Global and local asymptotics for the busy period of the M/G/1 queue

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Abstract

We consider an M/G/1 queue with subexponential service times. We give a simple derivation of the global and local asymptotics for the busy period. This analysis relies on the explicit formula for the joint distribution for the number of customers and the length of the busy period of the M/G/1 queue.

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1 Introduction

Let $A_1, A_2, \dots$ and $B_1, B_2, \dots$ be independent sequences of independent and identically distributed random variables. We call $\{A_i\}$ inter-arrival times and $\{B_i\}$ service times. We assume throughout that $\rho := \mathbf{E}\{B_1\}/\mathbf{E}\{A_1\} < 1$, so that the system is stable. Denote also $\xi_i = B_i - A_i$ and $S_n = \sum_{i=1}^{n} \xi_i$. We shall denote by A, B and ξ random variables with the same distributions as A_1, B_1 and ξ_1, respectively. We use the usual notation: we denote by $M/G/1$ the system with exponential service times A_i. We are interested in the asymptotics for the busy period

$$\tau = B_1 + \dots + B_\nu$$

where $\nu = \inf\{n : S_n < 0\}$. The random variable ν is the number of customers which arrive during the busy cycle.
In [11], it was shown that if B has a regularly varying distribution then

$$\mathbf{P}\{\tau > t\} \sim \mathbf{E}\{\nu\}\mathbf{P}\{B > (1-\rho)t\}, \quad t \to \infty. \tag{1}$$

This result has been generalized in [12] to the case of the $GI/G/1$ queue and under the assumption that the tail $\mathbf{P}\{B > t\}$ is of intermediate regular variation. Later on, it has been shown in [3] and [10], that asymptotics (1) hold for the $GI/G/1$ model for another subclass of heavy-tailed distributions which includes the Weibull distributions with parameter $\alpha < 1/2$. The tails of the distributions considered in [3] and [10] are heavier than $e^{-\sqrt{t}}$. As is shown in [2], the latter condition is crucial for asymptotics (1) to hold.

It was shown in [7] that in the $M/G/1$ queue

$$\mathbf{P}(\tau > t) \sim C\frac{\mathbf{P}(X_t > 0)}{t} \tag{2}$$

with a constant C, where $X_t = \sum_{i=0}^{N(t)} B_i - t$, and N is a Poisson process with intensity $\lambda = 1/\mathbf{E}A$. It was also shown in [7] that in the case when the tail of B is heavier than $e^{-\sqrt{t}}$ (and under some further conditions) the asymptotic equivalence (2) reduces to (1).
There are two aims of this note. The first one is to give a simple proof of (1) for the M/G/1 queue for some classes of distributions with tails lighter than $e^{-\sqrt{t}}$. This proof is based on the explicit formula for the joint distribution of τ and ν (see e.g. [6, (4.63)]):

$$\mathbf{P}(\nu = n, \tau > t) = \int_t^\infty \frac{(\lambda u)^{n-1}}{n!} e^{-\lambda u}\mathbf{P}(B_1 + \cdots + B_n \in du) \tag{3}$$

Conditions imposed on the right tail $\mathbf{P}\{B > t\}$ are close to minimal. These conditions include as particular cases all known conditions on the right tail that lead to (1).
The second (and probably more important) aim of this note consists in providing new results on local asymptotics for τ as well as for ν (the number of customers served during the busy period). These results are similar to (2):

$$\mathbf{P}(\tau \in (t, t+T]) \sim C\frac{\mathbf{P}(X_t \in (0,T])}{t}$$

with a constant C, if $0 < T < \infty$ is fixed.
The paper is organised as follows. In Section 2 we present a simple proof of (1) for the M/G/1. In Section 3 we give results on local asymptotics for τ and ν.

2 Global asymptotics for the busy period

Put $\overline{F}(x) = \mathbf{P}\{B > x\}$. We consider 2 subclasses of subexponential distribution.

Definition 2.1. *We say that distribution function F is intermediate regularly varying ($F \in \mathcal{IRV}$) if*

$$\limsup_{\kappa\downarrow 1}\limsup_{x\to\infty}\frac{\overline{F}(x)}{\overline{F}(\kappa x)} = 1.$$

If particular a regularly varying function F belongs to this class. Let $h_n = \sqrt{n}$. Let

$$\varepsilon(n) \quad\equiv\quad \sup_{x\ge 2h_n}\frac{\mathbf{P}\{\xi_1 > h_n, \xi_2 > h_n, S_2 > x\}}{\overline{F}(x)}.$$

We will use following conditions

- **(A)** $\mathbf{E}B < \infty$ and $F \in \mathcal{IRV}$
- **(B)** $\mathbf{E}B^2 < \infty$, $\overline{F}(x - \sqrt{x}) \sim \overline{F}(x)$ and

$$\varepsilon(n) = o(1/n).$$

The following proposition follows from the results of [8].

Proposition 2.1. *Let either condition (A) or (B) hold. Then*

$$\mathbf{P}\{S_n > 0\} \sim n\overline{F}(n|\mathbf{E}\xi|).$$

Note that square-root insensitivity condition $\overline{F}(x) \sim \overline{F}(x - \sqrt{x})$ implies that

$$\frac{-\ln \overline{F}(x)}{\sqrt{x}} \to 0.$$

A simple sufficient condition for (B) to hold is the following proposition (see [8] for the proof).

Proposition 2.2. *Let $\overline{F}(x) = x^{-2} e^{-g(x)}$ If $g(x)/x^{\delta}$ is eventually non-increasing for some $\delta < 1$, then $\varepsilon(n) = o(1/n)$.*

In particular, this Proposition implies that lognormal distributions and Weibull distributions with parameter $\alpha < 1/2$ satisfy (B).
We now state the main result of this Section.

Theorem 2.1. *Let either condition (A) or (B) hold. Then,*

$$\mathbf{P}\{\tau > t\} \sim \mathbf{E}\{\nu\}\mathbf{P}\{B > (1-\rho)t\}.$$

We also need the following result.

Proposition 2.3. *Let either condition (A) or (B) hold. Then,*

$$\mathbf{P}\{\nu > t\} \sim \mathbf{E}\{\nu\}\mathbf{P}\{B > |\mathbf{E}\{\xi\}|t\}, \quad t \to \infty. \tag{4}$$

Proof of Proposition 2.3. It follows from the result of [5] (proof can be found, for example, in [9]) that

$$\mathbf{P}\{\nu > n\} \sim \mathbf{E}\{\nu\}\frac{\mathbf{P}\{S_n > 0\}}{n} \tag{5}$$

if the sequence $a_n = \dfrac{\mathbf{P}\{S_n > 0\}}{n}$ is subexponential, i.e., if $\sum_{k=0}^{n} a_k a_{n-k} \sim 2a_n \sum_{k=0}^{\infty} a_k$ as $n \to \infty$. It is proved in [8] that under the assumptions of the Proposition,

$$\frac{\mathbf{P}\{S_n > 0\}}{n} \sim \overline{F}(|\mathbf{E}\xi|n), \quad n \to \infty.$$

It is also proved there that the sequence $\overline{F}(|\mathbf{E}\xi|n)$ is subexponential.

□

Proof of Theorem 2.1. For condition (B), the estimate from below is a corollary of the CLT and condition $\overline{F}(x - \sqrt{x}) \sim \overline{F}(x)$. The proof may be proved in [12]. For condition (A), the estimate follows similarly from the Law of Large Numbers.
Therefore, we will concentrate on the estimate from above. We will consider condition (B). The proof for condition (A) is similar. Note, that it follows from Proposition 2.3 and condition (B) that

$$\mathbf{P}(\nu > n) \sim \mathbf{P}(\nu > n - \sqrt{n}).$$

Therefore, there exists a function $l(n) \uparrow \infty$ such that

$$\mathbf{P}(\nu > n) \sim \mathbf{P}(\nu > n - \sqrt{n}l(n)).$$

By the total probability formula,

$$\begin{aligned}
\mathbf{P}(\tau > x) &= P_1 + P_2 + P_3 \\
&= \mathbf{P}(\tau > x, \nu \le (1-\varepsilon)\frac{x}{\mathbf{E}A}) \\
&+ \mathbf{P}(\tau > x, (1-\varepsilon)\frac{x}{\mathbf{E}A} < \nu \le \frac{x}{\mathbf{E}A} - \sqrt{x}l(x)) \\
&+ \mathbf{P}(\tau > x, \nu > \frac{x}{\mathbf{E}A} - \sqrt{x}l(x)).
\end{aligned}$$

First,

$$\begin{aligned} P_1 &= \mathbf{P}(B_1+\cdots+B_\nu > x, \nu \le (1-\varepsilon)\frac{x}{\mathbf{E}A}) \\ &\le \mathbf{P}(B_1+\cdots+B_{(1-\varepsilon)\frac{x}{\mathbf{E}A}} > x, A_1+\cdots+A_{(1-\varepsilon)\frac{x}{\mathbf{E}A}} > x) \\ &\le \mathbf{P}(B_1+\cdots+B_{\frac{x}{\mathbf{E}A}} > x)\mathbf{P}(A_1+\cdots+A_{(1-\varepsilon)\frac{x}{\mathbf{E}A}} > x) \\ &= o(1/x)\mathbf{P}(B_1+\cdots+B_{\frac{x}{\mathbf{E}A}} > x), \end{aligned}$$

where that latter equivalence follows from the fact that, when $\mathbf{E}A^2 < \infty$,

$$\mathbf{P}(\sum_{i=1}^{n}(A_i - \mathbf{E}A_i) > \varepsilon n) = o(1/n).$$

Then,

$$P_2 = \sum_{n=(1-\varepsilon)\frac{x}{\mathbf{E}A}}^{\frac{x}{\mathbf{E}A}-\sqrt{x}l(x)} \int_x^\infty \frac{\mathbf{P}(N(u)=n-1)}{n}\mathbf{P}(B_1+\cdots+B_n \in du).$$

For $u \ge n-1$, probability $\mathbf{P}(N(u)=n-1)$ is non-increasing in u. Therefore, $\mathbf{P}(N(u) = n-1) \le \mathbf{P}(N(x)=n-1)$, and this fact implies that

$$\begin{aligned} P_2 &\le \sum_{n=(1-\varepsilon)\frac{x}{\mathbf{E}A}}^{\frac{x}{\mathbf{E}A}-\sqrt{x}l(x)} \frac{\mathbf{P}(N(x)=n-1)}{n}\mathbf{P}(B_1+\cdots+B_n > x) \\ &\le \frac{\mathbf{P}(B_1+\cdots+B_{\frac{x}{\mathbf{E}A}} > x)}{(1-\varepsilon)\frac{x}{\mathbf{E}A}}\mathbf{P}\left((1-\varepsilon)\frac{x}{\mathbf{E}A} \le N(x) \le \frac{x}{\mathbf{E}A} - x^{1/\kappa}l(x)\right) \\ &\le \frac{\mathbf{P}(B_1+\cdots+B_{\frac{x}{\mathbf{E}A}} > x)}{(1-\varepsilon)\frac{x}{\mathbf{E}A}}\mathbf{P}\left((1-\varepsilon)\frac{x}{\mathbf{E}A} \le N(x) \le \frac{x}{\mathbf{E}A} - x^{1/2}l(x)\right) \end{aligned}$$

By the Central Limit Theorem for renewal processes,

$$\mathbf{P}\left((1-\varepsilon)\frac{x}{\mathbf{E}A} \le N(x) \le \frac{x}{\mathbf{E}A} - x^{1/2}l(x)\right) = o(1),$$

and, therefore,

$$P_2 = o(1/x)\mathbf{P}(B_1+\cdots+B_{\frac{x}{\mathbf{E}A}} > x).$$

As a result, we have

$$P_1 + P_2 = o(1/x)\mathbf{P}(B_1+\cdots+B_{\frac{x}{\mathbf{E}A}} > x) = o(\mathbf{P}(B > x - \rho x)).$$

Third term,

$$P_3 \le (1+o(1))\mathbf{P}(\nu > \frac{x}{\mathbf{E}A}) = (1+o(1))\mathbf{E}\{\nu\}\mathbf{P}(B > x-\rho x),$$

where the latter equivalence follows from (4).
Finally, we have

$$\mathbf{P}(\tau > x) \le (1+o(1))\mathbf{E}\{\nu\}\mathbf{P}(B > x - \rho x).$$

□

3 Local asymptotics for the busy period

It appears that explicit formula (3) is extremely convenient for the derivation of the local asymptotics. Recall that $S_n = \sum_{i=1}^{n}(B_i - A_i)$ and $X_t = \sum_{i=1}^{N(t)} B_i - t$. We will start with the following simple result.

Theorem 3.1. *Assume that random variable B is absolutely continuous. Then,*

$$\mathbf{P}\{\nu = n\} = \frac{f_{S_n}(0)}{\lambda n}. \tag{6}$$

and

$$f_\tau(t) = \frac{f_{X_t}(0)}{\lambda t}. \tag{7}$$

Here $f_\xi(x)$ denotes the density of random variable ξ at point x. Note that this Theorem holds not only for subexponential distributions, but for all distributions. It reduces the original problem to the simpler and well-studied problem of large deviations probabilities for random walks and Levy processes.
The next result holds for non-absolutely continuous random variables.

Theorem 3.2. *Let $T > 0$ be a fixed number and $\Delta = (0, T]$. Assume that $\mathbf{P}\{X_t \in \Delta\}e^{\delta t} \to \infty$ for any $\delta > 0$. Then,*

$$\mathbf{P}\{\tau \in t + \Delta\} \sim \frac{\mathbf{P}\{X_t \in \Delta\}}{\lambda t} \sim \frac{\mathbf{P}\{X_{[t]} \in \Delta\}}{\lambda [t]}, \quad as \quad t \to \infty,$$

where $[t]$ denotes the integer part of t.

We are going to give explicit asymptotics following from Theorem 3.1 and Theorem 3.2. First, we will give counterparts of condition (A) and (B). Fix $T > 0$ and put $\Delta = (0, T]$.

Definition 3.1. *We say that function $F(x + \Delta)$ is intermediate regularly varying if*

$$\limsup_{\kappa \downarrow 1} \limsup_{x \to \infty} \sup_{x \le y \le \kappa x} \left| \frac{F(y + \Delta)}{F(x + \Delta)} - 1 \right| = 0. \tag{8}$$

In particular, (8) holds when $F(x + \Delta)$ is regularly varying at infinity. Let $h_n = \sqrt{n}$. Let

$$\varepsilon_\Delta(n) \quad \equiv \quad \sup_{x \ge 2h_n} \frac{\mathbf{P}\{\xi_1 > h_n, \xi_2 > h_n, S_2 \in x + \Delta\}}{F(x + \Delta)}.$$

We will use the following conditions.

($\mathbf{A}_\Delta$) $\mathbf{E}B < \infty$ and $F(x + \Delta)$ is intermediate regularly varying

($\mathbf{B}_\Delta$) $\mathbf{E}B^2 < \infty$,

$$\sup_{y \le \sqrt{x}} \left| \frac{F(x - y + \Delta)}{F(x + \Delta)} - 1 \right| \to 0$$

and

$$\varepsilon_\Delta(n) = o(1/n).$$

A simple sufficient condition for (B) to hold can be stated analogously to Proposition 2.2. Then, one can see that $F(x+\Delta) \sim e^{-x^\beta}, \beta < 1/2$ satisfy condition (B) as well as $F(x+\Delta) \sim e^{-\ln^\beta x}, \beta > 1$. The following proposition follows from the results of [8].

Proposition 3.1. *Let either condition* (A_Δ) *or* (B_Δ) *hold. Then*

$$\mathbf{P}\{S_n \in \Delta\} \quad \sim \quad n\mathbf{P}\{\xi \in n|\mathbf{E}\xi| + \Delta\}$$

and hence (since $N(t)$ *is a Poisson process),*

$$\mathbf{P}\{X_{[t]} \in \Delta\} \sim [t]\mathbf{P}\{X_1 \in [t]|\mathbf{E}X_1| + \Delta\}.$$

A similar result holds for densities. Using Proposition 3.1, Theorem 3.1 and Theorem 3.2 immediately allows us to obtain local asymptotics:

Corollary 3.1. *Assume that* B *is absolutely continuous. Let one of conditions* (A_Δ) *or* (B_Δ) *hold for all* $T > 0$. *Then,*

$$\mathbf{P}\{\nu = n\} \sim (\lambda)^{-1} f(n\mathbf{E}(B - A))$$

and

$$f_\tau(t) \sim f(t - \rho t).$$

Corollary 3.2. *Let either condition* (A_Δ) *or* (B_Δ) *hold for some* $T > 0$. *Then,*

$$\mathbf{P}\{\tau \in t + \Delta\} \sim F(t - \rho t + \Delta).$$

Proof of Theorem 3.1. Put $a = \lambda^{-1} = \mathbf{E}A, b = \mathbf{E}B$. Then,

$$\mathbf{P}\{\nu = n\} = \int_0^\infty \frac{(\lambda u)^{n-1}}{n!} e^{-\lambda u} f_B^{*n}(u)du = \lambda^{-1} \int_0^\infty \frac{f_A^{*n}(u)}{n} f_B^{*n}(u)du = \lambda^{-1} \frac{f_{B-A}^{*n}(0)}{n}.$$

This implies the first statement of the Theorem.
Now we proceed to the local asymptotics for τ,

$$\begin{aligned} f_\tau(t) &= \sum_{n=1}^\infty \frac{(\lambda t)^{n-1}}{n!} e^{-\lambda t} f_B^{*n}(t) = \frac{1}{\lambda t} \sum_{n=1}^\infty \mathbf{P}\{N(t) = n\} f_B^{*n}(t) \\ &= \frac{1}{\lambda t} \sum_{n=1}^\infty \mathbf{P}\{N(t) = n\} \frac{\mathbf{P}\{B_1 + \cdots + B_n \in t + du\}}{du} \\ &= \frac{1}{\lambda t} \frac{\mathbf{P}\{\sum_{i=1}^{N(t)} B_i \in t + du\}}{du} = \frac{f_{X_t}(0)}{\lambda t}. \end{aligned}$$

□

Proof of Theorem 3.2. We will prove the asymptotic equivalence $\mathbf{P}\{\tau \in t + \Delta\} \sim \mathbf{P}\{X_t \in \Delta\}/t$. The proof of the equivalence $\mathbf{P}\{\tau \in t + \Delta\} \sim \mathbf{P}\{X_{[t]} \in \Delta\}/[t]$ may be given following the same lines. Fix any $\varepsilon > 0$. According to formula (3),

$$\begin{aligned} \mathbf{P}(\tau \in t + \Delta) &= \sum_{n=1}^\infty \int_t^{t+T} \frac{(\lambda u)^{n-1}}{n!} e^{-\lambda u} \mathbf{P}(B_1 + \ldots + B_n \in du) \\ &= \sum_{n=1}^\infty \int_t^{t+T} \frac{\mathbf{P}(N(u) = n)}{\lambda u} \mathbf{P}(B_1 + \ldots + B_n \in du) \\ &= \sum_{n=1}^{\lambda t - \varepsilon t} + \sum_{\lambda t - \varepsilon t}^{\lambda t + \varepsilon t} + \sum_{\lambda t + \varepsilon t}^{\infty} (\ldots) \equiv S_1 + S_2 + S_3. \end{aligned}$$

For the first sum we have:

$$\begin{aligned} S_1 &\leq \frac{1}{\lambda t}\sum_{n=1}^{\lambda t-\varepsilon t}\int_t^{t+T}\mathbf{P}(N(u)=n)\mathbf{P}(B_1+\ldots+B_n\in du) \\ &\leq \frac{1}{\lambda t}\mathbf{P}\left(N(t)\leq \lambda t-\varepsilon t\right)\sum_{n=1}^{\lambda t-\varepsilon t}\mathbf{P}\{B_1+\cdots+B_n\in t+\Delta\} = o\left(e^{-\delta t}\right) \end{aligned}$$

with some $\delta > 0$, since $N(x)$ is Poisson distributed with parameter λt. Indeed, it follows immediately from the exponential Chebyshev inequality that for some $\delta > 0$,

$$\mathbf{P}\{|N(t)-\lambda t|\geq \varepsilon t\} = o(e^{-\delta t}).$$

It can be shown in a similar way that $S_3 = o(e^{-\delta t})$. Let us now investigate the asymptotic behaviour of the remaining sum:

$$S_2 = (1+o(1))\frac{1}{\lambda t}\sum_{\lambda t-\varepsilon t}^{\lambda t+\varepsilon t}\frac{e^{-\lambda t}(\lambda t)^n}{n!}\int_t^{t+T}e^{-\lambda(u-t)}\left(1+\frac{u-t}{t}\right)^n\mathbf{P}(B_1+\ldots+B_n\in du).$$

We now consider

$$\begin{aligned} e^{-\lambda(u-t)}\left(1+\frac{u-t}{t}\right)^n &= e^{-\lambda(u-t)+n\log\left(1+\frac{u-t}{t}\right)} \leq \exp\left\{-\lambda(u-t)+n\frac{u-t}{t}\right\} \\ &\leq \exp\left\{-\lambda(u-t)+(\lambda+\varepsilon)(u-t)\right\} = \exp\left\{\varepsilon(u-t)\right\} \leq \exp\{\varepsilon T\}. \end{aligned}$$

We used here the facts that for the sum under consideration $n \leq \lambda(1+\varepsilon)t$, $u \leq t+T$ and the inequality $\log(1+t) \leq t$. In a similar way, with the help of the inequality $\log(1+t) \geq t/2, t \leq 1$ one can prove that $e^{-\lambda(u-t)}\left(1+\frac{u-t}{t}\right)^n \geq \exp\{-\varepsilon T/2\}$. Therefore,

$$\begin{aligned} S_2 &\leq (1+o(1))\frac{e^{\varepsilon T}}{\lambda t}\mathbf{P}(X_t\in\Delta, \lambda t-\varepsilon t\leq N(t)\leq \lambda t+\varepsilon t) \\ &\geq (1+o(1))\frac{e^{-\varepsilon T/2}}{\lambda t}\mathbf{P}(X_t\in\Delta, \lambda t-\varepsilon t\leq N(t)\leq \lambda t+\varepsilon t), \end{aligned}$$

where we denoted $X_t = \sum_{i=0}^{N(t)} B_i - t$. Note now that

$$\mathbf{P}(X_t\in\Delta, |N(t)-\lambda t|\geq \varepsilon t) \leq \mathbf{P}\left(|N(t)-\lambda t|\geq \varepsilon t\right) = o\left(e^{-\delta t}\right).$$

Hence, the sum under consideration is asymptotically equivalent to

$$\begin{aligned} \mathbf{P}\{\tau\in t+\Delta\} &\leq (1+o(1))\frac{e^{\varepsilon T}}{\lambda t}\mathbf{P}(X_t\in\Delta)+o(e^{-\delta t}) \\ &\geq (1+o(1))\frac{e^{-\varepsilon T/2}}{\lambda t}\mathbf{P}(X_t\in\Delta)+o(e^{-\delta t}). \end{aligned}$$

Using condition $e^{\delta t}\mathbf{P}\{X_t \in \Delta\} \to \infty, \delta > 0$ and then letting $\varepsilon \to 0$ we arrive at the conclusion of the Theorem.

□

Proofs of Corollary 3.1 and Corollary 3.2 are absolutely analogous. Therefore, we only give the proof of Corollary 3.2.

Proof of Corollary 3.2. It follows from Proposition 3.1 and Theorem 3.2 that

$$\mathbf{P}\{\tau\in t+\Delta\} \sim \lambda^{-1}\mathbf{P}\{X_1\in[t]|\mathbf{E}X_1|+\Delta\} = \lambda^{-1}\mathbf{P}\{X_1\in[t]-\rho[t]+\Delta\},$$

since $\mathbf{E}X_1 = EN(1)\mathbf{E}B - 1 = \rho - 1$. Now we should note that

$$\mathbf{P}\{X_1 \in [t] - \rho[t] + \Delta\} \sim \mathbf{P}\{\sum_{i=1}^{N(1)} B_i \in [t] - \rho[t] + \Delta\}$$
$$\sim \mathbf{E}N(1)\mathbf{P}(B \in [t] - \rho[t] + \Delta) \sim \mathbf{E}N(1)\mathbf{P}(B \in t - \rho t + \Delta),$$

by the theorem on local behaviour for randomly stopped sums from [1]. Indeed, either condition (A) or (B) implies that $F \in \mathcal{S}_\Delta$, see [1] for this Theorem and definitions.

□

Note that the Corollaries hold only for some classes of distributions such that $F(x + \Delta) \gg e^{-\sqrt{x}}$. It is also possible to obtain asymptotics for the distributions lighter than Weibull(1/2), but we should use results for large deviation of random walks other than Proposition 3.1. There are a lot of known results in that direction, we refer to the recent paper of Borovkov and Mogulskii [4] for some new results and review. We would also like to mention that the paper [4] contains a proof of the asymptotic equivalence $\mathbf{P}\{S_n \in \Delta\} \sim n\mathbf{P}\{\xi \in n|\mathbf{E}\xi| + \Delta\}$ for distributions such that $\mathbf{P}\{\xi > x\} = e^{-x^\beta l(x)}$ with $\beta < 1/2$, under some conditions on the function $l(x)$.

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