On some martingales for Markov processes

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December 5, 2006

Abstract

In this paper we present a martingale formula for Markov processes and their integrated process. This formula allows us to derive some new as well as some well-known martingales. We give some examples of their application in stochastic process theory.

1 Introduction

In applied probability nowadays martingales are considered to be very useful tools for studying stochastic processes. In particular the martingale convergence results, the martingale inequalities and last but not least the very useful optional stopping theorem can be applied once an appropriate martingale has been found. In this paper we consider martingales for the two dimensional process $Z_t = (X_t, Y_t)$, where X_t is a continuous time homogeneous Markov process, $Y_t = \int_0^t \xi(X_s)\,ds$ and ξ is some suitable function. By applying Dynkin's formula to the full generator of Z_t and a special class of functions in its domain we derive a quite general martingale M_t, which can be used to derive not only new martingales but also some well-known martingales. Among these are the Wald martingale and a special variant of the Kella-Whitt martingale for Lévy processes (section 3.4, see Asmussen and Kella [4], Kella and Whitt [25]), the exponential martingale used for exponential change of measure (section 3.2, see Palmowski and Rolski [36]) and a sum of powers resembling a stochastic Taylor series (section 3.3). We will give some examples of applications in the respective sections and refer for now to the tremendously rich literature about martingales and its applications. An introduction to martingales can be found in various textbooks of probability, e.g. in the book of Ross [43] or more elaborately in the book of Ethier and Kurtz [15]. The books of Asmussen [3] and Rolski et al. [41] provide many applications of martingale theory to applied probability models, including the calculation of expected stopping times, existence of stationary distributions, change of probability measure, evaluation of ruin probabilities. Further applications can be found in Kella and Stadje [24], Boxma et al. [9], Kinateder and Lee [27] and Rosenkrantz [42], the latter using a somewhat related approach to ours.

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Integrals of Markov processes are of interest in different areas of probability theory. In the special case of X_t being a Brownian motion process, Z_t is called a *Kolmogorov diffusion* (due to an early paper of Kolmogorov [28]). This process has attracted much interest (Goldman [16], Groeneboom et al. [17], Kendall and Price [26],McKean [34], Lachal [31], [30], Dufresne [13], Hesse [21], Lefebvre and Léonard [32]). Patie [37] investigates the integral of a generalized Ornstein-Uhlenbeck process.

If X_t is a Lévy process and $\xi(x) = e^{-x}$ then $Y_t(x) = \int_0^t e^{-X_s}\, ds$ is called the *exponential functional* of X_t. This process is of special importance e.g. in mathematical finance and much research has been done about on this topic in recent years (see Carmona, Petit and Yor [10], Bertoin and Yor [8], Maulik and Zwart [33], Guillemin et al. [18]). Some results concerning exponential functionals are given in section 3.4.

Our paper is organized as follows. We will, after a short introduction to the underlying concepts, present the main result, namely our formula for the martingale M_t. The third section then contains special cases of this martingale together with some applications and references to related literature. We close with an appendix containing a lemma which is used for the proof of the main result.

2 The main result

Let $(X_t)_{t\geq 0}$ be a homogeneous continuous time Markov process on a probability space $(\Omega, \mathcal{F}, P)$ adapted to a filtration $(\mathcal{F}_t)_{t\geq 0} \subseteq \mathcal{F}$ and with values in a state space $(E, \mathcal{E})$, where $\mathcal{E}$ is the Borel σ-field of E. We will assume that $E \subseteq \mathbb{R}^d$ for some $d \in \mathbb{N}$ and that X_t has right-continuous paths, though the results should also hold for more general cases. Let $\mathscr{M}(E)$ denote the set of measurable functions $E \to \mathbb{R}$. The full generator $\mathcal{A}$ is defined to be the operator assigning to some $f \in \mathscr{M}(E)$ a function $g \in \mathscr{M}(E)$, such that

$$f(X_t) - \int_0^t g(X_s)\, ds \tag{1}$$

becomes a martingale, the so called *Dynkin martingale*. If such a function exists then f belongs to the domain $\mathscr{D}(A)$ of the full generator $\mathcal{A}$ and we set $\mathcal{A}f = g$. The full generator is not necessarily uniquely defined, but all functions $g \in \mathscr{M}(E)$ for which the defining equation (1) holds differ only on a set $B \in \mathcal{E}$ of so called potential zero, which means that almost surely the total occupation time $\int_0^\infty 1_{\{X_s \in B\}}\, ds$ vanishes (see Palmowski and Rolski [36]).

Let $Y_t = \int_0^t \xi(X_s)\, ds$ where $\xi \in \mathscr{M}(E)$ is chosen in such a way that $E \int_0^t |\xi(X_s)|\, ds < \infty$. We consider the two dimensional process $(X_t, Y_t)_{t\geq 0}$ which is again a homogeneous continuous time Markov process, but with state space $E \times [0, \infty)$. In the special case when $\xi(x) \equiv 1$ we encounter the space-time process $Z_t = (X_t, t)$, which is known to have the generator $\mathcal{A}_Z g(x,y) = \mathcal{A} g_y(x) + \frac{\partial}{\partial y} g_x(y)$. Since in our more general case $Y_t - Y_0 \approx t \cdot \xi(X_t)$ for small t and the usual strong generator is defined as the limit $\left(E_x f(Z_t) - f(Z_0)\right)/t$ as t tends to zero one might expect that the full generator $\mathcal{A}_Z$ of Z_t is given by

$$\mathcal{A}_Z g(x,y) = \mathcal{A} g_y(x) + \xi(x) \cdot \frac{\partial}{\partial y} g_x(y), \tag{2}$$

where $g_x : y \mapsto g(x,y)$ and $g_y : x \mapsto g(x,y)$. Special cases of equation (2) can be found e.g. in Lachal [31, 30] for the Brownian motion case and in the work [39] of Peskir for more general diffusions. In Davis [12] and Rolski et al. [41] the $\xi(x) \equiv 1$ case is discussed for so called Piecewise deterministic Markov processes (see the section *time variation* in the book [12]). In this paper we merely provide a proof of formula (2) for the special case where $g(x,y) = f(x)h(y)$, see Lemma 4.1 in the appendix. Actually, we will not need equation (2) for more general functions g.

Next we define an operator $\mathcal{R}_\alpha : \mathscr{M}(E) \to \mathscr{M}(E)$ and its positive and negative powers. First let $\mathcal{R}_\alpha^0 f(x) = f(x)$ for all $f \in \mathscr{D}(\mathcal{R}_\alpha^0) := \mathscr{M}(E)$. For $k \geq 1$ the set $\mathscr{D}(\mathcal{R}_\alpha^k)$ is then defined to comprise all functions $f \in \mathscr{D}(\mathcal{R}_\alpha^{k-1})$ for which a function $g \in \mathscr{D}(\mathcal{A})$ exists such that

$$\mathcal{A} g(x) \quad = \quad \xi(x) \left(\alpha g(x) - \mathcal{R}_\alpha^{k-1} f(x) \right) \quad , \quad \forall x \in E. \tag{3}$$

In this case we define $\mathcal{R}_\alpha^k f(x) := g(x)$. Moreover, to define negative powers, we say that f is a member of $\mathscr{D}(\mathcal{R}_\alpha^{-k})$ if $f \in \mathscr{D}(\mathcal{R}_\alpha^{1-k})$, $\mathcal{R}_\alpha^{1-k} f \in \mathscr{D}(\mathcal{A})$ and for some $g \in \mathscr{M}(E)$

$$\mathcal{A} \mathcal{R}_\alpha^{1-k} f(x) \quad = \quad \xi(x) \left(\alpha \mathcal{R}_\alpha^{1-k} f(x) - g(x) \right). \tag{4}$$

Again we set $\mathcal{R}_\alpha^{-k} f(x) = g(x)$ if this is true. Note that for $\xi(x) \equiv 1$ and bounded $f \in \mathscr{M}(E)$ the operator $\mathcal{R}_\alpha$ is equal to the inverse operator of $(\alpha \cdot id - \mathcal{A})$ and thus is just the resolvent operator of the process (see Dynkin [14], Ethier and Kurtz [15], Kallenberg [22]). As before in the case of the full generator, our definition of $\mathcal{R}_\alpha^k f$ is not unique. Thus we agree that in any expression involving $\mathcal{R}_\alpha^k$ (or $\mathcal{R}_\alpha^{-k}$) these terms can be replaced by any choice of functions for which (3) (or (4)) holds.

The following result, the introduction of the martingale M_t, is our key result and the starting point for an exploration of some useful martingales. Let $\mathscr{C}^n(E)$, $n > 0$ denote the set of functions $f : E \to \mathbb{R}$ which are n times continuously differentiable. If $n < 0$ then $\mathscr{C}^n(E)$ comprises the n times integrable functions, in the sense that $f \in \mathscr{C}^n(E)$ if $g^{(n)} = f$ for some $g \in \mathscr{C}^{-n}(E)$.

Theorem 2.1. *Let $m, n, l \in \mathbb{Z}$ with $m \le n$. We assume that $h \in \mathscr{C}^{n+l}(E) \cap \mathscr{C}^{m+l-1}(E)$ and that $e^{-\alpha y} h^{(k)}(y)$ is bounded on the range of Y_t for $m+l-1 \le k \le n+l$. Moreover let $f \in \mathscr{D}(\mathcal{R}_\alpha^n) \cap \mathscr{D}(\mathcal{R}_\alpha^{m-1})$. Then*

$$M_t = e^{-\alpha Y_t} \sum_{k=m}^{n} \mathcal{R}_\alpha^k f(X_t) h^{(k+l-1)}(Y_t)$$
$$- \int_0^t \xi(X_s) e^{-\alpha Y_s} \left(\mathcal{R}_\alpha^n f(X_s) h^{(l+n)}(Y_s) - \mathcal{R}_\alpha^{m-1} f(X_s) h^{(l+m-1)}(Y_s) \right) ds \quad (5)$$

is a martingale.

Proof. We apply Lemma 4.1 from the appendix to the function

$$g(x, y) = \sum_{k=m}^{n} \mathcal{R}_\alpha^k f(x) \cdot e^{-\alpha y} h^{(k+l-1)}(y).$$

Note that since $f \in \mathscr{D}(\mathcal{R}_\alpha^k) \cap \mathscr{D}(\mathcal{R}_\alpha^{k-1})$,

$$\mathcal{A} \mathcal{R}_\alpha^k f(x) = \alpha \xi(x) \mathcal{R}_\alpha^k f(x) - \xi(x) \mathcal{R}_\alpha^{k-1} f(x),$$

according to the definition. Consequently

$$\begin{aligned}
\mathcal{A}_Z g(x, y) &= \sum_{k=m}^{n} \left(\alpha \xi(x) \mathcal{R}_\alpha^k f(x) - \xi(x) \mathcal{R}_\alpha^{k-1} f(x) \right) \cdot e^{-\alpha y} h^{(k+l-1)}(y) \\
&\quad + \xi(x) \sum_{k=m}^{n} \mathcal{R}_\alpha^k f(x) \cdot e^{-\alpha y} \left(h^{(k+l)}(y) - \alpha h^{(k+l-1)}(y) \right) \\
&= - \sum_{k=m}^{n} \xi(x) \mathcal{R}_\alpha^{k-1} f(x) \cdot e^{-\alpha y} h^{(k+l-1)}(y) \\
&\quad + \sum_{k=m+1}^{n+1} \xi(x) \mathcal{R}_\alpha^{k-1} f(x) \cdot e^{-\alpha y} h^{(k+l-1)}(y)
\end{aligned}$$

Thus cancelling out the common summands we obtain

$$\mathcal{A}_Z g(x, y) = \xi(x) \cdot e^{-\alpha y} \left(\mathcal{R}_\alpha^n f(x) h^{(l+n)}(y) - \mathcal{R}_\alpha^{m-1} f(x) h^{(l+m-1)}(y) \right).$$

Applying Dynkin's formula to $g(x, y)$ we find that

$$g(X_t, Y_t) - \int_0^t \mathcal{A}_Z g(X_s, Y_s)\, ds = M_t$$

is a martingale. $\square$

Some remarks are in order. First, the parameter l is somewhat dispensable in that (5) can be obtained by replacing h by $h^{(l)}$ in the $l = 0$ formula. However, we'll see that our formulation leads to more readable expressions. The variable α has been introduced because some frequently used martingales contain an exponential term. For some applications the sum term in (5) is superfluous and the choice of $m = n$ will reduce the formula to a more concise expression. We refer to section 3.1 for some simplified versions of (5).

By canceling the boundedness condition on h the process M_t becomes merely a *local martingale* instead of a true martingale and all occuring martingales should be replaced by their local versions. This can be verified by modifying Lemma 4.1 and observing that Y_t is a predictable process w.r.t. $\mathcal{F}_t$.

Something similar is true for the conditions on f. We will not discuss conditions for the membership of f to $\mathscr{D}(\mathcal{R}_\alpha^n) \cap \mathscr{D}(\mathcal{R}_\alpha^{m-1})$. In fact this could be a formidable difficulty for some processes. For other processes, like the Piecewise deterministic Markov processes (see Davis [12]) the domain of $\mathcal{A}$ is exactly known, so that one can reformulate the conditions of Theorem 2.1 in terms of absolute continuity and some integrability condition. Also the actual calculation of $\mathcal{R}_\alpha^k f(x)$ will be quite difficult in practice for most models and the respective equations will become more and more involved as k increases. Even for simple generators the calculations are not easy unless e.g. $\alpha = 0$ or $f(x) \equiv 0$. However, dealing with such practical problems will be the subject of subsequent work, where we will focus on specific models. For now we emphasize the existence of the martingale M_t under suitable conditions and invite researchers to apply it to their specific problem.

A final remark concerns the generator formula (2) for the two-dimensional process Z_t. As we mentioned before it is well-known that in the $\xi(x) \equiv 1$ case, when $Y_t = t$, the generator is given by $\mathcal{A}_Z g(x, y) = \mathcal{A} g_y(x) + \frac{\partial}{\partial y} g_x(y)$. If $\xi(x)$ is strictly positive then Y_t defines an additive functional and we can define a random time change by $\widetilde{Z}_t = Z_{t^\star}$, where $t^\star = \inf\{s \geq 0 | Y_s \geq t\}$. The generator of the new process is then obtained by dividing the generator of the original process by the function ξ (see Dynkin [14], Ethier and Kurtz [15], Gzyl [19]). With this information in mind we can give an alternative proof of formula (5) for the case of a strictly positive ξ. We start with our Markov process X_t. A time change yields the transformed process $X_t^\star$ with generator acting on functions f according to $\mathcal{A}^\star f(x) = \mathcal{A} f(x)/\xi(x)$, where $\mathcal{A}$ is as before the generator of X_t. Next we concentrate our attention to the two dimensional Markov process $Z_t^\star = (X_t^\star, t)$. Its generator is given by $\mathcal{A}_Z^\star f_y(x) + \frac{\partial}{\partial y} f_x(y)$, which equals $\frac{\mathcal{A}_Z f_y(x)}{\xi(x)} + \frac{\partial}{\partial y} f_x(y)$. The desired formula $\mathcal{A} f_y(x) + \xi(x) \cdot \frac{\partial}{\partial y} f_x(y)$ now follows by reverting the time change, causing a multiplication of the generator by $\xi(x)$. It follows from $Y_{t^\star} = t$ that the deterministic process t is re-transformed into Y_t, so that $Z_t^\star(X^\star, t)$ is changed to our process $Z_t = (X_t, Y_t)$.

3 Applications, prominent examples

At first glance formula (5) looks a bit awkward, but it reveals its qualities as soon as the appropriate functions and parameters are inserted. We will do this in the upcoming section.

3.1 Some simplifications of the martingale formula

The structure of M_t becomes more clear if we simplify it. For example if $m = n$ then the sum in (5) vanishes and

$$\begin{aligned} M_t &= e^{-\alpha Y_t}\mathcal{R}_\alpha^n f(X_t)h^{(n+l-1)}(Y_t) \\ &- \int_0^t \xi(X_s)e^{-\alpha Y_s}\Big(\mathcal{R}_\alpha^n f(X_s)h^{(l+n)}(Y_s) - \mathcal{R}_\alpha^{n-1} f(X_s)h^{(l+n-1)}(Y_s)\Big)\, ds \end{aligned} \tag{6}$$

remains. Letting e.g. $l = 1$ we get

$$\begin{aligned} M_t &= e^{-\alpha Y_t}\mathcal{R}_\alpha^n f(X_t)h^{(n)}(Y_t) \\ &- \int_0^t \xi(X_s)e^{-\alpha Y_s}\left(\mathcal{R}_\alpha^n f(X_s)h^{(n+1)}(Y_s) - \mathcal{R}_\alpha^{n-1} f(X_s)h^{(n)}(Y_s)\right)\, ds \end{aligned} \tag{7}$$

Further simplification can be achieved by letting $\xi(x) \equiv 1$. This means that $Y_t = t$, so that

$$\begin{aligned} M_t &= e^{-\alpha t}\mathcal{R}_\alpha^n f(X_t)h^{(n)}(t) \\ &- \int_0^t e^{-\alpha s}\left(\mathcal{R}_\alpha^n f(X_s)h^{(n+1)}(s) - \mathcal{R}_\alpha^{n-1} f(X_s)h^{(n)}(s)\right)\, ds. \end{aligned}$$

If we choose $\alpha = 0$ in (7) then we arrive at

$$\begin{aligned} M_t &= \mathcal{R}_0^n f(X_t)h^{(n)}(Y_t) \\ &- \int_0^t \xi(X_s)\left(\mathcal{R}_0^n f(X_s)h^{(n+1)}(Y_s) - \mathcal{R}_0^{n-1} f(X_s)h^{(n)}(Y_s)\right)\, ds. \end{aligned}$$

The $\xi(x) \equiv 1$ case is now given by

$$M_t = \mathcal{R}_0^n f(X_t)h^{(n)}(t) - \int_0^t \left(\mathcal{R}_0^n f(X_s)h^{(n+1)}(s) - \mathcal{R}_0^{n-1} f(X_s)h^{(n)}(s)\right)\, ds.$$

This formula can be used to obtain the simple martingale

$$M_t = f(X_t)h(t) - \int_0^t \left(f(X_s)h'(s) + \mathcal{A}f(X_s)h(s)\right)\, ds.$$

(compare with equation (9) in Athreya and Kurtz [5]). Clearly the $h(x) \equiv 1$ case is the Dynkin martingale (1).

A reasonable choice for the function h in (6) is an exponential function, since it is unchanged by differentiation. We choose $h(x) = e^{\beta x}$, $\beta \in \mathbb{R}$ and obtain the martingale

$$\begin{aligned}\beta^{-(n+l-1)} M_t &= e^{(\beta-\alpha)Y_t} \mathcal{R}_\alpha^n f(X_t) \\ &\quad - \int_0^t \xi(X_s) e^{(\beta-\alpha)Y_s} \left(\beta \mathcal{R}_\alpha^n f(X_s) - \mathcal{R}_\alpha^{n-1} f(X_s) \right) \, ds \end{aligned} \tag{8}$$

if $\beta^{n+l-1} \neq 0$. We will see in the next section, that this martingale has an important application in the theory of stochastic processes. The particular choice of $l = 1 - n$ and $\beta = 0$ leads to

$$M_t = e^{-\alpha Y_t} \mathcal{R}_\alpha^n f(X_t) + \int_0^t \xi(X_s) e^{-\alpha Y_s} \mathcal{R}_\alpha^{n-1} f(X_s) \, ds. \tag{9}$$

This martingale appears in a less general form, namely as

$$M_t = e^{-\alpha t} f(X_t) + \int_0^t e^{-\alpha s} \left(\alpha f(X_s) - \mathcal{A} f(X_s) \right) \, ds,$$

in the book of Ethier and Kurtz [15], Lemma 3.2. See also the papers of Stockbridge [45], Novikov et al. [35], Kou and Wang [29].

3.2 The exponential martingale, change of measure

Another martingale, also appearing in Lemma 3.2 of [15], is the exponential martingale

$$M_t = f(X_t) \cdot exp\left(- \int_0^t \frac{\mathcal{A} f(X_s)}{f(X_s)} \, ds \right) \tag{10}$$

for functions f in the domain which are bounded away from zero. The martingale property follows directly from (8) by inserting $n = 0$, $\beta = \alpha - 1$ and $\xi(x) = \mathcal{A} f(x) / f(x)$. Then $Y_t = \int_0^t \frac{\mathcal{A} f(X_s)}{f(X_s)} \, ds$ and $\mathcal{R}_\alpha^{-1} f(x) = (\alpha - 1) f(x)$. It follows that

$$\begin{aligned}\beta^{1-l} M_t &= e^{-Y_t} f(X_t) - \int_0^t \frac{\mathcal{A} f(X_s)}{f(X_s)} e^{-Y_s} \left((\alpha - 1) f(X_s) - \mathcal{R}_\alpha^{-1} f(X_s) \right) \, ds \\ &= e^{-Y_t} f(X_t).\end{aligned}$$

The technique of exponential change of measure (see Palmowski and Rolski [36]) is based on this martingale. More precisely, by setting $\tilde{P}_t(A) = E(M_t | A)$ for every $t \geq 0$ new probabilty measures $\tilde{P}_t$ are defined. It can be shown that under certain conditions there is a probability measure $\tilde{P}$ with $\tilde{P}_t = \tilde{P}|_{\mathcal{F}}$ having some nice properties, e.g. $\tilde{P}(A) = E(M_\tau | A)$ if τ is a finite stopping time (see also section 10.2.6. of Rolski et al. [41] for a general treatment). The Wald martingale is a special case of the exponential martingale (see the section 3.4 below).

3.3 A martingale resembling Taylor's formula

So far we haven not take any advantage of the sum term in (5). We will now derive some formulas which lead for example to a nice recursion formula for the moments of first hitting times.

In what follows let $\wp_k(x) = x^k/k!$ for $k \geq 0$ and $\wp_k(x) \equiv 0$ for $k < 0$. In particular $\wp_0(x) \equiv 1$ and $\wp_1(x) = x$. In addition let $\wp(x) \equiv 0$ denote the zero function. Negative derivatives $h^{(-k)}$ symbolize successive primitives, i.e. solutions of $g^{(k)} = h$.

Since higher derivatives of $\wp_j$ vanish it is tempting to insert these functions in place of h into our martingale (5). Doing so and letting $l = 1$ we obtain

$$M_t = e^{-\alpha Y_t} \sum_{k=m}^{n} \mathcal{R}_\alpha^k f(X_t) \frac{Y_t^{j-k}}{(j-k)!} - \int_0^t \xi(X_s) e^{-\alpha Y_s} \big(\mathcal{R}_\alpha^n f(X_s) \frac{Y_s^{j+n-1}}{(j+n-1)!} - \mathcal{R}_\alpha^{m-1} f(X_s) \frac{Y_s^{j-m}}{(j-m)!} \big)\, ds \tag{11}$$

for $j + n - 1 \geq 0$. We will apply this formula later in the context of exponential functionals of Lévy processes. For now we let $n = j = 0$ and $N = -m \geq 0$. Then our martingale is given by

$$M_t = e^{-\alpha Y_t} \sum_{k=0}^{N} \mathcal{R}_\alpha^{-k} f(X_t) \frac{Y_t^k}{k!} + \int_0^t \xi(X_s) e^{-\alpha Y_s} \mathcal{R}_\alpha^{-(N+1)} f(X_s) \frac{Y_s^N}{N!}\, ds. \tag{12}$$

In particular if we let $\xi(x) \equiv 1$ and $\alpha = 0$ we obtain from $\mathcal{R}_\alpha^{-k} f(x) = (-1)^k \mathcal{A}^k f(x)$ that

$$M_t = \sum_{k=0}^{N} \mathcal{A}^k f(X_t) \frac{(-t)^k}{k!} - \int_0^t \mathcal{A}^{N+1} f(X_s) \frac{(-s)^N}{N!}\, ds \tag{13}$$

which can be found in a recent article of Barrieu and Schoutens [6] (see also Chaumont and Yor [11]). Similar formulas have been present in the literature for a while; we mention a derivation of an equation akin to (13) for diffusion processes in the book of Karlin and Taylor [23], page 312, and for general Markov processes in the paper Athreya and Kurtz [5]. Stochastic Taylor-like formulas related to (12) also appear in Airault and Föllmer [2] and Helms [20].

To examine a further special case we let $l = -m$, $n = N + m$ with $N \geq 0$ and $h(x) = \wp_{N-1}(x)$ in (5). Then

$$M_t = e^{-\alpha Y_t} \sum_{k=0}^{N} \mathcal{R}_\alpha^{k+m} f(X_t) \frac{Y_t^{N-k}}{(N-k)!} + \int_0^t \xi(X_s) e^{-\alpha Y_s} \mathcal{R}_\alpha^{m-1} f(X_s) \frac{Y_s^N}{N!}\, ds. \tag{14}$$

Thus by inserting $m = 1$, $\xi(x) \equiv 1$ and $f(x) \equiv 1$ we arrive at

$$M_t \quad = \quad e^{-\alpha t} \sum_{k=0}^{N} \mathcal{R}_\alpha^{k+1} \wp_0(X_t) \frac{t^{N-k}}{(N-k)!} + \Gamma_{N,\alpha}(t), \tag{15}$$

where the remainder function $\Gamma_{N,\alpha}$ is given by

$$\Gamma_{N,\alpha}(t) = \int_0^t \frac{s^N}{N!} e^{-\alpha s}\, ds = \frac{1 - e^{-\alpha t} \sum_{i=0}^{N} \frac{(\alpha t)^i}{i!}}{\alpha^{N+1}} = \frac{(N+1)! - \Gamma(N+1, t\alpha)}{\alpha^{N+1}(N+1)!}.$$

Here $\Gamma(n,x) := e^{-x} \cdot (n-1)! \cdot \sum_{k=0}^{n-1} x^k/k!$ is the incomplete gamma function. Note that $\Gamma_N^\alpha(t) \to \wp_{N+1}(t)$ as $\alpha \to 0$, so the $\alpha = 0$ case of (15) is the nice martingale

$$M_t \quad = \quad \sum_{k=0}^{N} \mathcal{R}_0^k \wp_0(X_t) \frac{t^{N-k}}{(N-k)!}. \tag{16}$$

We demonstrate an application of (16) to the evaluation of expected hitting times. To this end let $\tau = \inf\{t > 0 | X_t = a\}$ be the first hitting time of some point $a \in E$. If optional stopping can be applied and the functions $\mathcal{R}_\alpha^k \wp_0$ exist we can recursively evaluate the higher moments of τ. For $N = 1$ we conclude from (16) that $M_t = t + \mathcal{R}_\alpha^1 \wp_0(X_t)$ is a martingale. Letting E_x denote the expectation w.r.t. the probability measure P_x concentrated on $\{X_0 = x\}$ we obtain the simple formula

$$E_x \tau \quad = \quad \mathcal{R}_\alpha^1 \wp_0(x) - \mathcal{R}_\alpha^1 \wp_0(a),$$

$\mathcal{R}_\alpha^1 \wp_0$ being some solution $f \in \mathscr{D}(\mathcal{A})$ of $\mathcal{A} f(x) \equiv 1$. Note that applicability of optional stopping has to be justified, for example by showing that τ is finite or that M_t is actually uniformly integrable.

For $N = 2$ equation (16) yields $M_t = t^2/2 + t\mathcal{R}_\alpha^1 \wp_0(X_t) + \mathcal{R}_\alpha^2 \wp_0(X_t)$, thus

$$E_x \tau^2 \quad = \quad 2\Big(\mathcal{R}_\alpha^2 \wp_0(x) - \mathcal{R}_\alpha^2 \wp_0(a) - \big(\mathcal{R}_\alpha^1 \wp_0(x) - \mathcal{R}_\alpha^1 \wp_0(a)\big) \cdot \mathcal{R}_\alpha^1 \wp_0(a)\Big).$$

For general N we obtain the recursion formula

$$E_x \tau^N = N! \left(\mathcal{R}_\alpha^N \wp_0(x) - \mathcal{R}_\alpha^N \wp_0(a) - \sum_{k=1}^{N-1} \mathcal{R}_\alpha^k \wp_0(a) \frac{E_x \tau^{N-k}}{(N-k)!} \right)$$

for the higher moments of τ. Furthermore a closed formula can be obtained from this recursion. Alas, the expression is not easy to evaluate and involves terms from partition theory:

$$E_x \tau^N = N! \sum_{k=0}^{N} \mathcal{R}_\alpha^{N-k} \wp_0(x) \sum_C \prod_{A \in C} (-\mathcal{R}_\alpha^{|A|} \wp_0(a)), \tag{17}$$

where C runs through all partitions of a set with k elements and $|A|$ denotes the number of elements in A.

Returning to the martingale (14) and letting $f(x) = \wp(x) \equiv 0$ we obtain the martingale

$$M_t \quad = \quad e^{-\alpha Y_t} \sum_{k=0}^{N} \mathcal{R}_\alpha^{k+1} \wp(X_t) \frac{Y_t^{N-k}}{(N-k)!}. \tag{18}$$

The $N = 0$ case is well-known. Remember that $\mathcal{R}_\alpha^1 \wp(x)$ is a solution of $\mathcal{A}f(x) = \xi(x)\alpha f(x)$. Equation (18) then states that

$$M_t \quad = \quad e^{-\alpha Y_t} \mathcal{R}_\alpha^1 \wp(X_t)$$

is a martingale. Applying optional stopping leads to a nice formula for the Laplace transform of the first hitting time, namely

$$E_x e^{-\alpha\tau} = \frac{\mathcal{R}_\alpha^1 \wp(x)}{\mathcal{R}_\alpha^1 \wp(a)}.$$

This method has been utilized e.g. in Kella and Stadje [24], see also Boxma et al. [9], and Pitman and Yor [40].

3.4 Lévy processes, exponential functionals

For Lévy processes in dimension one with no negative jumps and generating triplet (A, γ, ν) the infinitesimal generator is given by

$$\mathcal{A}f(x) \quad = \quad \frac{A}{2} \frac{d^2 f}{dx^2}(x) + \gamma f'(x) + \int\limits_{(0,\infty)} \left(f(x+y) - f(x) - 1_{\{|y| \le 1\}} y f'(x) \right) d\nu(y)$$

(see (31.11) of Sato [44], also Bertoin [7], we use the Sato-notation). By inserting the functions $f_u(x) = e^{-ux}$, $u \in \mathbb{R}$ it is easily seen that $\mathcal{A}f_u(x) = \phi(u) \cdot f_u(x)$, in other words each f_u is an eigenfunction of $\mathcal{A}$, the eigenvalue being just the Laplace exponent

$$\phi(u) = A\frac{u^2}{2} - \gamma u + \int_{(0,\infty)} \left(e^{-uy} - 1 + 1_{\{|y| \le 1\}} yu \right) \, d\nu(y).$$

Utilizing formula (10) and letting $\zeta \equiv 1$ we discover the celebrated Wald martingale given by

$$M_t = e^{-uX_t - \phi(u)t}$$

(see e.g. Asmussen [3] or Perry and Stadje [38] for applications). It follows immediately that $E_x e^{-uX_t} = exp\,(-t\phi(u))$, which is normally used as a definition for the Laplace exponent.

If we let $n = 0$ and $\alpha = u$ in equation (9) and if we assume that $\xi(x) \neq 0$ then it follows from

$$\mathcal{R}_\alpha^{-1} f_u(x) = u f_u(x) - \frac{\phi(u) f_u(x)}{\xi(x)}$$

that our martingale is given by

$$\begin{aligned} M_t &= e^{-uY_t} f_u(X_t) + \int_0^t \xi(X_s) e^{-uY_s} f_u(X_s) \Big(u - \frac{\phi(u)}{\xi(X_s)} \Big) \, ds \\ &= e^{-u(Y_t + X_t)} + u \int_0^t e^{-u(Y_s + X_s)} \, dY_s - \phi(u) \int_0^t e^{-u(Y_s + X_s)} \, ds. \end{aligned}$$

This is a version of the Kella-Whitt martingale; it is extensively used in the literature (see e.g. Kella and Whitt [25], Asmussen and Kella [4], Adan et al. [1]).

As we mentioned in the introduction, the so called exponential functional of a Lévy process given by $Y_t = \int_0^t e^{-uX_s} \, ds$ has been studied a lot. Obviously it corresponds to our $\xi(x) = f_u(x) = e^{-ux}$ case. Since $\mathcal{A} f_u(x) = \phi(u) \cdot f_u(x)$ one can show by induction that

$$\mathcal{R}_\alpha^k f_u(x) = \alpha^{-k} f_u(x) + k \alpha^{-(k+1)} \phi(u),$$

for all $k \in \mathbb{N}$ (even for $k \in \mathbb{Z}$). Equation (11) with $n = 0$, $n = -m$, $j \geq 0$ then yields the martingale

$$M_t = e^{-\alpha Y_t} \sum_{k=0}^{n} \Big(\alpha^k e^{-uX_t} - k \alpha^{k-1} \phi(u) \Big) \frac{Y_t^{j+k}}{(j+k)!}$$

$$- \int_0^t e^{-\alpha Y_s - uX_s} \Big(e^{-uX_s} \frac{1_{\{j \geq 1\}} Y_s^{j-1}}{(j-1)!} - \big(\alpha e^{-uX_s} - (n+1)\phi(u) \big) \frac{\alpha^n Y_s^{j+n}}{(j+n)!} \Big) \, ds.$$

This may be an interesting formula on its own, however, we let $\alpha = n = 0$ and obtain the martingale

$$j! M_t \quad = \quad e^{-uX_t} Y_t^j - \frac{1}{j+1} \phi(u) Y_t^{j+1} - j \int_0^t e^{-2uX_s} Y_s^{j-1} \, ds.$$

It follows from the martingale property that

$$E_x \left(e^{-uX_t} Y_t^j - \frac{1}{j+1} \phi(u) Y_t^{j+1} - j \int_0^t e^{-2uX_s} Y_s^{j-1} \, ds \right) = E_x(Y_0^j) = 1_{\{j=0\}}.$$

Consequently

$$E_x Y_t^{j+1} = \frac{1}{\phi(u)} \left(1_{\{j=0\}} - (j+1) E_x \big(e^{-uX_t} Y_t^j - j \int_0^t e^{-2uX_s} Y_s^{j-1} \, ds \big) \right). \quad (19)$$

In particular, for $j = 0$ we deduce that the mean of Y_t is given by

$$E_x Y_t \quad = \quad \frac{1 - e^{-t\phi(u)}}{\phi(u)}. \tag{20}$$

This equation can be validated for $u = 1$ by comparing it with the $j = 1$ case of the Laplace transform formula

$$\int_0^\infty e^{-qx} E_x Y_t^j \, dx = \frac{j!}{q\,(q + \phi(1)) \cdots (q + \phi(j))}$$

on page 194 of Bertoin and Yor [8]. We can also find a, even though less explicit, formula for the second moment. First note that the process $\tilde{Y}_t = \int_0^t e^{-2uX_s} \, ds$ is the $\xi(x)$-integral with respect to the doubled process $\tilde{X}_t = 2X_t$ which is easily seen to have the Laplace exponent $\tilde{\phi}(u) = \phi(2u)$. Thus

$$E_x \left(\int_0^t e^{-2uX_s} \, ds \right) = \frac{1 - e^{-t\phi(2u)}}{\phi(2u)}$$

according to equation (20). Consequently the $j = 1$ case of (19) is given by

$$E_x Y_t^2 \quad = \quad 2 \left(\frac{1 - e^{-t\phi(2u)}}{\phi(u)\phi(2u)} - \frac{E_x \left(e^{-uX_t} Y_t \right)}{\phi(u)} \right).$$

As $t \to \infty$ the known formula for the second moment of Y_∞ follows:

$$E_x Y_\infty^2 \quad = \quad \frac{2}{\phi(u)\phi(2u)}$$

(see e.g. (2.4) in Maulik and Zwart [33]).

4 Appendix

The following lemma is used to prove Theorem 2.1. It states that

$$\mathcal{A}_Z g(x, y) = \mathcal{A} g_y(x) + \xi(x) \cdot \frac{\partial}{\partial y} g_x(y) \tag{21}$$

holds in the special case where g is a product of two functions, one depending only on x, the other only on y.

Lemma 4.1. *Let $g(x, y) = f(x)h(y)$, $f \in \mathscr{D}(\mathcal{A})$ and $h \in \mathscr{M}(E)$, absolutely continuous and bounded on the range of Y_t. Then*

$$\mathcal{A}_Z g(x, y) = h(y)\mathcal{A} f(x) + f(x)\xi(x)h'(y),$$

i.e. (21) *holds.*

Proof. Since $W_t = f(X_t) - \int_0^t \mathcal{A}f(X_s)\,ds$ is a $\mathscr{C}^n(E)$ martingale it follows that $\int_0^t h(Y_s)\,dW_s$ is also a martingale (see [15], Problem 22 on page 92). We have

$$\int_0^t h(Y_s)\,dW_s = \int_0^t h(Y_s)\,df(X_s) - \int_0^t h(Y_s)\mathcal{A}f(X_s)\,ds$$

Using integration by parts we can write this as

$$h(Y_t)f(X_t) - \int_0^t f(X_s)\,dh(Y_s) - \int_0^t h(Y_s)\mathcal{A}f(X_s)\,ds$$

which is equal to

$$g(X_t, Y_t) - \int_0^t \big(f(X_s)\xi(X_s)h'(Y_s) + h(Y_s)\mathcal{A}f(X_s)\big)\,ds.$$

Thus $g \in \mathscr{D}(\mathcal{A}_Z)$ and $\mathcal{A}_Z g(x,y) = f(x)\xi(x)h'(y) + h(y)\mathcal{A}f(x)$. □

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