Factorial Designs and Harmonic Analysis on Finite Abelian Groups

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Abstract

In this paper factorial designs are discussed within the algebraic framework of harmonic analysis on finite groups. The runs in the full factorial designs are coded by the elements of a finite Abelian group. Harmonic analysis on the finite Abelian group is used to obtain a set of orthogonal contrasts and the canonical decomposition of the total sums of squares. Regular fractions can be defined as a coset of a finite Abelian group. This definition can be shown to be equivalent to other definitions given in the literature. Different properties of full factorial designs and regular fractions are derived within the proposed algebraic framework. Attention is paid to issues as the coding of the factor levels and the confounding relations.

Keywords: factorial designs, regular fraction, mixed designs, harmonic analysis.

1 Introduction

Many important results in the literature on factorial designs were derived using a group-theoretic approach. Fisher (1942) and Finney (1945) were the first to express the theory of symmetric factorial designs in terms of finite Abelian groups by labeling the runs in a symmetric full factorial design by the elements of an abstract group. The labeling proposed by Fisher (1942) and Finney (1945) can be easily extended to mixed factorial designs. If $n_1, n_2, \ldots, n_k$ denote the number of levels for the k factors in the experiment, then the runs in the mixed factorial design are labeled with the elements in the abstract Abelian group generated by k elements $a_1, a_2, \ldots, a_k$ and relations $a_1^{n_1} = a_2^{n_2} = \ldots = a_k^{n_k} = 1$. A realization of this abstract group was introduced in experimental design theory by Kempthorne (1947), who coded the runs in a full factorial design by the elements of the additive Abelian group $\mathbb{Z}/n_1\mathbb{Z} \oplus \mathbb{Z}/n_2\mathbb{Z} \oplus \ldots \oplus \mathbb{Z}/n_k\mathbb{Z}$. More recently, the coding introduced in Kempthorne (1947) was used by Dean and John (1975), John and Dean (1975), Lewis (1979), Bailey (1985), Kobilinsky (1985) and Collombier (1996). Bailey (1982a), Collombier (1996) and Pistone and Rogantin (2005) suggested to use the complex roots of unity to code the levels of the factors.

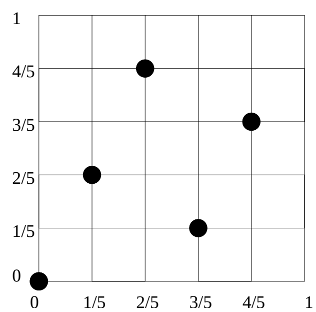
When this coding is used the runs of a full factorial experiment form a multiplicative Abelian group. Each of these ways of coding have been shown to be useful in the construction of factorial designs and the analysis of the data. However, some aspects of factorial designs are explained more easily using the coding as an additive Abelian group, while other aspects become more clear when the runs are coded by the elements of a multiplicative Abelian group. In this paper we take a general approach and define a full factorial design as a finite Abelian group on which statistical data is gathered. In this approach the character theory of the finite Abelian group gives a set of orthogonal contrasts. A canonical decomposition of the total sum of squares is obtained using harmonic analysis. Regular fractions are defined as a coset of a finite Abelian group. The confounding relations for regular fractions are easily obtained using the Poisson summation formula. The character theory of finite Abelian groups has proven to be useful in the search and construction of factorial designs and as a means to study the confounding of effects in fractional factorial designs (see Bailey (1982b), Bailey (1985), Kobilinsky (1985), El Mossadeq et al. (1985), Bailey (1990), Kobilinsky (1990) and Collombier (1996)). In this paper we study both the confounding of effects and the statistical analysis for factorial designs and regular fractions using character theory and harmonic analysis. The results presented in this paper apply to designs that have the structure of a finite Abelian group. Some of the results are also valid for non-Abelian groups. References for the analysis of statistical data structured on non-Abelian groups are Diaconis (1988) and Viana (2005).

The outline of this paper is as follows. First the notation and some definitions used in the paper are given in Section 2. Factorial designs are not the only designs that have the structure of a finite Abelian group. Some examples of other types of designs that are finite Abelian groups are given in Section 3. The basics of harmonic analysis on finite groups are explained in Section 4. Within the proposed algebraic framework the confounding relations on a regular fraction are found using the Poisson summation formula. This formula is presented in Section 5. In Section 6 we give an overview of the different definitions for regular fractions that have been proposed. In Section 7 we use character theory to find a set of defining equations for a regular fraction. The equivalence of the different definitions for regular fractions is shown in Section 8. In Section 9 we illustrate how the confounding relations on a regular fraction are obtained using the Poisson summation formula. The statistical inference for normal data structured on finite groups and regular fractions is discussed in Section 10.

2 Preliminaries

In this paper we use the following notation for factorial designs. A design is defined as a finite set which we denote by D. Throughout the paper we assume that our response is a real-valued random function on D and use $y : D \to \mathbb{R}$ to denote a realization of this response. When considering factorial experiments we let $X_1, X_2, \ldots, X_k$ denote the factors in the experiment and by n_j we denote the number of levels for factor X_j. If the levels of factor X_j are coded with the elements in the cyclic group $\mathbb{Z}/n_j\mathbb{Z}$, then the runs in the full factorial design are represented by the elements in $\mathbb{Z}/n_1\mathbb{Z} \oplus \mathbb{Z}/n_2\mathbb{Z} \oplus \ldots \oplus \mathbb{Z}/n_k\mathbb{Z}$, which is a group under addition modulo $(n_1, n_2, \ldots, n_k)$. We let Ω_n denote the set of all nth complex roots of unity. If the complex coding is used to code the full factorial design then the runs are represented by the elements of $\Omega_{n_1} \oplus \Omega_{n_2} \oplus \ldots \oplus \Omega_{n_k}$ which forms a group under elementwise multiplication. Because each full factorial designs is considered a group, we use G to refer to elements in this class of designs. A fractional factorial design or fraction is defined as a subset of a full

Figure 1: The lattice based design $\mathcal{L}_{(1,2),5}$



factorial design. Throughout the paper we use F to refer to a fraction. A factorial design is called symmetric if $n_1 = n_2 = \ldots = n_k$.

When discussing finite groups we assume that the group is multiplicative, except for some cases where an additive group is explicitly stated. An important role in the paper is played by the cosets of a group. Let H be a subgroup of the Abelian group G. The cosets of the subgroup H in G are the sets $aH = \{ah \mid h \in H\}$ where $a \in G$. Each element of G is contained in exactly one coset of H. Note that the aH and bH for different $a, b \in G$ may refer to the same coset. In some cases it is useful to have a unique representation for each coset. The notion of transversal is introduced for this purpose. A transversal of H in G is defined as a set containing exactly one element from each coset of H in G.

3 Examples of designs that are finite Abelian groups

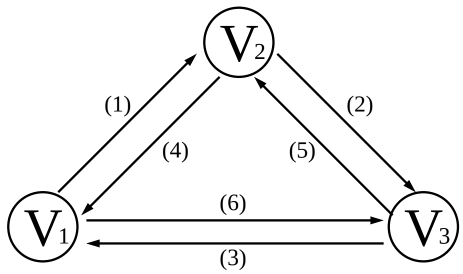
Full factorial designs are not the only designs that have the structure of a finite Abelian group. We briefly discuss some other designs that are finite Abelian groups. The lattice based designs for Fourier regression considered by Riccomagno et al. (1997) and Bates et al. (1998) have the structure of an Abelian group. The single-generator lattice for a sample size N and a generator $g = (g_1, g_2, \ldots, g_k) \in \mathbb{Z}^k$ is the set of points

$$\mathcal{L}_{g,N} = \left\{ \left(\frac{jg_1}{N}, \frac{jg_2}{N}, \ldots, \frac{jg_k}{N} \right) \mod (N) \,|\, j = 0, 1, \ldots, N-1 \right\},$$

where mod (N) means that the numerators jg_v are reduced mod (N). The elements in $\mathcal{L}_{g,N}$ form an Abelian group where the group-operation is the addition mod (N). Figure 1 shows the lattice based design with generator $g = (1, 2)$ and $N = 5$.

A second example is the special class of Latin squares formed by the so-called Cayley tables of the groups $\mathbb{Z}/n\mathbb{Z}$. The Cayley table of $\mathbb{Z}/n\mathbb{Z}$ is defined to be the Latin square with rows, columns and treatments coded by the elements of $\mathbb{Z}/n\mathbb{Z}$ such that the treatment in row r and column c is $r + c$. In the context of experimental design, the rows and columns usually represent two blocking factors. The data obtained on the Latin square can be indexed by elements in the set $F = \{(r, c, r + c) \mid r \in \mathbb{Z}/n\mathbb{Z}, c \in \mathbb{Z}/n\mathbb{Z}\}$. The set F is the subgroup of the Abelian group $(\mathbb{Z}/n\mathbb{Z})^3$ generated by $(1, 0, 1)$ and $(0, 1, 1)$. As an example, the Cayley table

Figure 2: A microarray experiment with dye swap



for $n = 4$ is given by

$$\begin{array}{cccc} 0 & 1 & 2 & 3 \\ 1 & 2 & 3 & 0 \\ 2 & 3 & 0 & 1 \\ 3 & 0 & 1 & 2 \end{array}.$$

Another example of a design with an Abelian group structure is the microarray experiment with dye swap that is presented in a schematic way in Figure 2. The aim of this experiment is to identify genes that are differently expressed in different regions of the brain. The brain regions under study are labeled V_1, V_2 and V_3 and the arrows between them correspond to hybridizations between two mRNA samples from the different regions. The mRNA sample at the tail of the arrow is labeled with a green dye and the sample at the head with a red dye. For each of the six hybridizations in the experiment a different microarray is used. All microarrays have the same n genes printed on them and on each microarray a measurement is made for each gene. More precisely, for each gene on each microarray the ratio of the red and green fluorescence intensities is determined. This ratio is indicative of the relative abundance of the corresponding DNA probe in the two mRNA samples. The observations can be indexed by elements in the set $S = A \times K$ where $A = \{1, 2, \ldots, 6\}$ and $K = \{1, 2, \ldots, n\}$ contain the indexes for the microarrays and genes, respectively. We define three commuting actions that can be performed on the set S. The first action is that of cyclicly relabeling the brain part regions. The corresponding group of actions is the group $G_1 = \{\mathrm{id}\,, (123)\,(456)\,, (132)\,(465)\}$. This group acts on S by $\sigma : (a, k) \to (\sigma a, k)$ for all $\sigma \in G_1$. The second action that we define is that of a dye swap. The group of actions is given by $G_2 = \{\mathrm{id}\,, (14)\,(25)\,(36)\}$ and acts on S by $\sigma : (a, k) \to (\tau a, k)$ for all $\tau \in G_2$. A third action is that of cyclicly relabeling the genes under the study. The corresponding group of actions is the group $G_3 = \{\mathrm{id}\,, \beta, \beta^2, \ldots, \beta^{n-1}\}$ generated by the permutation $\beta = (1\,2\,\ldots\,n)$. This group acts on S by $\gamma : (a, k) \to (a, \gamma k)$ for all $\gamma \in G_3$. We form the multiplicative Abelian group $G = G_1 \times G_2 \times G_3 = \{\sigma\tau\gamma \mid \sigma \in G_1, \tau \in G_2, \gamma \in G_3\}$ and let G act on S according to $\sigma\tau\gamma : (a, k) \to (\tau\sigma a, \gamma k)$. The group G has $3 \cdot 2 \cdot n$ elements (which equals $\mid S \mid$). If we pick an arbitrary element $s_0 = (a_0, k_0) \in S$ then each element in S can be obtained from s_0 by letting a unique element from G act on s_0. Hence, after fixing the reference $s_0 \in S$ the data obtained on S can be considered to be structured on the finite multiplicative Abelian group G.

4 Harmonic analysis on finite groups

In this section we give a short overview of harmonic analysis on finite groups. Serre (1977) and Terras (1999) give nice algebraic introductions to this topic. Statistical introductions can be found in Diaconis (1988) and Viana (2005). Both Diaconis (1988) and Viana (2005) consider the analysis of statistical data structured on groups. In this paper we extend their results to the case where the statistical data is structured on a coset of a finite Abelian group. In Section 4.1 we first consider harmonic analysis on finite groups in general. The special case of harmonic analysis on finite Abelian groups is considered in Section 4.2. In Section 4.3 we present an example that illustrates how harmonic analysis can be applied to find a decomposition of the sums of squares in a full factorial design.

4.1 The general case

In harmonic analysis a group is studied through its linear representations in a vector space $\mathcal{V}$. An exact definition of a linear representation will be given shortly, but first we introduce the general linear group $GL(\mathcal{V})$ of a vector space $\mathcal{V}$. The general linear group $GL(\mathcal{V})$ is the set of all isomorphisms of $\mathcal{V}$ onto itself. The elements of $GL(\mathcal{V})$ are, by definition, linear mappings of $\mathcal{V}$ into $\mathcal{V}$ which have an inverse. A linear representation is defined as follows.

Definition 4.1 *A linear representation ρ of a group G in a vector space $\mathcal{V}$ is a group homomorphism from G into $GL(\mathcal{V})$.*

Hence, a mapping $\rho : G \to GL(\mathcal{V})$ of a multiplicative group G is a representation if it satisfies $\rho(g_1 g_2) = \rho(g_1)\rho(g_2)$ for all $g_1, g_2 \in G$. The dimension of the representation ρ is defined to be the dimension of the vector space $\mathcal{V}$. For $\phi : \mathcal{V} \to \mathcal{V}$ a linear map, we let $\operatorname{tr}\phi$ denote the trace of the matrix representation of ϕ. The character of a representation is defined as follows.

Definition 4.2 *Given a linear representation $\rho : G \to GL(\mathcal{V})$ of a group G, the function $\chi_\rho : G \to \mathbb{C}$ defined by $\chi_\rho(g) = tr\,\rho(g)$ is called the character of the representation.*

Note that because the trace is basis free, the character does not depend on the basis that chosen for $\mathcal{V}$. Next we define the notion of irreducibility of a representation, but first we need to define the notion of a stable subspace.

Definition 4.3 *Let ρ be a representation of G in $GL(\mathcal{V})$. A linear subspace $\mathcal{W}$ of $\mathcal{V}$ is stable under ρ if for all $w \in \mathcal{W}$ and all $g \in G$ we have that that $\rho(g)w \in \mathcal{W}$.*

The irreducibility of a representation is defined as follows.

Definition 4.4 *A representation ρ of G in $GL(\mathcal{V})$ is irreducible if the only proper linear subspace of $\mathcal{V}$ that is stable under ρ is the null space.*

We refer to a the character of an irreducible representation as an irreducible character. All characters have the following nice property.

Lemma 4.5 *The character χ_ρ of a linear representation ρ of a group G satisfies $\chi_\rho\left(g^{-1}\right) = \overline{\chi_\rho}(g)$ for all $g \in G$.*

Proof See Proposition 1 (ii) in Serre (1977). $\square$

We now define equivalence for the linear representations of a group.

Definition 4.6 *Two linear representations $\rho_1 : G \to GL(\mathcal{V}_1)$ and $\rho_2 : G \to GL(\mathcal{V}_2)$ of a group G are equivalent if there exists an invertible linear map $f : \mathcal{V}_1 \to \mathcal{V}_2$ for which $f\rho_1(g)f^{-1} = \rho_2(g)$ for all $g \in G$.*

The characters of a group give us a very convenient equivalence criterion

Lemma 4.7 *Two linear representations of a group G are equivalent if and only if they have the same character.*

Proof See Corollary 2 in Serre (1977). $\square$

By $L^2(G)$ we denote the inner product space of all complex functions defined on G with the inner product

$$\langle f_1, f_2 \rangle_G = \frac{1}{\mid G \mid} \sum_{g \in G} f_1(g)\, \overline{f_2}(g).$$

In cases where there is no ambiguity about the finite group on which the inner product is defined, the inner product is simply denoted by $\langle f_1, f_2 \rangle$. Functions $f_1, f_2 \in L^2(G)$ are said to be orthogonal on G if $\langle f_1, f_2 \rangle_G = 0$. The characters of non-equivalent irreducible representations are orthogonal on the group G. This is shown in the next theorem.

Theorem 4.8 *If χ_1 and χ_2 are the characters of two non-equivalent irreducible representations of G, then $\langle \chi_1, \chi_2 \rangle_G = 0$.*

Proof See Theorem 3 in Serre (1977) and Theorem 2.5 in Viana (2005). $\square$

The characters also provide us with a very simple irreducibility criterion.

Lemma 4.9 *If χ_ρ is the character of a representation ρ of G, then $\langle \chi_\rho, \chi_\rho \rangle$ is a positive integer and $\langle \chi_\rho, \chi_\rho \rangle = 1$ if and only if ρ is irreducible.*

Proof See Theorem 5 in Serre (1977).

Two elements g' and $g \in G$ are called conjugate if there exists an element $h \in G$ such that $g' = hgh^{-1}$. It can be shown that conjugacy is an equivalence relation. Therefore it partitions the group G into equivalence classes. The equivalence class that contains the element g in G is $\{hgh^{-1} : h \in G\}$ and is usually referred to as the conjugacy class of g. A function $f : G \to \mathbb{C}$ satisfying $f\left(hgh^{-1}\right) = f(g)$ for all $h, g \in G$ is called a class function. By $\mathcal{C}(G)$ we denote the linear space of all class functions defined on a group G. The next lemma states that all characters are class functions.

Lemma 4.10 *Each character χ_ρ of a linear representation ρ of a group G is constant on the conjugacy classes of G.*

Proof Because ρ is a linear representation of G we have for all $g, h \in G$ that $\rho\left(hgh^{-1}\right) = \rho\left(h\right)\rho\left(g\right)\rho\left(h^{-1}\right)$. Hence, for all $h, g \in G$ we have

$$\chi_\rho\left(hgh^{-1}\right) = \text{tr}\ \left(\rho\left(hgh^{-1}\right)\right) = \text{tr}\ \left(\rho\left(h\right)\rho\left(g\right)\rho\left(h^{-1}\right)\right) = \text{tr}\ \left(\rho\left(h^{-1}\right)\rho\left(h\right)\rho\left(g\right)\right) =$$
$$\text{tr}\ \left(\left(\rho\left(h^{-1}h\right)\right)\rho\left(g\right)\right) = \text{tr}\ \left(\rho\left(1\right)\rho\left(g\right)\right) = \text{tr}\ \left(\rho\left(g\right)\right) = \chi_\rho\left(g\right),$$

from which we conclude that χ_ρ is constant on the conjugacy classes of G. □

The next lemma states that the distinct irreducible characters of a group G form an orthonormal basis for $\mathcal{C}\left(G\right)$.

Lemma 4.11 *The non-equivalent irreducible characters form an orthonormal basis for* $\mathcal{C}\left(G\right)$.

Proof See Theorem 6 in Serre (1977) or Theorem 2.7 in Viana (2005). □

Let ρ be a linear representation of G into $GL\left(\mathcal{V}\right)$ and $\rho_1, \rho_2, \ldots, \rho_w$ be the distinct non-equivalent irreducible representations of G, with corresponding characters $\chi_1, \chi_2, \ldots, \chi_w$. From Lemma 4.11 we know that $\{\chi_1, \chi_2, \ldots, \chi_w\}$ is an orthonormal basis for $\mathcal{C}\left(G\right)$. The Fourier expansion of $f \in \mathcal{C}\left(G\right)$ with respect to this basis is

$$f = \sum_{j=1}^{w} \langle f, \chi_j \rangle \chi_j.$$

The function $\widehat{f} : \widehat{G} \to C$ defined by $\widehat{f}\left(\chi_j\right) = \langle f, \chi_j \rangle$ is called the Fourier transform of f.

It can be shown that for all j we have that $m_j = \langle \chi_j, \chi_\rho \rangle$ is the number of irreducible representations equivalent to ρ_j in any decomposition of ρ. That is, the representation ρ is isomorphic to the direct sum

$$\rho = m_1\rho_1 \oplus m_2\rho_2 \oplus \ldots \oplus m_w\rho_w.$$

The next theorem gives the projection matrices associated with this decomposition.

Theorem 4.12 *Let* ρ *be a linear representation of* G *into* $GL\left(\mathcal{V}\right)$. *Let* $\rho_1, \rho_2, \ldots, \rho_w$ *be the distinct non-equivalent irreducible representations and let* $\chi_1, \chi_2, \ldots, \chi_w$ *and* $d_1, d_2, \ldots, d_w$ *be the corresponding characters and dimensions, respectively. Then*

$$P_j = \frac{d_j}{\mid G \mid} \sum_{g \in G} \overline{\chi_j}\left(g\right)\rho\left(g\right)$$

is a projection of $\mathcal{V}$ *onto a subspace* $\mathcal{V}_j$ *that is the sum of* m_j *isomorphic copies of the stable subspace associated with* $\rho_j, j = 1, 2, \ldots, w$. *Moreover,* $P_jP_k = 0$ *for all* $j \neq k$, $P_j^2 = P_j$ *for all* j *and* $\sum_{1 \leq j \leq w} P_j = I_v$, *where* $v = \dim \mathcal{V}$.

Proof See Theorem 8 in Serre (1977) and Theorem 2.8 in Viana (2005). □

The canonical decomposition given in the previous theorem does not depend on the chosen decomposition of ρ into irreducible representations. This follows from observing that the matrices P_j depend on the irreducible representations only through the irreducible characters which are equal for equivalent irreducible representations. In Section 10 we will use the decomposition of the identity matrix to find a decomposition of the total sum of squares into statistically independent parts. For that purpose we take ρ to be the regular representation, which is defined in the following way.

Definition 4.13 *Let $\mathcal{V}$ be a vector space of dimension N with a basis $\{e_g \mid g \in G\}$. For each $h \in G$ let $\rho(h)$ be the linear map of $\mathcal{V}$ into $\mathcal{V}$ defined by $\rho(h)\, e_g = e_{hg}$. Then ρ is a representation of G, which is called the regular representation.*

4.2 Harmonic analysis on finite Abelian groups

The irreducible representations of finite Abelian groups have several nice properties that we present here. The following theorem shows that when studying representations and characters of a finite Abelian group we may without loss of generality assume that the group under study is a direct product of cyclic groups.

Theorem 4.14 (Fundamental Theorem of Abelian groups) *Every finite Abelian group G is isomorphic to a direct product of cyclic groups, that is,*

$$G \cong \mathbb{Z}/n_1\mathbb{Z} \oplus \mathbb{Z}/n_2\mathbb{Z} \oplus \ldots \oplus \mathbb{Z}/n_k\mathbb{Z}.$$

Proof See Terras (1999), p. 163. □

Let $G = \mathbb{Z}/n_1\mathbb{Z} \oplus \mathbb{Z}/n_2\mathbb{Z} \oplus \ldots \oplus \mathbb{Z}/n_k\mathbb{Z}$. The set of all irreducible representations of G is $\{\rho_z \mid z \in G\}$ with ρ_z given by

$$\rho_z(g) = (\omega_{n_1})^{z_1 g_1} (\omega_{n_2})^{z_2 g_2} \ldots (\omega_{n_k})^{z_k g_k}, \tag{1}$$

where $\omega_{n_j} = e^{\frac{2\pi i}{n_j}}$. All irreducible representations of G are one-dimensional, which implies that the irreducible characters equal the irreducible representations. The set of all irreducible characters of the group G is $\{\chi_z \mid z \in G\}$ with χ_z given by

$$\chi_z(g) = (\omega_{n_1})^{z_1 g_1} (\omega_{n_2})^{z_2 g_2} \ldots (\omega_{n_k})^{z_k g_k}. \tag{2}$$

Note that the irreducible characters and representations are indexed by the elements in the Abelian group G. In addition, all irreducible characters are functions from G to the complex unit circle, which we will denote by $\mathbb{T}$. The set consisting of all irreducible characters of an Abelian group G is called the dual of the Abelian group G and is usually denoted by $\widehat{G}$.

Lemma 4.15 *A function $\chi : G \to \mathbb{T}$ is an irreducible character of the finite Abelian group G if and only if it is a homomorphism.*

Proof This follows directly from the fact that all irreducible representations of finite Abelian groups are one-dimensional. □

The set of the irreducible characters of a finite Abelian group has the structure of an Abelian group.

Lemma 4.16 *The irreducible characters of any finite Abelian group form a multiplicative Abelian group under the operation of pointwise multiplication.*

Proof Using Lemma 4.15 it is sufficient to show that the set of all homomorphisms from a finite Abelian group G into $\mathbb{T}$ is a group. The function $\chi : G \to \mathbb{T}$ given by $\chi(g) = 1$ is a homomorphism, since for all g, h in G we have $\chi(gh) = \chi(g)\,\chi(h) = 1$. This function is the identity element in the group. Assume that χ_1 and χ_2 are two homomorphisms from G into $\mathbb{T}$

then for all $g, h \in G$ we have that $\chi_1\chi_2(gh) = \chi_1(gh)\chi_2(gh) = \chi_1(g)\chi_1(h)\chi_2(g)\chi_2(h) = \chi_1\chi_2(g)\chi_1\chi_2(h)$ which proves that also $\chi_1\chi_2$ is a homomorphism from G into $\mathbb{T}$. Note that $\chi : G \to \mathbb{T}$ implies that $\chi(g)\overline{\chi(g)} = 1$ for all g. The inverse $\chi^{-1} = \overline{\chi}$ is a homomorphism from G into $\mathbb{T}$ because $\chi(gh) = \chi(g)\chi(h)$ for all $g, h \in G$ implies that $\overline{\chi}(gh) = \overline{\chi}(g)\overline{\chi}(h)$ for all $g, h \in G$. □

The dual $\widehat{G}$ of a finite Abelian group G is isomorphic to G. For the group $G = \mathbb{Z}/n_1\mathbb{Z} \oplus \mathbb{Z}/n_2\mathbb{Z} \oplus \ldots \oplus \mathbb{Z}/n_k\mathbb{Z}$ the isomorphism $\varphi : G \to \widehat{G}$ is given by $\phi(r) = \chi_r$ with χ_r defined as in (2). When $G \cong \widehat{G}$ we say that G is self-dual. The next lemma helps us find an orthonormal basis for $L^2(G)$ when G is a finite Abelian group.

Lemma 4.17 *Every function defined on an Abelian group is a class function.*

Proof If G is an Abelian group we have that $hgh^{-1} = g$ for all $h, g \in G$. Hence, for all $g \in G$ the conjugacy class containing g is the set $\{g\}$. As a result any function defined on G is constant on the conjugacy classes. □

Combining Lemmas 4.11 and 4.17 we find that the irreducible characters of any finite Abelian group G form an orthonormal basis for $L^2(G)$.

4.3 Example: a 3^3 factorial design analyzed using harmonic analysis

As an example of an application of harmonic analysis to factorial designs we consider the simplified seat-belt experiment that is analyzed in Section 5.1 of Wu and Hamada (2000). The goal of this experiment was to identify the factors that have an effect on the pull strength of truck seat belts following a crimping operation which joins an anchor and cable. The three factors considered are the hydraulic pressure of the crimping machine (X_1), die flat middle setting (X_2) and length of crimp (X_3). The design that was used is a 3^3 full factorial design with two replications. The observed strengths are given in Table 1. We analyze the data using harmonic analysis on the group $G = (\mathbb{Z}/3\mathbb{Z})^4$, where the fourth dimension corresponds to the replication. The set of irreducible representations for this group is $\{\rho_z \mid z \in G\}$ with ρ_z given in (1). The decomposition of the total sum of squares that was found using Theorem 4.12 and taking for ρ the regular representation is given in Table 2. The residual sum of squares is obtained by summing the sum of squares for the 54 irreducible representations $\rho_{(z_1,z_2,z_3,z_4)}$ with $z_4 \in \{1, 2\}$. Note that the sums of squares associated with an irreducible representation ρ_z and its complex-conjugate $\overline{\rho_z}$ are equal. This has some consequences for statistical inference that we discuss in Section 10.

From the decomposition given in Table 2 two other decompositions of the total sum of squares can be computed directly. The finest of these two is the decomposition into orthogonal components. The corresponding system of parametrization is called the orthogonal components system and is discussed in Section 5.3 of Wu and Hamada (2000). We mention it here only briefly. For a symmetric factorial design with k factors at n levels the component $X_1^{z_1}X_2^{z_2}\ldots X_k^{z_k}$ for $z \in (\mathbb{Z}/n\mathbb{Z})^k$ in this system represents the contrasts among the average response values observed on sets $C_0, \ldots, C_{n-2}, C_{n-1}$ where

$$C_j = \left\{(g_1, g_2, \ldots, g_k) \in (\mathbb{Z}/n\mathbb{Z})^k \mid z_1g_1 + z_2g_2 + \ldots + z_kg_k = j \mod n\right\}.$$

From Table 2 we find that the sum of squares for the components X_1X_2 and $X_1X_2^2$ are $2 \times 1.36373 \times 10^6 = 2.72746 \times 10^6$ and $2 \times 285397 = 570794$, respectively.

Table 1: Design matrix and response data of the Seat-Belt Experiment

	Factor					
Run	X_1	X_2	X_3	Strength		
1	0	0	0	5164	6615	5959
2	0	0	1	5356	6117	5224
3	0	0	2	3070	3773	4257
4	0	1	0	5547	6566	6320
5	0	1	1	4754	4401	5436
6	0	1	2	5524	4050	4526
7	0	2	0	5684	6251	6214
8	0	2	1	5735	6271	5843
9	0	2	2	5744	4797	5416
10	1	0	0	6843	6895	6957
11	1	0	1	6538	6328	4784
12	1	0	2	6152	5819	5963
13	1	1	0	6854	6804	6907
14	1	1	1	6799	6703	6792
15	1	1	2	6513	6503	6568
16	1	2	0	6473	6974	6712
17	1	2	1	6832	7034	5057
18	1	2	2	4968	5684	5761
19	2	0	0	7148	6920	6220
20	2	0	1	6905	7068	7156
21	2	0	2	6933	7194	6667
22	2	1	0	7227	7170	7015
23	2	1	1	7014	7040	7200
24	2	1	2	6215	6260	6488
25	2	2	0	7145	6868	6964
26	2	2	1	7161	7263	6937
27	2	2	2	7060	7050	6950

Table 2: Decomposition of the total sum of squares for the Seat-Belt Experiment

Effect	Orthogonal component	Irreducible representation	Dimension of representation	Sum of Squares
X_1	X_1	$\rho_{(1,0,0,0)}$	1	1.73109×10^7
		$\rho_{(2,0,0,0)}$	1	1.73109×10^7
X_2	X_2	$\rho_{(0,1,0,0)}$	1	469270
		$\rho_{(0,2,0,0)}$	1	469270
X_3	X_3	$\rho_{(0,0,1,0)}$	1	4.77474×10^6
		$\rho_{(0,0,2,0)}$	1	4.77474×10^6
$X_1 \times X_2$	X_1X_2	$\rho_{(1,1,0,0)}$	1	1.36373×10^6
		$\rho_{(2,2,0,0)}$	1	1.36373×10^6
	$X_1X_2^2$	$\rho_{(1,2,0,0)}$	1	285397
		$\rho_{(2,1,0,0)}$	1	285397
$X_1 \times X_3$	X_1X_3	$\rho_{(1,0,1,0)}$	1	1.4928×10^6
		$\rho_{(2,0,2,0)}$	1	1.4928×10^6
	$X_1X_3^2$	$\rho_{(1,0,2,0)}$	1	443294
		$\rho_{(2,0,1,0)}$	1	443294
$X_2 \times X_3$	X_2X_3	$\rho_{(0,1,1,0)}$	1	213607
		$\rho_{(0,2,2,0)}$	1	213607
	$X_2X_3^2$	$\rho_{(0,1,2,0)}$	1	10567
		$\rho_{(0,2,1,0)}$	1	10567
$X_1 \times X_2 \times X_3$	$X_1X_2X_3$	$\rho_{(1,1,1,0)}$	1	2.24646×10^6
		$\rho_{(2,2,2,0)}$	1	2.24646×10^6
	$X_1X_2X_3^2$	$\rho_{(1,1,2,0)}$	1	131508
		$\rho_{(2,2,1,0)}$	1	131508
	$X_1X_2^2X_3$	$\rho_{(1,2,1,0)}$	1	102768
		$\rho_{(2,1,2,0)}$	1	102768
	$X_1X_2^2X_3^2$	$\rho_{(1,2,2,0)}$	1	122720
		$\rho_{(2,1,1,0)}$	1	122720
residual				1.09226×10^7
total				6.88581×10^7

The other decomposition of the total sums of squares that we can compute directly from Table 2 is the standard ANOVA decomposition for a multi-way layout as discussed in Section 2.4 of Wu and Hamada (2000). For instance, in the seat-belt experiment the sum of squares for the interaction $X_1 \times X_2$ is $2 \times 1.36373 \times 10^6 + 2 \times 285397 = 3.29825 \times 10^6$.

5 Poisson summation formula

Good (1958) shows that the effects in a full factorial design can be efficiently calculated using a generalization of the algorithm of Yates (1937). In addition, he illustrates how the proposed algorithm can be used to speed up the computation of the discrete Fourier transform. In Good (1960), which is an addendum to Good (1958), the Poisson summation formula is introduced as a means to study the confounding of effects in factorial designs. Although Good (1960) presents the Poisson summation formula for the group $\mathbb{Z}/n_1\mathbb{Z} \oplus \mathbb{Z}/n_2\mathbb{Z} \oplus \ldots \oplus \mathbb{Z}/n_k\mathbb{Z}$ for arbitrary integers $n_1, n_2, \ldots, n_k$, its usefulness for the study of the confounding is only illustrated for the case where $n_1 = n_2 = \ldots = n_k = 2$. The Poisson summation formula is of great value when the data is obtained on a coset of a finite Abelian group and harmonic analysis is used for analyzing the data. The formula relates a sum of the function values f over a coset in a finite Abelian group G to a sum of the Fourier transforms $\widehat{f}$ over a coset in the dual $\widehat{G}$. Before we present the Poisson summation formula we introduce the quotient space and its dual. Let H be a subgroup of a group G. The quotient space G/H consists of the different cosets $gH = \{gh \mid h \in H\}$ of H in G.

Lemma 5.1 *Let H be a subgroup of a finite Abelian group G. The quotient space G/H is a finite Abelian group.*

Proof The group operation in G/H is the product given by

$$(aH)(bH) = \{ah_1bh_2 \mid h_1, h_2 \in H\} = \{abh_1h_2 \mid h_1, h_2 \in H\} = \{abh \mid h \in H\} = (ab)H.$$

From this we find that H is an identity element in G/H, that G/H is closed under multiplication and that $a^{-1}H$ is the inverse of aH. The associativity of the group operation follows by using that G is Abelian. This completes the proof. □

A concept that is related to the quotient space is that of the annihilator $\widehat{G}_H$ of a subgroup H of G (the hat in the notation for the annihilator will be justified by Lemma 5.3). The annihilator is defined as the set $\{\chi \in \widehat{G} \mid \chi(h) = 1 \text{ for all } h \in H\}$. This is the set of characters in $\widehat{G}$ that are constant on H and its cosets.

Lemma 5.2 *The annihilator $\widehat{G}_H$ of a subgroup H of a finite Abelian group G is a subgroup of $\widehat{G}$.*

Proof It is trivial to see that $1 \in \widehat{G}_H$. The set $\widehat{G}_H$ is closed under multiplication, since for all χ_1 and $\chi_2 \in \widehat{G}_H$ we have that $\chi_1\chi_2(h) = \chi_1(h)\chi_2(h) = 1$ for all $h \in H$. Let $\chi \in \widehat{G}_H$ and denote by χ^{-1} its inverse in $\widehat{G}$ then $\chi(h) = 1$ for all $h \in H$. From $\chi^{-1}(h)\chi(h) = 1$ for all $h \in G$ we find that $\chi^{-1}(h) = 1$ on H and $\chi^{-1} \in \widehat{G}_H$. Hence, we have shown that each annihilator $\widehat{G}_H$ is a subgroup in $\widehat{G}$. □

Combining Lemmas 5.1 and 4.16 we find that the dual of the quotient space is a multiplicative Abelian group. The dual of the quotient space G/H can be shown to be isomorphic to the annihilator $\widehat{G}_H$.

Lemma 5.3 *Let H be a subgroup of a finite Abelian group G. The dual of the quotient space G/H is isomorphic to $\widehat{G}_H$, that is, $\widehat{G/H} \cong \widehat{G}_H$.*

Proof Let T denote a transversal of H in G. We define the function $\varphi : \widehat{G}_H \to \widehat{G/H}$ by $(\varphi(\chi))(aH) = \chi(a)$ for all $a \in T$. We first show that the function φ is a homomorphism. This follows from observing that for all χ_1 and $\chi_2 \in \widehat{G}_H$ we have that $(\varphi(\chi_1\chi_2))(aH) = \chi_1\chi_2(a) = \chi_1(a)\chi_2(a) = (\varphi(\chi_1))(aH)(\varphi(\chi_2))(aH)$. Define the function $\psi : \widehat{G/H} \to \widehat{G}_H$ by $(\psi(\alpha))(g) = \alpha(aH)$ for all $g \in G$ where a is the unique element in T that satisfies $g \in aH$. Since $(\varphi(\psi(\alpha)))(aH) = (\psi(\alpha))(a) = \alpha(aH)$ for all $a \in T$ and $\alpha \in \widehat{G/H}$ it follows that φ is surjective. Injectivity follows from the fact that $(\psi(\varphi(\chi)))(g) = (\psi(\chi))(aH) = \chi(a) = \chi(a)\chi(h) = \chi(ah) = \chi(g)$ holds for all $a \in T$ and all $g \in aH$. Hence, the group homomorphism φ is an isomorphism. $\square$

In the proof for the Poisson summation formula we need the following lemma.

Lemma 5.4 *Let H be a subgroup of an Abelian group G. Then for $\chi \in \widehat{G}$ we have that*

$$\sum_{h \in H} \chi(h) = \begin{cases} \mid H \mid & \text{for } \chi \in \widehat{G}_H \\ 0 & \text{for } \chi \notin \widehat{G}_H \end{cases}.$$

Proof Let the function $\chi_{|H} : H \to \mathbb{T}$ be defined by $\chi_{|H}(h) = \chi(h)$ for all $h \in H$. For $\chi \in \widehat{G}_H$ we have by definition that $\chi_{|H} = 1$ on H, which implies that $\sum_{h \in H} \chi(h) = \mid H \mid$. The result for $\chi \notin \widehat{G}_H$ is obtained by first observing that $\chi_{|H}$ is a character on H. Since $\chi_{|H} \neq 1$ on H we must have that characters $\chi_{|H}$ and 1 are orthogonal on H. From this we find that $\sum_{h \in H} \chi(h) = 0$, which completes the proof. $\square$

We are now ready to present the Poisson summation formula.

Theorem 5.5 *Let H be a subgroup of a finite Abelian group G and $f : G \to \mathbb{C}$. Then for $a \in G$, $g \in G$ and $\alpha \in \widehat{G}$ we have*

$$\frac{1}{\mid H \mid} \sum_{h \in H} \overline{\alpha}(agh) f(agh) = \sum_{\chi \in \widehat{G}_H} \widehat{f}(\alpha\chi)\chi\left(ag^{-1}\right) \tag{3}$$

with important special cases

$$\frac{1}{\mid H \mid} \sum_{h \in H} \overline{\alpha}(ah) f(ah) = \sum_{\chi \in \widehat{G}_H} \widehat{f}(\alpha\chi)\chi(a) \tag{4}$$

and

$$\frac{1}{\mid H \mid} \sum_{h \in H} f(h) = \sum_{\chi \in \widehat{G}_H} \widehat{f}(\chi). \tag{5}$$

Proof Define the function $s : G \to \mathbb{C}$ by $s(g) = \sum_{h \in H} \overline{\alpha}(agh) f(agh)$. The Fourier transform $\widehat{s}(\chi)$ is

$$\widehat{s}(\chi) = \frac{1}{\mid G \mid} \sum_{g \in G} s(g)\chi(g) = \frac{1}{\mid G \mid} \sum_{g \in G} \sum_{h \in H} \overline{\alpha}(agh) f(agh) \overline{\chi}(g).$$

Since any character of the Abelian group G is also a representation of G we have that

$$\widehat{s}(\chi) = \frac{\overline{\chi}\left(a^{-1}\right)}{\mid G \mid} \sum_{g \in G} \sum_{h \in H} \overline{\alpha}(agh) f(agh) \overline{\chi}(agh) \overline{\chi}\left(h^{-1}\right).$$

After changing the order of summation we can rewrite $\widehat{s}(\chi)$ as

$$\widehat{s}(\chi) = \chi(a) \sum_{h \in H} \frac{\chi(h)}{\mid G \mid} \sum_{g \in G} f(agh) \overline{\alpha\chi}(agh) = \chi(a) \sum_{h \in H} \chi(h) \widehat{f}(\alpha\chi) = \widehat{f}(\alpha\chi) \chi(a) \sum_{h \in H} \chi(h).$$

Using Lemma 5.4 we find that $\widehat{s}(\chi) =\mid H \mid \widehat{f}(\alpha\chi) \chi(a)$ for $\chi \in \widehat{G}_H$ and $\widehat{s}(\chi) = 0$ for $\chi \notin \widehat{G}_H$. The Fourier-Bessel expansion of s is

$$s(g) = \sum_{\chi \in \widehat{G}} s(g) \overline{\chi}(g) = \sum_{\chi \in \widehat{G}_H} \mid H \mid \widehat{f}(\alpha\chi) \chi(a) \overline{\chi(g)} =\mid H \mid \sum_{\chi \in \widehat{G}_H} \widehat{f}(\alpha\chi) \chi\left(ag^{-1}\right).$$

From this we find (3) by substitution of $\sum_{h \in H} \overline{\alpha}(agh) f(agh)$ for $s(g)$. The form given in (4) follows by setting g equal to 1 in (3). Finally, if in (4) we set $\alpha = 1$ and $a = 1$ we find (5). □

The use of this formula to study confounding of effects is illustrated in Section 9.

6 Regular fractions of factorial designs

In this section we give an overview of the different definitions for regular fraction that have appeared in the literature. First, in Section 6.1, we introduce some terminology for factorial designs. In Section 6.2 the different definitions for regular fractions are presented.

6.1 Factors, partitions and interaction spaces

Consider a the full factorial design with its runs coded by the elements in the set $D = L_1 \oplus L_2 \oplus \ldots \oplus L_k$, where the n_i elements of L_i represent the distinct levels at which the factor X_i is set. The elements of L_i are called the factor levels or labels of X_i. Following Tjur (1984) we define a factor in a formal way as a mapping from an indexing set to a finite set of factor levels or labels.

Definition 6.1 *Let the runs in a full factorial design be coded by the elements of a set $D = L_1 \oplus L_2 \oplus \ldots \oplus L_k$. The factor X_i is the mapping $X_i : D \to L_i$ that maps each element $d \in D$ to its ith coordinate d_i.*

The factor X_i may be identified with the equivalence relation that it induces on D, elements of D being equivalent to each other if L_i takes the same value on them. This equivalence relation is also called the partition $\sigma_{\{i\}}$ of G induced by X_i. The interaction $X_i X_j$, where $i < j$, is defined as $X_i X_j : G \to L_i \oplus L_j$ with $X_i X_j(d) = (d_i, d_j)$. The interaction $X_i X_j$ induces a partition $\sigma_{\{i,j\}}$ of G in the same way as X_i does. The partition $\sigma_{\{i,j\}}$ is finer than $\sigma_{\{i\}}$ (or, equivalently, $\sigma_{\{i\}}$ nests $\sigma_{\{i,j\}}$ or $\sigma_{\{i\}}$ is coarser that $\sigma_{\{i,j\}}$). This is usually denoted by $\sigma_{\{i\}} > \sigma_{\{i,j\}}$. Generalization to higher order interactions is straightforward. By S we denote set containing all subsets of $\{1, 2, \ldots, k\}$ and we let $\Sigma = \{\sigma_I \mid I \subset S\}$. The set Σ is usually referred to as the complete factorial structure. An extensive treatment of partitions can be found in Bailey (1996).

Example 6.2 *Consider the $2^1 4^2$ factorial design coded as the Abelian group $G = \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/4\mathbb{Z}$. The complete factorial structure is $\Sigma = \{\sigma_\emptyset, \sigma_{\{1\}}, \sigma_{\{2\}}, \sigma_{\{1,2\}}\}$ where the equivalence classes given by*

$$\begin{array}{ll} \sigma_\emptyset & = \{(0,0), (0,1), (0,2), (0,3), (1,0), (1,1), (1,2), (1,3)\} \\ \sigma_{\{1\}} & = \{(0,0), (0,1), (0,2), (0,3)\} \cup \{(1,0), (1,1), (1,2), (1,3)\} \\ \sigma_{\{2\}} & = \{(0,0), (1,0)\} \cup \{(0,1), (1,1)\} \cup \{(0,2), (1,2)\} \cup \{(0,3), (1,3)\} \\ \sigma_{\{1,2\}} & = \{(0,0)\} \cup \{(0,1)\} \cup \{(0,2)\} \cup \{(0,3)\} \cup \{(1,0)\} \cup \{(1,1)\} \cup \{(1,2)\} \cup \{(1,3)\} \end{array}$$

form a partition of G. The partition $\sigma_\emptyset$ is usually referred to as the trivial partition.

A function $f : D \to \mathbb{C}$ is called a contrast on D if it satisfies $\langle f, 1\rangle_D := \sum_{d\in D} f(d) = 0$. A contrast defined on D may be associated with the factor X_i if it is constant on the classes of D induced by the partition $\sigma_{\{i\}}$ but not constant on D (that is the class induced by $\sigma_\emptyset$). Similarly, a function may be associated with the interaction $X_i X_j$ if its constant on the equivalence classes of D induced by the partition $\sigma_{\{i,j\}}$ but not constant on the classes of D induced by either $\sigma_{\{i\}}$ or $\sigma_{\{j\}}$. For I a non-empty subset in S we denote by X_I the effect $\prod_{i\in I} X_i$. In general, we associate a contrast f with an effect X_I if f is constant on the classes of D induced by σ_I but not constant on classes induced by σ_J for any $J \subset I$. A factorial effect is defined to be any function $f : D \to \mathbb{C}$ that is either constant on D or a contrast on D that can be associated with a factor or interaction.

The interaction space H_I, where $I \subset S$, is defined to be the subset of $L^2(D)$ that consists of all functions can be associated with the interaction X_I. The interaction spaces in $\{H_I \mid I \subset S\}$ are pairwise disjoint and together they span $L^2(D)$. When each set of levels $L_i, 1 \le i \le k$, is a cyclic group (that is, the group L_i is generated by a single element), we can use the characters of the group D to construct an orthogonal basis for each of the interaction spaces in $\{H_I \mid I \subset S\}$. To this end, let n_i denote the order of the group L_i. Without loss of generality we assume that each for each $1 \le i \le k$ we have that $L_i = \mathbb{Z}/n_i\mathbb{Z}$, in which case the full factorial design is given by the additive group $D = \mathbb{Z}/n_1\mathbb{Z} \oplus \mathbb{Z}/n_2\mathbb{Z} \oplus \ldots \oplus \mathbb{Z}/n_k\mathbb{Z}$. By e_i we denote the element in D whose jth coordinate, where $1 \le j \le k$, is given by

$$(e_i)_j = \begin{cases} 0 & i \ne j \\ 1 & i = j \end{cases}.$$

Note that since $\widehat{D} \cong D$ we can index the characters in $\widehat{D}$ by the elements of D, that is, $\widehat{D} = \{\chi_d \mid d \in D\}$. Each character of D can be expressed as a monomial in $\chi_{e_1}, \chi_{e_2}, \ldots, \chi_{e_k}$. More precisely, if $d \in D$ then $\chi_d = \prod_{i=1}^k (\chi_{e_i})^{d_i}$. For all $I \subseteq S$ we have that

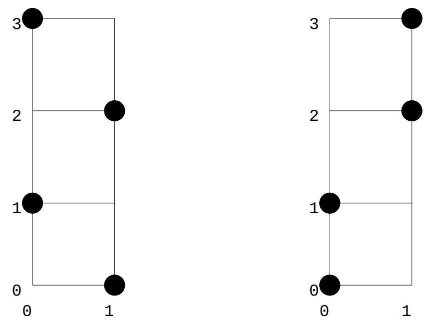
$$H_I = \operatorname{span}\{\chi_d \mid d_i \ne 0 \text{ for } i \in I \text{ and } d_i = 0 \text{ for } i \in S \setminus I\}.$$

Since all characters in $\widehat{D}$ are orthogonal, the set given by $\{\chi_d \mid d_i \ne 0 \text{ for } i \in I \text{ and } d_i = 0 \text{ for } i \in S \setminus I\}$ is an orthogonal basis for the interaction space H_I. The interaction space $H_\emptyset$ is given by $H_\emptyset = \operatorname{span}\{\chi_{(0,0,\ldots,0)}\}$

6.2 Definitions for regular fractions

Full factorial designs require that $n_1 n_2 \ldots n_k$ runs be performed, which implies that they can become quite costly when the number of factors is large. When running a complete full factorial design is too expensive, typically only a subset of the runs of the full factorial design are performed. We refer to such a subset F of a full factorial design D as a fractional factorial design or simply as a fraction.

Figure 3: A regular fraction F_1 (left) and a non-regular fraction F_2 (right) of a $2^1 4^1$ experiment



Example 6.3 *Two fractions F_1 and F_2 of the $2^1 4^1$ factorial design are given in Figure 3. The fraction F_1 is coded by the subset $\{(1,0),(0,1),(1,2),(0,3)\}$ of the Abelian group $G = \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/4\mathbb{Z}$. The fraction F_2 is coded by the subset $\{(0,0),(0,1),(1,2),(1,3)\}$ of G.*

A special class of fractions is formed by the regular fractions. Regular fractions are typically constructed using group-theoretic methods and have a confounding structure which is relatively simple. Several definitions for the regular fractions exist. In this section we give an overview of these definitions. However, we first introduce the definitions of orthogonality and (complete and partial) confounding on a fraction.

Definition 6.4 *Let the runs of a full factorial design be coded by the elements in the set $D = L_1 \oplus L_2 \oplus \ldots \oplus L_k$. The functions $f_1, f_2 : D \to \mathbb{C}$ are orthogonal on a fraction $F \subseteq D$ if*

$$\langle f_1, f_2 \rangle_F := \frac{1}{\mid F \mid} \sum_{d \in F} f_1(d) \overline{f_2}(d) = 0.$$

The functions f_1 and f_2 are completely confounded on F if there exists a non-zero constant $c \in \mathbb{C}$ such that $f_1(d) = c f_2(d)$ for all $d \in F$. The functions f_1 and f_2 are partially confounded on F if they are neither orthogonal nor completely confounded on F.

This definition is illustrated in the next example.

Example 6.5 *Consider again the fractions F_1 and F_2 of the $2^1 4^1$ factorial design given in Example 6.5. The characters of the Abelian group $G = \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/4\mathbb{Z}$ are given by*

$$\chi_z(g) = (-1)^{z_1 g_1} i^{z_2 g_2} \text{ for } z \in G.$$

The next table shows the characters $\chi_z, z \in G$, evaluated on F_1.

g	$\chi_{(0,0)}(g)$	$\chi_{(0,1)}(g)$	$\chi_{(0,2)}(g)$	$\chi_{(0,3)}(g)$	$\chi_{(1,0)}(g)$	$\chi_{(1,1)}(g)$	$\chi_{(1,2)}(g)$	$\chi_{(1,3)}(r)$
$(1,0)$	1	1	1	1	-1	-1	-1	-1
$(0,1)$	1	i	-1	$-i$	1	i	-1	$-i$
$(1,2)$	1	-1	1	1	-1	-1	-1	1
$(0,3)$	1	$-i$	-1	i	1	$-i$	-1	i

On the fraction F_1 we have that $\chi_{(1,2)} = -\chi_{(0,0)}$, which means that $\chi_{(1,2)}$ is completely confounded with $\chi_{(0,0)}$ on F_1. In addition we have that $\chi_{(1,3)} = -\chi_{(0,1)}, \chi_{(1,0)} = -\chi_{(0,2)}$ and

$\chi_{(0,3)} = -\chi_{(1,1)}$. *The characters that are not completely confounded on* F_1 *can be shown to be orthogonal on* F_1. *To illustrate partial confounding we consider the fraction* F_2. *The characters* $\chi_r, r \in G$, *evaluated on* F_2 *are given in the next table.*

g	$\chi_{(0,0)}(g)$	$\chi_{(0,1)}(g)$	$\chi_{(0,2)}(g)$	$\chi_{(0,3)}(g)$	$\chi_{(1,0)}(g)$	$\chi_{(1,1)}(g)$	$\chi_{(1,2)}(g)$	$\chi_{(1,3)}(r)$
$(0,0)$	1	1	1	1	1	1	1	1
$(0,1)$	1	i	-1	$-i$	1	i	-1	$-i$
$(1,2)$	1	1	1	1	-1	1	-1	-1
$(1,3)$	1	$-i$	-1	i	-1	i	1	$-i$

Observe that $\langle \chi_{(0,0)}, \chi_{(1,1)} \rangle_{F_2} = 2-2i$ *and that there does not exist an element* $c \in \mathbb{C}$ *for which* $\chi_{(0,0)} = c\chi_{(1,1)}$. *This means that the characters* $\chi_{(0,0)}$ *and* $\chi_{(1,1)}$ *are partially confounded on* F_2.

Different definitions of regular fraction are given in Collombier (1996), Wu and Hamada (2000), Pistone and Rogantin (2005) and Dey and Mukerjee (1999). Each of these definitions is briefly discussed here.

In Chapter 5 of Collombier (1996) the runs of a full factorial design are identified with the elements of an (arbitrary) finite Abelian group G. The corresponding definition of a regular fraction is as follows.

Definition 6.6 (Collombier) *Let the runs of a full factorial design be coded by the elements of the finite Abelian group* G. *A fraction* $F \subseteq G$ *is regular if it is a coset of the full factorial design* G.

The fraction F_1 that we considered in Example 6.3 is a coset in $G = \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/4\mathbb{Z}$. The fraction F_2 that was considered in the same example is not a coset in G. Hence, according to Definition 6.6 F_1 is a regular fraction of G and F_2 is a non-regular fraction. Definition 6.6 appears to be the most convenient definition for regular fraction when considering factorial designs within the framework of harmonic analysis. It is considered in detail in Sections 7, 8, 9 and 10.2.

The definition given in the introduction of Section 5 of Wu and Hamada (2000) has to be associated with a specific set of factorial effects.

Definition 6.7 (Wu and Hamada) *Let the runs of a full factorial design be coded by the elements in* $D = L_1 \oplus L_2 \oplus \ldots \oplus L_k$. *Let* $\{f_1, f_2, \ldots, f_r\}$ *with* $f_i : D \to \mathbb{C}$ *be the set containing all factorial effects of interest. A fraction* $F \subseteq D$ *is regular if for all* i *and* j, $1 \le i \le j \le r$, *the functions* f_i *and* f_j *are either orthogonal or completely confounded on* F.

Equivalently, a fraction is regular if there is no partial confounding of the functions in the set $\{f_1, f_2, \ldots, f_r\}$ on the fraction F. Note that in Example 6.5 the characters are not partially confounded on F_1, but on F_2 they are. Hence, when the set of factorial effects under consideration is the set of characters $\widehat{G}$ of the group $G = \mathbb{Z}/2 \oplus \mathbb{Z}\mathbb{Z}/4\mathbb{Z}$ then fraction F_1 is regular and fraction F_2 is non-regular.

Pistone and Rogantin (2005) consider regular fractions of symmetric factorial designs. In presenting their definition we assume that all k factors are set at n levels. Pistone and Rogantin (2005) code the runs in the full factorial design by the elements in the multiplicative Abelian group $D = (\Omega_n)^k$. Their definition for a regular fractions is the following.

Definition 6.8 (Pistone and Rogantin) *Let the runs of a full factorial design be coded by the elements of the finite Abelian group* $D = (\Omega_n)^k$. *A fraction* $F \subseteq D$ *is regular if there exists a subgroup* L *of* $(\mathbb{Z}/n\mathbb{Z})^k$ *and a homorphism* $\psi : L \to \Omega_n$ *for which*

$$F = \{(d_1, d_2, \ldots, d_k) \in D \mid d_1^{\ell_1} d_2^{\ell_2} \ldots d_k^{\ell_k} = \psi(\ell) \text{ for all } \ell \in L\}.$$

An equivalent definition of a regular fraction is given in terms of the indicator function. The indicator function was first introduced in the literature on design of experiments in Fontana et al. (2000) as a means to characterize fractions of two-level factorial designs. The coefficients of its polynomial expansion provide useful information regarding orthogonality of effects and design properties as regularity and resolution. The representation of the fraction by its indicator function was generalized to two-level factorial designs with replications by Ye (2003) and multilevel factorial experiments using orthogonal polynomials and integer coding of levels by Cheng and Ye (2004). Pistone and Rogantin (2005) use the indicator function for multilevel factorial experiments with the factor levels coded by the complex roots of unity. In this paper we only consider the case in which there are no replications. We apply the indicator function to the case where G is a finite Abelian group and use its expansion with respect to the orthogonal basis for $L^2(G)$ formed by the characters in $\widehat{G}$. For G a full factorial design coded as a finite Abelian group, the indicator function G of a fraction $F \subseteq G$ is the function $\mathcal{I}_F : G \to \{0, 1\}$ defined by

$$\mathcal{I}_F(g) = \begin{cases} 1 & \text{if } g \in F \\ 0 & \text{if } g \notin F \end{cases}.$$

The indicator function is expressed in a unique way as a linear combination of the characters in $\widehat{G}$ using the Fourier-Bessel expansion,

$$\mathcal{I}_F = \sum_{\chi \in \widehat{G}} \widehat{\mathcal{I}_F}(\chi)\, \chi \text{ where } \widehat{\mathcal{I}_F}(\chi) = \langle \mathcal{I}_F, \chi \rangle_G = \frac{\mid F \mid}{\mid G \mid} \langle 1, \chi \rangle_F. \tag{6}$$

Let for $g \in (\mathbb{Z}/n\mathbb{Z})^k$ the function $\chi_g : D \to \Omega_n$ be defined as $\chi_g(d) = d_1^{g_1} d_2^{g_2} \ldots d_k^{g_k}$. The functions in the set $\{\chi_g \mid g \in (\mathbb{Z}/n\mathbb{Z})^k\}$ form an orthonormal basis for $L^2(D)$. This follows from observing that the functions in $\{\chi_g \mid g \in (\mathbb{Z}/n\mathbb{Z})^k\}$ are exactly the irreducible characters of the Abelian group D. A fraction F of D is regular if its indicator function $\mathcal{I}_F : D \to \{0, 1\}$ expressed in terms of the orthonormal basis $\{\chi_g \mid g \in (\mathbb{Z}/n\mathbb{Z})^k\}$ has the form

$$\mathcal{I}_F = \frac{1}{\mid L \mid} \sum_{\ell \in L} \overline{\psi(\ell)} \chi_\ell,$$

for L a subgroup of $(\mathbb{Z}/n\mathbb{Z})^k$ and $\psi : L \to \Omega_n$ a homomorphism. A similar result was obtained independently by Ye (2004). He also used the multiplicative Abelian group $D = (\Omega_n)^k$ to code the runs in a symmetric full factorial design. Ye (2004) shows that a fraction $F \subseteq D$ is regular if and only if for all coefficients $\widehat{\mathcal{I}_F}(\chi_g)$ in the expansion

$$\mathcal{I}_F = \sum_{g \in (\mathbb{Z}/n\mathbb{Z})^k} \widehat{\mathcal{I}_F}(\chi_g)\, \chi_g$$

we have that $\left(\frac{\widehat{\mathcal{I}_F}(\chi_g)}{\widehat{\mathcal{I}_F}(\chi_0)}\right)^n = 1$ or $\widehat{\mathcal{I}_F}(\chi_g) = 0$.

The definition of Dey and Mukerjee (1999) is based on the finite field approach introduced in Bose (1947). This approach is restricted to symmetric factorials where the number of levels n is a prime power. Only for these values of n there exists a field $GF(n)$ with n elements. The field $GF(n)$ is called the Galois field of order n and is unique up to isomorphism. If n is prime, then $GF(n) \cong \mathbb{Z}/n\mathbb{Z}$. If $n = p^q$ for some prime number p and integer $q > 1$, then $GF(n)$ can be represented by p-ary polynomials modulo an irreducible polynomial of degree q. Within this approach the full factorial design is coded by the elements in $D = (GF(n))^k$ and a regular fraction is defined as follows.

Definition 6.9 (Dey and Mukerjee) *Let the runs of a full factorial design be coded by the vectors in $D = (GF(n))^k$. A fraction $F \subseteq D$ is regular if there exist a $c \in (GF(n))^k$ and a $p \times k$ matrix A ($p \leq k$) with entries in $GF(n)$ for which*

$$F = \{z \in (GF(n))^k \mid Az = c\}$$

Pistone and Rogantin (2005) showed that for n^k factorial designs with n a prime power Definition 6.8 was equivalent to Definition 6.9.

7 Defining equations for cosets

In this section we adopt Definition 6.6 for regular fraction. We use character theory to show how any regular fraction can be defined as the solution of a set of equations. The multiplicative defining relations for regular fractions are found easily using this definition. The approach is similar to that described in Section 5.1 of Collombier (1996).

Lemma 7.1 *Given an Abelian group G, a non-empty subset $S \subseteq \widehat{G}$ and an element $a \in G$. The set defined by $\{g \in G \mid \chi(g) = \chi(a) \text{ for all } \chi \in S\}$ is a coset in G. In particular, the set defined by $\{h \in G \mid \chi(h) = 1 \text{ for all } \chi \in S\}$ is a subgroup of G*

Proof Let $C = \{g \in G \mid \chi(g) = \chi(a) \text{ for all } \chi \in S\}$. Define H by $H := a^{-1}C$. We will show that H is a subgroup of G. Note that

$$\begin{aligned} H = a^{-1}C &= \{a^{-1}g \in G \mid \chi(g) = \chi(a) \text{ for all } \chi \in S\} \\ &= \{a^{-1}g \in G \mid \chi(a^{-1})\chi(g) = \chi(a^{-1})\chi(a) \text{ for all } \chi \in S\} \\ &= \{a^{-1}g \in G \mid \chi(a^{-1}g) = 1 \text{ for all } \chi \in S\} \\ &= \{h \in G \mid \chi(h) = 1 \text{ for all } \chi \in S\}. \end{aligned}$$

We show that the set H is a subgroup of G. First because $\chi(1) = 1$ for all $\chi \in \widehat{G}$ we have that $1 \in H$. In addition for $h_1, h_2 \in H$ and $\chi \in S$ we have that $\chi(h_1h_2) = \chi(h_1)\chi(h_2) = 1$. This implies that for h_1 and $h_2 \in H$ we have that $h_1h_2 \in H$. Now it remains to show that every element $h \in H$ has its inverse in H. Clearly, $h^{-1} \in G$ and for all $\chi \in S$ we have that $\chi(h^{-1}) = \chi(h^{-1})\chi(h) = \chi(h^{-1}h) = \chi(1) = 1$. This implies that $h^{-1} \in H$ and, hence, we have shown that H is a subgroup of G. The result that C is a coset follows from observing that $C = aH$. □

Lemma 7.1 gives a way of defining subgroups and cosets of an Abelian group G. The set S can in terms of experimental design be thought of as a set of contrasts that are chosen to be completely confounded with the mean.

Lemma 7.2 *Let H be a subgroup of an Abelian group G and $a \in G$. Then $aH = \{g \in G \mid \chi(g) = \chi(a)$ for all $\chi \in \widehat{G}_H\}$. In particular, $H = \{h \in G \mid \chi(h) = 1$ for all $\chi \in \widehat{G}_H\}$.*

Proof By the definition of $\widehat{G}_H$ we have for all $\chi \in \widehat{G}_H$ that $\chi(h) = 1$ for each $h \in H$. Then for any $a \in G$ we have that $\chi(ah) = \chi(a)$. From this $aH \subseteq \{h \in G \mid \chi(h) = \chi(a)$ for all $\chi \in \widehat{G}_H\}$. For the converse, recall from Lemma 5.3 that $\widehat{G}_H$ is isomorphic to the dual of the quotient space. From this we find that the elements in $\widehat{G}_H$ form an orthonormal basis for the linear space of all complex-valued functions that are constant on the cosets of H in G. Now assume that there exists an element $gh \notin aH$ where $h \in H$ for which $\chi(gh) = \chi(a)$ for all $\chi \in \widehat{G}_H$. Then using $\chi(a) = \chi(gh) = \chi(g)\chi(h) = \chi(g)$ we find that for all $\chi \in \widehat{G}_H$ and all $h \in aH \cup gH$ (where aH and gH are different cosets) we have that $\chi(h) = \chi(a)$. This contradicts that $\widehat{G}_H$ is an orthogonal basis for the linear space of complex-valued functions defined on G that are constant on the cosets of H. Hence, $\{h \in G \mid \chi(h) = \chi(a)$ for all $\chi \in \widehat{G}_H\} \subseteq aH$. This completes the proof for $aH = \{g \in G \mid \chi(g) = \chi(a)$ for all $\chi \in \widehat{G}_H\}$. The statement $H = \{h \in G \mid \chi(h) = 1$ for all $\chi \in \widehat{G}_H\}$ follows directly by choosing $a = 1$. □

Lemma 7.2 shows how any coset aH of an Abelian group G can be described as the solution of a set of equations. In literature on design of experiments the subgroup $\widehat{G}_H$ is usually referred to as the defining contrasts subgroup. The coset aH is the the fraction defined by the equation. When $a = 1$ the fraction is called a principal fraction. The theory is illustrated in the next example. We use an example from the well-studied class of two-level factorial designs to illustrate how the multiplicative defining equations arise within the proposed algebraic framework.

Example 7.3 *Consider a full 2^4 factorial design coded as the additive Abelian group $G = (\mathbb{Z}/2\mathbb{Z})^4$. The set of irreducible characters for this group is given by $\{\chi_z \mid z \in G\}$ where*

$$\chi_z(g) = (-1)^{z_1 g_1}(-1)^{z_2 g_2}(-1)^{z_3 g_3}(-1)^{z_4 g_4}.$$

Let $S = \{\chi_z \mid \ z \in \{(1,1,1,0), (0,1,1,1)\}\}$. The subgroup $H = \{g \in G \mid \chi(g) = 1$ for all $\chi \in S\} = \{(0,0,0,0), (0,1,1,0), (1,0,1,1), (1,1,0,1)\}$ is a regular fraction of G. The subgroup H by definition contains exactly those elements $g \in G$ that satisfy the following equations

$$\begin{aligned} \chi_{(1,1,1,0)}(g) &= (-1)^{g_1}(-1)^{g_2}(-1)^{g_3} = 1, \\ \chi_{(0,1,1,1)}(g) &= (-1)^{g_2}(-1)^{g_3}(-1)^{g_4} = 1. \end{aligned}$$

If we define factors $I : G \to \{1\}$ by $I' := \chi_{(0,0,0,0)}$ and $X_1', X_2', X_3', X_4' : G \to \Omega_2$ by $X_j' := \chi_{e_j}$, then the two equations can be written as $X_1'X_2'X_3' = I$ and $X_2'X_3'X_4' = I$. These equations are called multiplicative defining relations for the fraction H. Note that the fraction $H = \{g \in G \mid \chi(g) = 1$ for all $\chi \in \widehat{G}_H\}$ where $\widehat{G}_H = \{\chi_z \mid z \in \{(0,0,0,0), \{(1,1,1,0), (0,1,1,1), (1,0,0,1)\}$. The fraction H is defined as the set of all elements $g \in G$ that satisfy $\chi_z(g) = 1$ for all $z \in \{(0,0,0,0), \{(1,1,1,0), (0,1,1,1), (1,0,0,1)\}$. More specifically, H consists of all $g \in G$ that satisfy the following equations

$$\begin{aligned} \chi_{(0,0,0,0)}(g) &= 1, \\ \chi_{(1,1,1,0)}(g) &= (-1)^{g_1}(-1)^{g_2}(-1)^{g_3} = 1, \\ \chi_{(0,1,1,1)}(g) &= (-1)^{g_2}(-1)^{g_3}(-1)^{g_4} = 1, \\ \chi_{(1,0,0,1)}(g) &= (-1)^{g_1}(-1)^{g_4} = 1, \end{aligned}$$

which gives the relation

$$1 = (-1)^{g_1}(-1)^{g_2}(-1)^{g_3} = (-1)^{g_2}(-1)^{g_3}(-1)^{g_4} = (-1)^{g_1}(-1)^{g_4}.$$

Hence, H consists of all $g \in G$ on which functions $X_1'X_2'X_3'$, $X_2'X_3'X_4'$, $X_1'X_4'$ are equal to 1. That is, it is the set of consisting of all solutions in G for the system $I = X_1'X_2'X_3' = X_2'X_3'X_4' = X_1'X_4'$ of equations.

The cosets $a + H$ of H in G are regular fractions. On the fraction $F = a + H$ we have that $\chi(g) = \chi(a)$ for all $\chi \in \widehat{G}_H$, more specifically, for all $g \in F$ and $a = (a_1, a_2, a_3, a_4)$ we have

$$\begin{array}{rcl} \chi_{(0,0,0,0)}(g) & = & 1, \\ \chi_{(1,1,1,0)}(g) & = & (-1)^{g_1}(-1)^{g_2}(-1)^{g_3} \;=\; (-1)^{a_1}(-1)^{a_2}(-1)^{a_3}, \\ \chi_{(0,1,1,1)}(g) & = & (-1)^{g_2}(-1)^{g_3}(-1)^{g_4} \;=\; (-1)^{a_2}(-1)^{a_3}(-1)^{a_4}, \\ \chi_{(1,0,0,1)}(g) & = & (-1)^{g_1}(-1)^{g_4} \;=\; (-1)^{a_1}(-1)^{a_4}. \end{array}$$

For $a = (1, 0, 0, 0)$ we find that for all $g \in F$,

$$\begin{array}{rclcr} \chi_{(0,0,0,0)}(g) & = & 1, & & \\ \chi_{(1,1,1,0)}(g) & = & (-1)^{g_1}(-1)^{g_2}(-1)^{g_3} & = & -1, \\ \chi_{(0,1,1,1)}(g) & = & (-1)^{g_2}(-1)^{g_3}(-1)^{g_4} & = & 1, \\ \chi_{(1,0,0,1)}(g) & = & (-1)^{g_1}(-1)^{g_4} & = & -1. \end{array}$$

The fraction $(1, 0, 0, 0) + H$ is the set of all solutions in G to the system $I = -X_1'X_2'X_3' = X_2'X_3'X_4' = -X_1'X_4'$ of equations. Note that the functions X_1', X_2', X_3', X_4' are the factors when the runs in the design are coded by the elements in the multiplicative group $(\Omega_2)^4$.

8 Equivalence of definitions

In this section we present the main results of this paper. We show the equivalence of the definitions for regular fractions given by Collombier (1996) (Definition 6.6), Wu and Hamada (2000) (Definition 6.7) and Pistone and Rogantin (2005) (Definition 6.8). The equivalence is shown using the character theory of finite Abelian groups.

The following theorem states an important property of the characters of a finite Abelian group. The theorem is a slight extension of the results stated in Propositions 1.4 and 1.5 in Chapter 5 of Collombier (1996).

Theorem 8.1 *Let H be a subgroup of an Abelian group G and $a \in G$. The irreducible characters $\chi_1, \chi_2 \in \widehat{G}$ are either orthogonal or completely confounded on the coset aH. More specifically, the irreducible characters χ_1 and χ_2 are orthogonal on aH if they belong to different cosets of $\widehat{G}_H$ and are completely confounded on aH if they belong to the same coset of $\widehat{G}_H$.*

Proof Choose an irreducible character $\alpha \in \widehat{G}$ and consider the coset $\alpha\widehat{G}_H$ in $\widehat{G}$. This is the set

$$\alpha\widehat{G}_H = \{\chi \in \widehat{G} \mid \chi(h) = \alpha(h) \text{ for all } h \in H\}.$$

Using that all characters $\chi \in \widehat{G}$ satisfy $\chi(g^{-1}) = \overline{\chi}(g)$ and $\chi(gh) = \chi(g)\chi(h)$ for all $g, h \in G$, we find that

$$\begin{array}{rcl} \alpha\widehat{G}_H & = & \{\chi \in \widehat{G} \mid \chi(ah) = \chi(a)\alpha(h) \text{ for all } h \in H\} \\ & = & \{\chi \in \widehat{G} \mid \chi(ah) = \chi(a)\alpha(a^{-1})\alpha(ah) \text{ for all } h \in H\}\;. \\ & = & \{\chi \in \widehat{G} \mid \chi(ah) = \chi(a)\overline{\alpha}(a)\alpha(ah) \text{ for all } h \in H\}, \end{array}$$

From this we find that $\chi(d) = c\alpha(d)$ for all $d \in aH$ where $c = \overline{\alpha}\,\chi(a)$. Hence, all characters in $\alpha\widehat{G}_H$ are completely confounded with the character $\alpha \in \widehat{G}$ on any coset aH.

In order to prove that the characters in different cosets of $\widehat{G}_H$ in $\widehat{G}$ are orthogonal on cosets of H in G, assume that α is chosen such that $\alpha\widehat{G}_H \neq \widehat{G}_H$. Then for $\chi_1, \chi_2 \in \widehat{G}_H$ the inner product $\langle \chi_1, \alpha\chi_2 \rangle_{aH}$ can be shown to equal zero in the following way

$$\langle \chi_1, \alpha\chi_2 \rangle_{aH} = \frac{1}{|\,H\,|} \sum_{h \in H} \chi_1(ah)\,\overline{\alpha\chi_2}(ah) = \frac{\chi_1(a)\,\overline{\alpha\chi_2}(a)}{|\,H\,|} \sum_{h \in H} \chi_1(h)\,\overline{\alpha}(h)\,\overline{\chi_2}(h).$$

Since $\chi_1, \chi_2 \in \widehat{G}_H$ we find that

$$\langle \chi_1, \alpha\chi_2 \rangle_{aH} = \frac{\chi_1(a)\,\overline{\alpha\chi_2}(a)}{|\,H\,|} \sum_{h \in H} \overline{\alpha}(h).$$

Using Lemma 5.4 and $\overline{\alpha} \notin \widehat{G}_H$ we find that $\sum_{h \in H} \overline{\alpha}(h) = 0$. Hence, we have shown that when α is chosen such that $\alpha\widehat{G}_H \neq \widehat{G}_H$ characters χ_1 and $\alpha\chi_2$ are orthogonal on any coset aH in G. The result generalizes to $\langle \chi_1, \chi_2 \rangle_{aH} = 0$ for χ_1 and χ_2 in different cosets of $\widehat{G}_H$. The inner product satisfies $\langle \alpha\chi_1, \alpha\chi_2 \rangle_{aH} = \langle \chi_1, \chi_2 \rangle_{aH}$ for all $\alpha \in G$. Hence, we can always multiply the characters χ_1 or χ_2 by some $\alpha \in \widehat{G}$ such that either $\alpha\chi_1$ or $\alpha\chi_2 \in \widehat{G}_H$. This concludes the proof. $\square$

From Theorem 8.1 we know that there is no partial confounding of the characters in $\widehat{G}$ on the cosets in G. The definition of regular fraction by Wu and Hamada (2000) (Definition 6.7) refers to the non-existence of partial confounding. This property has to be associated with a specific set of factorial effects. Here, we consider regularity with respect to set of factorial effects formed by the characters in $\widehat{G}$. In order to prove that for this specific basis the definition of regular fraction given by Collombier (1996) (Definition 6.6) is equal to the definition given by Wu and Hamada (2000) (Definition 6.7), it remains to be shown that the cosets in a finite Abelian group G are the only subsets of G on which there is no partial confounding of characters. But before we can state and prove this result, we first need a lemma.

Lemma 8.2 *Let G be a Abelian group and $F \subseteq G$ a fraction of G. Then for all $\chi_1, \chi_2 \in \widehat{G}$ and $a \in G$ the following statements hold.*

1. *χ_1 and χ_2 are completely confounded on aF if and only if they are completely confounded on F;*
2. *χ_1, χ_2 are orthogonal on aF if and only if they are orthogonal on F;*
3. *χ_1, χ_2 are partially confounded on aF if and only if they are partially confounded on F.*

Proof

1. If characters χ_1 and $\chi_2 \in \widehat{G}$ satisfy $\chi_1(f) = c\chi_2(f)$ for all $f \in F$ and some non-zero $c \in \mathbb{C}$, then for all $f \in F$ we have that $\chi_1(af) = \chi_1(a)\,\chi_1(f) = c\chi_1(a)\,\chi_2(f) = c\chi_1(a)\,\chi_2\left(a^{-1}\right)\chi_2(af) = c\chi_1\overline{\chi_2}(a)\,\chi_2(af)$. Hence, if χ_1 and χ_2 are completely confounded on F then they are completely confounded on aF. Since this statement holds for any $a \in G$ the converse is also true.

2. If χ_1 and χ_2 are orthogonal on F then we have

$$\begin{aligned}\langle\chi_1,\chi_2\rangle_{aF} &= \tfrac{1}{|F|}\sum_{f\in F}\chi_1(af)\,\overline{\chi_2}(bf) = \tfrac{1}{|F|}\sum_{f\in F}\chi_1(a)\,\chi_1(f)\,\overline{\chi_2}(a)\,\overline{\chi_2}(f)\\ &= \tfrac{\chi_1(a)\overline{\chi_2}(a)}{|F|}\sum_{f\in F}\chi_1(f)\,\overline{\chi_2}(f) = \chi_1\overline{\chi_2}(a)\,\langle\chi_1,\chi_2\rangle_F = 0.\end{aligned}$$

   Hence, orthogonality of χ_1 and χ_2 on F implies orthogonality of χ_1 and χ_2 on aF. This statement holds for any $a \in G$. Hence, its converse is also true.

3. This follows from parts 1 and 2 and Definition 6.4 of partial confounding. □

We will now show that the cosets of G are the only fractions on which there is no partial confounding of characters in $\widehat{G}$.

Theorem 8.3 *Let G be a finite Abelian group and $F \subseteq G$ a fraction of G. If there is no partial confounding of characters in $\widehat{G}$ on F, then F is a coset in G.*

Proof Let $a \in F$ and consider $H = a^{-1}F$. Then we have that $1 \in H$. Denote by $\widehat{G}_F$ the set of characters that are constant on F, that is, the characters that on F are completely confounded with 1. From Lemma 8.2 part 2 we know that $\widehat{G}_H = \widehat{G}_F$. Since there is no partial confounding of characters in $\widehat{G}$ on F, we know that all characters in $\widehat{G} \setminus \widehat{G}_F$ are orthogonal to 1 on F. From Lemma 8.2 we find that this can only be true if and only if all characters in $\widehat{G} \setminus \widehat{G}_F = \widehat{G} \setminus \widehat{G}_H$ are orthogonal to $1 \in \widehat{G}_H$ on H. We will show that H is a subgroup using the indicator function $\mathcal{I}_H$. Note that for all $\chi \in \widehat{G}_H$ we have that $\chi(h) = \chi(1) = 1$ for all $h \in H$. The characters $\chi \in \widehat{G} \setminus \widehat{G}_H$ are orthogonal to 1 on H. The Fourier coefficients of the indicator function $\mathcal{I}_H$ are given by

$$\langle\mathcal{I}_H,\chi\rangle = \begin{cases} \frac{|H|}{|G|} & \chi \in \widehat{G}_H \\ 0 & \chi \notin \widehat{G}_H \end{cases}$$

Denote by $\langle H\rangle$ the subgroup in G generated by H. Since $\langle H\rangle$ is a subgroup of G we can use Theorem 8.1 to obtain the Fourier coefficients of the indicator function. These are

$$\langle\mathcal{I}_{\langle H\rangle},\chi\rangle = \begin{cases} \frac{|\langle H\rangle|}{|G|} & \chi \in \widehat{G}_{\langle H\rangle} \\ 0 & \chi \notin \widehat{G}_{\langle H\rangle} \end{cases}$$

The result now follows if we could prove that for any $H \subseteq G$ satisfying $1 \in H$ the equality $\widehat{G}_{\langle H\rangle} = \widehat{G}_H$ holds. The proof for this is as follows. If χ is constant on the set H then $\chi(h) = \chi(1) = 1$ for all $h \in H$. In that case we find that for all g and h in H we have that $\chi(gh) = \chi(g)\,\chi(h) = 1$. Hence, $\chi(h) = 1$ for all $h \in H$ implies that $\chi(h) = 1$ for all $h \in \langle H\rangle$. Clearly, any character $\chi \in \widehat{G}$ that is not constant on H is also not constant on $\langle H\rangle$. This proves that $\widehat{G}_{\langle H\rangle} = \widehat{G}_H$, which implies that

$$\mathcal{I}_H = \frac{|\ H\ |}{|\ \langle H\rangle\ |}\mathcal{I}_{\langle H\rangle}.$$

Both $\mathcal{I}_H$ and $\mathcal{I}_{\langle H\rangle}$ are indicator functions. This forces $\frac{|H|}{|\langle H\rangle|}$ to equal 1. Hence, we find that $\mathcal{I}_H = \mathcal{I}_{\langle H\rangle}$, which proves that H is a subgroup in G. As a result $F = aH$ is a coset in G. □

Theorems 8.1 and 8.3 show that the given for a regular fraction by Collombier (1996) (Definition 6.6) of regular fraction is equivalent to that given by Wu and Hamada (2000) (Definition 6.7) when the set of factorial effects that is under consideration is given by the characters $\widehat{G}$ of the group G.

We now show that definition of regular fractions given in Collombier (1996) (Definition 6.6) and Pistone and Rogantin (2005) (Definition 6.8) are equivalent. In the proof we need the following result.

Lemma 8.4 *Let H be a subgroup of a finite Abelian group G. If T is a transversal of $\widehat{G}_H$ in $\widehat{G}$, then $\widehat{H} = \{\chi_{|H} \mid \chi \in T\}$.*

Proof Let the representation ρ correspond to an irreducible character in T and denote by $\rho_{|H}$ its restriction to H. Then $\rho_{|H}$ is an irreducible representation of H. That $\rho_{|H}$ is a representation of H follows from

$$\rho_{|H}(h_1 h_2) = \rho(h_1 h_2) = \rho(h_1)\,\rho(h_2) = \rho_{|H}(h_1)\,\rho_{|H}(h_2) \text{ for all } h_1, h_2 \in H.$$

The representation $\rho_{|H}$ must be irreducible since it is one-dimensional. Hence, the elements in the set $\{\chi_{|H} \mid \chi \in T\}$ are all irreducible characters of H. The representations that correspond to these irreducible characters are all non-equivalent. Let $\alpha_1, \alpha_2 \in T$ and assume that $\alpha_1(h) = \alpha_2(h)$ for all $h \in H$. Then $\alpha_1 \widehat{G}_H = \{\chi \in H \mid \chi(h) = \alpha_1(h) \text{ for all } h \in H\} = \{\chi \in H \mid \chi(h) = \alpha_2(h) \text{ for all } h \in H\} = \alpha_2 \widehat{G}_H$ which contradicts $\alpha_1, \alpha_2 \in T$. Hence, for different elements $\alpha_1, \alpha_2 \in T$ we have that $\alpha_1(h) \neq \alpha_2(h)$ for at least one $h \in H$. Using Lemma 4.7 we find that the set of irreducible representations for H obtained from the characters in T is a set of non-equivalent representations. Using that $\widehat{G}_H$ is isomorphic to the dual of the quotient space G/H (see Lemma 5.3) and that G/H is Abelian we find that the number of elements of $\widehat{G}_H$ is given by

$$\mid \widehat{G}_H \mid = \mid \widehat{G/H} \mid = \mid G/H \mid = \frac{\mid G \mid}{\mid H \mid}.$$

Since $\mid \widehat{G} \mid = \mid G \mid$ the number of cosets of $\widehat{G}_H$ in $\widehat{G}$ equals $\mid H \mid$. From this we find that $\mid T \mid = \mid \widehat{H} \mid$ and that the elements in $\widehat{H} = \{\chi_{|H} \mid \chi \in T\}$ form a complete set of non-equivalent irreducible representations for H. $\square$

The definition for a regular fraction given by Pistone and Rogantin (2005) (Definition 6.8) for symmetric factorial designs also defines a coset. We state the next lemma.

Lemma 8.5 *Let L be a subgroup of $(\mathbb{Z}/n\mathbb{Z})^k$ and $\psi : L \to \Omega_n$ a homomorphism. The set $\{(d_1, d_2, \ldots, d_k) \in (\Omega_n)^k \mid d_1^{\ell_1} d_2^{\ell_2} \ldots d_k^{\ell_k} = \psi(\ell) \text{ for all } \ell \in L\}$ is a coset in $(\Omega_n)^k$.*

Proof To simplify notation we let $G = (\mathbb{Z}/n\mathbb{Z})^k$ and $D = (\Omega_n)^k$. The morphisms between the relevant groups are given in the following commutative diagram.

$$\begin{tikzcd}
L \arrow[r, hook] \arrow[d, "\psi = \chi_{d|L}"] & G = (\mathbb{Z}/n\mathbb{Z})^k \arrow[r, shift left, "\varphi"] \arrow[d, "\chi_d"] & D = (\Omega_n)^k \arrow[l, shift left, "\varphi^{-1}"] \arrow[dl, "\nu_g"] \\
\Omega_n \arrow[r, hook] & \mathbb{T} &
\end{tikzcd}$$

From Lemma 8.4 we know that any homomorphism $\psi : L \to \mathbb{T}$ can be seen as the restriction to L of some homomorphism $\chi : G \to \mathbb{T}$. The set of all homomorphisms from G into $\mathbb{T}$ is given by $\widehat{G} = \{\chi_d \mid d \in D\}$ where $\chi_d(g) = d_1^{g_1} d_2^{g_2} \dots d_k^{g_k}$. Hence, we have that $\psi = \chi_{a|L}$ for some $a \in D$. Consider the functions $\nu_g : D \to \mathbb{T}$ defined by $\nu_g(d) = \chi_d(g)$. For all $g \in G$ we have that ν_g is a character on D. As a result we find that

$$\begin{array}{ll} \{d \in D \mid d_1^{\ell_1} d_2^{\ell_2} \dots d_k^{\ell_k} = \psi(\ell) \text{ for all } \ell \in L\} & = \\ \{d \in D \mid \nu_\ell(d) = \chi_a(\ell) \text{ for all } \ell \in L\} & = \\ \{d \in D \mid \nu_\ell(d) = \nu_\ell(a) \text{ for all } \ell \in L\} & = \\ \{d \in D \mid \nu(d) = \nu(a) \text{ for all } \nu \in S\}, & \end{array}$$

where $S = \{\nu_\ell \mid \ell \in L\} \subseteq \widehat{D}$. The result now follows from Lemma 7.1. □

The final lemma in this section states that any coset in $D = (\Omega_n)^k$ is regular according to the definition given by Pistone and Rogantin (2005) (Definition 6.8).

Lemma 8.6 *Let H be a subgroup of $(\Omega_n)^k$ and $a \in (\Omega_n)^k$. There exists a subgroup L of $(\mathbb{Z}/n\mathbb{Z})^k$ and a homomorphism $\psi : L \to \Omega_n$ such that $aH = \{(d_1, d_2, \dots, d_k) \in (\Omega_n)^k \mid d_1^{\ell_1} d_2^{\ell_2} \dots d_k^{\ell_k} = \psi(\ell)$ for all $\ell \in L\}$.*

Proof A commutative diagram containing the morphisms between the relevant groups is given in the proof of Lemma 8.5. Again we let $D = (\Omega_n)^k$ and $G = (\mathbb{Z}/n\mathbb{Z})^k$. Consider the subgroup L of G defined by $L = \{\ell \in G \mid h_1^{\ell_1} h_2^{\ell_2} \dots h_k^{\ell_k} = 1 \text{ for all } h \in H\}$. Then $\widehat{G}_H = \{\nu_\ell \mid \ell \in L\}$ where $\nu_\ell : D \to \mathbb{T}$ is defined as $\nu_\ell(d) = d_1^{\ell_1} d_2^{\ell_2} \dots d_k^{\ell_k}$ for all $d \in D$. The isomorphism $\varphi : G \to D$ is given by $\varphi(g) = (\omega_n^{g_1}, \omega_n^{g_2}, \dots, \omega_n^{g_k})$. Let $\psi : L \to \Omega_n$ be defined as $\psi(\ell) = \nu_{\varphi^{-1}(a)}(\varphi(\ell))$. We show that ψ is a homomorphism of L. Using that both φ and $\nu_{\varphi^{-1}(a)}$ are homomorphisms we find that

$$\psi(\ell_1 + \ell_2) = \nu_{\varphi^{-1}(a)}(\varphi(\ell_1 + \ell_2)) = \nu_{\varphi^{-1}(a)}(\varphi(\ell_1)\varphi(\ell_2)) =$$

$$\nu_{\varphi^{-1}(a)}(\varphi(\ell_1))\,\nu_{\varphi^{-1}(a)}(\varphi(\ell_2)) = \psi(\ell_1)\,\psi(\ell_2).$$

For arbitrary $a \in D$ and $\ell \in L$ let $b = \varphi^{-1}(a)$ and $m = \varphi(\ell)$. Then the following equality holds

$$\nu_\ell(a) = {a_1}^{\ell_1} {a_2}^{\ell_2} \dots {a_k}^{\ell_k} = \left(\omega_n^{b_1}\right)^{\ell_1} \left(\omega_n^{b_2}\right)^{\ell_2} \cdots \left(\omega_n^{b_k}\right)^{\ell_k}$$

$$= \left(\omega_n^{\ell_1}\right)^{b_1} \left(\omega_n^{\ell_2}\right)^{b_2} \cdots \left(\omega_n^{\ell_k}\right)^{b_k} = {m_1}^{b_k} {m_2}^{b_2} \dots {m_k}^{b_k} = \nu_b(m) = \nu_{\varphi^{-1}(a)}(\varphi(\ell)).$$

Using this we find that

$$\begin{array}{lll} \{d \in D \mid d_1^{\ell_1} d_2^{\ell_2} \dots d_k^{\ell_k} = \psi(\ell) \text{ for all } \ell \in L\} & = & \\ \{d \in D \mid \nu_\ell(d) = \nu_{\varphi^{-1}(a)}(\varphi(\ell)) \text{ for all } \ell \in L\} & = & \\ \{d \in D \mid \nu_\ell(d) = \nu_\ell(a) \text{ for all } \ell \in L\} & = & \\ \{d \in D \mid \nu(d) = \nu(a) \text{ for all } \nu \in \widehat{G}_H\} & = & aH, \end{array}$$

where the last equality follows from Lemma 7.2. □

From Lemmas 8.5 and 8.6 we find that the regular fractions that Pistone and Rogantin (2005) define are exactly the cosets of the multiplicative group $(\Omega_n)^k$.

9 Confounding within the framework of harmonic analysis

We introduce the expectation function for the data observed on the group G. This function $\mu : G \to \mathbb{R}$ is defined by $\mu(g) = \mathrm{E}[y(g)]$. At this point no assumptions about the distribution of y are needed. The function μ has a Fourier expansion given by

$$\mu = \sum_{\chi \in \widehat{G}} \widehat{\mu}(\chi)\,\chi.$$

Each Fourier coefficient $\widehat{\mu}(\chi)$ can be unbiasedly estimated by the Fourier coefficient $\widehat{y}(\chi)$. For G a finite Abelian group we present a lemma that gives the confounding relations for the factorial effects in $\widehat{G}$. More precisely, it states which linear combinations of the Fourier coefficients $\{\widehat{\mu}(\chi), \chi \in \widehat{G}\}$ can be estimated unbiasedly and how each of these linear combinations can be estimated. The lemma is based on the Poisson summation formula. Good (1960) was the first to use the Poisson summation formula to find the confounding relations on a regular fraction.

Lemma 9.1 *Let H be a subgroup of a finite Abelian group G, $a \in G$ and $\alpha \in \widehat{G}$. For all response functions $y : G \to \mathbb{R}$ and the expectation function $\mu : G \to \mathbb{R}$ defined by $\mu(g) = E[y(g)]$ we have that*

$$E\left[\frac{1}{\mid H \mid}\sum_{g \in aH} \overline{\alpha}(g)\,y(g)\right] = \sum_{\chi \in \alpha\widehat{G}_H} \overline{\alpha}\chi(a)\,\widehat{\mu}(\chi) \tag{7}$$

Proof The result follows directly from (4) by substituting y for f and taking the expected value at both sides of the equation. □

From Lemma 8.4 we know that $\widehat{H} = \{\overline{\alpha_{|H}} \mid \alpha \in T\}$ is a set of $\mid H \mid$ orthogonal characters of the group H. Any function $f : H \to \mathbb{C}$ can be expressed as a unique linear combination of the elements in $\widehat{H}$. Since different characters in $\alpha \in T$ correspond to different cosets $\alpha\widehat{G}_H$ in $\widehat{G}$, we find that each linear combination that appears as a right-hand side of (7) can only be estimated unbiasedly as a whole. The next example illustrates the confounding in a 2^{3-1} fractional factorial experiment.

Example 9.2 *We consider the full 2^3 factorial design coded as the additive Abelian group $G = (\mathbb{Z}/2\mathbb{Z})^3$. Let $H = \{(0,0,0), (1,1,0), (1,0,1), (0,1,1)\}$. Then the subgroup H is $\{g \in G \mid \chi(g) = 1$ for all $\chi \in \widehat{G}_H\}$ where $\widehat{G}_H = \{\chi_r \mid r \in \{(0,0,0), (1,1,1)\}\}$. We are now interested in the exact confounding relations on the regular fraction $F = (1,0,0) + H$. The cosets of $\widehat{G}_H$ in $\widehat{G}$ are*

$$\begin{aligned}
\widehat{G}_H &= \{\chi_{(0,0,0)}, \chi_{(1,1,1)}\};\\
\chi_{(1,0,0)}\widehat{G}_H &= \{\chi_{(1,0,0)}, \chi_{(0,1,1)}\};\\
\chi_{(0,1,0)}\widehat{G}_H &= \{\chi_{(0,1,0)}, \chi_{(1,0,1)}\};\\
\chi_{(0,0,1)}\widehat{G}_H &= \{\chi_{(0,0,1)}, \chi_{(1,1,0)}\}.
\end{aligned}$$

Let $y : F \to \mathbb{R}$ be the function that maps each element of the fraction to the response value that is observed in the corresponding run. Using Lemma 9.1 we find that

$$\begin{aligned}
E\left[\tfrac{1}{4}\left(y(1,0,0) + y(0,1,0) + y(0,0,1) + y(1,1,1)\right)\right] &= \widehat{\mu}\left(\chi_{(0,0,0)}\right) - \widehat{\mu}\left(\chi_{(1,1,1)}\right);\\
E\left[\tfrac{1}{4}\left(-y(1,0,0) + y(0,1,0) + y(0,0,1) - y(1,1,1)\right)\right] &= \widehat{\mu}\left(\chi_{(1,0,0)}\right) - \widehat{\mu}\left(\chi_{(0,1,1)}\right);\\
E\left[\tfrac{1}{4}\left(y(1,0,0) - y(0,1,0) + y(0,0,1) - y(1,1,1)\right)\right] &= \widehat{\mu}\left(\chi_{(0,1,0)}\right) - \widehat{\mu}\left(\chi_{(1,0,1)}\right);\\
E\left[\tfrac{1}{4}\left(y(1,0,0) + y(0,1,0) - y(0,0,1) - y(1,1,1)\right)\right] &= \widehat{\mu}\left(\chi_{(0,0,1)}\right) - \widehat{\mu}\left(\chi_{(1,1,0)}\right).
\end{aligned}$$

If the Fourier coefficients $\widehat{\mu}\left(\chi_{(1,1,1)}\right), \widehat{\mu}\left(\chi_{(0,1,1)}\right), \widehat{\mu}\left(\chi_{(1,0,1)}\right)$ and $\widehat{\mu}\left(\chi_{(1,1,0)}\right)$ are assumed to be zero, then unbiased estimators for $\widehat{\mu}\left(\chi_{(0,0,0)}\right), \widehat{\mu}\left(\chi_{(1,0,0)}\right), \widehat{\mu}\left(\chi_{(0,1,0)}\right)$ and $\widehat{\mu}\left(\chi_{(0,0,1)}\right)$ can directly be read from the previous equations.

Another form of confounding occurs when there are factors that in the experiment are set at more than two levels. In that case some of the irreducible characters are non-real on G. If χ is a irreducible character that is non-real of a finite Abelian group G then $\overline{\chi}$ is an irreducible character of G that is not equivalent to χ. For such χ we have that $\widehat{\mu}\left(\overline{\chi}\right) = \overline{\widehat{\mu}\left(\chi\right)}$ when we assume μ to be real-valued. This type of confounding is discussed in more detail in Section 10 and is related to the equality of the sums of squares for conjugated pairs of irreducible representations that we found in the example considered in Section 4.3.

10 Statistical inference

In this section we assume that the observations that are made on the finite group are independently distributed normal variables. In Section 10.1 a general finite group is considered. The general procedure in the case of independent random variables with equal variance is to decompose the total sums of squares using the decomposition of the identity matrix given in Theorem 4.12 for the regular representation. We show that if some of the characters are not real-valued then the sums of squares that appear in the decomposition are not independent. A new decomposition that is based on the decomposition in Theorem 4.12 is proposed. The parts of this new decomposition are shown to be independent. In Section 10.2 we illustrate the analysis of statistical data structured on a coset of a finite Abelian group or, equivalently, data obtained in a regular fractional factorial design.

10.1 Statistical inference for normal data on finite groups

In this section we consider the set of observations made on a finite group G as a vector in $\mathbb{R}^{|G|}$. To this end we let $\{e_g \mid g \in G\}$ denote the standard basis for $\mathcal{V} = \mathbb{R}^{|G|}$ indexed by the elements in G. The vector of observations with respect to this basis is given by $y = \sum_{g \in G} y\left(g\right) e_g$. We assume that this vector has a multivariate normal distribution, more precisely $y \sim \mathcal{N}_N\left(\mu, \sigma^2 I_N\right)$, where $N = \mid G \mid$ and $\mu = \sum_{g \in G} \mu\left(g\right) e_g$. No assumptions are made on the expectation function $\mu : G \rightarrow \mathbb{R}$. The decomposition of $y^T y = \sum_{g \in G} \left(y\left(g\right)\right)^2$ based on the canonical decomposition of the identity matrix I_N is given in Theorem 4.12 is

$$y^T y = \sum_{j=1}^{w} SS_j \text{ where } SS_j = y^T P_j y. \tag{8}$$

Let $\tilde{y}$ denote the vector of length $\mid G \mid$ with each of its elements equal to the average of all observations made on G. The total sum of squares that is usually considered in analysis of variance is given by $SS_T = \left(y - \tilde{y}\right)^T \left(y - \tilde{y}\right) = y^T y - \mid G \mid \tilde{y}^T \tilde{y}$. The decomposition in (8) also gives us a decomposition for SS_T. To see this, assume without loss of generality that the irreducible representation ρ_1 in Theorem 4.12 is the trivial representation, that is, ρ_1 is the one-dimensional representation given by $\rho_1\left(g\right) = 1$ for all $g \in G$. The corresponding projection matrix is $P_1 = \frac{1}{|G|} \sum_{j=1}^{w} \rho\left(g\right)$. The matrices $\rho\left(g\right)$ for ρ the regular representation are permutation matrices with $\rho\left(g\right)_{hj} = 1$ if and only if $h = gj$. Since for each pair $h, j \in G$ there is a unique g for which $h = gj$ we find that $\sum_{j=1}^{w} \rho\left(g\right)$ is the all-one matrix. This implies that $y^T P_1 y =\mid G \mid \tilde{y}^T\tilde{y}$ and, hence, a decomposition of SS_T is simply $\sum_{j=2}^{w} SS_j$ with SS_j defined as in (8). For the purpose of statistical inference we are interested in the distribution of the sums of squares SS_j that appear in this decomposition for SS_T. The distribution of quadratic forms in normal random variables has been widely studied. Let A be a real symmetric matrix and assume that $y \sim \mathcal{N}_N(\mu, V)$ with V nonsingular. A theorem by Driscoll (1999) states that the quadratic form $y^T Ay$ has a chi-square distribution if and only if AV is idempotent. The chi-square distribution has rank (A) degrees of freedom and non-centrality parameter $\mu^T A\mu$. This is an extension of an earlier result by Pearson. Another important theorem that is known as Craig's theorem states a necessary and sufficient condition for the independence of two quadratic forms. If $y \sim \mathcal{N}_N(\mu, V)$ with V nonsingular and A_1 and A_2 are real symmetric matrices then the condition that is necessary and sufficient for $y^T A_1 y$ and $y^T A_2 y$ to be statistically independent is $A_1 V A_2 = 0$. A detailed proof can be found in Driscoll and Krasnicka (1995). However, because of the property $\sum_{j=1}^{w} P_j = I_N$ the most direct way to find the distribution of the SS_j is to use the Fisher-Cochran theorem.

Theorem 10.1 (Fisher-Cochran theorem) *Let $y \sim \mathcal{N}_N(\mu, I_N)$ and let $P_1, P_2, \ldots, P_w$ be w real symmetric $N \times N$ matrices, such that*

$$I_N = \sum_{j=1}^{w} P_j.$$

Then a necessary and sufficient condition for $y^T P_j y \sim \chi^2(\nu_j, \lambda_j)$ with $y^T P_j y$ and $y^T P_i y$ independent for $i \neq j$ is

$$N = \sum_{j=1}^{w} \mathit{rank}\ (P_j)\,,$$

in which case $\nu_j = \mathit{rank}\ (P_j)$ and $\lambda_j = \mu P_j \mu$.

Proof See Rao (1973), p. 185. □

The condition that all matrices $P_1, P_2, \ldots, P_w$ are idempotent is sufficient for the requirement $N = \sum_{j=1}^{w}$ rank (P_j) to be satisfied. This follows easily using the next lemma.

Lemma 10.2 *The rank of an idempotent matrix is equal to its trace.*

Proof See Rao (1973), p. 28 □

Using Lemma 10.2 we find that in the case where all matrices $P_1, P_2, \ldots, P_w$ in the decomposition $I_N = \sum_{j=1}^{w} P_j$ are idempotent we have that

$$N = \mathrm{tr}\,(I_N) = \mathrm{tr}\left(\sum_{j=1}^{w} P_j\right) = \sum_{j=1}^{w} \mathrm{tr}\ (P_j) = \sum_{j=1}^{w} \mathrm{rank}\ (P_j)\,.$$

In the cases where the projection matrices P_j have non-real entries Theorem 10.1 cannot be applied directly. Recall that in the example of the 3^3 factorial design that we discussed in Section 4.3 we encountered projection matrices with non-real entries. In that example we found that the sum of squares associated with an irreducible representation ρ_z was equal to that associated with its complex-conjugate $\overline{\rho_z}$. This equality of sum of squares can be shown to occur for all possible realizations of the random vector y and, hence, the sums of squares appearing in the decomposition given in (8) cannot all be pairwise independent. We propose a method for statistical inference where equal sums of squares are combined. The resulting decomposition consists of mutually independent sum of squares. Before we present these results we first need some lemmas that state some properties of the matrices P_j that appear in the canonical decomposition that were not mentioned in Theorem 4.12.

Lemma 10.3 *Each matrix P_j that appears in the canonical decomposition for the regular representation is Hermitian.*

Proof Using Lemma 4.5 we find that $\chi_j(g) = \overline{\chi_j}\left(g^{-1}\right)$ for all irreducible characters χ_j and elements $g \in G$. The matrices $\rho(g), g \in G$, are permutation matrices and satisfy $\rho\left(g^{-1}\right) = (\rho(g))^T$. Hence, we have

$$P_j^T = \frac{d_j}{\mid G \mid} \sum_{g \in G} \overline{\chi_j}(g)^T \rho(g) = \frac{d_j}{\mid G \mid} \sum_{g \in G} \chi_j\left(g^{-1}\right)\left(\rho\left(g^{-1}\right)\right) = \frac{d_j}{\mid G \mid} \sum_{g \in G} \chi_j(g)(\rho(g)) = \overline{P_j},$$

which completes the proof. □

The next lemma guarantees that the decomposition of the total sum of squares given in (8) when applied to the regular representation is a decomposition into real parts that are all non-negative.

Lemma 10.4 *Each matrix P_j that appears in the canonical decomposition for the regular representation is positive semidefinite.*

Proof Using that P_j is idempotent and Hermitian we find that

$$y^T P_j y = y^T P_j^2 y = y^T \overline{P_j}^T P_j y = (P_j y, P_j y),$$

where $(P_j y, P_j y)$ denotes the standard inner product on $\mathbb{C}^N$. Note that $(P_j y, P_j y) \geq 0$ and $(P_j y, P_j y) = 0$ only if $P_j y = 0$. We find that P_j is positive semidefinite by observing that this implies that $y^T P_j y \in \mathbb{R}$ and $y^T P_j y \geq 0$ for all $y \in \mathbb{R}^N$. The matrix P_j is not positive definite because it need not be full rank and, hence, $P_j y = 0$ does not imply $y = 0$. □

We propose a decomposition of the $N \times N$ identity matrix into real symmetric matrices that is based on the canonical decomposition for the regular representation. To this end we use that for every non-real matrix that appears in the canonical decomposition also its complex conjugate appears in the decomposition.

Lemma 10.5 *If a matrix P_j in the canonical decomposition for the regular representation has non-real entries, then there exists a $i, 1 \leq i \leq h$, such that $P_j = \overline{P_i}$.*

Proof If the character $\chi_j = \operatorname{tr} \rho_j$ is real-valued on G, then because all $\rho(g)$ of the regular representation are permutation matrices we have that all entries in P_j are real-valued. Hence, without loss of generality we assume that χ_j is not real-valued on G. We now show that if ρ_j is an irreducible representation of G then also $\rho_i = \overline{\rho_j}$ is an irreducible representation of G. Clearly $\rho_j(gh) = \rho_j(g)\rho_j(h)$ for all $g, h \in G$ implies $\overline{\rho_j}(gh) = \overline{\rho_j}(g)\overline{\rho_j}(h)$ for all $g, h \in G$ from which we find that ρ_i is a representation of G. Let χ_j denote the character of ρ_j and χ_i the character of ρ_i. Because ρ_j is irreducible we have by Lemma 4.9 that $\langle \chi_j, \chi_j \rangle = 1$. The character χ_i satisfies $\chi_i = \overline{\chi_j}$ which implies that $\langle \chi_i, \chi_i \rangle = \langle \overline{\chi_j}, \overline{\chi_j} \rangle = \langle \chi_j, \chi_j \rangle = 1$. Using Lemma 4.9 we find that also ρ_i is irreducible. From the assumption that χ_j is not real-valued on G we have that $\chi_j(g) \neq \overline{\chi_j}(g) = \chi_i(g)$ for some $g \in G$. Lemma 4.7 now implies that the representations ρ_j and ρ_i are non-equivalent. From $\rho_i = \overline{\rho_j}$ it follows that the dimensions d_j and d_i are equal. The projection matrix P_i for the irreducible representation ρ_i in the canonical decomposition is

$$P_i = \frac{d_i}{\mid G \mid} \sum_{g \in G} \overline{\chi_i}(g) \rho(g) = \frac{d_j}{\mid G \mid} \sum_{g \in G} \chi_j(g) \rho(g) = \overline{P_j},$$

which concludes the proof. □

The last lemma tells us that if a matrix P_j in the canonical decomposition has complex entries, then its complex conjugate $\overline{P_j} = P_i$ is also in the decomposition. The corresponding sums of squares SS_j and SS_i are equal. This follows from

$$SS_j = y^T P_j y = \left(y^T P_j y\right)^T = y^T P_j^T y = y^T \overline{P_j} y = y^T P_i y = SS_i,$$

where we have used that P_j is Hermitian. Our strategy now is to replace each complex conjugate pair of matrices P_j and P_i in the decomposition by their sum $P_j + P_i$. This way we find a new decomposition $I_N = Q_1 + Q_2 + \ldots + Q_m$. The corresponding decomposition of $y^T y$ is

$$y^T y = \sum_{j=1}^{m} SS_j^\star \text{ where } SS_j^\star = y^T Q_j y. \tag{9}$$

Lemma 10.6 *The matrices Q_j in the new decomposition are real, symmetric and idempotent.*

Proof It is trivial to see that all entries in the matrix $Q_j = P_j + \overline{P_j}$ are real. The symmetry of the matrix $Q_j = P_j + \overline{P_j}$ is obtained from the fact that P_j is Hermitian in the following way

$$Q_j^T = \left(P_j + \overline{P_j}\right)^T = P_j^T + \overline{P_j}^T = \overline{P_j} + P_j = Q_j.$$

From Lemma 10.5 we have that $\overline{P_j} = P_i$ for some $i \neq j$. In addition, from Theorem 4.12 we know that $P_j^2 = P_j$, $P_i^2 = P_i$, $P_j P_i = 0$ and $P_i P_j = 0$. From these two results we find that

$$Q_j^2 = \left(P_j + \overline{P_j}\right)^2 = (P_j + P_i)^2 = P_j^2 + P_j P_i + P_i P_j + P_i^2 = P_j + P_i = P_j + \overline{P_j} = Q_j.$$

Hence, we have shown that the matrices Q_j are real, symmetric and idempotent. □

Since all Q_j in decomposition (9) are idempotent we find using Lemma 10.2 that

$$N = \mathrm{tr}(I_N) = \mathrm{tr}\left(\sum_{j=1}^{m} Q_j\right) = \sum_{j=1}^{m} \mathrm{tr}\ (Q_j) = \sum_{j=1}^{m} \mathrm{rank}\ (Q_j). \tag{10}$$

This result is used in the proof of the next theorem.

Theorem 10.7 *If $y \sim \mathcal{N}_N\left(\mu, \sigma^2 I_N\right)$ then we have that the random variables*

$$\frac{SS_j^\star}{\sigma^2} = \frac{y^T Q_j y}{\sigma^2}, \quad j = 1, 2, \ldots, m,$$

in (9) are independently distributed according to a $\chi^2\left(\nu_j^\star, \lambda_j^\star\right)$ distribution with $\nu_j^\star = \mathrm{rank}\ (Q_j)$ and $\lambda_j^\star = \mu^T Q_j \mu$.

Proof Define the random vector $x = \frac{y}{\sigma}$ and note that $x \sim \mathcal{N}_N\left(\frac{\mu}{\sigma}, I_N\right)$. Because the matrices $Q_1, Q_2, \ldots, Q_m$ are real and symmetric and satisfy $I_N = \sum_{j=1}^{m} Q_j$ and $N = \sum_{j=1}^{m} \text{rank}\,(Q_j)$ we can apply Theorem 10.1 (the Fisher-Cochran theorem) and find that the random variables

$$x^T Q_j x = \frac{SS_j^\star}{\sigma^2}, \quad j = 1, 2, \ldots, m,$$

are independently distributed according to a $\chi^2\left(\nu_j^\star, \lambda_j^\star\right)$ distribution with $\nu_j^\star = \text{rank}\,(Q_j)$ and $\lambda_j^\star = \frac{\mu^T Q_j \mu}{\sigma^2}$. □

If for some of the non-centrality parameters we can assume that $\lambda_k^\star = 0$ then the usual F-test can be applied to test the hypothesis $\lambda_j^\star = 0$ against $\lambda_j^\star \neq 0$ for all j of interest. The next lemma and corollary illustrate that if the representation ρ_j is one-dimensional with character χ_j then an equivalent hypothesis can be stated in terms of the Fourier coefficients $\widehat{\mu}\,(\chi_j)$ and $\widehat{\mu}\,(\overline{\chi_j})$.

Lemma 10.8 *Let the function $y : G \to \mathbb{C}$ be represented as a vector by $y = \sum_{g \in G} y\,(g)\, e_g$. If ρ is taken to be the regular representation, then for all one-dimensional irreducible representations ρ_j and the matrices P_j defined in Theorem 4.12 we have that*

$$y^T P_j y = d_j \mid G \mid \widehat{y}\,(\chi_j)\, \widehat{y}\,(\overline{\chi_j})\,.$$

Proof The product $\widehat{y}\,(\chi_j)\, \widehat{y}\,(\overline{\chi_j})$ can be expressed as

$$\begin{aligned}
\widehat{y}\,(\chi_j)\, \widehat{y}\,(\overline{\chi_j}) &= \tfrac{1}{|G|^2} \textstyle\sum_{g \in G} \sum_{h \in G} \overline{\chi_j}\,(g)\, y\,(g)\, \chi_j\,(h)\, y\,(h) \\
&= \tfrac{1}{|G|^2} \textstyle\sum_{g \in G} \sum_{h \in G} \overline{\chi_j}\,(g)\, y\,(g)\, \overline{\chi_j}\left(h^{-1}\right) y\,(h) \\
&= \tfrac{1}{|G|^2} \textstyle\sum_{g \in G} \sum_{h \in G} \overline{\chi_j}\left(gh^{-1}\right) y\,(g)\, y\,(h) \\
&= \tfrac{1}{|G|^2} \textstyle\sum_{g^\star \in G} \sum_{h \in G} \overline{\chi_j}\,(g^\star)\, y\,(g^\star h)\, y\,(h) \\
&= \tfrac{1}{|G|^2} \textstyle\sum_{g^\star \in G} \overline{\chi_j}\,(g^\star) \sum_{h \in G} y\,(g^\star h)\, y\,(h)
\end{aligned}$$

The matrix $\rho\,(g)$ satisfies $\rho\,(g)\, e_h = e_{gh}$ for all $h \in G$. With respect to the basis $\{e_g \mid g \in G\}$ the matrix $\rho\,(g^\star)$ has entries $(\rho\,(g^\star))_{rh} = 1$ if $g^\star h = r$ and $(\rho\,(g^\star))_{rh} = 0$ if $g^\star h \neq r$. This implies that

$$\begin{aligned}
d_j \mid G \mid \widehat{y}\,(\chi_j)\, \widehat{y}\,(\overline{\chi_j}) &= \tfrac{d_j}{|G|} \textstyle\sum_{g^\star \in G} \overline{\chi_j}\,(g^\star)\, y^T \rho\,(g^\star)\, y \\
&= y^T \left(\tfrac{d_j}{|G|} \textstyle\sum_{g^\star \in G} \overline{\chi_j}\,(g^\star)\, \rho\,(g^\star)\right) y \\
&= y^T P_j y,
\end{aligned}$$

which completes the proof. □

The next corollary follows directly from Theorem 10.7 using the previous lemma.

Corollary 10.9 *Let the function $\mu : G \to \mathbb{R}$ be represented as a vector by $\mu = \sum_{g \in G} \mu\,(g)\, e_g$. If ρ_j is a one-dimensional irreducible representation of G, then the non-centrality parameter of the distribution of the quadratic form $SS_j^\star = y^T Q_j y$ is*

$$\lambda_j^\star = \mu^T Q_j \mu = 2\,\frac{d_j \mid G \mid}{\sigma^2} \widehat{\mu}\,(\chi_j)\, \widehat{\mu}\,(\overline{\chi_j})\,.$$

Table 3: ANOVA Table for the Seat-Belt Experiment

Orthogonal component	Degrees of freedom	Combined Sum of Squares	Mean Squares	F	p-value
X_1	2	3.46217×10^7	1.73109×10^7	85.58	0.000
X_2	2	938539	469270	2.32	0.108
X_3	2	9.54948×10^6	4.77474×10^6	23.61	0.000
X_1X_2	2	2.72745×10^6	1.36373×10^6	6.74	0.002
$X_1X_2^2$	2	570795	285397	1.41	0.253
X_1X_3	2	2.98559×10^6	1.4928×10^6	7.38	0.001
$X_1X_3^2$	2	886587	443294	2.19	0.122
X_2X_3	2	427214	213607	1.06	0.355
$X_2X_3^2$	2	21134	10567	0.05	0.949
$X_1X_2X_3$	2	4.49293×10^6	2.24646×10^6	11.11	0.000
$X_1X_2X_3^2$	2	263016	131508	0.65	0.526
$X_1X_2^2X_3$	2	205537	102768	0.51	0.605
$X_1X_2^2X_3^2$	2	245439	122720	0.61	0.549
residual	54	1.09226×10^7	202270		
total	80	6.88581×10^7			

Note that we have that $\lambda_j^\star = 0$ if and only if $\widehat{\mu}(\chi_j) = 0$ (which is equivalent to $\widehat{\mu}(\overline{\chi_j}) = 0$). Hence, testing the hypothesis $\lambda_j^\star = 0$ against $\lambda_j^\star \neq 0$ is equivalent to simultaneously testing whether $\widehat{\mu}(\chi_j) = 0$ and $\widehat{\mu}(\overline{\chi_j}) = 0$ against the alternative hypothesis that at least one of these Fourier coefficients is not equal to 0. If the alternative hypothesis is true then both Fourier coefficients are not equal to 0 because $\widehat{\mu}(\overline{\chi_j}) = \overline{\widehat{\mu}(\chi_j)}$.

In the special case that the finite group G on which the data is structured is Abelian, all irreducible representations are one-dimensional. In that case each hypothesis of the form $\lambda_j^\star = 0$ against $\lambda_j^\star \neq 0$ has an equivalent hypothesis in terms of the Fourier coefficient $\widehat{\mu}(\overline{\chi_j})$ if χ_j is real-valued or the Fourier coefficients $\widehat{\mu}(\chi_j)$ and $\widehat{\mu}(\overline{\chi_j})$ if χ_j is not real-valued.

Example 10.10 *We continue the example of the simplified seat-belt experiment in Section 4.3. The statistical analysis is presented in Table 3. The sum of squares for complex conjugate pairs of irreducible representations are added to form a single sum of squares of two degrees of freedom. The results are similar to those obtained by Wu and Hamada (2000). Only the F-values corresponding to the orthogonal complements X_1, X_3, X_1X_2, X_1X_3 and $X_1X_2X_3$ are significant at a 0.05 significance level. The significance of the F-value for the test regarding the orthogonal complement X_1 implies that the hypothesis $\widehat{\mu}\left(\chi_{(1,0,0,0)}\right) = \widehat{\mu}\left(\chi_{(2,0,0,0)}\right) = 0$ is rejected in favor of the alternative hypothesis that both $\widehat{\mu}\left(\chi_{(1,0,0,0)}\right)$ and $\widehat{\mu}\left(\chi_{(2,0,0,0)}\right)$ are different from 0. Since the F-value corresponding to the orthogonal component X_2 is not significant, the hypothesis $\widehat{\mu}\left(\chi_{(0,1,0,0)}\right) = \widehat{\mu}\left(\chi_{(0,2,0,0)}\right) = 0$ is accepted.*

10.2 Statistical inference for normal data on a coset of a finite Abelian group

The data obtained on a regular fractional factorial designs can be viewed as data structured on a coset of a finite Abelian group. We consider the statistical analysis of such data. Let

H be a subgroup of the finite Abelian group G and a an element of G. We assume that the data is obtained on the coset aH in G and let N denote the number of elements of the coset. In order to find a decomposition of the total sum of squares we need a decomposition of the $N \times N$ identity matrix.

First we consider the case where the statistical data is obtained on the principal fraction H of the full factorial design G. In that case we can directly view the data as being structured on the group H and use the theory presented in Section 10.1 to analyze the data. By $\mu_{|H} : H \to \mathbb{R}$ we denote the function $\mu : G \to \mathbb{R}$ restricted to H. The theory in Section 10.1 gives us a method for testing the hypothesis $\widehat{\mu_{|H}}(\alpha) = \widehat{\mu_{|H}}(\overline{\alpha}) = 0$ (where $\alpha \in \widehat{H}$) against the alternative hypothesis that both of these Fourier coefficients are not equal to 0. The exact relationship between the Fourier coefficients $\widehat{\mu_{|H}} : \widehat{H} \to \mathbb{C}$ and $\widehat{\mu} : \widehat{G} \to \mathbb{C}$ is found using the Poisson summation formula. From Lemma 8.4 we know that all irreducible characters of H can be viewed as an irreducible character of G that is restricted to H. If $\alpha \in \widehat{G}$ we find that the Fourier coefficients $\widehat{\mu_{|H}}\left(\alpha_{|H}\right)$ and $\widehat{\mu}(\chi)$, $\chi \in \alpha \widehat{G}_H$, are related according to

$$\widehat{\mu_{|H}}\left(\alpha_{|H}\right) = \frac{1}{\mid H \mid} \sum_{h \in H} \mu_{|H}(h)\, \overline{\alpha_{|H}}(h) = \frac{1}{\mid H \mid} \sum_{h \in H} \mu(h)\, \overline{\alpha}(h) = \sum_{\chi \in \alpha \widehat{G}_H} \widehat{\mu}(\chi),$$

where the last equality is obtained using (4) with $a = 1$. Hence, when we are testing the hypothesis $\widehat{\mu_{|H}}\left(\alpha_{|H}\right) = \widehat{\mu_{|H}}\left(\overline{\alpha_{|H}}\right) = 0$ against the alternative hypothesis that at least one of these Fourier coefficients is different from 0 we are in fact testing the hypothesis

$$\begin{gathered} \textstyle\sum_{\chi \in \alpha \widehat{G}_H} \widehat{\mu}(\chi) = \sum_{\chi \in \overline{\alpha} \widehat{G}_H} \widehat{\mu}(\overline{\chi}) = 0 \\ \text{against} \\ \textstyle\sum_{\chi \in \alpha \widehat{G}_H} \widehat{\mu}(\chi) \neq 0 \text{ and} \sum_{\chi \in \overline{\alpha} \widehat{G}_H} \widehat{\mu}(\chi) \neq 0. \end{gathered} \tag{11}$$

All hypotheses that can be tested within the framework of harmonic analysis are of the form given in (11).

When the fractional design on which the data is obtained is not a principal fraction, an additional step is needed. Assume that the statistical data is obtained on the coset aH of G. We denote by $y : G \to \mathbb{R}$ the response function. We only observe the values of this function for $g \in aH$. The expectation function is given by $\mu : G \to \mathbb{R}$ and is defined by $\mu(g) = E[y(g)]$ for all $g \in G$. We define the functions $y_a : G \to \mathbb{R}$ and $\mu_a : G \to \mathbb{R}$ by $y_a(g) = y(ag)$ and $\mu_a(g) = \mu(ag)$, respectively. These functions satisfy $y_a(h) = y(ah)$ and $\mu_a(h) = \mu(ah)$ for all $h \in H$. The observed data in $\{y_a(h) \mid h \in H\}$ is now structured on the group H and we can apply the theory from Section 10.1 using the function $y_{a|H} : H \to \mathbb{R}$ as the function that gives the observed values. The Fourier coefficients of the function $\mu_{a|H}$ can be expressed as

$$\begin{aligned} \widehat{\mu_{a|H}}\left(\alpha_{|H}\right) &= \tfrac{1}{H} \textstyle\sum_{h \in H} \overline{\alpha_{|H}}(h)\, \mu_{a|H}(h) = \tfrac{1}{H} \sum_{h \in H} \overline{\alpha}(h)\, \mu_a(h) = \tfrac{1}{H} \sum_{h \in H} \overline{\alpha}(h)\, \mu(ah) \\ &= \tfrac{\alpha(a)}{H} \textstyle\sum_{h \in H} \overline{\alpha}(ah)\, \mu(ah) = \alpha(a) \sum_{\chi \in \alpha \widehat{G}_H} \widehat{\mu}(\chi)\, \overline{\alpha}\chi(a), \end{aligned}$$

where the last equality follows using (4). Testing the hypothesis $\widehat{\mu_{a|H}}\left(\alpha_{|H}\right) = \widehat{\mu_{a|H}}\left(\overline{\alpha_{|H}}\right) = 0$ against the alternative hypothesis that at least one of these Fourier coefficients is equal to 0 is equivalent to testing the hypothesis

$$\begin{gathered} \textstyle\sum_{\chi \in \alpha \widehat{G}_H} \widehat{\mu}(\chi)\, \overline{\alpha}\chi(a) = \sum_{\chi \in \overline{\alpha} \widehat{G}_H} \widehat{\mu}(\chi)\, \overline{\alpha}\chi(a) = 0 \\ \text{against} \\ \textstyle\sum_{\chi \in \alpha \widehat{G}_H} \widehat{\mu}(\chi)\, \overline{\alpha}\chi(a) \neq 0 \text{ and } \sum_{\chi \in \overline{\alpha} \widehat{G}_H} \widehat{\mu}(\chi)\, \overline{\alpha}\chi(a) \neq 0. \end{gathered} \tag{12}$$

Table 4: Design matrix and response data of a 3^{3-1} factorial design

	Factor			
Run	X_1	X_2	X_3	Response
1	0	0	2	3070
2	0	1	0	5547
3	0	2	1	5735
4	1	0	0	6843
5	1	1	1	6799
6	1	2	2	4968
7	2	0	1	6905
8	2	1	2	6215
9	2	2	0	7145

When the data is obtained on the coset aH in G then all hypotheses that can be tested within the framework of harmonic analysis are of the form given in (12). The theory is illustrated in the next example.

Example 10.11 *Consider the regular fraction of the 3^3 factorial design given in Table 4. The fraction is the coset $F = (0,0,2) + H$ in $G = (\mathbb{Z}/3\mathbb{Z})^3$ where the subgroup H is given by*

$$H = \{(0,0,0), (0,1,1), (0,2,2), (1,1,2), (1,2,0), (1,0,1), (2,2,1), (2,0,2), (2,1,0)\}.$$

The annihilator for H in G is $\widehat{G}_H = \{\chi_{(0,0,0)}, \chi_{(1,1,2)}, \chi_{(2,2,1)}\}$. We have that $\chi_{(0,0,0)}(g) = 1$, $\chi_{(1,1,2)}(g) = \omega$ and $\chi_{(2,2,1)}(g) = \omega^2$ for all $g \in F$. Hence, using the results in Section 7 this fraction can be characterized by the defining equation $I' = \omega^2 X_1' X_2' (X_3')^2 = \omega (X_1')^2 (X_2')^2 X_3'$. The decomposition of the total sum of squares that is obtained using harmonic analysis is presented in Table 5. The cosets of $\widehat{G}_H$ in $\widehat{G}$ that correspond to the orthogonal component X_1 are $\chi_{(1,0,0)}\widehat{G}_H = \{\chi_{(1,0,0)}, \chi_{(2,1,2)}, \chi_{(0,2,1)}\}$ and $\chi_{(2,0,0)}\widehat{G}_H = \{\chi_{(2,0,0)}, \chi_{(0,1,2)}, \chi_{(1,2,1)}\}$. Let $\omega = e^{\frac{2\pi i}{3}}$. The F-test that corresponds to X_1 tests the hypothesis $\widehat{\mu}(\chi_{(1,0,0)}) + \omega\widehat{\mu}(\chi_{(2,1,2)}) + \omega^2\widehat{\mu}(\chi_{(0,2,1)}) = \widehat{\mu}(\chi_{(2,0,0)}) + \omega\widehat{\mu}(\chi_{(0,1,2)}) + \omega^2\widehat{\mu}(\chi_{(1,2,1)}) = 0$ against the alternative hypothesis that both $\widehat{\mu}(\chi_{(1,0,0)}) + \omega\widehat{\mu}(\chi_{(2,1,2)}) + \omega^2\widehat{\mu}(\chi_{(0,2,1)}) \neq 0$ and $\widehat{\mu}(\chi_{(2,0,0)}) + \omega\widehat{\mu}(\chi_{(0,1,2)}) + \omega^2\widehat{\mu}(\chi_{(1,2,1)}) \neq 0$. The p-value for this test is 0.085 *and at a significance level of* 0.05 *the null-hypothesis stating that $\widehat{\mu}(\chi_{(1,0,0)}) + \omega\widehat{\mu}(\chi_{(2,1,2)}) + \omega^2\widehat{\mu}(\chi_{(0,2,1)}) = \widehat{\mu}(\chi_{(2,0,0)}) + \omega\widehat{\mu}(\chi_{(0,1,2)}) + \omega^2\widehat{\mu}(\chi_{(1,2,1)}) = 0$ is accepted. Note that if the Fourier coefficients that correspond to the orthogonal components $X_1 X_2^2 X_3$ and $X_2 X_3$ are zero, that is, $\widehat{\mu}(\chi_{(1,2,1)}) = \widehat{\mu}(\chi_{(2,1,2)}) = \widehat{\mu}(\chi_{(0,1,2)}) = \widehat{\mu}(\chi_{(0,2,1)}) = 0$, then the hypothesis that is tested reduces to $\widehat{\mu}(\chi_{(1,0,0)}) = \widehat{\mu}(\chi_{(2,0,0)}) = 0$ against both $\widehat{\mu}(\chi_{(1,0,0)}) \neq 0$ and $\widehat{\mu}(\chi_{(2,0,0)}) \neq 0$.*

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Table 5: ANOVA Table for the data in Table 4

Orthogonal component	Degrees of freedom	Combined Sum of Squares	Mean Squares	F	p-value
X_1	2	6.20368×10^6	3.10184×10^6	10.71	0.085
X_2	2	511924	255962	0.8835	0.539
X_3	2	6.08926×10^6	3.04463×10^6	10.51	0.087
residual	2	579423	289711		
total	8	1.33843×10^7			

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