

WEAK COUPLING LIMIT OF A POLYMER PINNED AT INTERFACES

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ABSTRACT. We consider a simple random walk of length N denoted by $(S_i)_{i \in \{1, \dots, N\}}$, and we define independently a double sequence $(\gamma_i^j)_{i \geq 1, j \geq 1}$ of i.i.d. random variables and $(w_i)_{i \geq 1}$ a sequence of centered i.i.d. random variables. We set $\beta \geq 0$, $\lambda \geq 0$, $h \geq 0$ and $K \in \mathbb{N}$ and transform the measure of each random walk trajectory with the Hamiltonian $\lambda \sum_{i=1}^N (w_i + h) \text{sign}(S_i) + \beta \sum_{j=-K}^K \sum_{i=1}^N \gamma_i^j \mathbf{1}_{\{S_i=j\}}$. This new path measure describes an hydrophobic(philic) copolymer interacting with a layer of width $2K$ around an interface between oil and water.

In this article we prove the convergence at weak coupling (namely when λ , h and β go to 0) of this discrete model towards its continuous counterpart. To that aim we develop a technique of coarse graining introduced by Bolthausen and den Hollander in [3]. This result shows in particular that the randomness of the pinning around the interface vanishes as the coupling becomes weaker.

We also introduce a new model of polymer interacting with infinitely many horizontal interfaces located at heights $(P_k)_{k \in \mathbb{Z}}$ through the Hamiltonian $\beta \sum_{i=1}^N \sum_{j \in \mathbb{Z}} \gamma_i^j \mathbf{1}_{\{S_i=P_k\}}$ and we extend the former convergence result to a particular case of this model, namely when the widths between successive interfaces are equal.

Keywords: Polymers, Localization-Delocalization Transition, Pinning, Random Walk, Weak Coupling.

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1. DISCRETE MODELS

1.1. A single interface model. We consider a copolymer of N monomers, and an interface separating two solvents (for example oil and water). This interface is given by the x axis.

- **Configurations.** The possible configurations of the polymer are given by the 2^N different trajectories of a simple random walk (S) of length N . Let $(X_i)_{i \geq 1}$ be i.i.d. bernoulli trials satisfying $P(X_1 = \pm 1) = 1/2$. Let $S_0 = 0$ and $S_n = \sum_{i=1}^n X_i$ for $n \geq 1$. Let $\Lambda_i = \text{sign}(S_i)$ if $S_i \neq 0$ and $\Lambda_i = \Lambda_{i-1}$ otherwise.
- **Pinning potential.** We define a pinning potential in a layer of finite width around the interface. For every $j \in \{-K, -K+1, \dots, K-1, K\}$, we let $(\gamma_i^j)_{i \geq 1}$ be i.i.d. random variables, satisfying

$$\mathbb{E}(\exp(\beta|\gamma_1^j|)) < \infty \quad \text{for every } \beta \geq 0.$$

- **Copolymer.** Let $\lambda \geq 0$, $h \geq 0$, $\beta \geq 0$ and let $(w_i)_{i \geq 1}$ be i.i.d., bounded and symmetric random variables, that are independent of γ and satisfy $\mathbb{E}(w_1^2) = 1$. These variables define the rate of hydrophobicity of each monomer. Indeed, the higher w_i is, the more hydrophobic monomer i is. We remark that the disorders γ and w are defined under the law \mathbb{P} .

- **Hamiltonian.** For each trajectory of the random walk, we define the following Hamiltonian

$$H_{N,\beta,h}^{w,\gamma,\lambda}(S) = \lambda \sum_{i=1}^N (w_i + h) \Lambda_i + \beta \sum_{j=-K}^K \sum_{i=1}^N \gamma_i^j \mathbf{1}_{\{S_i=j\}}. \quad (1.1)$$

1.2. A multi-interfaces model. We consider an homopolymer of N monomers, and a medium composed of a solvent and infinitely many horizontal interfaces that interact with the monomers.

In what follows and for every $x \in \mathbb{R}$ we will denote $\lfloor x \rfloor = \max\{n \in \mathbb{Z} : n \leq x\}$.

- **Configurations.** The possible configurations of the polymer are still given by the trajectories of a simple random walk (S) .
- **Interfaces.** Let $c > 0$ and let $(p_i)_{i \in \mathbb{Z} \setminus \{0\}}$ be a sequence of real numbers satisfying $p_i \geq c$ for every $i \in \mathbb{Z} \setminus \{0\}$. Then, define $(P_j)_{j \in \mathbb{Z}}$ as

$$P_0 = 0, \quad P_j = \sum_{i=1}^j p_i \quad \text{if } j > 0 \quad \text{and} \quad P_j = - \sum_{i=-1}^j p_i \quad \text{if } j < 0. \quad (1.2)$$

The quantity $\lfloor P_j \rfloor$ gives the height of the j^{th} interface. The sequence $(p_i)_{i \in \mathbb{Z} \setminus \{0\}}$ will be considered as a parameter of the model and denoted by p .

- **Pinning potential.** Under the law \mathbb{P} , we define a pinning potential along each interface $\lfloor P_k \rfloor$. Therefore, for every $k \in \mathbb{Z}$, we introduce independently of all the other variables the i.i.d sequence $(\gamma_i^k)_{i \geq 1}$ such that there exists an $M \in \mathbb{N} \setminus \{0\}$ satisfying, $\gamma_1^k \in \{-M, \dots, M\}$ for every $k \in \mathbb{Z}$. We assume also that $\mathbb{E}(\gamma_1^k) = \mathbb{E}(\gamma_1^1) > 0$ for every $k \in \mathbb{Z}$.
- **Parameter.** Let $\beta \geq 0$ be a coupling constant, namely the inverse temperature.
- **Hamiltonian.** For each trajectory of the random walk, we define the Hamiltonian

$$H_{N,\beta,p}^\gamma(S) = \beta \sum_{k \in \mathbb{Z}} \sum_{i=1}^N \gamma_i^k \mathbf{1}_{\{S_i = \lfloor P_k \rfloor\}}. \quad (1.3)$$

To avoid heavy notation, and to enounce properties which are verified by the two models we use general notation for the disorders and the families of parameters. Thus, we denote by χ the disorders w, γ (model 1) and γ (model 2), and by θ the families of parameters β, λ, h (model 1) and β, p (model 2). Therefore, the Hamiltonian is denoted by $H_{N,\theta}^\chi(S)$ and we perturb the law of the random walk as follow

$$\frac{dP_{N,\theta}^\chi(S)}{dP}(S) = \frac{\exp(H_{N,\theta}^\chi(S))}{Z_{N,\theta}^\chi}. \quad (1.4)$$

This new measure $P_{N,\theta}^\chi$ is called polymer measure of size N .

1.3. Discrete free energies.

Definition 1. For every $N \in \mathbb{N}$, we introduce the free energy of the system in size N , denoted by $\Phi_N(\theta)$ and defined as

$$\mathbb{E} \left[\frac{1}{N} \log Z_{N,\theta}^\chi \right] = \Phi_N(\theta). \quad (1.5)$$

Proposition 2. *For every θ , there exists a real number, denoted by $\Phi(\theta)$, which satisfies*

$$\lim_{N \rightarrow \infty} \Phi_N(\theta) = \Phi(\theta).$$

The limit $\Phi(\theta)$ is called free energy of the model.

In the single interface case (model 1.1), this proposition has been proved in different articles (see [8] or [9] for example) for some quantities similar to $Z_N^{w,\gamma}$. In our case, the difference comes from the fact that the disorder is spread on a layer of finite width around the interface, but the proof remains essentially the same and is left to the reader. We notice also that $\Phi(\beta, \lambda, h)$ is continuous, convex in each variable, and non decreasing in β .

In the multi-interface case, we give a complete proof of Proposition 2. Notice that, contrary to what happens in the single interface case, this proof is not based on a Kingman Theorem.

2. CONTINUOUS MODELS

We define in this section the continuous counterparts of the models 1.1 and 1.2.

2.1. A single interface model.

- **Configurations.** In the continuous case, the configurations of the polymer will be given by the set of trajectories of the Brownian motion $(B_s)_{s \in [0,t]}$. We denote by \tilde{P} the law of B , and by Λ_s the sign of B_s .
- **Pinning potential.** The pinning potential of this model will be given by the local time spent at 0 by B between 0 and t . It will be denoted by L_t^0 or L_t when there is no ambiguity.
- **Copolymer.** Let $\lambda \geq 0$, $h \geq 0$, $\beta \geq 0$ and independently of B , let $(R_s)_{s \geq 0}$ be a standard Brownian motion with law $\tilde{\mathbb{P}}$. We consider dR_s an elementary variation of R at position s . This quantity gives the hydrophobicity of the polymer around the position s , and plays the role of w_i in the discrete model.
- **Hamiltonian:** for a fixed trajectory of R we define, for every trajectory of B , the following Hamiltonian

$$\tilde{H}_{\lambda,h,\beta}^{R,t}(B) = \lambda \int_0^t \Lambda(s)(dR_s + hds) + \beta L_t. \quad (2.1)$$

2.2. A multi-interfaces model.

- **Configurations.** The configurations of the polymer are still given by the trajectories of the Brownian motion $(B_s)_{s \in [0,t]}$.
- **Interfaces.** Recall that the sequence $(P_k)_{k \in \mathbb{Z}}$ is defined in (1.2). The horizontal interfaces are located at the heights $\{P_k : k \in \mathbb{Z}\}$. Along the k^{th} interface, the pinning reward is given by $L_t^{P_k}$, where L_t^x is the local time spent by B at level x between times 0 and t .
- **Parameter.** Let $\beta \geq 0$ be a coupling constant, namely the inverse temperature.
- **Hamiltonian.** For each trajectory of the random walk, we define the Hamiltonian

$$\tilde{H}_{\beta,t}(B) = \beta \sum_{j \in \mathbb{Z}} L_t^{P_j}. \quad (2.2)$$

As in the discrete case, we use the notations χ for the disorder R (model 2.1). We use θ for the families of parameters (β, λ, h) (model 2.1) and (β, p) (model 2.2). Therefore, the Hamiltonian is denoted by $\tilde{H}_{t,\theta}^\chi(B)$ and we define the polymer measure of length t by perturbing the law of the Brownian motion as follows

$$\frac{d\tilde{P}_{t,\theta}^\chi}{d\tilde{P}}(B) = \frac{\exp(\tilde{H}_{t,\theta}^\chi(B))}{\tilde{Z}_{t,\theta}^\chi}. \quad (2.3)$$

2.3. Continuous free energies.

Proposition 3. *For every θ , there exists a real number, denoted by $\tilde{\Phi}(\theta)$, which satisfies*

$$\lim_{t \rightarrow \infty} \tilde{\mathbb{E}} \left[\frac{1}{t} \log \tilde{Z}_{t,\theta}^\chi \right] = \tilde{\Phi}(\theta).$$

This limit is called free energy of the model.

In the single interface case (model 2.1), a proof of Proposition 3 in the case $\beta = 0$ is available in [8]. This prove is adapted in [15] to cover the case $\beta > 0$.

In the multi-interface case (model 2.2), a proof of Proposition 3 can be obtained without difficulty by adapting the proof of Proposition 2, that we give in section 8.1, to the continuous case. For this reason this proof is left to the reader. Finally we notice that in both single and multi-interface models, the free energy is continuous convex and non decreasing in each variable.

3. PHYSICAL MOTIVATIONS

3.1. Single interface model: a more realistic model of interface. Models of polymer pinned at an interface have attracted a lot of interest in the last years (see [12], [1], [14]). One of the physical situations that can be modelled by such systems is a polymer put in the neighborhood of an interface between two solvents (see [3]). It gives opportunities to study the localization of the polymer with respect to the interface. Nevertheless, these models do not take into account that such an interface has a width, that is to say a small layer in which the two solvents are more or less mixed together. The model that we develop in this chapter gives a more realistic image of an interface. It allows us also to consider a case, in which microemulsions of a third solvent are spread in a thin layer around the interface.

3.2. Multi-interface model. Until now, the mathematical works about directed polymers in interaction with a medium have been concentrated on two general families of environments. On one hand, the media composed by two phases and separated by a flat interface, on the other hand the i.i.d. media for which every site of \mathbb{Z}^2 is associated with a reward (these rewards are i.i.d). Recently, new models have been developed to study the behavior of a polymer in other environments. For instance, in [5] a copolymer is placed in a media composed by successive horizontal layers of oil and water and in [6], a model of copolymer in emulsions is investigated. In this article we introduce a model of polymer pinned by a random potential along infinitely many interfaces. These interfaces are horizontal and the widths between them are not equal. This has not been studied yet, but should be a consistent way to investigate the motion of a particle in a medium composed by horizontal traps. For instance, it allows us to model the motion of a dust particle in an atmosphere which is striated by thin pollution layers.

4. SINGLE INTERFACE MODEL

4.1. Localization criterium. In the discrete and continuous single interface cases, the free energy gives us a tool to decide, for every (β, λ, h) , whether the system is localized or not. Indeed, in the discrete case we denote by D_N the subset $\{S : S_i > K \forall i \in \{K+1, \dots, N\}\}$, and we restrict the computation of Φ to D_N . Then, since $P(D_N) = (1 + o(1))c/\sqrt{N}$ as $N \uparrow \infty$, we obtain

$$\begin{aligned} \Phi(\beta, \lambda, h) &\geq \liminf_{N \rightarrow \infty} \frac{1}{N} \log E \left[\exp \left(\lambda \sum_{i=1}^N (w_i + h) + \beta \sum_{i=1}^K \gamma_i^i \right) \mathbf{1}_{\{D_N\}} \right] \\ &\geq \lambda h + \liminf_{N \rightarrow \infty} \frac{\lambda \sum_{i=1}^N w_i}{N} + \liminf_{N \rightarrow \infty} \frac{\beta \sum_{i=1}^K \gamma_i^i}{N} + \liminf_{N \rightarrow \infty} \frac{\log(P(D_N))}{N} \geq \lambda h. \end{aligned} \quad (4.1)$$

We will say that the polymer is delocalized when $\Phi(\beta, \lambda, h) = \lambda h$ because the trajectories of D_N give the whole free energy, and localized when $\Phi(\beta, \lambda, h) > \lambda h$. The (β, λ, h) -space is divided into a localized phase, denoted by \mathcal{L} , and a delocalized phase denoted by \mathcal{D} .

In the continuous case, we consider the subset $\tilde{D}_t = \{B : B_s > 0 \forall s \in [1, t]\}$. A computation similar to (4.1) shows that the localization condition remains the same, i.e. $\tilde{\Phi} > \lambda h$.

The separation between the localized and delocalized phases has an interpretation in terms of trajectories of the polymer. This issue has been closely studied recently and we refer to [19] or [10] for precise estimates about it. We mention here a result of [2] concerning the delocalized phase. It shows that the proportion of time spent by the polymer under an arbitrary level $L > 0$ is equal to 0, namely

$$\lim_{N \rightarrow \infty} \frac{1}{N} E_N^{w, \gamma} (\#\{i \in \{1, \dots, N\} : S_i \leq L\}) = 0, \quad \mathbb{P}\text{-a.s. in } w.$$

In the localized phase, since Φ is convex in β , a simple computation shows that the polymer comes back to the layer around the origin at a positive density of sites.

4.2. Critical curve. For γ, K and β fixed, both the discrete and continuous single interface models undergo a critical curve denoted by $\lambda \rightarrow h_c^\beta(\lambda)$ ($\tilde{h}_c^\beta(\lambda)$ in the continuous case), which divides the (λ, h) -space into the two phases \mathcal{L} and \mathcal{D} . Namely, $\Phi(\lambda, h, \beta) > \lambda h$ when $h < h_c^\beta(\lambda)$, and $\Phi(\lambda, h, \beta) = \lambda h$ when $h \geq h_c^\beta(\lambda)$. The existence of this curve is proved in [3] for the case $\beta = 0$, and can be easily adapted to our case. For this reason we will not give the details in this article.

It has been proved in [3] that, when $\beta = 0$, the critical curve $\tilde{h}_c^0(\lambda)$ of the continuous model is a straight line of slope K_c^0 . This is still true when we add a pinning term. Indeed, the critical curve satisfies $\tilde{h}_c^{\lambda\beta}(\lambda) = \lambda K_c^\beta$. Moreover, since $\tilde{\Phi}$ is non decreasing in β , K_c^β is non decreasing in β , and we give in Appendix 1 a short proof of the convexity of K_c^β . As a consequence, K_c^β is continuous in β as long as it is finite.

4.3. Preview and results. In this article, we prove that the discrete single interface model converges (in a sense that will be specified) toward its continuous counterpart when the parameters $(\lambda, h$ and $\beta)$ tend to zero at a certain speed. Such a convergence has been proved in [3] without pinning (i.e. $\beta = 0$). However, when $\beta > 0$, we know that some zones, in the interacting layer around the origin, concentrate a large number of high rewards and play a particular role from the localization point of view. Indeed, the chain can target when it goes back to the origin in order to maximize the rewards. Consequently, some zones favor the localization of the polymer more than others (see [1] and [14]). We

wonder here whether the passage to a very weak coupling conserves the randomness of these rewards or leads to a complete averaging of the disorder?

We answer this question in Theorems 4 and 5. Indeed, we show a convergence, in terms of free energy, of the discrete model to the continuous model, when the parameters tend to 0 at appropriate speeds. The associated continuous model has a pinning term at the interface, given by the local time at 0 of a the Brownian motion B . Therefore, the randomness of the pinning term vanishes in the weak coupling limit.

Theorem 4. *Let β , λ and h be non negative constants, and let $\Sigma = \sum_{j=-K}^K \mathbb{E}(\gamma_1^j)$. We have the following convergence*

$$\lim_{a \rightarrow 0} \frac{1}{a^2} \Phi(a\beta, a\lambda, ah) = \tilde{\Phi}(\beta\Sigma, \lambda, h). \quad (4.2)$$

This Theorem will be deduced from Theorem 5.

Remark 1. We define the quantities $\Psi_N(\beta, \lambda, h) = \Phi_N(\beta, \lambda, h) - \lambda h$ and $\tilde{\Psi}_t(\beta, \lambda, h) = \tilde{\Phi}_t(\beta, \lambda, h) - \lambda h$. They converge respectively to $\Psi(\beta, \lambda, h) = \Phi(\beta, \lambda, h) - \lambda h$ and $\tilde{\Psi}(\beta, \lambda, h) = \tilde{\Phi}(\beta, \lambda, h) - \lambda h$, which are called *excess free energies* of the polymer. Therefore, to decide whether the polymer is localized or not, it suffices to compare Ψ or $\tilde{\Psi}$ with zero. Moreover, since $\sum_{i=1}^N (w_i + h) = hN + o(N)$ when $N \uparrow \infty$, we can subtract this quantity from the former single interface Hamiltonian and associate Ψ_N with

$$\tilde{H}_{N,\beta,\lambda,h}^{w,\gamma} = -2\lambda \sum_{i=1}^N (w_i + h)\Delta_i + \beta \sum_{j=-K}^K \sum_{i=1}^N \gamma_i^j \mathbf{1}_{\{S_i=j\}},$$

with $\Delta_i = 1$ if $\Lambda_i = -1$ and $\Delta_i = 0$ otherwise. Similarly, $\tilde{\Psi}_t(\beta, \lambda, h)$ is associated with

$$\tilde{H}_{t,\beta}^R = -2\lambda \int_0^t \mathbf{1}_{\{B_s < 0\}} (dR_s + hds) + \beta L_t^0,$$

and Ψ and $\tilde{\Psi}$ are convex and continuous in each of the three variables, non decreasing in β and non increasing in h . We emphasize also the fact that, proving Theorem 4 with Φ and $\tilde{\Phi}$ or Ψ and $\tilde{\Psi}$ is equivalent.

Remark 2. Stating Theorem 5 requires a slight modification of the Hamiltonian. Indeed, let β_1 and β_2 be two non-negative numbers and define

$$I_1 = \{j \in \{-K, \dots, K\} : \mathbb{E}(\gamma_1^j) > 0\} \quad \text{and} \quad I_2 = \{j \in \{-K, \dots, K\} : \mathbb{E}(\gamma_1^j) < 0\}.$$

Then, if $\mathbb{E}(\gamma_1^j) \neq 0$ for every $j \in \{-K, \dots, K\}$, we define

$$H_{N,\beta_1,\beta_2,\lambda,h}^{w,\gamma} = \beta_1 \sum_{j \in I_1} \sum_{i=1}^N \gamma_i^j \mathbf{1}_{\{S_i=j\}} + \beta_2 \sum_{j \in I_2} \sum_{i=1}^N \gamma_i^j \mathbf{1}_{\{S_i=j\}} + \lambda \sum_{i=1}^N (w_i + h)\Lambda_i. \quad (4.3)$$

The associated free energy $\Psi(\beta_1, \beta_2, \lambda, h)$ is defined as in Proposition 2, and satisfies $\Psi(\beta, \lambda, h) = \Psi(\beta, \beta, \lambda, h)$. Thus, in what follows, we will use the notation $\Psi(\beta_1, \beta_2, \lambda, h)$ if $\beta_1 \neq \beta_2$, otherwise we will use $\Psi(\beta, \lambda, h)$. We let $\Sigma = \Sigma_1 + \Sigma_2$, with $\Sigma_1 = \sum_{j \in I_1} \mathbb{E}(\gamma_1^j)$ and $\Sigma_2 = \sum_{j \in I_2} \mathbb{E}(\gamma_1^j)$.

Theorem 5. *Suppose $\mathbb{E}(\gamma_1^j) \neq 0$ for every $j \in \{-K, \dots, K\}$. If $\beta_1 > 0$, $\beta_2 > 0$, and $(\mu_1, \mu_2) \in \mathbb{R}^2$ satisfy*

$$\mu_1 > \beta_1 \Sigma_1 + \beta_2 \Sigma_2 > \mu_2,$$

and $\rho > 0$, $h > 0$, $h' \geq 0$, $\lambda > 0$ satisfy $(1 + \rho)h' < h$, then there exists $a_0 > 0$ such that for every $a < a_0$

$$\begin{aligned} \frac{1}{a^2} \Psi(a\beta_1, a\beta_2, a\lambda, ah) &\leq (1 + \rho) \tilde{\Psi}(\mu_1, \lambda, h') \\ \tilde{\Psi}(\mu_2, \lambda, h) &\leq \frac{1 + \rho}{a^2} \Psi(a\beta_1, a\beta_2, a\lambda, ah'). \end{aligned} \quad (4.4)$$

This result gives also the convergence of the slope of the discrete critical curve at 0 to its continuous counterpart. We give the details in the next corollary.

Corollary 6. *For every $\beta \geq 0$, and even if there exists $j \in \{-K, \dots, K\}$ such that $\mathbb{E}(\gamma_i^j) = 0$, then*

$$\lim_{\lambda \rightarrow 0} \frac{h_c^{\lambda\beta}(\lambda)}{\lambda} = K_c^{\beta\Sigma}.$$

5. A PARTICULAR CASE: THE HOMOPOLYMER

By fixing $\lambda = 1$ and $w_i \equiv 0$ for every $i \geq 1$, we can model a homopolymer instead of a copolymer. Indeed, in this case the polymer only consists of hydrophobic monomers, and its related Hamiltonian is given by

$$h \sum_{i=1}^N \Delta_i + \beta \sum_{j=-K}^K \sum_{i=1}^N \gamma_i^j \mathbf{1}_{\{S_i=j\}}. \quad (5.1)$$

This type of model, that we call h -model, with a pinning term at the interface in competition with a repulsion effect (given here by $h \sum_{i=1}^N \Delta_i$), has already been investigated in the literature (see [11], or [4]). It has been proved for instance, that some properties of the h -model can be extended to the wetting model by letting the parameter h tend to ∞ (see [15]).

The free energy of the h -model is denoted by $\Phi(\beta, h)$ and the localization condition remains: $(\beta, h) \in \mathcal{L}$ when $\Phi(\beta, h) > h$ and $(\beta, h) \in \mathcal{D}$ when $\Phi(\beta, h) = h$. The critical curve of the h -model, which separates the (h, β) -plane into a localized and a delocalized phase is denoted by $h_c(\beta)$. This curve is also increasing, convex and satisfies $h_c(0) = 0$. Another particularity of this system comes from the simplicity of its continuous limit. Indeed, applied to this case, Theorem 4 implies that the continuous Hamiltonian is given by

$$h \int_0^t \Delta_s ds + \beta \Sigma L_t. \quad (5.2)$$

Thus, the disorder disappears and we can compute explicitly some quantities related to this limit. If we denote by $\tilde{\Phi}(\beta\Sigma, h)$ the continuous free energy which is associated with (5.2), then we obtain the following proposition.

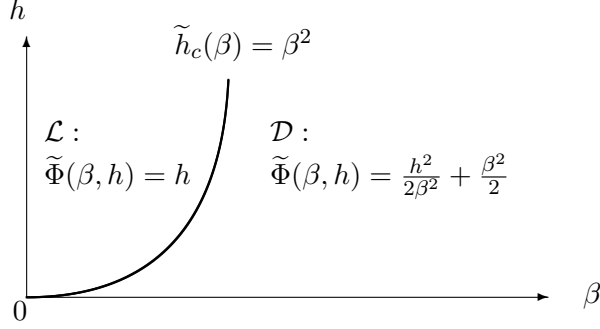
For simplicity, we state the proposition for the case $\Sigma = 1$.

Proposition 7. *Let $\beta \geq 0$ and $h \geq 0$. Then,*

$$\tilde{\Phi}(\beta, h) = h \quad \text{if } h \geq \beta^2, \quad \text{and} \quad \tilde{\Phi}(\beta, h) = \frac{h^2}{2\beta^2} + \frac{\beta^2}{2} \quad \text{if } h < \beta^2.$$

Since $h^2/(2\beta^2) + \beta^2/2 > h$ when $h < \beta^2$, we obtain the continuous critical curve, i.e., $\tilde{h}_c(\beta) = \beta^2$ for $\Sigma = 1$ (see Fig 1).

Fig. 1:



With the general h -model, i.e. with Σ not necessarily equal to 1, we can deduce from Theorem 4, the behavior of some quantities linked to the discrete model as β tends to zero. For instance, we can compute the slope in 0 of the discrete critical curve. It gives

$$\lim_{\beta \rightarrow 0} \frac{h_c(\beta)}{\beta^2} = \Sigma^2. \quad (5.3)$$

This limit is conform to the intuition, to the extend that a stronger pinning along the interface enlarges the localized area, and consequently, increases the slope of the critical curve at the origin. It is also confirmed by the bounds of the critical curve found in [15].

With Proposition 7, we differentiate $\tilde{\Phi}(h, \beta)$ with respect to β and we find the asymptotic behavior of the reward average in the weak coupling limit. Indeed, when $h < \beta^2$, by convexity of Φ_N in β , we can write that, a.s. in γ ,

$$\lim_{a \rightarrow 0} \lim_{N \rightarrow \infty} \frac{1}{aN} E_{N,a\beta}^{a^2 h, w} \left[\sum_{j=-K}^K \sum_{i=1}^N \gamma_i^j \mathbf{1}_{\{S_i=j\}} \right] = \beta - \frac{h^2}{\beta^3}.$$

The same derivative with respect to h gives an approximation, for a small, of the time proportion spent by the polymer under the interface, i.e.,

$$\lim_{a \rightarrow 0} \lim_{N \rightarrow \infty} E_{N,a\beta}^{a^2 h, w} \left[\frac{\sum_{i=1}^N \Delta_i}{N} \right] = \frac{\beta^2 - h}{2\beta^2}.$$

6. MULTI-INTERFACE CASE

6.1. Preview and results. In this section, we want to extend the convergence of the discrete model towards the continuous model to the case of a polymer interacting with an infinite number of interfaces. However, for technical reasons, we will consider the regular interfaces model, i.e., when the widths between successive interfaces are all equal to a constant $r > 0$. Therefore, for every $k \in \mathbb{Z}$ we have $P_k = kr$. Actually, some parts of the proofs of Theorems 8 and 9 are satisfied by the model with irregular interfaces (i.e. satisfying the assumptions (1.2)). This is the case for instance of Lemma 11 which is one of the key of the coarse graining. For this reason, it seems that these two theorems should also be verified with irregular interfaces, but we do not prove it in this article.

Theorem 8. *For every $\beta \geq 0$, we have the convergence*

$$\lim_{a \rightarrow 0} \frac{1}{a^2} \Phi(a\beta, r/a) = \tilde{\Phi}(\beta, r). \quad (6.1)$$

Similarly to what we did in the single interface case, Theorem 8 is actually the consequence of a stronger theorem that we introduce now.

Theorem 9. *Let $\rho > 0$, $\alpha > 0$, $\beta' > \beta$ and $\beta'' < \frac{\beta}{1+\rho}$. Then for a small enough we obtain the two inequalities*

$$\frac{1}{a^2} \Phi(a\beta, r/a) \leq \frac{1}{1+\rho} \tilde{\Phi}((1+\rho)\beta', r) + \alpha, \quad \text{and} \quad \tilde{\Phi}(\beta'', r) \leq \frac{1}{(1+\rho)a^2} \Phi(a\beta, r/a) + \alpha. \quad (6.2)$$

The main difference between this Theorem and Theorem 5 comes from the parameter α that we introduce here. Notice that we did not use it in Theorem 5, otherwise we could not have proved the convergence of the slopes in corollary 6.

7. PREPARATION

7.1. Technical Lemma.

Lemma 10. *For every $K \in \mathbb{N}$ and every $(f_{-K}, f_{-K+1}, \dots, f_K)$ in $(\mathbb{R}^+)^{2K+1}$ the following convergence occurs:*

$$\lim_{N \rightarrow \infty} E \left[\exp \left(\frac{1}{\sqrt{N}} \sum_{j=-K}^K f_j \sum_{i=1}^N \mathbf{1}_{\{S_i=j\}} \right) \right] = E \left[\exp \left(\left(\sum_{j=-K}^K f_j \right) L_1^0 \right) \right], \quad (7.1)$$

where L_1^0 is the local time in 0 of a Brownian motion $(B_s)_{s \geq 0}$ between 0 and 1.

Proof. First, we prove the following intermediate result. For every $K \in \mathbb{N}$

$$\frac{1}{\sqrt{N}} \sum_{j=-K}^K f(j) \sum_{i=1}^N \mathbf{1}_{\{S_i=j\}} \xrightarrow{Law} \left(\sum_{j=-K}^K f_j \right) L_1^0. \quad (7.2)$$

For simplicity, we only prove that $\frac{1}{\sqrt{N}} (\sum_{i=1}^N \mathbf{1}_{\{S_i=0\}}, \sum_{i=1}^N \mathbf{1}_{\{S_i=1\}})$ converges in law to (L_1^0, L_1^0) as $N \uparrow \infty$. The proof for $2K+1$ levels is exactly the same. For this convergence in law, we use a result of [16], saying that we can build, on the same probability space (Ω, \mathcal{A}, P) , a simple random walk $(S_i)_{i \geq 0}$ and a Brownian motion $(B_s)_{s \geq 0}$ such that P almost surely

$$\lim_{n \rightarrow \infty} \sup_{j \in \{0,1\}} \frac{1}{\sqrt{n}} |U_n^j - L_n^j| = 0 \quad (7.3)$$

with $U_n^j = \sum_{i=1}^n \mathbf{1}_{\{S_i=j\}}$ and L_n^x the local time in x of B between 0 and n . The equation (7.3) implies that $\frac{1}{\sqrt{n}} (U_n^0 - L_n^0)$ and $\frac{1}{\sqrt{n}} (U_n^1 - L_n^1)$ tend a.s. to 0 as $n \uparrow \infty$. Therefore, the proof of (7.2) will be completed if we show that $\frac{1}{\sqrt{n}} (L_n^0, L_n^1)$ converges in law to (L_1^0, L_1^0) . By the scaling property of Brownian motion, we obtain that, for every $n \geq 1$, $\frac{1}{\sqrt{n}} (L_n^0, L_n^1)$ has the same law as $(L_1^0, L_1^{1/\sqrt{n}})$. Thus, since L_1^x is a.s. continuous in $x = 0$, we obtain immediately the a.s. convergence of $(L_1^0, L_1^{1/\sqrt{n}})$ towards (L_1^0, L_1^0) . This a.s. convergence implies the convergence in law and (7.2) is proved.

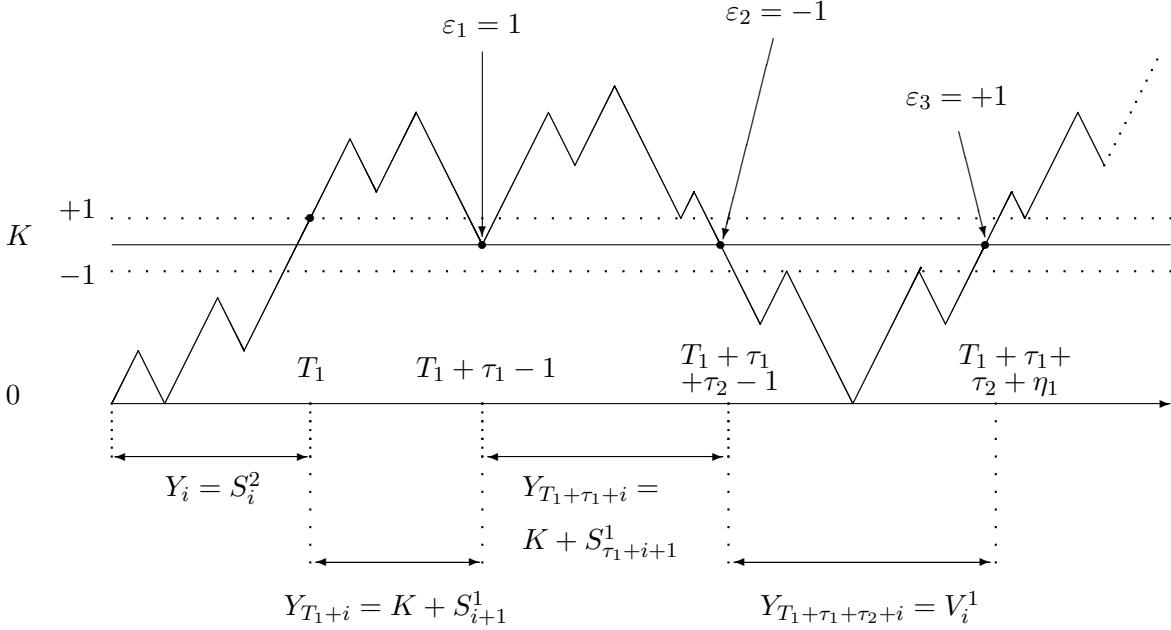
Since the function $\exp(x)$ is continuous, (7.2) gives us the convergence in law of $W_N = \exp(\frac{1}{\sqrt{N}} \sum_{j=-K}^K f_j \sum_{i=1}^N \mathbf{1}_{\{S_i=j\}})$ to $\exp((\sum_{j=-K}^K f_j) L_1^0)$ as $N \uparrow \infty$. The uniform integrability of the sequence $(W_N)_{N \geq 1}$ will therefore be sufficient to complete the proof of Lemma 10. To that aim we will use the following construction.

Let $(S_n^1)_{n \geq 0}$ be a reflected simple random walk, and denote by k_N the number of return to the origin before time N and $\tau_1, \tau_2, \dots, \tau_{k_N}, N - \tau_1 - \dots - \tau_{k_N}$ the length of the corresponding excursions out of the origin until time N . Independently, we let $(S_n^2)_{n \geq 0}$

be a reflected simple random walk starting in 0 and we denote by T_1 her first passage time in $K + 1$. Next, for every $i \geq 1$, we let $(V_n^i)_{n \geq 0}$ be a reflected simple random walk, independent of all the others, which satisfies $V_0^i = K - 1$. We denote by η_i the first passage time in $K + 1$ of V_n^i . Finally, we define a sequence $(\epsilon_i)_{i \geq 1}$ of independent Bernoulli trials satisfying $P(\epsilon_i = \pm 1) = 1/2$.

At this stage we build a new process (see Fig 2), denoted by $(Y_i)_{i \geq 1}$, such that $Y_i = S_i^2$ for every $i = 0, 1, \dots, T_1$. Thus, $Y_{T_1} = K + 1$, and we set $Y_{T_1+i} = K + S_{i+1}^1$ for every $i = 0, \dots, \tau_1 - 1$, so that $Y_{T_1+\tau_1-1} = K$. At this stage, either $\epsilon_1 = 1$ and we set $Y_{T_1+\tau_1+i} = K + S_{\tau_1+i+1}^1$ for every $i = 0, \dots, \tau_2 - 1$ and then $Y_{T_1+\tau_1+\tau_2-1} = K$, or $\epsilon_1 = -1$ and $Y_{T_1+\tau_1+i} = V_1^i$ for every $i = 0, \dots, \eta_1$ and $Y_{T_1+\tau_1+\eta_1} = K + 1$. We go on like this, namely, after the j^{th} excursion of H above K , if $\epsilon_j = 1$, Y describes above K the next excursion of $(S_n^1)_{n \geq 0}$, otherwise Y describes an excursion between 0 and K until it reaches $K + 1$. At this moment, Y describes above K the next excursion of $(S_n^1)_{n \geq 0}$ and so on...

Fig. 2:



We denote by k_N^1 the number of excursions between 0 and K done by Y before time N , and by j_N the number of steps that Y does between 0 and K before N . It comes easily that $k_N^1 \leq k_N$, and that

$$j_N \leq k_N + \sum_{j=1}^{k_N^1} \eta_j + T_1 \leq k_N + \sum_{j=1}^{k_N} \eta_j + T_1. \quad (7.4)$$

We denote $F = \max\{f_{-K}, f_{-K+1}, \dots, f_K\}$ and to prove the uniform integrability of K_N , it suffices to show that $V_N = \exp(Fj_N/\sqrt{N})$ is bounded from above in L^2 norm, independently of N . By definition, $(\zeta_i)_{i \geq 1}, T_1$ and k_N are independent, and by using the Jensen's

inequality we obtain

$$\begin{aligned} E(V_N^2) &\leq E\left[\exp\left(\frac{2F}{\sqrt{N}}(k_N + \sum_{j=1}^{k_N} \eta_j + T_1)\right)\right] = E\left[E\left[\exp\left(\frac{2F(1+\eta_1)}{\sqrt{N}}\right)\right]^{k_N}\right] E\left[\exp\left(\frac{2FT_1}{\sqrt{N}}\right)\right] \\ &\leq E\left[\exp\left[\frac{k_N}{\sqrt{N}}(\log E(\exp(2F + 2F\eta_1)))\right]\right] E\left[\exp(2FT_1)\right]. \end{aligned} \quad (7.5)$$

To complete our proof, it just remains to prove that for every $b > 0$ the sequence $(E[\exp(bk_N/\sqrt{N})])_{N \geq 0}$ is bounded from above independently of N . To that aim, we notice that $k_N \leq k_{2N} \leq N$ and write the obvious inequality

$$E\left[\exp(bk_{2N}/\sqrt{N})\right] \leq \sum_{k=0}^{\lfloor \sqrt{N/2} \rfloor} e^{\sqrt{2}b(k+1)} P(k_{2N} \in [k\sqrt{2N}, (k+1)\sqrt{2N}]). \quad (7.6)$$

With the help of [7] we can compute an upper bound of $P(k_{2N} \in [k\sqrt{2N}, (k+1)\sqrt{2N}])$. Indeed, for every $k \leq \lfloor \sqrt{N/2} \rfloor$ we obtain

$$P(k_{2N} \in [k\sqrt{2N}, (k+1)\sqrt{2N}]) \leq \sum_{j=\lfloor k\sqrt{2N} \rfloor}^{\max(\lfloor (k+1)\sqrt{2N} \rfloor, N)} P(S_{2N} = 0) \frac{(1-\frac{1}{N}) \dots (1-\frac{j-1}{N})}{(1-\frac{1}{2N}) \dots (1-\frac{j-1}{2N})}. \quad (7.7)$$

The function $x \rightarrow \log(1-x) + x$ is decreasing on $[0, 1)$ and consequently, for every $j \in \{\lfloor k\sqrt{2N} \rfloor, \dots, \max(\lfloor (k+1)\sqrt{2N} \rfloor, N)\}$, we have $\log(1-j/N) - \log(1-j/2N) \leq -j/2N$. Therefore,

$$\frac{(1-\frac{1}{N}) \dots (1-\frac{j-1}{N})}{(1-\frac{1}{2N}) \dots (1-\frac{j-1}{2N})} \leq \exp\left(\sum_{i=1}^{j-1} -\frac{i}{2N}\right) = \exp\left(-\frac{j(j-1)}{4N}\right) \leq \exp\left(-\frac{(k-1)^2}{2}\right).$$

Moreover $\lfloor (k+1)\sqrt{2N} \rfloor - \lfloor k\sqrt{2N} \rfloor \leq \sqrt{2N} + 1$ and there exists a constant $c > 0$ such that $P(S_{2N} = 0) \leq c/\sqrt{2N}$ for every $N \geq 1$. That is why, the equation (7.7) becomes

$$P(k_{2N} \in [k\sqrt{2N}, (k+1)\sqrt{2N}]) \leq 2c \exp(-(k-1)^2/2).$$

This results allows us to rewrite (7.6) as

$$E\left[\exp(bk_{2N}/\sqrt{N})\right] \leq \sum_{k=0}^{\infty} 2c e^{b(k+1)} e^{-\frac{(k-1)^2}{2}},$$

and the r.h.s. of this inequality is the sum of a convergent series. Therefore, the sequence $(W_N)_{N \geq 0}$ is uniformly integrable and the proof of Lemma 10 is completed.

Before asserting Lemma 11, we recall that $c > 0$, and that $(p_i)_{i \in \mathbb{Z} \setminus \{0\}}$ is a sequence of real numbers which satisfy $p_i \geq c$ for every $i \in \mathbb{Z} \setminus \{0\}$. We recall also that $P_0 = 0$, $P_j = \sum_{i=1}^j p_i$ if $j > 0$, and $P_j = -\sum_{i=-1}^j p_i$ if $j < 0$. \square

Lemma 11. *For every sequence $(p_i)_{i \in \mathbb{Z} \setminus \{0\}}$ and $\lambda > 0$ small enough the following convergence occurs*

$$\lim_{N \rightarrow \infty} E\left[\exp\left(\frac{\lambda}{\sqrt{N}} \sum_{i=1}^N \sum_{k \in \mathbb{Z}} \mathbf{1}_{\{S_i = \lfloor P_k \sqrt{N} \rfloor\}}\right)\right] = E\left[\exp\left(\lambda \sum_{k \in \mathbb{Z}} L_1^{P_k}\right)\right] \quad (7.8)$$

Proof. We denote

$$T_N^\lambda = \frac{\lambda}{\sqrt{N}} \sum_{i=1}^N \sum_{k \in \mathbb{Z}} \mathbf{1}_{\{S_i = \lfloor P_k \sqrt{N} \rfloor\}},$$

and $M_N^\lambda = \exp(T_N^\lambda)$. Similarly to what we did in Lemma 10, this proof is divided in two steps. In the first step, we prove the convergence in law of M_N^λ towards $\exp(\lambda \sum_{k \in \mathbb{Z}} L_1^{P_k})$ as $N \uparrow \infty$. In the second step, we prove that $(M_N^\lambda)_{N \geq 1}$ is uniformly integrable for λ small enough and this completes the proof of the lemma.

7.2. Step 1. We begin with proving that T_N^λ converges in law to $\lambda \sum_{k \in \mathbb{Z}} L_1^{P_k}$ as $N \uparrow \infty$. This will be sufficient to complete the proof of this step because the exponential is continuous. For every $k \in \mathbb{Z}$ we have $|P_k| \geq |k|c$. Therefore,

$$\frac{1}{\sqrt{N}} \#\{i \in \{1, \dots, N\} : |S_i| \geq N^{3/5}\} \geq \frac{1}{\sqrt{N}} \sum_{i=1}^N \sum_{|k| > c^{-1}N^{1/10}} \mathbf{1}_{\{S_i = \lfloor P_k \sqrt{N} \rfloor\}}. \quad (7.9)$$

Moreover, the law of the iterated logarithm gives us that P a.s. in S , $|S_n|/n^{3/5}$ tends to 0 as $n \uparrow \infty$. Hence, P a.s. in S , the set $\{n \in \mathbb{N} : |S_n| \geq n^{3/5}\}$ is finite. This entails that, P a.s. in S , both the left and the right hand sides of (7.9) tend to 0 when $N \uparrow \infty$. Thus, the Slutsky's lemma allows us to restrict T_N^λ to the sum over $|k| \leq c^{-1}N^{1/10}$ instead of $k \in \mathbb{Z}$.

At this stage we use a coupling result, due to Revesz in [16], which can be enounced as follows

Theorem 12. *For every $\alpha > 0$ and $n \geq 1$, one can build, on the same probabilistic space (Ω, \mathcal{A}, Q) , an n -steps simple random walk $(S_i^{(n)})_{i \in \{1, \dots, n\}}$ and a local time of Brownian motion between 0 and 1 $(L_1^{x, (n)})_{x \in \mathbb{R}}$, such that Q a.s.*

$$\frac{1}{n^{1/4+\alpha}} \sup_{x \in \mathbb{Z}} \left| \sqrt{n} L_1^{x/\sqrt{n}, (n)} - \#\{i \in \{1, \dots, n\} : S_i^{(n)} = x\} \right| \xrightarrow{n \rightarrow \infty} 0.$$

By using this theorem with $\alpha = 1/8$, we can build $L^{(n)}$ and $S^{(n)}$ such that they satisfy

$$\sup_{x \in \mathbb{Z}} \left| L_1^{x/\sqrt{n}, (n)} - \#\{i \in \{1, \dots, n\} : S_i^{(n)} = x\} / \sqrt{n} \right| = \xi(n) n^{-1/8},$$

where $\xi(n)$ tends Q a.s. to 0 as $n \uparrow \infty$. Thus, we obtain

$$\sum_{|k| \leq c^{-1}N^{1/10}} \left| \#\{i \in \{1, \dots, N\} : S_i^{(N)} = \lfloor P_k \sqrt{N} \rfloor\} / \sqrt{N} - L_1^{\lfloor P_k \sqrt{N} \rfloor / \sqrt{N}, (N)} \right| \leq \frac{2\xi(N)N^{1/10}}{cN^{1/8}}. \quad (7.10)$$

Moreover, for every $N \geq 1$ and $\gamma < 1/2$, the local time $L^{(N)}$ can be chosen γ Hölderian Q a.s.. Therefore, there exists $C > 0$ such that Q a.s.

$$\sum_{|k| \leq c^{-1}N^{1/10}} \left| L_1^{\lfloor P_k \sqrt{N} \rfloor / \sqrt{N}, (N)} - L_1^{P_k, (N)} \right| \leq \sum_{|k| \leq c^{-1}N^{1/10}} \frac{1}{N^{\gamma/2}} \leq CN^{1/10-\gamma/2}. \quad (7.11)$$

At this stage we choose $\gamma = 1/4$, so that the r.h.s. of (7.11) tends to 0 as $N \uparrow \infty$. Moreover, the r.h.s. of (7.10) converges also to 0 when $N \uparrow \infty$. We notice finally that

$$\sum_{|k| \leq c^{-1}N^{1/10}} L_1^{P_k, (N)} \xrightarrow[N \rightarrow \infty]{\text{law}} \sum_{k \in \mathbb{Z}} L_1^{P_k, (1)}, \quad (7.12)$$

which entails, by using (7.10), (7.11), (7.12) and the Slutsky's lemma, that T_N^λ converges in law to $\lambda \sum_{k \in \mathbb{Z}} L_1^{P_k, (1)}$ when $N \uparrow \infty$. This completes the proof of Step 1.

7.3. Step 2. To prove the uniform integrability of M_N^λ for λ small enough, we bound from above the quantities

$$A_{j,N} = P\left(\sum_{k \in \mathbb{Z}} \#\{i \in \{1, \dots, N\} : S_i = \lfloor P_k \sqrt{N} \rfloor\} \geq j\sqrt{N}\right).$$

At this stage we recall that the width between two successive interfaces is always larger than $\lfloor c\sqrt{N} \rfloor$, because $p_k \geq c$ for every $k \in \mathbb{Z} \setminus \{0\}$. Therefore, by applying a coupling method, we show easily that

$$A_{j,N} \leq P\left(\sum_{k \in \mathbb{Z}} \#\{i \in \{1, \dots, N\} : S_i = k \lfloor c\sqrt{N} \rfloor\} \geq j\sqrt{N}\right). \quad (7.13)$$

Associated with a simple random walk starting from x , we denote by $R_x^{r\sqrt{N}}$ the law of τ , which is the first passage time of the random walk through $-\lfloor r\sqrt{N} \rfloor$, $\lfloor r\sqrt{N} \rfloor$ or 0. Then, we can bound $A_{j,N}$ from above by

$$A_{j,N} \leq \sum_{\tau_1 + \dots + \tau_{\lfloor j\sqrt{N} \rfloor + 1} \leq N} \prod_{v=1}^{\lfloor j\sqrt{N} \rfloor + 1} R_0^{c\sqrt{N}}(\tau_v) \leq \left(\sum_{l=1}^N R_0^{c\sqrt{N}}(\tau) \right)^{j\sqrt{N}+1} \leq (R_0^{c\sqrt{N}}(\tau \leq N))^{j\sqrt{N}}. \quad (7.14)$$

Thus, we need to bound from above the quantity $R_0^{c\sqrt{N}}(\tau \leq N)$, but for symmetry reasons, τ is independent of the first step of S . Therefore,

$$R_0^{c\sqrt{N}}(\tau \leq N) = R_1^{c\sqrt{N}}(\tau \leq N-1),$$

so that τ becomes the first passage time of S (a simple random walk starting in $x = 1$) through $\mathbb{N} \times \{0\}$ or $\mathbb{N} \times \{\lfloor c\sqrt{N} \rfloor\}$. It gives

$$R_1^{c\sqrt{N}}(\tau \leq N-1) \leq R_1^{c\sqrt{N}}(S_\tau = \lfloor c\sqrt{N} \rfloor) + R_1^{c\sqrt{N}}(\tau \leq N-1; S_\tau = 0) \quad (7.15)$$

$$= 1 - R_1^{c\sqrt{N}}(l > N-1; S_\tau = 0). \quad (7.16)$$

Then, it remains to find a lower bound of $R_1^{c\sqrt{N}}(\tau > N-1; S_\tau = 0)$ (denoted by $D(c, N)$) in the following. To that aim, we consider the quantities $R_1^{c\sqrt{N}}(\tau = k; S_\tau = 0)$ (denoted by $U_{1,k}^{c\sqrt{N}}$), which are computed in [7] (page 322). We obtain

$$D(c, N) \geq \sum_{k=N+1}^{\infty} U_{1,k}^{c\sqrt{N}} \geq \sum_{k=N+1}^{\infty} \lfloor c\sqrt{N} \rfloor^{-1} \sum_{\nu=1}^{\lfloor c\sqrt{N} \rfloor - 1} \cos^{k-1} \left(\frac{\Pi\nu}{\lfloor c\sqrt{N} \rfloor} \right) \sin^2 \left(\frac{\Pi\nu}{\lfloor c\sqrt{N} \rfloor} \right). \quad (7.17)$$

We perform the sum over k , and after simplifications (7.17) becomes

$$D(c, N) \geq 2 \lfloor \sqrt{cN} \rfloor^{-1} \sum_{\nu=1}^{\lfloor \sqrt{cN} \rfloor - 1} \cos^2 \left(\frac{\Pi\nu}{2\lfloor \sqrt{cN} \rfloor} \right) \cos^N \left(\frac{\Pi\nu}{\lfloor \sqrt{cN} \rfloor} \right).$$

By using the equalities $\cos(\Pi - x) = -\cos(x)$ and $\cos^2(\Pi/2 - x) = \sin^2(x)$, we can write

$$D(c, N) \geq 2 \lfloor \sqrt{cN} \rfloor^{-1} \sum_{\nu=1}^{(\lfloor \sqrt{cN} \rfloor - 1)/2} \cos^N \left(\frac{\Pi\nu}{\lfloor \sqrt{cN} \rfloor} \right) \left[\cos^2 \left(\frac{\Pi\nu}{2\lfloor \sqrt{cN} \rfloor} \right) + (-1)^N \sin^2 \left(\frac{\Pi\nu}{2\lfloor \sqrt{cN} \rfloor} \right) \right]. \quad (7.18)$$

Thus, since $\cos^2(x) \geq \sin^2(x)$ for every $x \in [0, \Pi/4]$, we obtain

$$D(c, N) \geq 2 \lfloor \sqrt{cN} \rfloor^{-1} \cos^N \left(\frac{\Pi}{\lfloor \sqrt{cN} \rfloor} \right) \left[\cos^2 \left(\frac{\Pi}{2\lfloor \sqrt{cN} \rfloor} \right) - \sin^2 \left(\frac{\Pi}{2\lfloor \sqrt{cN} \rfloor} \right) \right]. \quad (7.19)$$

Therefore, since the term into brackets in (7.19) tends to 1 as $N \uparrow \infty$, and since $\cos^N(\Pi/\lfloor \sqrt{cN} \rfloor)$ tends to $\exp(-\Pi^2/2c)$, we obtain that there exists $c' > 0$ such that for N large enough, $D(c, N) > c'/\sqrt{N}$. Therefore, recalling (7.15) and (7.14), we can write

$$R_0^{c\sqrt{N}}(l \leq N) \leq \left(1 - \frac{c'}{\sqrt{N}} \right)^{j\sqrt{N}} = \exp(j\sqrt{N} \log(1 - c'/\sqrt{N})), \quad (7.20)$$

so that there exists $c'' > 0$ such that for N large enough and independent of j , $A_{j,N} \leq \exp(-c''j)$. This completes the proof of Step 2. \square

8. PROOF OF THEOREMS AND PROPOSITIONS

8.1. Proof of Proposition 2. We prove Proposition 2 in the multi-interface discrete case. To that aim we recall the notations $Z_{2n,\beta,p}^\gamma = E[\exp(H_{2n,\beta,p}^\gamma)]$ and $\Phi_{2n}(\beta,p) = \mathbb{E}[\log(Z_{2n,\beta,p}^\gamma)/2n]$. For simplicity we will not recall every time the dependence of Z , Φ and H in β and p .

We also define the partition function and the free energy when the polymer is pinned at the interface $P_0 = 0$ at both extremities, i.e., $Z_{2n}^{\gamma,c} = E[\exp(H_{2n}^\gamma \mathbf{1}_{\{S_{2n}=0\}})]$ and $\Phi_{2n}^c = \mathbb{E}[\log(Z_{2n}^{\gamma,c})/2n]$

The sequence $(2n\Phi_{2n}^c)_{n \geq 1}$ is superadditive and therefore Φ_{2n}^c converges to $\sup_{n \geq 1} \{\Phi_{2n}^c\}$ as $n \uparrow \infty$. Moreover, since the variables γ are bounded by M , the sequence $(\Phi_{2n}^c)_{n \geq 1}$ is bounded from above by βM and its limit is finite. At this stage, proving that $(\Phi_{2n}^c)_{n \geq 1}$ and $(\Phi_{2n})_{n \geq 1}$ have the same limit will be sufficient to complete the proof of Proposition 2. This will be a consequence of the next lemma.

Lemma 13. *For every $n \geq 1$, we have the inequality*

$$\Phi_{4n}^c = \frac{1}{4n} \mathbb{E} \left[\log E \left[e^{\beta H_{4n}^\gamma} \mathbf{1}_{\{S_{4n}=0\}} \right] \right] \geq \frac{1}{2n} \mathbb{E}[\log(Z_{2n}^\gamma)] - \xi_{2n} = \Phi_{2n} - \xi_{2n} \quad (8.1)$$

where ξ is a positive function, that tends to 0 as $n \uparrow \infty$.

To simplify the proof of this lemma, we denote $I_n^M = \{-2nM, \dots, 2nM\}$ and $V_{n_1, n_2}^\gamma(l, k, r) = \{S_{n_1} = l, S_{n_2} = k, H_{n_2}^\gamma - H_{n_1}^\gamma = r\}$. By taking into account the position of S_{2n} and by recalling that the variables γ are bounded by $M \in \mathbb{N} \setminus \{0\}$, we can write

$$Z_{2n}^\gamma = E[e^{\beta H_{2n}^\gamma}] = \sum_{k=-2n}^{2n} \sum_{r_1 \in I_n^M} \sum_{r_2 \in I_n^M} P(V_{0,2n}^\gamma(0, k, r_1)) e^{\beta r_1} \frac{P(V_{2n,4n}^\gamma(k, 0, r_2))^{1/2} e^{\beta r_2/2}}{\sum_{r_2=0}^{2n} P(V_{2n,4n}^\gamma(k, 0, r_2))^{1/2} e^{\beta r_2/2}}$$

We apply the Cauchy-Schwartz inequality and obtain

$$\begin{aligned} (Z_{2n}^\gamma)^2 &\leq \sum_{k=-2n}^{2n} \sum_{r_1 \in I_n^M} \sum_{r_2 \in I_n^M} P(V_{0,2n}^\gamma(0, k, r_1)) e^{\beta r_1} P(V_{2n,4n}^\gamma(k, 0, r_2)) e^{\beta r_2} \quad (8.2) \\ &\quad \times \sum_{k=-2n}^{2n} \sum_{r_1 \in I_n^M} \sum_{r_2 \in I_n^M} \frac{P(V_{0,2n}^\gamma(k, 0, r_1)) e^{\beta r_1}}{\sum_{r_2 \in I_n^M} P(V_{2n,4n}^\gamma(k, 0, r_2)) e^{\beta r_2}}. \end{aligned}$$

Then we notice that the first term of the r.h.s. of (8.2) is equal to $E[e^{\beta H_{4n}^\gamma} \mathbf{1}_{\{S_{4n}=0\}}]$, therefore, by using the Jensen inequality we obtain

$$\begin{aligned} \mathbb{E}[\log(Z_{2n}^\gamma)^2] &\leq \mathbb{E}[\log E[e^{\beta H_{4n}^\gamma} \mathbf{1}_{\{S_{4n}=0\}}]] \\ &\quad + \log \sum_{k=-2n}^{2n} \sum_{r_2 \in I_n^M} \mathbb{E} \left[\sum_{r_1 \in I_n^M} \frac{P(V_{0,2n}^\gamma(k, 0, r_1)) e^{\beta r_1}}{\sum_{r_2=0}^{2n} P(V_{2n,4n}^\gamma(k, 0, r_2)) e^{\beta r_2}} \right]. \quad (8.3) \end{aligned}$$

Moreover, by recalling that $(\gamma_i^k)_{i \in \{1, 2n\}}^{k \in \mathbb{Z}}$ is independent of $(\gamma_i^k)_{i \in \{2n+1, 4n\}}^{k \in \mathbb{Z}}$, and by applying the Jensen inequality with the concave function $x \rightarrow 1/x$ we obtain

$$\mathbb{E} \left[\sum_{r_1 \in I_n^M} \frac{P(V_{0,2n}^\gamma(k, 0, r_1)) e^{\beta r_1}}{\sum_{r_2 \in I_n^M} P(V_{2n,4n}^\gamma(k, 0, r_2)) e^{\beta r_2}} \right] \leq \frac{\sum_{r_1 \in I_n^M} \mathbb{E}[P(V_{0,2n}^\gamma(k, 0, r_1))] e^{\beta r_1}}{\sum_{r_2 \in I_n^M} \mathbb{E}[P(V_{2n,4n}^\gamma(k, 0, r_2))] e^{\beta r_2}} \leq 1. \quad (8.4)$$

Thus, the relations (8.3) and (8.4) allow us to write

$$\frac{1}{4n} \mathbb{E} \left[\log E \left[e^{\beta H_{4n}^\gamma} \mathbf{1}_{\{S_{4n}=0\}} \right] \right] \geq \frac{1}{2n} \mathbb{E} [\log(Z_{2n}^\gamma)] - \frac{1}{4n} \log 2n^2$$

and the proof of Lemma 13 is completed.

For all $n \geq 1$ we have $\Phi_{2n} \geq \Phi_{2n}^c$, hence $\liminf_{n \rightarrow \infty} \Phi_{2n} \geq \lim_{n \rightarrow \infty} \Phi_{2n}^c$. Moreover, Lemma 13 gives $\limsup_{n \rightarrow \infty} \Phi_{2n} \leq \lim_{n \rightarrow \infty} \Phi_{2n}^c$ and the proof of Proposition 2 is completed.

8.2. Proof of corollary 6. In this section, we assume that Theorem 5 is satisfied.

We prove this corollary by applying Theorem 5 with particular parameters. We denote $\rho = 1/n$, $\mu_1 = \beta\Sigma + 1/n$, $h = (1 + 2/n)K_c^{\mu_1}$, $h' = K_c^{\mu_1}$, $\beta_1 = \beta_2 = \beta$, and $\lambda = 1$. For a small enough, the first inequality of Theorem 5 gives

$$\frac{1}{a^2} \Psi \left(a\beta, a, a \left(1 + \frac{2}{n}\right) K_c^{\beta\Sigma + \frac{1}{n}} \right) \leq \left(1 + \frac{1}{n}\right) \tilde{\Psi} \left(\beta\Sigma + \frac{1}{n}, 1, K_c^{\beta\Sigma + \frac{1}{n}} \right). \quad (8.5)$$

By definition of $K_c^{(\cdot)}$, the right hand side of (8.5) is equal to zero. Therefore, we have the inequality $\liminf_{a \rightarrow \infty} h_c^{a\beta}(a)/a \leq (1 + 2/n)K_c^{\beta\Sigma + 1/n}$. Then, we let $n \uparrow \infty$ and since $K_c^{(\cdot)}$ is continuous in β , the former inequality becomes $\liminf_{a \rightarrow \infty} h_c^{a\beta}(a)/a \leq K_c^{\beta\Sigma}$. It remains to prove the opposite inequality. To that aim, we apply the second inequality of Theorem 5 with the parameters $\rho = 1/n$, $\mu_2 = \beta\Sigma - 1/n$, $h = (1 + 2/n)K_c^{\mu_2}$, $h' = K_c^{\mu_2}$, $\beta_1 = \beta_2 = \beta$, and $\lambda = 1$. For a small enough we obtain

$$\tilde{\Psi} \left(\beta\Sigma - \frac{1}{n}, 1, K_c^{\beta\Sigma - \frac{1}{n}} - \frac{1}{n} \right) \leq \frac{1+1/n}{a^2} \tilde{\Psi} \left(a\beta, a, \frac{a}{1+1/n} \left(K_c^{\beta\Sigma - \frac{1}{n}} - \frac{2}{n} \right) \right). \quad (8.6)$$

Therefore, we can write $\limsup_{a \rightarrow \infty} h_c^{a\beta}(a)/a \geq (K_c^{\beta\Sigma - 1/n} - 2/n)/(1 + 1/n)$ because the l.h.s. of (8.6) is strictly positive. Finally, since $\beta \rightarrow K_c^{(\beta)}$ is continuous, we let $n \uparrow \infty$ and it completes the proof of the corollary.

8.3. Proof of Theorem 4. In this section, we prove that Theorem 4 is a consequence of Theorem 5. This proof is divided into 3 steps. In the first step, we show that Theorem 4 is satisfied when $\lambda > 0$, $h > 0$, and every pinning reward γ_1^j has a non zero average. In the second step, we prove that the result can be extended to the case in which some γ_1^j have a zero average. Finally, in the last step, we will consider the case $h = 0$.

Step 1. First, we consider the case $\lambda > 0$, $h > 0$ and $\mathbb{E}(\gamma_1^j) \neq 0$ for every $j \in \{-K, \dots, K\}$. We can apply the first inequality of Theorem 5 with the parameters $\rho = 1/n$, $h' = h/(1 + 1/n)^2$, $\beta_1 = \beta_2 = \beta$ and $\mu_1(v) = \beta(1 + 1/v)\Sigma_1 + \beta(1 - 1/v)\Sigma_2$ (n and $v \in \mathbb{N} - \{0\}$). It gives, for every integers n and v strictly positive, that

$$\limsup_{a \rightarrow 0} \frac{1}{a^2} \Psi(a\beta, a\lambda, ah) \leq \left(1 + \frac{1}{n}\right) \tilde{\Psi}(\mu_1(v), \lambda, \frac{h}{(1+1/n)^2}). \quad (8.7)$$

At this stage, we let successively n and v tend to ∞ , and, by continuity of $\tilde{\Psi}$ in h and β we obtain $\limsup_{a \rightarrow 0} 1/a^2 \Psi(a\beta, a\lambda, ah) \leq \tilde{\Psi}(\beta\Sigma, \lambda, h)$. The lower bound is proved with the second inequality of Theorem 5. Indeed, if we choose $\mu_2(v) = \beta(1 - 1/v)\Sigma_1 + \beta(1 + 1/v)\Sigma_2$ and keep the other notations, we obtain

$$\tilde{\Psi}(\mu_2, \lambda, h(1 + \frac{1}{n})^2) \leq \left(1 + \frac{1}{n}\right) \liminf_{a \rightarrow 0} \frac{1}{a^2} \Psi(a\beta, a\lambda, ah). \quad (8.8)$$

We let $n \uparrow \infty$, and after, we let $v \uparrow \infty$. In that way, we can conclude that $\lim_{a \rightarrow 0} 1/a^2 \Psi(a\beta, a\lambda, ah) = \tilde{\Psi}(\beta\Sigma, \lambda, h)$ which implies Theorem 4.

Step 2. We prove Theorem 4 when there exists $j \in \{-K, \dots, K\}$ such that $\mathbb{E}(\gamma_1^j) = 0$. For that, we choose $\mu > 0$ and small enough, such that, $\mathbb{E}(\gamma_i^j + \mu) \neq 0$ for every $j \in \{-K, \dots, K\}$. With these new variables we can use the result of Step 1 with $\Sigma_\mu = \Sigma + (2K + 1)\mu$. Therefore, we can apply Theorem 5 and since the free energy Φ_μ associated with the variables $\gamma_i^j + \mu$ is larger than Φ , we obtain

$$\limsup_{a \rightarrow 0} \frac{1}{a^2} \Phi(a\beta, a\lambda, ah) \leq \lim_{a \rightarrow 0} \frac{1}{a^2} \Phi_\mu(a\beta, a\lambda, ah) = \tilde{\Phi}(\beta(\Sigma + (2K + 1)\mu), \lambda, h).$$

As $\tilde{\Phi}$ is continuous in β , we let $\mu \downarrow 0$ and write $\limsup_{a \rightarrow 0} 1/a^2 \Phi(a\beta, a\lambda, ah) \leq \tilde{\Phi}(\beta\Sigma, \lambda, h)$. Thus, it suffices to do the same computation with $-\mu < 0$, and we obtain the other inequality, i.e.,

$$\liminf_{a \rightarrow 0} \frac{1}{a^2} \Phi(a\beta, a\lambda, ah) \leq \lim_{\mu \rightarrow 0} \tilde{\Phi}(\beta(\Sigma - (2K + 1)\mu), \lambda, h) = \tilde{\Phi}(\beta\Sigma, \lambda, h).$$

Therefore, we can say that $\liminf_{a \rightarrow 0} \frac{1}{a^2} \Phi(a\beta, a\lambda, ah) = \tilde{\Phi}(\beta\Sigma, \lambda, h)$.

Step 3. It remains to prove Theorem 4 when $h = 0$. Since Ψ and $\tilde{\Psi}$ are non increasing in h , Theorem 4 with strictly positive parameters (proved in Step 2) implies

$$\liminf_{a \rightarrow 0} \frac{1}{a^2} \Phi(a\beta, a\lambda, 0) = \liminf_{a \rightarrow 0} \frac{1}{a^2} \Psi(a\beta, a\lambda, 0) \geq \tilde{\Psi}(\beta\Sigma, \lambda, 0) = \tilde{\Phi}(\beta\Sigma, \lambda, 0).$$

To prove the opposite inequality, we just notice that Φ is non decreasing in h . Effectively

$$\frac{\partial \Phi}{\partial h} \Big|_{(\beta, \lambda, 0)} = \lim_{N \rightarrow \infty} \mathbb{E} \left[\sum_{(\Lambda_1, \dots, \Lambda_N) \in \{-1, 1\}^N} P(\Lambda_1, \dots, \Lambda_N) \frac{\lambda \sum_{i=1}^N \Lambda_i}{N} \exp(\lambda \sum_{i=1}^N w_i \Lambda_i + \beta \dots) \right], \quad (8.9)$$

and by symmetry of the laws of the random walk and of the variables $\{w_i\}_{i=1,2,\dots}$, we can transform w_i in $-w_i$, and $(\Lambda_1, \dots, \Lambda_N)$ in $(-\Lambda_1, \dots, -\Lambda_N)$, without changing (8.9). It gives

$$\begin{aligned} \frac{\partial \Phi}{\partial h} \Big|_{\beta, \dots} &= \lim_{N \rightarrow \infty} \mathbb{E} \left[\sum_{\Lambda_1, \dots, \Lambda_N} P(-\Lambda_1, \dots, -\Lambda_N) \frac{-\lambda \sum_{i=1}^N \Lambda_i}{N} \exp(-\lambda \sum_{i=1}^N -w_i \Lambda_i + \beta \dots) \right] \\ &= \mathbb{E} \left[\sum_{\Lambda_1, \dots, \Lambda_N} P(\Lambda_1, \dots, \Lambda_N) \frac{-\lambda \sum_{i=1}^N \Lambda_i}{N} \exp(\lambda \sum_{i=1}^N w_i \Lambda_i + \beta \dots) \right] = -\frac{\partial \Phi}{\partial h} \Big|_{(\beta, \lambda, 0)}. \end{aligned}$$

Therefore, this derivative is equal to 0 and since Φ is convex in h , Φ is also non-decreasing in h . We let n and v tend to ∞ in (8.7) and we add λh on both sides. It gives, for $h > 0$, that $\limsup_{a \rightarrow 0} \frac{1}{a^2} \Phi(a\beta, a\lambda, ah) \leq \tilde{\Phi}(\beta\Sigma, \lambda, h)$. Since Φ is non-decreasing in h , the former inequality implies, $\limsup_{a \rightarrow 0} \frac{1}{a^2} \Phi(a\beta, a\lambda, 0) \leq \tilde{\Phi}(\beta\Sigma, \lambda, h)$. Then we let $h \downarrow 0$ and the proof of Theorem 4 is completed.

9. PROOF OF THEOREM 5

9.1. Coarse graining. First, we define a relation (previously introduced in [3]), which is very useful to carry out the proof.

Definition 14. *let $f_{t,\varepsilon,\delta}(a, h, \beta_1, \beta_2)$ and $g_{t,\varepsilon,\delta}(a, h, \beta_1, \beta_2)$ be real-valued functions. The relation $f \ll g$ occurs if for every $\beta_3 > \beta_1$, $\beta_2 > \beta_4$, $\rho > 0$, and $h > h' \geq 0$ satisfying*

$(1 + \rho)h' < h$, there exists δ_0 such that for $0 < \delta < \delta_0$ there exists $\varepsilon_0(\delta)$ such that for $0 < \varepsilon < \varepsilon_0$ there exists $a_0(\varepsilon, \delta)$ satisfying

$$\limsup_{t \rightarrow \infty} f_{t,\varepsilon,\delta}(a, h, \beta_1, \beta_2) - (1 + \rho)g_{t(1+\rho)^2, \varepsilon(1+\rho)^2, \delta(1+\rho)^2}(a(1 + \rho), h', \beta_3, \beta_4) \leq 0$$

for $0 < a < a_0$ (9.1)

In this proof we consider some functions of the form

$$F_{t,\varepsilon,\delta}(a, h, \beta_1, \beta_2) = \mathbb{E} \left[\frac{1}{t} \log E \left(\exp(aH_{t,\varepsilon,\delta}(a, h, \beta_1, \beta_2)) \right) \right],$$

and we denote

- $F_{t,\varepsilon,\delta}^1(a, h, \beta_1, \beta_2) = \frac{1}{a^2} \Psi_{\lfloor t/a^2 \rfloor}(a\beta_1, a\beta_2, a, ah)$
- $F_{t,\varepsilon,\delta}^7(a, h, \beta_1, \beta_2) = \tilde{\Psi}_t(\beta_1 \Sigma_1 + \beta_2 \Sigma_2, 1, h)$.

The proof of (4.4) will consist in showing that $F^1 \ll F^7$ and $F^7 \ll F^1$ (denoted by $F^1 \sim F^7$). To that aim, we will create the intermediate functions F_2, \dots, F_6 associated with slight modifications of the Hamiltonian to transform, step by step, the discrete Hamiltonian into the continuous one. As the relation \sim is transitive, we will prove at every step that $F^i \sim F^{i+1}$, to conclude finally that $F^1 \sim F^7$.

9.2. Scheme of the proof. To show that $F^i \ll F^{i+1}$ we let $H^i = H^I + H^{II}$ and, by the Hölder inequality, we can bound F^i from above as follows

$$F_{t,\varepsilon,\delta}^i(a, h, \beta) \leq \frac{1}{t(1+\rho)} \mathbb{E} \left[\log E \left(\exp(a(1+\rho)H^I) \right) \right] + \frac{1}{t(1+\rho^{-1})} \mathbb{E} \left[\log E \left(\exp(a(1+\rho^{-1})H^{II}) \right) \right].$$

Thus, if we choose $H^I = H_{t(1+\rho)^2, \varepsilon(1+\rho)^2, \delta(1+\rho)^2}^{i+1}(a(1 + \rho), h', \beta_3, \beta_4)$, we obtain

$$\begin{aligned} F_{t,\varepsilon,\delta}^i(a, h, \beta_1, \beta_2) - (1 + \rho)F_{t(1+\rho)^2, \varepsilon(1+\rho)^2, \delta(1+\rho)^2}^{i+1}(a(1 + \rho), h', \beta_3, \beta_4) \\ \leq \frac{1}{t(1+\rho^{-1})} \mathbb{E} \left[\log E \left(\exp(a(1 + \rho^{-1})H^{II}) \right) \right]. \end{aligned}$$

Then, it suffices to prove that $\limsup_{t \rightarrow \infty} 1/t \log \mathbb{E} E \left(\exp(a(1 + \rho^{-1})H^{II}) \right) \leq 0$ for a, ε and δ small enough.

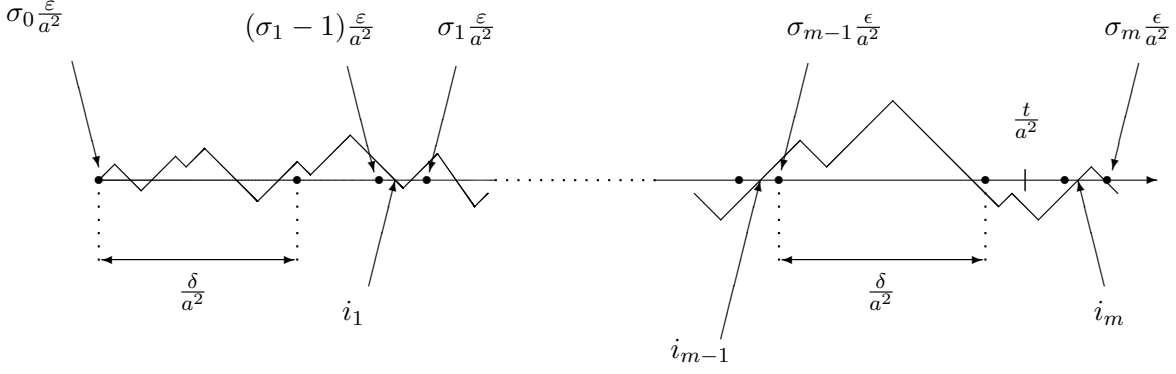
9.3. Step 1. The first Hamiltonian that we consider in this proof is given by

$$H_{t,\varepsilon,\delta}^{(1)}(a, h, \beta_1, \beta_2) = -2 \sum_{i=1}^{t/a^2} \Delta_i(w_i + ah) + \beta_1 \sum_{j=I_1}^{t/a^2} \sum_{i=1} \gamma_i^j \mathbf{1}_{\{S_i=j\}} + \beta_2 \sum_{j=I_2}^{t/a^2} \sum_{i=1} \gamma_i^j \mathbf{1}_{\{S_i=j\}},$$

with $\Delta_i = 1$ if $\Lambda_i = -1$ and $\Delta_i = 0$ if $\Lambda_i = 1$.

We define some notation to build the intermediate Hamiltonians (see Fig. 3).

- $\sigma_0 = 0$, $i_0^v = 0$ and $i_{k+1}^v = \inf \{ n > \sigma_k \varepsilon / a^2 + \delta / a^2 : S_n = 0 \}$
- $m = \inf \{ k \geq 1 : i_m > t / a^2 \}$
- $i_k = i_k^v$ for $k < m$ and $i_m = t / a^2$
- $\sigma_{k+1} = \inf \{ n \geq 0 : i_{k+1} \in](n-1)\varepsilon / a^2, n\varepsilon / a^2] \}$,
- $\bar{I}_k =](\sigma_{k-1} + 1)\varepsilon / a^2, \sigma_k \varepsilon / a^2] \cap]0, t / a^2]$, $s_{k+1} = \text{sign}(\Delta_{i_{k+1}-1})$



We define the first transformation of the Hamiltonian

$$H_{t,\varepsilon,\delta}^{(2)}(a, h, \beta_1, \beta_2) = -2 \sum_{k=1}^m s_k \left[\sum_{i \in \bar{I}_k} w_i + ah |\bar{I}_k| \right] + \sum_{i=1}^{t/a^2} \beta_1 \sum_{j \in I_1} \gamma_i^j \mathbf{1}_{\{S_i=j\}} + \beta_2 \sum_{j \in I_2} \gamma_i^j \mathbf{1}_{\{S_i=j\}}$$

and we want to show that $F_1 \ll F_2$. To that aim, we denote

$$H^{II} = -2 \sum_{i=1}^{t/a^2} \Delta_i (w_i + ah) + 2 \sum_{k=1}^m s_k \left(\sum_{i \in \bar{I}_k} w_i + a(1 + \rho) h' |\bar{I}_k| \right) \\ + (\beta_1 - \beta_3) \sum_{j \in I_1} \sum_{i=1}^{t/a^2} \gamma_i^j \mathbf{1}_{\{S_i=j\}} + (\beta_2 - \beta_4) \sum_{j \in I_2} \sum_{i=1}^{t/a^2} \gamma_i^j \mathbf{1}_{\{S_i=j\}}, \quad (9.2)$$

and it remains to prove that $\limsup_{t \rightarrow \infty} \frac{1}{t} \log \mathbb{E} \mathbb{E}(\exp(a(1 + \rho^{-1}) H^{II})) \leq 0$. We integrate over the disorder γ and the third and fourth terms of the right hand side of (9.2) give some contributions of the form

$$\exp\left(\sum_{j \in I_p} \sum_{i=1}^{t/a^2} \log \mathbb{E}[\exp((\beta_p - \beta_{2+p}) a(1 + \rho^{-1}) \gamma_i^j)] \mathbf{1}_{\{S_i=j\}}\right) \quad \text{for } p = 1 \text{ and } p = 2.$$

Since $\mathbb{E}(\exp(\lambda |\gamma_1^j|)) < \infty$ for every $j \in \{-K, \dots, K\}$ and $\lambda > 0$, we can write a first order Taylor expansion of $\log \mathbb{E}(\exp(Aa \gamma_1^j))$ when $a \downarrow 0$. It gives

$$\log \mathbb{E}(\exp(Aa \gamma_1^j)) = Aa \mathbb{E}(\gamma_1^j) + o(a). \quad (9.3)$$

We assume in this proof that $\mathbb{E}(\gamma_1^j) \neq 0$ for every $j \in \{-K, \dots, K\}$ (see the assumptions of Theorem 5) and therefore $\{-K, \dots, K\} = I_1 \cup I_2$. For every $i \in I_1$, $\mathbb{E}(\gamma_1^j) > 0$, and $\beta_1 - \beta_3 < 0$. Thus, by (9.3), we obtain, for a small enough, that

$$\sum_{j \in I_1} \sum_{i=1}^{t/a^2} \log \mathbb{E}((\beta_1 - \beta_3) a(1 + \rho^{-1}) \gamma_i^j) \mathbf{1}_{\{S_i=j\}} \leq 0. \quad (9.4)$$

The sum over I_2 satisfies the same inequality for a small enough because $\beta_2 - \beta_4 > 0$ and $\mathbb{E}(\gamma_1^j) < 0$ when $j \in I_2$. Therefore, we can remove the third and fourth terms of H^{II} in (9.2) and by rewriting $\sum_{i=1}^{t/a^2}$ as $\sum_{k=1}^{m_{t/a^2}} \sum_{i \in \bar{I}_k}$, we can rewrite H^{II} as

$$H^{II} = -2 \sum_{k=1}^{m_{t/a^2}} \sum_{i \in \bar{I}_k} w_i (\Delta_i - s_k) - 2a(1 + \rho) h' \sum_{k=1}^{m_{t/a^2}} \sum_{i \in \bar{I}_k} (\Delta_i - s_k) - 2a(h - (1 + \rho) h') \sum_{k=1}^{m_{t/a^2}} \sum_{i \in \bar{I}_k} \Delta_i$$

Thus, we integrate over the disorder w which is independent of the random walk. But, since $\mathbb{E}(w_i) = 0$ and $\mathbb{E}(\exp(\lambda |w_1|)) < \infty$ for every $\lambda > 0$, a second order expansion gives that for every $c \in \mathbb{R}$ there exists $A > 0$ such that for a small enough

$$\log \mathbb{E}(\exp(c a w_i (\Delta_i - s_k))) \leq A a^2 |\Delta_i - s_k|. \quad (9.5)$$

Finally, we have to prove, for $A > 0$ and $B > 0$, that

$$\limsup_{t \rightarrow \infty} \frac{1}{t} \log E \left[\exp \left(Aa^2 \sum_{k=1}^{m_{t/a^2}} \sum_{i \in \bar{I}_k} |s_k - \Delta_i| - Ba^2 \sum_{i=1}^{t/a^2} \Delta_i \right) \right] \leq 0. \quad (9.6)$$

This is explicitly proved in [3] (page 1355), and completes the Step 1 because the proof of $F_2 \ll F_1$ is very similar and consists essentially in showing (9.6).

9.4. Step 2. In this step we aim at transforming the disorder w into a sequence $(\hat{w}_i)_{i \geq 1}$ of independent random variables of law $\mathcal{N}_{0,1}$. To that aim, we use a coupling method developed in [17] to define on the same probabilistic space and for every $j \in \mathbb{N} \setminus \{0\}$ the variables $(w_i)_{i \in \{(j-1)\varepsilon/a^2+1, \dots, j\varepsilon/a^2\}}$ and some independent variables of law $\mathcal{N}_{0,1}$, denoted by $(\hat{w}_i)_{i \in \{(j-1)\varepsilon/a^2+1, \dots, j\varepsilon/a^2\}}$, such that for every $p > 2$ and $x > 0$

$$\mathbb{P} \left(\left| \sum_{i=(j-1)\varepsilon/a^2+1}^{j\varepsilon/a^2} w_i - \hat{w}_i \right| \geq x \right) \leq \frac{(Ap)^p \varepsilon}{x^p a^2} \mathbb{E}(w_1^p). \quad (9.7)$$

These constructions are made independently on every blocs $\{(j-1)\varepsilon/a^2+1, \dots, j\varepsilon/a^2\}$. Thus, we can form the third Hamiltonian as follow

$$H_{t,\varepsilon,\delta}^{(3)}(a, h, \beta_1, \beta_2) = -2 \sum_{k=1}^m s_k \left[\sum_{i \in \bar{I}_k} \hat{w}_i + ah |\bar{I}_k| \right] + \sum_{i=1}^{t/a^2} \beta_1 \sum_{j \in I_1} \gamma_i^j \mathbf{1}_{\{S_i=j\}} + \beta_2 \sum_{j \in I_2} \gamma_i^j \mathbf{1}_{\{S_i=j\}}.$$

To prove that $F^2 \ll F^3$, we need the Hamiltonian H^{II} . It takes the value

$$H^{II} = H_{t,\varepsilon,\delta}^{(2)}(a, h, \beta_1, \beta_2) - H_{t(1+\rho)^2, \varepsilon(1+\rho)^2, \delta(1+\rho)^2}^{(3)}(a(1+\rho), h', \beta_3, \beta_4). \quad (9.8)$$

As in Step 1 (see (9.4)) we delete the two pinning terms in H^{II} and it is sufficient to consider

$$\begin{aligned} H^{II} &= -2 \sum_{k=1}^m s_k \sum_{i \in \bar{I}_k} (w_i - \hat{w}_i) + 2a \sum_{k=1}^m s_k (h - (1+\rho)h') |\bar{I}_k| \\ &\leq 2 \sum_{k=1}^m s_k \left(\sum_{j=\sigma_{k-1}+1}^{\sigma_k} \left| \sum_{i=j\varepsilon/a^2+1}^{(j+1)\varepsilon/a^2} w_i - \hat{w}_i \right| - (h - (1+\rho)h') \frac{\varepsilon}{a} \right). \end{aligned}$$

We want to prove that $\limsup_{t \rightarrow \infty} 1/t \log \mathbb{E} E (\exp(a(1+\rho^{-1})H^{II})) \leq 0$. By independence of (w, \hat{w}) on each blocs $\{(j-1)\varepsilon/a^2+1, \dots, j\varepsilon/a^2\}$, it suffices to show that for every $C > 0$ and $B > 0$

$$\mathbb{E} \left[\exp \left(Ca \left| \sum_{i=1}^{\varepsilon/a^2} w_i - \hat{w}_i \right| - B\varepsilon \right) \right] \leq 1 \text{ for } \varepsilon, \text{ and } a \text{ small enough.} \quad (9.9)$$

We prove this point as follows,

$$\mathbb{E} \left[\exp \left(Ca \left| \sum_{i=1}^{\varepsilon/a^2} w_i - \hat{w}_i \right| \right) \right] \leq \sum_{k=N}^{+\infty} e^{Ca(k+1)\frac{\varepsilon}{\sqrt{a}}} \mathbb{P} \left(\left| \sum_{i=1}^{\varepsilon/a^2} w_i - \hat{w}_i \right| \geq k \frac{\varepsilon}{\sqrt{a}} \right) + e^{CN\sqrt{a\varepsilon}}. \quad (9.10)$$

By using (9.7) and the fact that $\mathbb{E}(w_1^k) \leq R^k$, we obtain that for every j and $k \geq 1$

$$\mathbb{P} \left(\left| \sum_{i=(j-1)\varepsilon/a^2+1}^{j\varepsilon/a^2} w_i - \hat{w}_i \right| \geq \frac{k\varepsilon}{\sqrt{a}} \right) \leq \frac{(AR\sqrt{a})^k}{\varepsilon^{k-1} a^2}. \quad (9.11)$$

We consider (9.10) with $N = 5$, and we use (9.11) to obtain

$$\mathbb{E} \left[\exp \left(Ca \left| \sum_{i=1}^{\varepsilon/a^2} w_i - \hat{w}_i \right| \right) \right] \leq e^{5C\sqrt{a\varepsilon}} + \varepsilon \frac{e^{C\sqrt{a\varepsilon}}}{a^2} \sum_{k=5}^{+\infty} \left(e^{C\sqrt{a\varepsilon}} \frac{AR\sqrt{a}}{\varepsilon} \right)^k.$$

Therefore, for $\varepsilon > 0$ fixed, there exists $K(\varepsilon, a) > 0$ which tends to zero when a tends to zero, and satisfies

$$\mathbb{E} \left[\exp \left(Ca \left| \sum_{i=1}^{\varepsilon/a^2} w_i - \hat{w}_i \right| \right) \right] \leq (1 + K(\varepsilon, a)) e^{5C\varepsilon\sqrt{a}}.$$

This implies (9.9), and completes the Step 2 because the proof of $F^3 \ll F^2$ is exactly the same.

9.5. Step 3. In this step, we make a link between the discrete and the continuous models. For that, we take into account the number of return to the origin of the random walk, and the local time of the Brownian motion. We define, independently of the random walk, an i.i.d. sequence $(l_1^k)_{k \geq 0}$ of local times spent in 0 by a Brownian motion between 0 and 1. The law of this sequence is denoted by χ . Then, we build the new Hamiltonian

$$H_{t,\varepsilon,\delta}^{(4)}(a, h, \beta_1, \beta_2) = -2 \sum_{k=1}^m s_k \left(\sum_{i \in \bar{I}_k} \hat{w}_i + ah |\bar{I}_k| \right) + \frac{(\beta_1 \Sigma_1 + \beta_2 \Sigma_2) \sqrt{\delta}}{a} \sum_{k=1}^m l_1^k. \quad (9.12)$$

As usual, to prove that $F_3 \ll F_4$, we consider H^{II} , in which we can already remove the term $-2a(h - (1 + \rho)h') \sum_{k=1}^m s_k |\bar{I}_k|$ because it is negative. Therefore we can bound H^{II} from above as follows

$$\begin{aligned} H^{II} &\leq \beta_1 \sum_{k=1}^m \sum_{j \in I_1} \sum_{i=i_{k-1}+1}^{i_k} \gamma_i^j \mathbf{1}_{\{S_i=j\}} - \frac{\beta_3 \Sigma_1 \sqrt{\delta} \Sigma}{a} \sum_{k=1}^m l_1^k \\ &\quad + \beta_2 \sum_{k=1}^m \sum_{j \in I_2} \sum_{i=i_{k-1}+1}^{i_k} \gamma_i^j \mathbf{1}_{\{S_i=j\}} - \frac{\beta_4 \Sigma_2 \sqrt{\delta} \Sigma}{a} \sum_{k=1}^m l_1^k. \end{aligned}$$

To prove that $\limsup_{t \rightarrow \infty} \frac{1}{t} \log E_{P_{\otimes \chi}} \mathbb{E}(\exp(a(1 + \rho^{-1})H^{II})) \leq 0$, we first apply the Hölder inequality (with the coefficients $p = q = 2$), and then we integrate over the disorder γ . Therefore, it remains to prove for $x = 1$ and 2 that

$$\begin{aligned} \limsup_{t \rightarrow \infty} \frac{1}{t} \log E_{P_{\otimes \chi}} \left[\exp \left(\sum_{k=1}^m \sum_{j \in I_x} \sum_{i=i_{k-1}+1}^{i_k} \right. \right. \\ \left. \left. \log \mathbb{E} \left(\exp \left(2a\beta_x (1 + \rho^{-1}) \gamma_i^j \right) \right) \mathbf{1}_{\{S_i=j\}} - 2\beta_{x+2} \sqrt{\delta} \Sigma_x (1 + \rho^{-1}) l_1^k \right) \right] \leq 0. \end{aligned} \quad (9.13)$$

For simplicity, in what follows we will use E instead of $E_{P_{\otimes \chi}}$. We begin with the proof of (9.13) in the case $x = 1$. To that aim, we recall (9.3), that gives

$$\log \mathbb{E} \left(\exp \left(2a\beta_1 (1 + \rho^{-1}) \gamma_i^j \right) \right) = 2\mathbb{E}(\gamma_1^j) a \beta_1 (1 + \rho^{-1}) + o(a). \quad (9.14)$$

Therefore, we can choose β'' such that $\beta_1 < \beta'' < \beta_3$ and a small enough to obtain for every $j \in I_1$ the inequality $\log \mathbb{E}(\exp(2a\beta_1(1 + \rho^{-1})\gamma_i^j)) \leq 2a\beta''(1 + \rho^{-1})\mathbb{E}(\gamma_1^j)$. Finally, since $\mathbb{E}(\gamma_1^j) > 0$ for every j , we can replace $(i_k)_{k \in \{1, \dots, m\}}$ by $(i_k^v)_{k \in \{1, \dots, m\}}$ (see the notation

at the beginning of Step 1, and it remains to prove that for $B > A > 0$

$$\limsup_{t \rightarrow \infty} \frac{1}{t} \log E \left[\exp \left(\sum_{k=1}^m \left(Aa \sum_{j \in I_1} \mathbb{E}(\gamma_1^j) \sum_{i=i_{k-1}^v+1}^{i_k^v} \mathbf{1}_{\{S_i=j\}} - B\sqrt{\delta} \Sigma_1 l_1^k \right) \right) \right] \leq 0. \quad (9.15)$$

For simplicity, we will use the notation $\mathbb{E}(\gamma_1^j) = f(j)$, and consequently $\Sigma_1 = \sum_{j \in I_1} f(j)$. For every N , we build a new filtration, i.e., $F_N = \sigma(A_{i_N^v} \cup \sigma(l_1^{0,1}, \dots, l_1^N))$ with $A_k = \sigma(X_1, \dots, X_k)$ and the random variable

$$M_N = \frac{\exp \left(\sum_{k=1}^N Aa \sum_{j \in I_1} f(j) \#\{v \in \{i_{k-1}^v+1, i_k^v\} : S_v=j\} - B\sqrt{\delta} \Sigma_1 \sum_{k=1}^N l_1^k \right)}{\mu^N E \left(\exp \left(Aa \sum_{j \in I_1} \#\{i \in \{0, \frac{\delta+\epsilon}{a^2}\} : S_i=j\} - B\sqrt{\delta} \Sigma_1 l_1^1 \right) \right)^N}$$

where μ is a constant > 1 . We will precise the value of μ later, to make sure that M_N is a positive surmartingale with respect to $(F_N)_{N \geq 0}$. To that aim, for every $j \in \{-K, \dots, K\}$ we introduce $P_N^j = \#\{u \in \{i_{N-1}^v+1, i_N^v\} : S_u=j\}$, and we define the new filtration $(G_N)_{N \geq 1}$ by $G_{N-1} = \sigma(F_{N-1} \cup \sigma(X_{i_{N-1}^v+1}, \dots, X_{i_{N-1}^v+(\delta+\epsilon)/a^2}, l_1^N))$. Then, we consider the quantity $E(M_N | F_{N-1})$ and by independence of the random walk excursions out of the origin we obtain

$$E(M_N | F_{N-1}) = M_{N-1} \frac{\mu^{-1} E(\exp(Aa \sum_{j \in I_1} f(j) P_N^j - B\sqrt{\delta} \Sigma_1 l_1^N) | F_{N-1})}{E(\exp(Aa \sum_{j \in I_1} f(j) \#\{i \in \{0, \frac{\delta+\epsilon}{a^2}\} : S_i=j\} - B\sqrt{\delta} \Sigma_1 l_1^1))}. \quad (9.16)$$

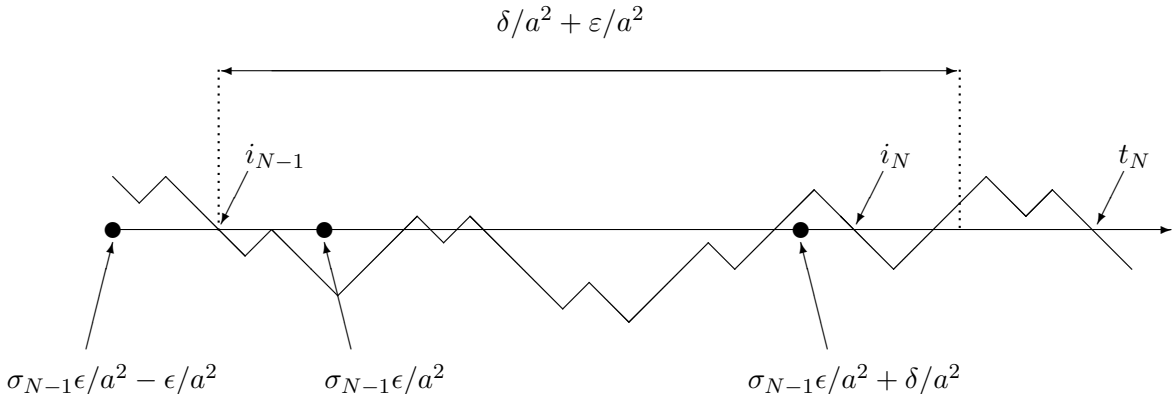
We define $t_N = \inf\{i > i_{N-1}^v + (\delta + \epsilon)/a^2 : S_i = 0\}$ and notice that $t_N \geq i_N^v$ (see Fig 4 for an example in which $t_N > i_N^v$). Therefore, we can write $P_N^j \leq B_{1,N}^j + B_{2,N}^j$ with

$$B_{1,N}^j = \#\{v \in \{i_{N-1}^v+1, \dots, i_{N-1}^v + \frac{\delta+\epsilon}{a^2}\} : S_v=j\} \text{ and } B_{2,N}^j = \#\{v \in \{i_{N-1}^v + \frac{\delta+\epsilon}{a^2} + 1, \dots, t_N\} : S_v=j\}. \quad (9.17)$$

We denote by C the quantity $E[\exp(Aa \sum_{j \in I_1} f(j) P_N^j - B\sqrt{\delta} \Sigma_1 l_1^N) | F_{N-1}]$. Thus, since $B_{1,N}^j$ is measurable with respect to G_{N-1} and since $F_{N-1} \subset G_{N-1}$ we can write

$$C \leq E \left[\exp \left(Aa \sum_{j \in I_1} f(j) B_{1,N}^j - B\sqrt{\delta} \Sigma_1 l_1^N \right) E \left[\exp \left(Aa \sum_{j \in I_1} f(j) B_{2,N}^j \right) \middle| G_{N-1} \right] \middle| F_{N-1} \right].$$

Fig. 4:



If we denote by Υ the quantity $E(\exp(Aa \sum_{j \in I_1} f(j) B_{2,N}^j) | G_{N-1})$, the fact that the local times (l_1^1, \dots, l_1^N) are independent of the random walk allows us to write the equality $\Upsilon = E(\exp(Aa \sum_{j \in I_1} f(j) B_{2,N}^j) | A_{i_{N-1}+(\delta+\varepsilon)/a^2})$. The strong Markov property can be applied here. Indeed, if $(V_n)_{n \geq 0}$ is a simple random walk with $V_0 = S_{i_{N-1}+(\delta+\varepsilon)/a^2}$, and if $s = \inf\{n > 1 : V_n = 0\}$, we can write

$$\Upsilon = E_V[\exp(Aa \sum_{j \in I_1} f(j) \# \{i \in \{1, \dots, s\} : V_i = j\})].$$

Thus, if we denote $f = \max_{j \in I_1} \{f_j\}$, we can bound Υ from above as

$$\Upsilon \leq E_V[\exp(Aaf \# \{i \in \{1, \dots, s\} : V_i \in \{-K, \dots, K\}\})]. \quad (9.18)$$

We want to find an upper bound of Υ independent of the starting point $S_{i_{N-1}+(\delta+\varepsilon)/a^2}$. The r.h.s. of (9.18) is even with respect to the starting point, therefore we can consider that V is a reflected random walk. That is why it suffices to bound from above the quantities $W(x, a) = E_x(\exp(Aaf \# \{i \in \{1, \dots, s\} : |V_i| \in \{0, \dots, K\}\}))$ with $x \in \mathbb{N}$. Moreover, the Markov property implies that $W(x, a) = W(K, a)$ for every $x \geq K$, and $W(x, a) < W(K, a)$ if $x < K$ because the random walk starting in K touches necessarily in x before reaching 0. Therefore, we can write an upper bound of C , i.e.,

$$C \leq E\left[\exp\left(Aa \sum_{j \in I_1} f(j) B_{1,N}^j - B\sqrt{\delta} \Sigma_1 l_1^N\right) | F_{N-1}\right] W(K, a),$$

and since the excursion of a random walk are independent we can assert that $B_{1,N}^j$ is independent of F_{N-1} . Hence,

$$\begin{aligned} E\left[\exp\left(Aa \sum_{j \in I_1} f(j) B_{1,N}^j - B\sqrt{\delta} \Sigma_1 l_1^N\right) | F_{N-1}\right] = \\ E\left[\exp\left(Aa \sum_{j \in I_1} f(j) \# \{i \in \{0, \frac{\delta+\varepsilon}{a^2}\} : S_i = j\} - B\sqrt{\delta} \Sigma_1 l_1^N\right)\right], \end{aligned}$$

and (9.16) becomes $E(M_N | F_{N-1}) \leq M_{N-1} W(K, a) / \mu$. But $W(K, a)$ tends to 1 as $a \downarrow 0$ and becomes smaller than μ for a small enough. That is why for a small enough $(M_N)_{N \geq 0}$ is a surmartingale. Since the stopping time m_{t/a^2} is bounded from above by t/a^2 , we can apply a stopping time theorem and say that $E(M_m) \leq E(M_1) \leq 1$. Then, to complete the proof of (9.15), it suffices to show that, for δ, ε, a small enough the quantity $V_{\delta, \varepsilon, a}$, defined in (9.19), is smaller than 1.

$$V_{\delta, \varepsilon, a} = \mu E\left[\exp\left(Aa \sum_{j \in I_1} f(j) \# \{i \in \{0, \frac{\delta+\varepsilon}{a^2}\} : S_i = j\} - B\sqrt{\delta} \Sigma_1 l_1^1\right)\right]. \quad (9.19)$$

We recall that the random walk and the local time l_1^1 are independent. Therefore,

$$V_{\delta, \varepsilon, a} = \mu E\left[\exp\left(Aa \sum_{j \in I_1} f(j) \# \{i \in \{0, \frac{\delta+\varepsilon}{a^2}\} : S_i = j\}\right)\right] E\left[\exp(-B\sqrt{\delta} \Sigma_1 l_1^1)\right].$$

By Lemma 10, we know that

$$\lim_{a \rightarrow 0} V_{\delta, \varepsilon, a} = \mu E\left[\exp(A\sqrt{\delta} + \varepsilon \Sigma_1 l_1^1)\right] E\left[\exp(-B\sqrt{\delta} \Sigma_1 l_1^1)\right].$$

Since Σ_1 is fixed, it enters in the constants A and B without changing the fact that $B > A$. For every x in \mathbb{R} we denote $f(x) = E(\exp(x l_1^1))$. The law of l_1^1 is known (see [18]), and the derivative of f in 0 satisfies $f'(0) = E(l_1^1) > 0$. Therefore, a first order development of f gives $f(A\sqrt{\delta} + \varepsilon) = 1 + f'(0)A\sqrt{\delta} + \varepsilon + o(\sqrt{\delta} + \varepsilon)$ and $f(-B\sqrt{\delta}) = 1 - f'(0)B\sqrt{\delta} + o(\sqrt{\delta})$. If we take $\varepsilon \leq \delta^2$, we obtain

$$f(A\sqrt{\delta} + \varepsilon)f(-B\sqrt{\delta}) \leq 1 + f'(0)\sqrt{\delta}(A\sqrt{1 + \delta} - B) + o(\sqrt{\delta}). \quad (9.20)$$

Since $B > A$, the right hand side of (9.20) is strictly smaller than 1 for δ small enough. For such a δ , for $\epsilon \leq \delta^2$ and for $\mu > 1$ but small enough we obtain $\lim_{a \rightarrow 0} V_{\delta, \epsilon, a} < 1$. As a consequence, for a small enough, $V_{\delta, \epsilon, a} < 1$. This completes the proof of (9.15), and therefore, the proof of (9.13) for $x = 1$.

The proof of (9.13) for $x = 2$, is easier than the former one. Indeed, $\mathbb{E}(\gamma_i^j) < 0$ for every $j \in I_2$, and therefore, if we choose β'' such that $\beta_2 > \beta'' > \beta_4$, the first order development of (9.3) gives, for a small enough,

$$\log \mathbb{E}[\exp(2a\beta_2(1 + \rho^{-1})\gamma_1^j)] \leq 2a\beta''(1 + \rho^{-1})\mathbb{E}(\gamma_1^j).$$

By following the scheme of the former proof (for $x=1$), we notice that it suffices to replace $\{u \in \{i_{k-1}^v + 1, i_k^v\} : S_u = j\}$ by $\{u \in \{i_{k-1}^v + 1, i_{k-1}^v + \delta + \epsilon/a^2\} : S_u = j\}$ in the definition of M_N . Moreover, there is no need to introduce $\mu > 1$ in the definition of M_N , which is in this case a positive martingale. The rest of the proof is similar to the case $x = 1$.

The proof of $F_4 \ll F_3$ is almost the same, we just exchange the role of β_1, β_2 and β_3, β_4 in the definition of H^{II} . Consequently, the role of A and $-B$ in (9.15) are also exchanged, and, as in the former proof, Lemma 10 implies the result.

9.6. Step 4. We notice that the quantities $m, \sigma_1, \sigma_2, \dots, \sigma_m, s_1, s_2, \dots, s_m$ can also be defined for a Brownian motion on the interval $[0, t]$. Indeed, we denote $\sigma_0 = 0, z_0 = 0$, and recursively $z_{k+1} = \inf\{s > \sigma_k \epsilon + \delta : B_s = 0\}$ while σ_{k+1} is the unique integer satisfying $z_{k+1} \in ((\sigma_{k+1} - 1)\epsilon, \sigma_{k+1}\epsilon]$ and $s_{k+1} = 1$ if the excursion ending in z_{k+1} is in the lower half-plan, $s_{k+1} = 0$ otherwise. Finally, we let $m_t = \inf\{k \geq 1 : z_k > t\}$ and $z_m = t$. At this stage, we want to transform the random walk that gives the possible trajectories of the polymer into a Brownian motion. For that (as in [3]), we denote by Q the measure of $(m_{t/a^2}, \sigma_1, \sigma_2, \dots, \sigma_m, s_1, s_2, \dots, s_m)$ associated with the random walk on $[0, t/a^2]$ and by \tilde{Q} the measure of $(m_t, \sigma_1, \sigma_2, \dots, \sigma_m, s_1, s_2, \dots, s_m)$ associated with the Brownian motion on $[0, t]$.

As proved in [3] (page 1362) Q and \tilde{Q} are absolutely continuous and their Radon-Nikodým derivative satisfies that there exists a constant $K'_{a, \epsilon, \delta} > 0$ such that for every $\delta > 0$

$$\lim_{\epsilon \rightarrow 0} \limsup_{a \rightarrow 0} K'_{a, \epsilon, \delta} = 0 \quad \text{and} \quad (1 - K')^m \leq \frac{d\tilde{Q}}{dQ} \leq (1 + K')^m. \quad (9.21)$$

We recall that χ is the law of the local times $(l_1^1, l_1^2, \dots, l_0^{1,m})$, which are independent of the random walk and consequently of Q . Moreover, $|\bar{I}_k| = (\sigma_k - \sigma_{k-1})\epsilon/a^2$. Hence, the equation (9.12) gives that $H_{t, \epsilon, \delta}^{(4)}(a, h, \beta)$ depends only on $(m_{t/a^2}, \sigma_1, \sigma_2, \dots, \sigma_m, s_1, s_2, \dots, s_m)$ and $(l_1^1, l_1^2, \dots, l_0^{1,m})$. That is why, we can write

$$F_{t, \epsilon, \delta}^4(a, h, \beta_1, \beta_2) = \frac{1}{t} \log E_{\chi \otimes Q}[\exp(H_{t, \epsilon, \delta}^{(4)}(a, h, \beta))].$$

At this stage, we define F_5 by replacing the random walk by a Brownian motion, namely by integrating over $\chi \otimes \tilde{Q}$ instead of $\chi \otimes Q$. We define

$$H_{t, \epsilon, \delta}^{(5)}(a, h, \beta_1, \beta_2) = H_{t, \epsilon, \delta}^{(4)}(a, h, \beta_1, \beta_2) + \log(d\tilde{Q}/dQ),$$

and therefore,

$$F_{t, \epsilon, \delta}^5(a, h, \beta_1, \beta_2) = \frac{1}{t} \log E_{\chi \otimes \tilde{Q}}[e^{H_{t, \epsilon, \delta}^{(4)}(a, h, \beta_1, \beta_2)}] = \frac{1}{t} \log E_{R \otimes Q}[e^{H_{t, \epsilon, \delta}^{(5)}(a, h, \beta_1, \beta_2)}].$$

Now, we aim at proving that $F^4 \ll F^5$. To that aim, we calculate H^{II} , i.e.,

$$\begin{aligned} H^{II} &= H_{t,\varepsilon,\delta}^{(4)}(a, h, \beta_1, \beta_2) - H_{t(1+\rho)^5, \varepsilon(1+\rho)^2, \delta(1+\rho)^2}^{(5)}(a(1+\rho), h', \beta_3, \beta_4) \\ &= -\frac{2}{a}(h - (1+\rho)h') \sum_{k=1}^m s_k(\sigma_k - \sigma_{k-1})\varepsilon \\ &\quad + ((\beta_1 - \beta_3)\Sigma_1 + (\beta_2 - \beta_4)\Sigma_2) \frac{\sqrt{\delta}}{a} \sum_{k=1}^m l_1^k - \frac{1}{a(1+\rho)} \log \frac{d\tilde{Q}}{dQ} \\ &\leq -\frac{2}{a}(h - (1+\rho)h') \sum_{k=1}^m s_k(\sigma_k - \sigma_{k-1})\varepsilon - \frac{1}{a(1+\rho)} \log \frac{d\tilde{Q}}{dQ}. \end{aligned}$$

We do not give the details of the end of this step because it is done in [3] (page 1361–1362). To prove that $F_5 \ll F_4$, we consider the density $dQ/d\tilde{Q}$ in H^{II} , and (9.21) can also be applied. It completes the Step 4.

9.7. Step 5. From now on, we integrate over $\chi \otimes \tilde{Q}$ in F^5 and consequently the term $\log(d\tilde{Q}/dQ)$ does not appear in $H^{(5)}$ any more. In this step, transform the local times $(l_1^1, \dots, l_1^k, \dots)$ into the local times of the Brownian motion that determines \tilde{Q} . We recall that L_t is the local time spent at 0 by $(B_s)_{s \geq 0}$ between the times 0 and t .

But before, we define $(R_s)_{s \geq 0}$ a Brownian motion, independent of B , and we emphasize the fact that, for every $k \in \{1, \dots, m\}$,

$$a \sum_{i \in \bar{I}_k} \hat{w}_i \stackrel{D}{=} R_{\sigma_k \varepsilon} - R_{\sigma_{k-1} \varepsilon} \quad \text{and} \quad a^2 |\bar{I}_k| = (\sigma_k - \sigma_{k-1})\varepsilon. \quad (9.22)$$

Then, we can rewrite the fifth Hamiltonian as

$$H_{t,\varepsilon,\delta}^{(5)}(a, h, \beta_1, \beta_2) = -\frac{2}{a} \sum_{k=1}^{m_t} \left[s_k (R_{\sigma_k \varepsilon} - R_{\sigma_{k-1} \varepsilon} + h(\sigma_k - \sigma_{k-1})\varepsilon) - \frac{\beta_1 \Sigma_1 + \beta_2 \Sigma_2}{2} \sqrt{\delta} l_1^k \right]. \quad (9.23)$$

We define the sixth Hamiltonian as,

$$H_{t,\varepsilon,\delta}^{(6)}(a, h, \beta_1, \beta_2) = -\frac{2}{a} \sum_{k=1}^{m_t} \left[s_k (R_{\sigma_k \varepsilon} - R_{\sigma_{k-1} \varepsilon} + h(\sigma_k - \sigma_{k-1})\varepsilon) \right] + \frac{\beta_1 \Sigma_1 + \beta_2 \Sigma_2}{a} L_t.$$

At this stage, we notice that F^5 and F^6 do not depend on a anymore. Hence, to simplify the following steps, we transform a bit the general scheme of the proof. Indeed, from now on, we will denote, for $i = 5, 6$ or 7 ,

$$F_{t,\varepsilon,\delta}^i(h, \beta_1, \beta_2) = \frac{1}{t} \log E_{\tilde{Q}} \left[\exp(\bar{H}_{t,\varepsilon,\delta}^i(h, \beta_1, \beta_2)) \right] \quad (9.24)$$

with $\bar{H}_{t,\varepsilon,\delta}^i(h, \beta_1, \beta_2) = a H_{t,\varepsilon,\delta}^i(h, \beta_1, \beta_2)$. Therefore, to prove that $F^i \ll F^j$ we use

$$H^{II} = \bar{H}_{t,\varepsilon,\delta}^i(h, \beta_1, \beta_2) - \frac{1}{1+\rho} \bar{H}_{t(1+\rho)^2, \varepsilon(1+\rho)^2, \delta(1+\rho)^2}^j(h', \beta_3, \beta_4), \quad (9.25)$$

and we show that $\limsup_{t \rightarrow \infty} 1/t \log \tilde{\mathbb{E}} E(\exp((1+\rho^{-1})H^{II})) \leq 0$.

We want to prove that $F^5 \ll F^6$ but, by the scaling property of Brownian motion, it is not difficult to show that for $i = 5$ or 6

$$\bar{H}_{t(1+\rho)^2, \varepsilon(1+\rho)^2, \delta(1+\rho)^2}^i(h, \beta_1, \beta_2) = (1+\rho) \bar{H}_{t,\varepsilon,\delta}^i((1+\rho)h, \beta_1, \beta_2). \quad (9.26)$$

Therefore, by (9.25), we can write $H^{II} = \overline{H}_{t,\varepsilon,\delta}^4(h, \beta_1, \beta_2) - \overline{H}_{t,\varepsilon,\delta}^5((1+\rho)h', \beta_3, \beta_4)$. Thus, since $(1+\rho)h' < h$ and $-\sum_{k=1}^m s_k(\sigma_k - \sigma_{k-1})\varepsilon < 0$, we obtain

$$\begin{aligned} H^{II} &\leq \beta_1 \Sigma_1 \sqrt{\delta} \sum_{k=1}^m l_1^k - \beta_3 \Sigma_1 \sum_{k=1}^m L_{z_k^v} - L_{z_{k-1}^v} \\ &\quad + \beta_3 \Sigma_1 (L_{t+\delta} - L_t) + \beta_2 \Sigma_2 \sqrt{\delta} \sum_{k=1}^m l_1^k - \beta_4 \Sigma_2 \sum_{k=1}^m L_{z_k} - L_{z_{k-1}} \end{aligned}$$

with $z_j^v = z_j$ for every $j < m$ and $z_m^v = \inf\{t > \sigma_{m-1}\varepsilon + \delta : B_t = 0\}$. Finally, by the Hölder inequality, it suffices to prove, for $B > A$, that

$$\limsup_{t \rightarrow \infty} \frac{1}{t} \log E \left[\exp \left(A \sum_{k=1}^m \sqrt{\delta} l_1^k - B \sum_{k=1}^m L_{z_k^v} - L_{z_{k-1}^v} \right) \right] \leq 0, \quad (9.27)$$

and

$$\limsup_{t \rightarrow \infty} \frac{1}{t} \log E \left[\exp \left(A \sum_{k=1}^m L_{z_k^v} - L_{z_{k-1}^v} - B \sum_{k=1}^m \sqrt{\delta} l_1^k \right) \right] \leq 0, \quad (9.28)$$

and

$$\limsup_{t \rightarrow \infty} \frac{1}{t} \log E[\exp(B(L_{t+\delta} - L_t))] = 0. \quad (9.29)$$

We denote by C_t the first time of return to the origin after time t . Proving (9.29) is immediate because C_t is a stopping time with respect to the natural filtration of B , we can therefore apply the strong Markov property to obtain, for every $u \in [t, t + \delta]$, the equality $E(\exp(B(L_{t+\delta} - L_u)) | C_t = u) = E[\exp(B(L_{t+\delta-u} - L_u))]$. Thus, we can write

$$E[\exp(B(L_{t+\delta} - L_t))] = \int_t^{t+\delta} E[\exp(B(L_{t+\delta-u} - L_u))] dC_t(u) \leq E[\exp(BL_\delta)]. \quad (9.30)$$

This implies (9.29), and it remains to prove (9.27), and (9.28). We define a new filtration, $F_N = \sigma(\sigma((B_s)_{s \leq z_N^v}) \cup \sigma(l_1^1, \dots, l_1^N))$. We notice that $(z_N^v)_{N \geq 0}$ is a sequence of increasing stopping times, and consequently, F_N is an increasing filtration. We denote by M_N the quantity

$$M_N = \frac{\exp \left(A \sum_{k=1}^N \sqrt{\delta} l_1^k - B \sum_{k=1}^N L_{z_k^v} - L_{z_{k-1}^v} \right)}{E[\exp(-BL_\delta + A\sqrt{\delta}l_1^1)]^N}, \quad (9.31)$$

which is a surmartingale with respect to F_N . Indeed, L and $(l_1^k)_{k \geq 1}$ are independent, $(L_s)_{s \geq z_N^v}$ is independent of F_N (because $B_{z_N^v} = 0$) and $L_{z_{N+1}^v} - L_{z_N^v} \geq L_{z_N^v + \delta} - L_{z_N^v}$. Thus, since $E(\exp(-B(L_{z_N^v + \delta} - L_{z_N^v}))) = E(\exp(-B(L_\delta)))$, we obtain $E(M_{N+1} | F_N) \leq M_N$. Moreover, m_t is a stopping time with respect to F_N and is bounded from above by t/δ . Therefore, to prove (9.27), it suffices to show (as in Step 3) that for $B > A$ and δ small enough, $V = E[\exp(A\sqrt{\delta}l_1^1 - BL_\delta)] \leq 1$. Moreover, L_δ and $\sqrt{\delta}l_1^1$ have the same law and are independent. That is why we can write $V = E[\exp(A\sqrt{\delta}l_1^1)]E[\exp(-B\sqrt{\delta}l_1^1)]$, which is strictly smaller than 1 for δ small enough (as proved in Step 3).

We prove (9.28) in a very similar way. Effectively, since $L_{z_{N+1}^v} - L_{z_N^v} \leq L_{z_N^v + \delta + \varepsilon} - L_{z_N^v}$, we prove that the inequality (9.27) is still satisfied when A and $-B$ are exchanged. Therefore, the proof of $F^5 \ll F^6$ is completed. To end this step, we notice that (9.28) and (9.27) imply directly that $F^6 \ll F^5$. Thus, the proof of Step 5 is completed.

9.8. Step 6. Let $\mu_1 = \beta_1 \Sigma_1 + \beta_2 \Sigma_2$ and $\mu_3 = \beta_3 \Sigma_1 + \beta_4 \Sigma_2$. This step is the last one, therefore, the following Hamiltonian is the one of the continuous model, i.e.,

$$\overline{H}_{t,\varepsilon,\delta}^{(7)}(h, \beta_1, \beta_2) = -2 \int_0^t \mathbf{1}_{\{B_s < 0\}} (dR_s + h ds) + \mu_1 L_t.$$

For simplicity, we define $(\phi_s)_{s \in [0,t]}$ by $\phi_s = s_k$ for every $s \in (\sigma_{k-1}\varepsilon, \sigma_k\varepsilon]$. In that way, $\sum_{k=1}^m s_k (R_{\sigma_k\varepsilon} - R_{\sigma_{k-1}\varepsilon}) + h(\sigma_m - \sigma_0) = \int_0^t \phi_s (dR_s + h ds)$. Moreover, the scaling property of Brownian motion gives, for $i = 6$ or 7 ,

$$\overline{H}_{t(1+\rho)^2, \varepsilon(1+\rho)^2, \delta(1+\rho)^2}^{(i)}(h, \beta) \stackrel{D}{=} \overline{H}_{t,\varepsilon,\delta}^{(i)}((1+\rho)^2 h, (1+\rho)\beta_1, (1+\rho)\beta_2).$$

Hence, to show that $F^6 \ll F^7$, we consider (as in Step 5) the difference

$$H^{II} = \overline{H}_{t,\varepsilon,\delta}^{(6)}(h, \beta_1, \beta_2) - \frac{1}{1+\rho} \overline{H}_{t(1+\rho)^2, \varepsilon(1+\rho)^2, \delta(1+\rho)^2}^{(7)}(h', (1+\rho)\beta_3, (1+\rho)\beta_4),$$

which is equal to $\overline{H}_{t,\varepsilon,\delta}^{(6)}(h, \beta_1, \beta_2) - \overline{H}_{t,\varepsilon,\delta}^{(7)}((1+\rho)h', (1+\rho)\beta_3, (1+\rho)\beta_4)$. Thus, we can bound H^{II} from above as follows

$$\begin{aligned} H^{II} &= -2 \int_0^t (\phi_s - \mathbf{1}_{\{B_s < 0\}}) dR_s - 2 \int_0^t (h\phi_s - (1+\rho)h'\mathbf{1}_{\{B_s < 0\}}) ds + (\mu_1 - \mu_3)L_t \\ H^{II} &\leq -2 \int_0^t (\phi_s - \mathbf{1}_{\{B_s < 0\}}) dR_s - 2h \int_0^t (\phi_s - \mathbf{1}_{\{B_s < 0\}}) ds + (\mu_1 - \mu_3)\Sigma L_t. \end{aligned}$$

We want to prove that $\limsup_{t \rightarrow \infty} \frac{1}{t} \log \widetilde{\mathbb{E}} E(\exp((1+\rho^{-1})H^{II})) \leq 0$ and after the integration over $\widetilde{\mathbb{E}}$, it remains to prove that for $A > 0$ and $B > 0$

$$\limsup_{t \rightarrow \infty} \frac{1}{t} \log \widetilde{\mathbb{E}} E[\exp(A \int_0^t |\phi_s - \mathbf{1}_{\{B_s < 0\}}| ds - BL_t)] \leq 0.$$

As in Step 3 (see Fig. 4), we notice that between z_{k-1} and z_k , if we find an excursion of length larger than $\delta + \varepsilon$, it is necessarily the one which ends at z_k and gives the value of s_k . It means that, apart eventually from the very beginning of such an excursion (between z_{k-1} and $\sigma_{k-1}\varepsilon$), s_k and ϕ_s have the same value along the excursion. Finally, we obtain

$$\int_0^t |\mathbf{1}_{\{B_s < 0\}} - \phi_s| ds \leq P_{0,t,\delta,\varepsilon} + m\varepsilon,$$

where $P_{u,v,\delta,\varepsilon}$ is the sum between u and v of the excursion lengths which are smaller than $\delta + \varepsilon$. The term $m\varepsilon$ allows us to take into account the formerly mentioned situation between z_{k-1} and $\sigma_{k-1}\varepsilon$.

With this upper bound, we can write $H^{II} \leq AP_{0,t,\delta,\varepsilon} + Am\varepsilon - BL_t$ with $A > 0$ and $B > 0$. Therefore, to complete the proof we must show that for δ and ε small enough the inequality $\limsup_{t \rightarrow \infty} \frac{1}{t} \log E(\exp(1/(1+\rho)H^{II})) \leq 0$ occurs. Thus, by applying the Hölder inequality, it suffices to prove that, for two strictly positive constants A and B , we have

$$\limsup_{t \rightarrow \infty} \frac{1}{t} \log E[\exp(A\varepsilon m - BL_t)] \leq 0 \quad \text{and} \quad \limsup_{t \rightarrow \infty} \frac{1}{t} \log E[\exp(AP_{0,t,\delta,\varepsilon} - BL_t)] \leq 0. \quad (9.32)$$

We begin with the proof of the first inequality of (9.32). To that aim, we recall that, for every $k < m$, we have $z_k > z_{k-1} + \delta$. Therefore, we can write

$$A\varepsilon m - BL_t \leq A\varepsilon m - B \sum_{k=1}^m L_{z_{k-1} + \delta} - L_{z_{k-1}} + B(L_{t+\delta} - L_t).$$

From the equation (9.29) and the Hölder inequality we deduce that the term $B(L_{t+\delta} - L_t)$ does not change the result. For this reason we just have to consider the quantity $1/t \log E[\exp(\sum_{k=1}^m A\varepsilon - B(l_{z_{k-1}+\delta} - L_{z_k}))]$ when $t \uparrow \infty$. As in (9.58), we define the martingale

$$M_N = \frac{1}{(V_{\varepsilon,\delta})^N} \exp\left(\sum_{k=1}^N A\varepsilon - B(L_{z_{k-1}+\delta} - L_{z_k})\right) \text{ with } V_{\varepsilon,\delta} = E[\exp(A\varepsilon - BL_\delta)]. \quad (9.33)$$

Since m is a stopping time bounded from above by t/δ , it is sufficient to show that $V_{\varepsilon,\delta} < 1$ for δ, ε small enough. It is the case because $E[\exp(-BL_\delta)] < 1$ for every $B > 0$. Therefore, we take ε small enough and it completes the proof.

It remains to prove the second part of (9.32). Notice that $P_{0,t,\delta,\varepsilon} = \sum_{k=1}^m P_{z_{k-1},z_k,\delta,\varepsilon}$ and that for every $k \leq m$ $P_{z_{k-1},z_k,\delta,\varepsilon} \leq 2(\delta + \varepsilon)$ (still because there can not be more than one excursion larger than $\delta + \varepsilon$ between z_{k-1} and z_k). Therefore, we obtain the following upper bound

$$AP_{0,t,\delta,\varepsilon} - BL_t \leq 2A(\delta + \varepsilon)m - B\sum_{k=1}^m L_{z_{k-1}+\delta} - L_{z_{k-1}} + B(L_{t+\delta} - L_t).$$

As in (9.29) the term $B(L_{t+\delta} - L_t)$ is removed, and it remains to consider $1/t \log E[\sum_{k=1}^m A(\varepsilon + \delta) - B(l_{z_{k-1}+\delta} - L_{z_k})]$ when $t \uparrow \infty$. To that aim, we build again the martingale

$$M_N = \frac{1}{(D_{\varepsilon,\delta})^N} \exp\left(\sum_{k=1}^N A(\varepsilon + \delta) - B(l_{z_{k-1}+\delta} - L_{z_k})\right) \quad (9.34)$$

with $D_{\varepsilon,\delta} = E[\exp(A(\delta + \varepsilon) - BL_\delta)]$. The term m is a bounded stopping time, therefore, it suffices to show, for δ, ε small enough, that $D_{\varepsilon,\delta} < 1$. To that aim, we choose $\varepsilon \leq \delta$, and it remains to consider the quantity $E[\exp(2A\delta - BL_\delta)]$. Moreover, $L_\delta = {}_D\sqrt{\delta}L_1$, and if we denote $f(x) = E[\exp(xL_1)]$, we can use a first order development of f in 0. It gives $f(-B\sqrt{\delta}) = 1 - f'(0)B\sqrt{\delta} + \xi_1(\delta)\sqrt{\delta}$ with $f'(0) > 0$ and $\lim_{x \rightarrow 0} \xi_1(x) = 0$. We also know that, $\exp(2A\delta) = 1 + 2A\delta + \xi_2(\delta)\delta$ with $\lim_{x \rightarrow 0} \xi_2(x) = 0$. Hence, for $\varepsilon \leq \delta$ and δ small enough, we obtain $E(\exp(2A\delta - BL_\delta)) = \exp(2A\delta)f(-B\sqrt{\delta}) < 1$. The proof of $F_6 \ll F_5$ is exactly the same and the Step 6 is completed.

9.9. Proof of Proposition 7.

Proof. The computation of $\tilde{\Phi}$ is based on the fact that $\tilde{\Phi}(\beta, h)$ is equal to the quantity $h + \lim_{t \rightarrow \infty} 1/t \log E(\exp(-2h\Gamma^-(t) + \beta L_t^0))$, where $\Gamma^-(t) = \int_0^t \mathbf{1}_{\{B_s < 0\}} ds$. Moreover, the joint law of $(\Gamma^-(t), L_t)$ is available in [13] and takes the value

$$dP_{(\Gamma^-(t), L_t^0)}(\tau, b) = \mathbf{1}_{\{0 < \tau < t\}} \mathbf{1}_{\{b > 0\}} \frac{bt \exp\left(-\frac{tb^2}{8\tau(t-\tau)}\right)}{4\pi\tau^{\frac{3}{2}}(t-\tau)^{\frac{3}{2}}} db d\tau. \quad (9.35)$$

This allows to perform completely the computation of $\tilde{\Phi}$. □

9.10. Proof of Theorem 8. Theorem 8 is a consequence of Theorem 9. Indeed, considering the first equation of (6.2), we let ρ and α tend to 0 and β' tend to β . Then, since $\tilde{\Phi}$ is continuous in β we obtain $\limsup_{a \rightarrow 0} \frac{1}{a^2} \Phi(a\beta, r/a) \leq \tilde{\Phi}(\beta, r)$. The other inequality is proved with the second inequality of (6.2), namely, we let α and ρ tend to 0 and β'' to β to obtain $\limsup_{a \rightarrow 0} \frac{1}{a^2} \Phi(a\beta, r/a) \geq \tilde{\Phi}(\beta, r)$. This completes the proof of Theorem 8.

9.11. Proof of Theorem 9. Recall that in this proof the sequence $(p_i)_{i \in \mathbb{Z} \setminus \{0\}}$ is fixed and equal to $(kr)_{k \in \mathbb{Z}}$. We need to transform some aspects of the former coarse graining to perform this proof. Indeed, we transform the Definition 14 as follows. We introduce the relation $f \preceq g$, for functions of type $f_{t,\varepsilon,\delta}^r(a, \beta)$. Two functions f and g of this form satisfy $f \preceq g$ if for every $\beta' > \beta$, $\rho > 0$ and $\alpha > 0$ there exists ν_0 such that for $0 < \nu < \nu_0$ there exists $\varepsilon_0(\nu)$ such that for $0 < \varepsilon < \varepsilon_0$ there exists $a_0(\varepsilon, \nu)$ which satisfies

$$\limsup_{t \rightarrow \infty} f_{t,\varepsilon,\nu}^r(a, \beta) - (1 + \rho) g_{t(1+\rho)^2, \varepsilon(1+\rho)^2, \nu(1+\rho)^2}^r(a(1+\rho), \beta') \leq \alpha$$

for $0 < a < a_0$. (9.36)

We will still consider functions of the form

$$F_{t,\varepsilon,\nu}^r(a, \beta) = \mathbb{E} \left[\frac{1}{t} \log E \left[\exp(a H_{t,\varepsilon,\nu}^r(a, \beta)) \right] \right],$$

and we denote

$$F_{t,\varepsilon,\delta}^{1,r}(a, \beta) = \frac{1}{a^2} \Phi_{\lfloor t/a^2 \rfloor}(a\beta, r/a) \quad F_{t,\varepsilon,\delta}^{4,r}(a, \beta) = \tilde{\Phi}_t(\beta, r). \quad (9.37)$$

Thus, if we can show that $F^1 \preceq F^4$ and $F^4 \preceq F^1$ (denoted by $F^1 \sim F^4$), the proof of Theorem 9 will be completed. Indeed, for every $\rho > 0$, $\alpha > 0$ and $\beta' > \beta$ the relation $F^1 \preceq F^4$ entails, for a small enough, that

$$\frac{1}{a^2} \Phi(a\beta, r/a) \leq (1 + \rho) \tilde{\Phi}(\beta', r(1 + \rho)) + \alpha. \quad (9.38)$$

Moreover, by the scaling property of Brownian motion we notice that

$$\sum_{k \in \mathbb{Z}} L_{t(1+\rho)^2}^{(1+\rho)P_k} =_D (1 + \rho) \sum_{k \in \mathbb{Z}} L_t^{P_k} \quad (9.39)$$

which implies that $(1 + \rho)^2 \tilde{\Phi}(\beta', r(1 + \rho)) = \tilde{\Phi}((1 + \rho)\beta', r) + \alpha$. This last equality and (9.38) entail the first inequality of Theorem 9. The other inequality is given by the relation $F^4 \preceq F^1$. Indeed, since $\beta'' < \beta/(1 + \rho)$ the relation (9.36) gives directly

$$\tilde{\Phi}(\beta'', r) \leq \frac{1}{(1 + \rho)a^2} \Phi(a\beta, r/a) \quad (9.40)$$

and it completes the proof of Theorem 9.

At this stage, it remains to prove that $F^1 \sim F^4$. We perform this proof through 3 steps and we introduce 2 intermediate Hamiltonians, i.e., H^2 and H^3 . We modify also the scheme of the proof given in the former coarse graining (see (9.2)). Indeed, since we introduced a parameter $\alpha > 0$ in (9.36), the relation $F^i \ll F^{i+1}$ will be obtained if we can prove, for every $\alpha > 0$, that $\limsup_{t \rightarrow \infty} \frac{1}{t} \log E \left[\exp(a(1 + \rho^{-1})H^{II}) \right] \leq \alpha$ when ν , ε and a are small enough. In this coarse graining, H^{II} is given by

$$H_{t,\varepsilon,\nu}^{i,r}(a, \beta) - H_{t(1+\rho)^2, \varepsilon(1+\rho)^2, \nu(1+\rho)^2}^{i+1,r(1+\rho)}(a(1+\rho), \beta').$$

Before starting with Step 1, we define some notation.

Definition 15. For every $A \in \mathbb{R}$ and $\nu > 0$ we define the quantity $f(A, \nu)$ by

$$f(A, \nu) = \tilde{E} \left[\exp \left(A \sum_{k \in \mathbb{Z}} L_\nu^{kr} \right) \right] = \tilde{E} \left[\exp \left(A \sqrt{\nu} \sum_{k \in \mathbb{Z}} L_1^{kr/\sqrt{\nu}} \right) \right]. \quad (9.41)$$

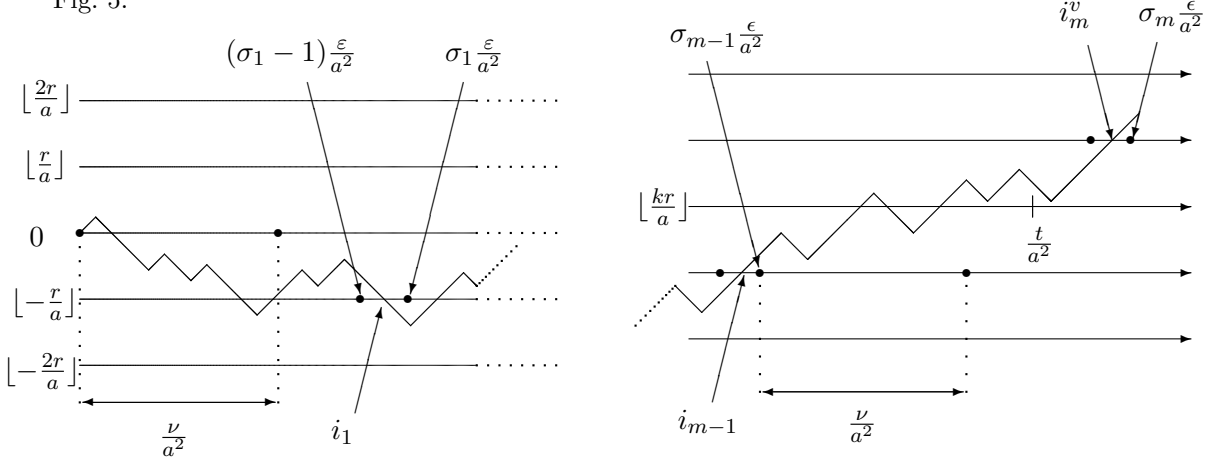
Notice that the second equality of (9.41) is given by the scaling property of Brownian motion. We define also some notation, which will help us to introduce the intermediate Hamiltonians (see Fig. 5).

- $\sigma_0 = 0$, $i_0^v = 0$ and $i_{k+1}^v = \inf \{ n > \sigma_k \varepsilon / a^2 + \nu / a^2 : \exists l \in \mathbb{Z} : S_n = \lfloor lr/a \rfloor \}$

- $m = \inf \{k \geq 1 : i_m^v > t/a^2\}$
- $\sigma_{k+1} = \inf \{n \geq 0 : i_{k+1} \in](n-1)\varepsilon/a^2, n\varepsilon/a^2]\}$
- $i_k = i_k^v$ for $k < m$ and $i_m = t/a^2$
- $s_k = 1$ if $\exists l \in \mathbb{Z}$ such that $S_{i_{k-1}} = \lfloor lr/a \rfloor + 1$ and $s_k = -1$ if $S_{i_{k-1}} = \lfloor lr/a \rfloor - 1$.

Finally, for simplicity, we assume that $\mathbb{E}(\gamma_i^k) = 1$ for every $k \in \mathbb{Z}$.

Fig. 5:



9.12. **Step 1.** The function F^1 is defined in (9.37), and its associated Hamiltonian is

$$H_{t,\varepsilon,\nu}^{1,r}(a, \beta) = \sum_{j=1}^m \sum_{i=i_{j-1}+1}^{i_j} \left[\beta \sum_{k \in \mathbb{Z}} \gamma_i^k \mathbf{1}_{\{S_i = \lfloor kr/a \rfloor\}} \right]. \quad (9.42)$$

In this step, we perform the first modification of the Hamiltonian. Indeed, on every interval $\{i_{j-1}, \dots, i_j\}$ we replace the discrete number of contacts between the random walk and the interfaces (i.e., $\sum_{k \in \mathbb{Z}} \gamma_i^k \mathbf{1}_{\{S_i = \lfloor kr/a \rfloor\}}$) by some continuous local times. To that aim, we define $(B_j)_{j \geq 1}$, a sequence of independent Brownian motions and we denote by $L_1^{x,j}$ their local time spent at x between 0 and 1. Thus, we define the first intermediate Hamiltonian as,

$$H_{t,\varepsilon,\nu}^{2,p}(a, \beta) = \frac{\beta}{a} \sum_{j=1}^m \left[\sqrt{\nu} \sum_{k \in \mathbb{Z}} L_1^{kr/\sqrt{\nu},j} \right] \quad (9.43)$$

and it satisfies the relation $H_{t(1+\rho)^2, \varepsilon(1+\rho)^2, \nu(1+\rho)^2}^{2,r(1+\rho)}(a(1+\rho), \beta) \stackrel{D}{=} H_{t,\varepsilon,\nu}^{2,r}(a, \beta)$. Therefore, we chose $\beta' > \beta$ and H^{II} takes the value

$$H^{II} = \sum_{j=1}^m \left[\sum_{i=i_{j-1}+1}^{i_j} \left(\beta \sum_{k \in \mathbb{Z}} \gamma_i^k \mathbf{1}_{\{S_i = \lfloor kr/a \rfloor\}} \right) - \frac{\beta'}{a} \sqrt{\nu} \sum_{k \in \mathbb{Z}} L_1^{kr/\sqrt{\nu},j} \right]. \quad (9.44)$$

Recall that the γ variables are bounded by M , that $i_n^v = i_n$ for all $n \leq m-1$ and that $i_m^v > i_m$. We want to substitute i_j^v to i_j for $j \in \{1, \dots, m\}$. Then, by definition of i_m^v we can write

$$\sum_{i=i_{m-1}+1}^{i_m} \left(\beta \sum_{k \in \mathbb{Z}} \gamma_i^k \mathbf{1}_{\{S_i = \lfloor kr/a \rfloor\}} \right) \leq \sum_{i=i_{m-1}+1}^{i_m^v} \left(\beta \sum_{k \in \mathbb{Z}} \gamma_i^k \mathbf{1}_{\{S_i = \lfloor kr/a \rfloor\}} \right) + \gamma \beta M / a^2. \quad (9.45)$$

Therefore, we can replace i_j by i_j^v in the r.h.s. of (9.44) without changing the value of $\limsup_{t \rightarrow \infty} \frac{1}{t} \log \mathbb{E} E[\exp(a(1+\rho^{-1})H^{II})]$.

At this stage we must show that $\limsup_{t \rightarrow \infty} \frac{1}{t} \log \mathbb{E} E[\exp(a(1+\rho^{-1})H^{II})] \leq 0$, but by independence of the variables γ_i^j with respect to all the other random variables, we can

integrate first with respect to \mathbb{E} . The pinning rewards become $\log \mathbb{E}[\exp(a(1 + 1/\rho)\beta\gamma_i^j)]$. Then we introduce β'' which satisfies $\beta < \beta'' < \beta'$ and since $\mathbb{E}[\gamma_i^j] = 1$, a first order Taylor expansion gives us that for a small enough $\log \mathbb{E}[\exp(a(1 + 1/\rho)\beta\gamma_i^j)] \leq a(1 + 1/\rho)\beta''$.

As we did in the former coarse graining we use a surmartingale method. Thus, we build a new filtration, i.e., $F_N = \sigma(A_{i_N^v} \cup \sigma((B_1(s), \dots, B_N(s))_{s \in [0,1]}))$ with $A_k = \sigma(X_1, \dots, X_k)$ and the random variable

$$M_N = \frac{\prod_{j=1}^N \exp \left[a(1 + \frac{1}{\rho})\beta'' \sum_{i=i_{j-1}^v+1}^{i_j^v} (\sum_{k \in \mathbb{Z}} \mathbf{1}_{\{S_i = \lfloor kr/a \rfloor\}}) - \beta'(1 + \frac{1}{\rho})\sqrt{\nu} \sum_{k \in \mathbb{Z}} L_1^{kr/\sqrt{\nu}, j} \right]}{E \left[\exp \left(a(1 + \frac{1}{\rho})\beta'' \sum_{i=1}^{\frac{\nu+\varepsilon}{a^2}} \sum_{k \in \mathbb{Z}} \mathbf{1}_{\{S_i = \lfloor kr/a \rfloor\}} \right) - (1 + \frac{1}{\rho})\beta' \sqrt{\nu} \sum_{k \in \mathbb{Z}} L_1^{kr/\sqrt{\nu}, 1} \right)^N \right]}.$$
(9.46)

With respect to the filtration $(F_N)_{N \geq 1}$, the sequence $(M_N)_{N \geq 1}$ is a surmartingale. Indeed, for every $N \geq 1$, M_N is measurable with respect to F_N and $E[M_N | F_{N-1}] = M_{N-1}U$ with

$$U = \frac{E \left[\exp \left((1 + \frac{1}{\rho}) \left[a\beta'' \sum_{i=i_{N-1}^v+1}^{i_N^v} (\sum_{k \in \mathbb{Z}} \mathbf{1}_{\{S_i = \lfloor kr/a \rfloor\}}) - \beta' \sqrt{\nu} \sum_{k \in \mathbb{Z}} L_1^{kr/\sqrt{\nu}, N} \right] \right) \middle| F_{N-1} \right]}{E \left[\exp \left(a(1 + \frac{1}{\rho})\beta'' \sum_{i=1}^{\frac{\nu+\varepsilon}{a^2}} \sum_{k \in \mathbb{Z}} \mathbf{1}_{\{S_i = \lfloor kr/a \rfloor\}} \right) - (1 + \frac{1}{\rho})\beta' \sqrt{\nu} \sum_{k \in \mathbb{Z}} L_1^{kr/\sqrt{\nu}, 1} \right]}.$$

The local time L^N is independent of F_{N-1} and i_{N-1}^v satisfies $S_{i_{N-1}^v} = \lfloor kr/a \rfloor$ for some $k \in \mathbb{Z}$, therefore by Markov property and since by definition

$$\sum_{i=i_{N-1}^v+1}^{i_N^v} \sum_{k \in \mathbb{Z}} \mathbf{1}_{\{S_i = \lfloor kr/a \rfloor\}} \leq \sum_{i=i_{N-1}^v+1}^{i_{N-1}^v + \frac{\nu+\varepsilon}{a^2}} \sum_{k \in \mathbb{Z}} \mathbf{1}_{\{S_i = \lfloor kr/a \rfloor\}}$$

we obtain that $U \leq 1$. Moreover, m_{t/a^2} is a stopping time with respect to F_N and is bounded. Therefore, by applying a standard stopping time theorem, it is sufficient to prove that for ν , ε and a small enough,

$$E \left[\exp \left(a(1 + 1/\rho)\beta'' \sum_{i=1}^{\frac{\nu+\varepsilon}{a^2}} \sum_{k \in \mathbb{Z}} \mathbf{1}_{\{S_i = \lfloor kr/a \rfloor\}} \right) \right] E \left[\exp \left(- (1 + 1/\rho)\beta' \sqrt{\nu} \sum_{k \in \mathbb{Z}} L_1^{kr/\sqrt{\nu}, 1} \right) \right] \leq 1.$$
(9.47)

At this stage we apply Lemma 11 which tells that the first expectation of (9.47) converges to $f((1 + 1/\rho)\beta'', \sqrt{\nu + \varepsilon})$ when a tends to 0. As a consequence, it suffices to show that for $A < B$ the quantity $f(A, \sqrt{\nu + \varepsilon})f(B, \sqrt{\nu})$ becomes smaller than 1 as soon as ν and ε are small enough.

Lemma 16. *For every $A \in \mathbb{R}$, we can write*

$$f(A, \nu) = E \left[\exp \left(A\sqrt{\nu} \sum_{k \in \mathbb{Z}} L_1^{kr/\sqrt{\nu}, 1} \right) \right] = 1 + A\sqrt{\nu} E[L_1^{0,1}] + \xi(\nu)\sqrt{\nu} \quad (9.48)$$

where $\xi(\nu)$ tends to 0 with ν .

Proof. We prove in Appendix 2 that $E[\sum_{k \in \mathbb{Z}} L_1^{kr/\sqrt{\nu}, 1}]$ tend to $E[L_1^{0,1}]$ when $\nu \downarrow 0$ and in Appendix 3 that there exists $\lambda_0 > 0$ and $M > 0$ satisfying $E[\exp(\lambda_0 \sum_{k \in \mathbb{Z}} L_1^{kr/\sqrt{\nu}, 1})] \leq M$ for every $\nu \geq 1$. Thus, the fact that for every $x \geq 0$, $\exp(x) - 1 - x = x^2 f(x)$ with $|f(x)| \leq \exp(x)$ is sufficient to complete the proof of the lemma. \square

At this stage, we choose $\varepsilon \leq \nu^2$ and with the help of Lemma 16, we obtain the equality $f(A, \nu + \varepsilon)f(-B, \nu) = 1 + (A - B)\sqrt{\nu} + \xi_2(\nu)\sqrt{\nu}$ with $\xi_2(\nu) \downarrow 0$ when $\nu \downarrow 0$. Therefore, for ν small enough, the quantity $f(A, \nu + \varepsilon)f(-B, \nu)$ becomes smaller than 1 and it completes the proof of $F^1 \preceq F^2$.

To show that $F^2 \preceq F^1$ we follow the proof of $F^1 \preceq F^2$ and we exchange the terms $\sqrt{\nu} \sum_{k \in \mathbb{Z}} L_1^{kr/\sqrt{\nu}, j}$ and $\sum_{i=i_{j-1}^v+1}^{i_j^v} (\sum_{k \in \mathbb{Z}} \mathbf{1}_{\{S_i = \lfloor kr/a \rfloor\}})$ in (9.46). Then, the lemmas 11 and 16 allow us to conclude and the proof of Step 1 is completed.

9.13. Step 2. As we did in the fourth step of the former coarse graining, we will use this step to integrate with respect to the law of the brownian motion B instead of the random walk S . For that we define the continuous counterparts of $(m, \sigma_1, \dots, \sigma_m, s_1, \dots, s_m)$. Indeed, we denote $\sigma_0 = 0$, $z_0^v = 0$ and recursively $z_{j+1}^v = \inf\{s > \sigma_j \epsilon + \nu : \exists k \in \mathbb{Z} : B_s = kr\}$ while σ_{j+1} is the unique integer satisfying $z_{j+1}^v \in ((\sigma_{j+1} - 1)\epsilon, \sigma_{j+1}\epsilon]$. We let $m_t = \inf\{j \geq 1 : z_j^v > t\}$ and for $j < m$ we let $z_j = z_j^v$ whereas $z_m = t$. Finally we let s_{j+1} be equal to 1 if B is above the interface that it touches in z_{j+1}^v and $s_{j+1} = -1$ if it is below. At this stage, we want to transform the random walk that gives the possible trajectories of the polymer into a Brownian motion. For that, we denote by Q the measure of $(m_t/a^2, \sigma_1, \sigma_2, \dots, \sigma_m, s_1, s_2, \dots, s_m)$ associated with the random walk on $[0, \lfloor t/a^2 \rfloor]$ and by \tilde{Q} the measure of $(m_t, \sigma_1, \sigma_2, \dots, \sigma_m, s_1, s_2, \dots, s_m)$ associated with the Brownian motion on $[0, t]$.

At this stage, we introduce the second intermediate Hamiltonian. It is given by

$$H_{t, \varepsilon, \nu}^{3, r}(a, \beta) = \frac{\beta}{a} \sum_{j=1}^m \left[\sqrt{\nu} \sum_{k \in \mathbb{Z}} L_1^{kr/\sqrt{\nu}, j} \right] + \frac{1}{a} \log \frac{d\tilde{Q}}{dQ}. \quad (9.49)$$

Therefore, we can bound H^{II} from above as follows

$$H^{II} = \frac{\beta - \beta'}{a} \sqrt{\nu} \sum_{j=1}^m \sum_{k \in \mathbb{Z}} L_1^{kr/\sqrt{\nu}, j} - \frac{1}{a(1+\rho)} \log \frac{d\tilde{Q}}{dQ} \leq \frac{1}{a(1+\rho)} \log \frac{dQ}{d\tilde{Q}} \quad (9.50)$$

The next lemma gives us an important tool to estimate the Radon-Nykodim density $dQ/d\tilde{Q}$. Then for every $\varepsilon > 0$, $a > 0$, $l \in \mathbb{N}$ and $y \in \{-\frac{\varepsilon}{a^2}, 0\}$ we set

$$Q_{l, \varepsilon}^{y, a} = P(\{\inf\{i \geq \frac{\nu}{a^2} - y : \exists k \in \mathbb{Z} : S_i = \lfloor \frac{kr}{a} \rfloor\} \in I_l\} \cap \{s_1 = 1\}) \quad (9.51)$$

and for $\tilde{y} \in [-\varepsilon, 0]$, the continuous counterpart, i.e.,

$$\tilde{Q}_{l, \varepsilon}^{\tilde{y}} = P(\{\inf\{s \geq \nu - \tilde{y} : \exists k \in \mathbb{Z} : B_s = kr\} \in I_l\} \cap \{s_1 = 1\}) \quad (9.52)$$

Lemma 17. *For every $\alpha > 0$ and $\eta > 0$, there exists an ε_0 such that for all $\varepsilon \leq \varepsilon_0$ there exists an $a_0(\varepsilon)$ such that for all $a < a_0(\varepsilon)$,*

$$\sup_{y \in \{-\varepsilon/a^2, \dots, 0\}} \sup_{\tilde{y} \in [-\varepsilon, \dots, 0]} \max \left\{ \left| \frac{Q_{l, \varepsilon}^{y, a}}{\tilde{Q}_{l, \varepsilon}^{\tilde{y}}} \right|, \left| \frac{\tilde{Q}_{l, \varepsilon}^{\tilde{y}}}{Q_{l, \varepsilon}^{y, a}} \right| \right\} \leq (1 + \eta) e^{\alpha \varepsilon} \quad (9.53)$$

We give a proof of Lemma 17 in the Appendix 4.

Then, since $m \leq t/\nu$, the inequality (9.50) and Lemma 17 allow us to write

$$\tilde{E}[e^{a(1+\rho^{-1})H^{II}}] \leq \tilde{E}[\exp(a \frac{1+\rho^{-1}}{1+\rho} (m \log(1+\eta) + \sum_{i=1}^m \alpha l_i \varepsilon))] \leq e^{(\frac{A}{\nu} \log(1+\eta) + \alpha)t}.$$

Therefore $\limsup_{t \rightarrow \infty} \frac{1}{t} \log \tilde{E}[\exp(a(1+\rho^{-1})H^{II})] \leq \frac{A}{\nu} \log(1+\eta) + \alpha$ and the r.h.s. of this inequality is smaller than 2α for η small enough. This completes the proof of $F^2 \preceq F^3$.

The proof of $F^3 \preceq F^2$ is very similar, indeed the only difference with respect to the former proof is that the quantity $dQ/d\tilde{Q}$ in (9.50) is replaced by $d\tilde{Q}/dQ$, but Lemma 17 can still be applied. This completes the proof of Step 2.

9.14. **Step 3.** By integrating over \tilde{P} instead of P we can rewrite $F_{t,\varepsilon,\nu}^{3,r}(a, \beta)$ under the form $\frac{1}{t} \log \tilde{E}[\exp(\overline{H}_{t,\varepsilon,\nu}^{3,r}(\beta))]$ with $\overline{H}_{t,\varepsilon,\nu}^{3,r}(\beta) = \beta \sum_{j=1}^m [\sqrt{\nu} \sum_{k \in \mathbb{Z}} L_1^{kr/\sqrt{\nu},j}]$. Therefore, as we did in the former coarse graining we do not consider the quantity a any more. At this stage, we introduce the final Hamiltonian, i.e.,

$$\overline{H}_{t,\varepsilon,\nu}^{4,r}(\beta) = \beta \sum_{k \in \mathbb{Z}} L_t^{kr}, \quad (9.54)$$

and the difference H^{II} is given by $\overline{H}_{t,\varepsilon,\nu}^{3,r}(\beta) - \frac{1}{1+\rho} \overline{H}_{t(1+\rho)^2, \varepsilon(1+\rho)^2, \nu(1+\rho)^2}^{4,r}(\beta')$. Since the scaling property of Brownian motion gives the equality $(1+\rho) \sum_{k \in \mathbb{Z}} L_t^{kr} =_D \sum_{k \in \mathbb{Z}} L_{t(1+\rho)^2}^{kr(1+\rho)}$, we obtain

$$H^{II} = \beta \sum_{j=1}^m \sum_{k \in \mathbb{Z}} L_\nu^{kr,j} - \beta' \sum_{k \in \mathbb{Z}} L_t^{kr} = \sum_{j=1}^m \left[\beta \sum_{k \in \mathbb{Z}} L_\nu^{kr,j} - \beta' \sum_{k \in \mathbb{Z}} (L_{z_j}^{kr} - L_{z_{j-1}}^{kr}) \right]. \quad (9.55)$$

Moreover, by definition of z_m^v we have $\sum_{k \in \mathbb{Z}} (L_{t+\nu}^{kr} - L_t^{kr}) \geq \sum_{k \in \mathbb{Z}} (L_{z_m^v}^{kr} - L_t^{kr})$. Therefore,

$$H^{II} \leq \sum_{j=1}^m \left[\beta \sum_{k \in \mathbb{Z}} L_\nu^{kr,j} - \beta' \sum_{k \in \mathbb{Z}} (L_{z_j^v}^{kr} - L_{z_{j-1}^v}^{kr}) \right] + \beta' \sum_{k \in \mathbb{Z}} (L_{t+\nu}^{kr} - L_t^{kr}). \quad (9.56)$$

It remains to prove that $\limsup_{t \rightarrow \infty} \frac{1}{t} \log \tilde{E}(\exp((1+\rho^{-1})H^{II})) \leq 0$. Trough an Hölder inequality, we can remove the second term of the r.h.s. of (9.56) and it remains to show that for $A < B$ we have $\limsup_{t \rightarrow \infty} \frac{1}{t} \log R_t \leq 0$ with

$$R_t = \tilde{E}[\exp(\sum_{j=1}^m [A \sum_{k \in \mathbb{Z}} L_\nu^{kr,j} - B \sum_{k \in \mathbb{Z}} (L_{z_j^v}^{kr} - L_{z_{j-1}^v}^{kr})])] \leq 0. \quad (9.57)$$

To that aim, we use a martingale method, as we did in Step 1. We recall that for every $j \geq 1$ the local time L^j is associated with the Brownian motion B_j . Then, we define for all $N \geq 1$, $F_N = \sigma(\sigma((B_s)_{s \leq z_N^v}) \cup \sigma((B_s^1, \dots, B_s^N)_{s \geq 0}))$ such that $(F_N)_{N \geq 1}$ is an increasing filtration and we denote by M_N the quantity

$$M_N = \frac{\exp(\sum_{j=1}^N A \sum_{k \in \mathbb{Z}} L_\nu^{kr,j} - B \sum_{k \in \mathbb{Z}} (L_{z_j^v}^{kr} - L_{z_{j-1}^v}^{kr}))}{\tilde{E}[\exp(A \sum_{k \in \mathbb{Z}} L_\nu^{kr,1} - B \sum_{k \in \mathbb{Z}} L_\nu^{kr})]^N}. \quad (9.58)$$

Since L^N is independent of F_{N-1} and B we obtain

$$E[M_N | F_{N-1}] = M_{N-1} \frac{\tilde{E}[\exp(-B \sum_{k \in \mathbb{Z}} L_{z_N^v}^{kr} - L_{z_{N-1}^v}^{kr}) | F_{N-1}]}{\tilde{E}[\exp(-B \sum_{k \in \mathbb{Z}} L_\nu^{kr})]}.$$

Moreover, $z_N^v \geq z_{N-1}^v + \nu$ and $B_{z_{N-1}^v} = kr$ for some $k \in \mathbb{Z}$. Therefore by applying the Markov property we obtain that $E[M_N | F_{N-1}] \leq M_{N-1}$. As a consequence $(M_N)_{N \geq 1}$ is a surmartingale with respect to F_N . The term m_t is a stopping time with respect to $(F_N)_{N \geq 1}$ and is bounded. Thus, to complete the proof of $F^3 \preceq F^4$ we must show that for ν small enough the quantity

$$\tilde{E}[\exp(A \sum_{k \in \mathbb{Z}} L_\nu^{kr,1} - B \sum_{k \in \mathbb{Z}} L_\nu^{kr})] \quad (9.59)$$

is smaller than 1. However, the two local times of (9.59) comes from independent Brownian motions and therefore it suffices to prove that $f(A, \nu)f(-B, \nu) \leq 1$ for ν small enough. This comes directly from Lemma 16.

The proof of $F^4 \preceq F^3$ is similar to the former and we do not give it in details here.

10. APPENDIX

10.1. **Appendix 1.** We prove the convexity of K_c^β in β . For that, we consider β_1 and β_2 such that $K_c^{\beta_1}$ and $K_c^{\beta_2}$ are finite. Then, we denote $\alpha \in [0, 1]$, and obtain

$$\tilde{H}_{1, \alpha K_c^{\beta_1} + (1-\alpha)K_c^{\beta_2}, \alpha\beta_1 + (1-\alpha)\beta_2}^{t,R} = \alpha \tilde{H}_{1, K_c^{\beta_1}, \beta_1}^{t,R} + (1-\alpha) \tilde{H}_{1, K_c^{\beta_2}, \beta_2}^{t,R}.$$

We apply the Hölder inequality (with $p = 1/\alpha$ and $q = 1/(1-\alpha)$), and we let $t \uparrow \infty$, it gives

$$\begin{aligned} \tilde{\Phi}(1, \alpha K_c^{\beta_1} + (1-\alpha)K_c^{\beta_2}, \alpha\beta_1 + (1-\alpha)\beta_2) &\leq \alpha \tilde{\Phi}(1, K_c^{\beta_1}, \beta_1) + (1-\alpha) \tilde{\Phi}(1, K_c^{\beta_2}, \beta_2) \\ &\leq \alpha K_c^{\beta_1} + (1-\alpha) K_c^{\beta_2}. \end{aligned}$$

Since $\tilde{\Phi}(1, h, \beta) \geq h$ (see remark 2), we have

$$\tilde{\Phi}(1, \alpha K_c^{\beta_1} + (1-\alpha)K_c^{\beta_2}, \alpha\beta_1 + (1-\alpha)\beta_2) = \alpha K_c^{\beta_1} + (1-\alpha) K_c^{\beta_2},$$

which implies that

$$\alpha K_c^{\beta_1} + (1-\alpha) K_c^{\beta_2} \geq K_c^{\alpha\beta_1 + (1-\alpha)\beta_2},$$

and then the convexity is proved.

10.2. **Appendix 2.** We give an exact expression of the quantity $N_1(\nu) = E[\sum_{k \in \mathbb{Z}} L_1^{kr/\sqrt{\nu}}]$ and we prove that $N_1(\nu)$ tends to $E[L_1^0]$ when $\nu \downarrow 0$. Let τ_k be the first passage time of the Brownian motion B at $kr/\sqrt{\nu}$.

$$N_1(\nu) = \sum_{k \in \mathbb{Z}} E[\mathbf{1}_{\{\tau_k < 1\}} E[L_1^{kr/\sqrt{\nu}} | \mathcal{F}_{\tau_k}]] = E[L_1^0] + 2 \sum_{k \in \mathbb{Z} \setminus \{0\}} \int_0^1 E[L_{1-u}^0] dP_{\tau_k}(u) \quad (10.1)$$

$$(10.2)$$

Since the density of τ_k is known and takes the value $dP_{\tau_k}(u) = \frac{kr}{\sqrt{2\pi u^3 \nu}} \exp(-\frac{k^2 r^2}{2u\nu})$ we can write

$$N_1(\nu) = E[L_1^0] \left[1 + \sum_{k \in \mathbb{Z} \setminus \{0\}} \frac{kr}{\sqrt{2\pi \nu}} \int_0^1 \frac{\sqrt{1-u}}{u^{3/2}} e^{-\frac{k^2 r^2}{2u\nu}} du \right] \quad (10.3)$$

and then, by setting $v = \frac{1}{u} - 1$ we obtain

$$N_1(\nu) = E[L_1^0] \left[1 + \frac{1}{2\pi} \sum_{k \in \mathbb{Z} \setminus \{0\}} \frac{kr}{\sqrt{\nu}} e^{-\frac{k^2 r^2}{2\nu}} \int_0^\infty \frac{\sqrt{v}}{1+v} e^{-\frac{k^2 r^2 v}{2\nu}} dv \right]. \quad (10.4)$$

By considering $\nu \leq 1$, we can bound for every $k \in \mathbb{Z} \setminus \{0\}$ the quantity $\int_0^\infty \frac{\sqrt{v}}{1+v} e^{-\frac{k^2 r^2 v}{2\nu}} dv$ by a constant $C > 0$ independent of ν and k . Moreover the function $x \rightarrow x \exp(-x^2/2)$ is decreasing on $[1, \infty)$ and therefore for every $\nu \leq r^2$ we obtain

$$\sum_{k \in \mathbb{Z} \setminus \{0\}} \frac{kr}{\sqrt{\nu}} e^{-\frac{k^2 r^2}{2\nu}} \int_0^\infty \frac{\sqrt{v}}{1+v} e^{-\frac{k^2 r^2 v}{2\nu}} dv \leq C \sum_{k \in \mathbb{Z} \setminus \{0\}} \frac{kr}{\sqrt{\nu}} e^{-\frac{(kr)^2}{2\nu}}. \quad (10.5)$$

The r.h.s. of (10.5) tends to 0 when $\nu \downarrow 0$ and this is sufficient to conclude that $N_1(\nu)$ tends to $E[L_1^0]$ when $\nu \downarrow 0$.

10.3. Appendix 3. We denote $c > 0$ and $r \geq c > 0$. We prove that there exist $\lambda_0 > 0$ and $M > 0$ satisfying $\tilde{E}[\exp(\lambda_0 \sum_{k \in \mathbb{Z}} L_1^{kr})] \leq M$ for every $r \geq c$.

Proof. We denote $H_N = \frac{1}{\sqrt{N}} \sum_{i=1}^N \sum_{k \in \mathbb{Z}} \mathbf{1}_{\{S_i = \lfloor kr\sqrt{N} \rfloor\}}$ and $\tilde{H} = \sum_{k \in \mathbb{Z}} L_1^{kr}$. Moreover, for every $j \in \mathbb{Z}$ we denote $V_{j,N} = E[\exp(\lambda H_N) \mathbf{1}_{\{H_N \in [j, j+1)\}}]$ and $P_j = \tilde{E}[\exp(\lambda \tilde{H}) \mathbf{1}_{\{\tilde{H} \in [j, j+1)\}}]$. We proved in Lemma 11 that H_N converges in law towards \tilde{H} . This entails that $V_{j,N}$ tends to P_j when $N \uparrow \infty$. Moreover, we showed in the second step of the proof of Lemma 11 that there exists $c' > 0$ depending only on c such that for N large enough and independent of j we have $P(H_N \geq j) \leq \exp(-c'j)$. Therefore, by the Fatou's property we obtain

$$\sum_{j \in \mathbb{Z}} \liminf_{N \rightarrow \infty} V_{j,N} \leq \liminf_{N \rightarrow \infty} \sum_{j \in \mathbb{Z}} V_{j,N} \quad (10.6)$$

$$\sum_{j \in \mathbb{Z}} P_j \leq \liminf_{N \rightarrow \infty} \sum_{j \in \mathbb{N}} e^{\lambda_0(j+1)} P(H_N \geq j) \leq \sum_{j \in \mathbb{N}} e^{\lambda_0(j+1) - c'j} = M < \infty. \quad (10.7)$$

This completes the proof of the property. \square

10.4. Appendix 4. We give here the frame of the proof of lemma 17 by asserting two facts that are sufficient to complete the proof. However, we will not prove in details these two facts because the computations that are required are too heavy to be exposed here. For this reason, we assert these two facts and show how they allow us to prove Lemma 17. Moreover, for simplicity we will consider only the quantity Q/\tilde{Q} in (9.53) because the proof of (9.53) for \tilde{Q}/Q is completely similar.

We let α and η be two strictly positive constants. We let also ξ be strictly positive and satisfy $(1 + \xi)^2 / (1 - \xi) < 1 + \eta$.

Fact 18. *For every $\varepsilon > 0$ there exists $c_\varepsilon > 0$, $C_\varepsilon > 0$, $l_0(\varepsilon) \in \mathbb{N} \setminus \{0\}$ and $a_0(\varepsilon) > 0$ such that for $l \geq l_0(\varepsilon)$ and $a < a_0(\varepsilon)$ we obtain*

$$c_\varepsilon \exp\left(\frac{l\varepsilon}{a^2} \log\left(\cos\left(\frac{\pi a}{r}\right)\right)\right) \leq Q_{l,\varepsilon}^{y,a} \leq C_\varepsilon \exp\left(\frac{l\varepsilon}{a^2} \log\left(\cos\left(\frac{\pi a}{r}\right)\right)\right) \quad \forall y \in \left\{-\frac{\varepsilon}{a^2}, \dots, 0\right\}$$

$$\text{and } c_\varepsilon \exp\left(-\frac{l\varepsilon\pi^2}{2r^2}\right) \leq \tilde{Q}_{l,\varepsilon}^{\tilde{y},a} \leq C_\varepsilon \exp\left(-\frac{l\varepsilon\pi^2}{2r^2}\right) \quad \forall \tilde{y} \in [-\varepsilon, 0].$$

Then, by using Fact 18 we obtain for $l \geq l_0(\varepsilon)$ and $a \leq a_0(\varepsilon)$ that

$$M_{l,\varepsilon}^a = \sup_{y \in \{-\varepsilon/a^2, \dots, 0\}} \sup_{\tilde{y} \in [-\varepsilon, 0]} \left| \frac{Q_{l,\varepsilon}^{y,a}}{\tilde{Q}_{l,\varepsilon}^{\tilde{y},a}} \right| \leq \frac{C_\varepsilon}{c_\varepsilon} e^{l\varepsilon \left(\frac{1}{a^2} \log\left(\cos\left(\frac{\pi a}{r}\right)\right) + \frac{\pi^2}{2r^2} \right)}, \quad (10.8)$$

but we notice that $\frac{1}{a^2} \log\left(\cos\left(\frac{\pi a}{r}\right)\right) + \frac{\pi^2}{2r^2} \rightarrow 0$ as $a \downarrow 0$. Therefore, there exists $a_1(\alpha, \varepsilon) \leq a_0(\varepsilon)$ such that for $a \leq a_1(\alpha, \varepsilon)$ and $l \geq l_0(\varepsilon)$ we get $M_{l,\varepsilon}^a \leq [(C_\varepsilon/c_\varepsilon)e^{-\alpha l \varepsilon/2}]e^{\alpha l \varepsilon}$. Therefore, there exists $l_1(\alpha, \varepsilon) \geq l_0(\varepsilon)$ such that for $l \geq l_1(\alpha, \varepsilon)$ the quantity $(C_\varepsilon/c_\varepsilon)e^{-\alpha l \varepsilon/2}$ becomes smaller than 1. Thus $M_{l,\varepsilon}^a \leq e^{\alpha l \varepsilon}$ for $l \geq l_1(\alpha, \varepsilon)$ and $a \leq a_1(\alpha, \varepsilon)$.

At this stage, it remains to prove that there exists $\varepsilon_1(\alpha, \eta) > 0$ such that for all $\varepsilon < \varepsilon_1$ there exists $a_2(\alpha, \varepsilon) > 0$ such that $a \leq a_2$ entails $M_{l,\varepsilon}^a \leq (1 + \eta)e^{\alpha l \varepsilon}$ for all $l \in \{1, l_1(\varepsilon)\}$.

To prove this we begin with recalling that for every $l \geq 1$, by the Donsker theorem the quantity $Q_{l,\varepsilon}^{-\varepsilon/a^2, a}$ tends to $\tilde{Q}_{l,\varepsilon}^{-\varepsilon}$ as $a \downarrow 0$. Therefore, there exists $a_3(\alpha, \varepsilon)$ such that $a \leq a_3(\alpha, \varepsilon)$ and $l \leq l_1(\alpha, \varepsilon)$ entails $|Q_{l,\varepsilon}^{-\varepsilon/a^2, a} / \tilde{Q}_{l,\varepsilon}^{-\varepsilon}| \leq 1 + \xi$. Then, with the following fact, we can complete the proof of Lemma 17.

Fact 19. *There exists $\varepsilon_1 > 0$ such that for every $\varepsilon \leq \varepsilon_1$ there exists $a'(\varepsilon) > 0$ such that $a \leq a'(\varepsilon)$ entails*

$$\begin{aligned} |Q_{l,\varepsilon}^{y,a} - Q_{l,\varepsilon}^{-\varepsilon/a^2,a}| &\leq \xi Q_{l,\varepsilon}^{-\varepsilon/a^2,a} \quad \text{for every } l \geq 1 \text{ and } y \in \{-\varepsilon/a^2, \dots, 0\} \\ |\tilde{Q}_{l,\varepsilon}^{\tilde{y}} - \tilde{Q}_{l,\varepsilon}^{-\varepsilon}| &\leq \xi \tilde{Q}_{l,\varepsilon}^{-\varepsilon} \quad \text{for every } l \geq 1 \text{ and } \tilde{y} \in [-\varepsilon, 0]. \end{aligned}$$

Therefore, we set $a_2(\alpha, \varepsilon) = \min\{a'(\varepsilon), a_3(\alpha, \varepsilon)\}$ and for $\varepsilon \leq \varepsilon_1$, $a \leq a_2$, $y \in \{-\varepsilon/a^2, \dots, 0\}$, $\tilde{y} \in [-\varepsilon, 0]$ and $l \leq l_1(\alpha, \varepsilon)$ we can write

$$\left| \frac{Q_{l,\varepsilon}^{y,a}}{Q_{l,\varepsilon}^{\tilde{y}}} \right| \leq \left| \frac{Q_{l,\varepsilon}^{-\varepsilon/a^2,a}}{\tilde{Q}_{l,\varepsilon}^{-\varepsilon}} \right| \frac{1 + |Q_{l,\varepsilon}^{y,a} - Q_{l,\varepsilon}^{-\varepsilon/a^2,a}| / Q_{l,\varepsilon}^{-\varepsilon/a^2,a}}{1 - |Q_{l,\varepsilon}^{\tilde{y}} - \tilde{Q}_{l,\varepsilon}^{-\varepsilon}| / \tilde{Q}_{l,\varepsilon}^{-\varepsilon}} \leq (1 + \xi) \frac{1 + \xi}{1 - \xi} \leq 1 + \eta. \quad (10.9)$$

This completes the proof of Lemma 17.

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