 - tz}\, dt \tag{6.8}$$

is the analytic continuation of ${}_2F_1(1, 1, c; z)$ to $\mathbb{C} \setminus [1, \infty)$, and of course ${}_2F_1(1, 1, c; 1) < \infty$ whenever $c > 2$. It follows that a_z, b_z, p_z, q_z and hence U_z, X_z, Y_z have analytic continuation (determined by (6.8)) to the region $\Delta \equiv \mathbb{C} \setminus \{(-\infty, -1] \cup [1, \infty)\}$. In addition (6.8) shows that for $z \in [0, 1]$

$$F_z \leq \frac{|\mathcal{S}|}{|\mathcal{S}| - 1}, \qquad \text{with equality only when } \ z = 1. \tag{6.9}$$

Also, (6.8) shows that $F_{-z} > 0$ for $z \in [-1, 1]$. It is easy to show that $\binom{|\mathcal{S}|+n}{|\mathcal{S}|} \equiv (|\mathcal{S}| + 1)_n/(1)_n$ is an increasing sequence and thus taking the first two terms in the series for F_{-1} (the remainder is guaranteed to be positive) we have

$$F_{-1} = \sum_{n=0}^{\infty}(-1)^n \frac{(1)_n}{(|\mathcal{S}| + 1)_n} \begin{cases} \leq 1, \\ \geq 1 - \dfrac{1}{|\mathcal{S}| + 1} > \dfrac{|\mathcal{S}| - 2}{|\mathcal{S}| - 1}. \end{cases} \tag{6.10}$$

One of Gauss' relations for contiguous functions is that for $z \in \mathbb{C} \setminus [1, \infty)$,

$$\gamma(1-z) \cdot {}_2F_1(\alpha, \beta, \gamma; z) - \gamma \cdot {}_2F_1(\alpha - 1, \beta, \gamma; z) + (\gamma - \beta)z \cdot {}_2F_1(\alpha, \beta, \gamma + 1; z) = 0. \tag{6.11}$$

At $\alpha = \beta = 1$ and $\gamma = |\mathcal{S}|$ we have

$$|\mathcal{S}|(1-z)F_z^* - |\mathcal{S}| + (|\mathcal{S}| - 1)zF_z = 0. \qquad \therefore \quad \frac{|\mathcal{S}|F_z^* - (|\mathcal{S}| - 1)F_z}{|\mathcal{S}| - (|\mathcal{S}| - 1)F_z} = \frac{1}{1-z}. \tag{6.12}$$

This is equivalent to (4.11), with q_z, U_z, X_z and Y_z in (6.6)–(6.7). It may be of interest to see what formulae the relation (4.11) gives for $|z| < 1$ for more general choices of reinforcement. Of course one can also obtain the relation $(X_z + Y_z)/(U_z - |\mathcal{S}|q_z) = (1-z)^{-1}$ for all $z \in \Delta$ using the fact that both sides of this inequality have analytic continuation to Δ (provided $U_z - |\mathcal{S}|q_z \neq 0$ in this region) and they agree on $0 < z < 1$.

When $|\mathcal{S}| = 2$ (i.e. the 1-dimensional nearest-neighbour model), since

$$F_z^* = \int_0^1 \frac{1}{1-tz}\, dt = \frac{1}{z} \log \frac{1}{1-z} \qquad (z < 1), \tag{6.13}$$

while F_1, F_{-1} and F_{-1}^* all converge, by (6.7) we have

$$\left.\begin{aligned} X_z &= -F_zF_{-z} + F_z^*F_{-z} + F_{-z}^*F_z \\ Y_z &= F_z^*F_{-z} - F_{-z}^*F_z \end{aligned}\right\} \sim \frac{F_{-z}}{z} \log \frac{1}{1-z}, \qquad \text{as } z \to 1 \text{ in } \Delta. \tag{6.14}$$

6.2 Proof of Proposition 2.7

Fix $d \geq 2$ for the first claim. Then $|\mathcal{S}| > 2$, and (2.2) holds as we noted at the beginning of Section 6. By (6.6)–(6.7) and (6.10) with $F_1 = |\mathcal{S}|/(|\mathcal{S}| - 1)$ and $F_1^* = (|\mathcal{S}| - 1)/(|\mathcal{S}| - 2)$, we have

$$\frac{v}{\sigma^2} = \frac{\frac{|\mathcal{S}|}{2}(F_1 - F_{-1})}{-(|\mathcal{S}| - 1)F_1F_{-1} + |\mathcal{S}|F_1^*F_{-1}} = \frac{|\mathcal{S}| - 2}{2}\left(\frac{|\mathcal{S}|}{F_{-1}(|\mathcal{S}| - 1)} - 1\right) \in (0, 1). \tag{6.15}$$

To prove the final claim, we apply the following trivial extension (to the case where there are 2 singularities) of a result of Flajolet and Odlyzko [4].

Proposition 6.1 (Flajolet-Odlyzko). *Suppose that $f(z) = \sum_{n \geq 0} z^n f_n$ has analytic continuation in the region $\Delta = \mathbb{C} \setminus \{(-\infty, -1] \cup [1, \infty)\}$, and that*

$$f(z) = \begin{cases} K(1-z)^{-2}(\log \frac{1}{1-z})^{-1}\Big(1 + \mathcal{O}\big((\log \frac{1}{1-z})^{-1}\big)\Big) & (z \to 1), \\ \mathcal{O}(\log(1+z)) & (z \to -1). \end{cases} \tag{6.16}$$

Then $f_n \sim Kn/\log n$.

To verify that we can apply Proposition 6.1, it suffices to prove the following Lemma. Once the lemma is proved, the integral (6.8) with $c = 3$ $(\equiv 2 + 1)$ and $z = \pm 1$ shows that $F_1/F_{-1} = (2\log 2 - 1)^{-1}$ and that the constant below in (6.17) takes its value of (2.3).

Lemma 6.2. *Let $f(m) = m$. Then $-\nabla^2 \hat{G}_z(0)$ has analytic continuation in the region $\Delta \equiv \mathbb{C} \setminus \{(-\infty, -1] \cup [1, \infty)\}$. In particular, for the 1-dimensional nearest-neighbour model,*

$$-\nabla^2 \hat{G}_z(0) = \begin{cases} \frac{F_1 - F_{-1}}{2F_{-1}}(1-z)^{-2}(\log \frac{1}{1-z})^{-1}\Big(1 + \mathcal{O}\big((\log \frac{1}{1-z})^{-1}\big)\Big) & (z \to 1 \text{ in } \Delta), \\ \mathcal{O}(\log(1+z)) & (z \to -1 \text{ in } \Delta). \end{cases} \tag{6.17}$$

Proof. Observe that $U_z - |\mathcal{S}|q_z = (-(|\mathcal{S}|-1)F_z + |\mathcal{S}|)F_{-z}$ is non-zero for $z \in (-1, 1)$ and is the product of two non-zero complex numbers (and hence non-zero) when z is not real-valued. The fact that $-\nabla^2 \hat{G}_z(0)$ has analytic continuation in Δ then follows from (5.2), (6.6)–(6.7) and the fact that (6.8) gives analyticity of F_z and F_z^* in $\mathbb{C} \setminus [1, \infty)$.

Now consider the 1-dimensional nearest-neighbour model. From (5.2) and using the parity of q_z, U_z, X_z and Y_z, we have

$$-\nabla^2 \hat{G}_{-z}(0) = \frac{-(2q_z X_z + U_z Y_z)}{U_z + 2q_z} = \mathcal{O}\big(\log(1-z)\big), \tag{6.18}$$

due to (6.14) and the fact that the denominator converges to $4q_1 > 0$ as $z \to 1$. This verifies the limit in (6.17) as $z \to -1$.

To prove the other limit in (6.17), we use (5.3)–(5.4) for $z \in \Delta$: $-\nabla^2 \hat{G}_z(0) = c_z(1-z)^{-2}$, where

$$c_z = \frac{2q_1}{X_z + Y_z} + \frac{2(q_z - q_1)}{X_z + Y_z} + \frac{Y_z(1-z)}{X_z + Y_z}. \tag{6.19}$$

It is immediate from (6.14) that the first term is $\mathcal{O}((\log \frac{1}{1-z})^{-1})$, while the last term is $\mathcal{O}(1-z)$. The second term contains $q_z - q_1 = \frac{1}{2}((F_z - F_1) - (F_{-z} - F_{-1}))$, due to (6.6). Using (6.8) for $z \in \Delta$, we have

$$F_z - F_1 = (1-z)\int_0^1 \frac{-2t}{1-zt}\, dt = (1-z)\,\mathcal{O}\big(\log(1-z)\big), \tag{6.20}$$

$$F_{-z} - F_{-1} = (1-z)\int_0^1 \frac{2t(1-t)}{(1+zt)(1+t)}\, dt = \mathcal{O}(1-z), \tag{6.21}$$

so that the second term in (6.19) is also $\mathcal{O}(1-z)$. Therefore, the first term is the slowest term. Using (6.14) and isolating the main factor $F_z^* = \frac{1}{z}\log\frac{1}{1-z}$, we can rewrite c_z in (6.19) as

$$c_z = \frac{2q_1}{2F_z^* F_{-z} - F_z F_{-z}} + \mathcal{O}(1-z) = \frac{q_1}{F_{-1}}\frac{1}{F_z^*}\Big(1 + \mathcal{O}\big((\log \tfrac{1}{1-z})^{-1}\big)\Big) + \mathcal{O}(1-z). \tag{6.22}$$

The proof of (6.17) is now completed by using (6.6) at $z = 1$. This completes the proof of Proposition 2.7. $\square$

Acknowledgements

This work was supported by the Netherlands Organization for Scientific Research (NWO) and by the European Science Foundation (ESF). We thank Philippe Flajolet and Mat Rogers for helpful correspondence and Remco van der Hofstad, Wouter Kager, Vlada Limic and Silke Rolles for their encouragement and suggestions.

References

[1] D. Coppersmith and P. Diaconis. Random walk with reinforcement. Unpublished manuscript, 1986.

[2] B. Davis. Reinforced random walk. *Probab. Theory Relat. Fields.*, 84:203–229, 1990.

[3] R. Durrett, H. Kesten, and V. Limic. Once edge-reinforced random walk on a tree. *Probab. Theory Relat. Fields.*, 122, 2001.

[4] P. Flajolet and A. Odlyzko. Singularity analysis of generating functions. *J. Disc. Math.*, 3:216–240, 1990.

[5] L. Horváth and Q. Shao. Limit distributions of directionally reinforced random walks. *Adv. in Math.*, 134:367–383.

[6] M. Keane and S. Rolles. Edge reinforced random walk on finite graphs. In *Infinite dimensional stochastic analysis (Amsterdam, 1999)*, pages 217–234. R. Neth. Acad. Arts Sci., Amsterdam, 2000.

[7] V. Limic. Attracting edge property for a class of reinforced random walks. *Ann. Probab.*, 31:1615–1654, 2003.

[8] V. Limic and P. Tarrès. Attracting edge and strongly edge reinforced walks. *Ann. Probab.*, to appear, 2006.

[9] N. Madras and G. Slade. *The Self-Avoiding Walk.* Birkhäuser, Boston, 1993.

[10] R.D. Mauldin, M. Monticino, and Weizsäcker. Directionally reinforced random walks. *Adv. in Math.*, 117:239–252, 1996.

[11] F. Merkl and S. Rolles. A random environment for linearly edge-reinforced random walks on infinite graphs. *Probab. Theory Relat. Fields.*, To appear, 2006.

[12] H. Renlund. Reinforced random walk. Technical report, Department of Mathematics, Uppsala University.

[13] T. Selke. Reinforced random walks on the d-dimensional integer lattice. Preprint.