A Random Multiple Access Protocol with Spatial Interactions

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Abstract: We analyse an ALOHA type access protocol where users have local interactions. We establish that the fluid model of the system workload satisfies a differential equation. We exhibit a sufficient condition on the stability of this differential equation and deduce a sufficient condition for the stability of the protocol. We discuss the necessary condition.
Keywords: ALOHA protocol, Spatial interactions, Stability of processes, Fluid limits.

1 Introduction and Stability Result

1.1 A Spatial ALOHA

We consider a random spatial service system governed by an ALOHA-type algorithm. More precisely, time is slotted, during each time slot n a random number ξ_n of users arrive in the system, and at each slot every user in the system requires service (transmission) with a certain probability (transmission probability) independently of all others. Usually a sequence $\{\xi_n\}$ is assumed to be i.i.d.

The ALOHA multi-access algorithm was first proposed by Abramson [1]. The slotted scheme was introduced by Roberts [15]. We consider the latter setting. In the conventional slotted ALOHA model, there is one server. If, at the beginning of a time slot n, a total number W_n of users in the system is known, each of them asks for service (transmission) with probability $\frac{1}{W_n}$ independently of the other users. If two or more users require transmissions simultaneously, then transmissions collide, the users stay in the system and try to transmit later. All service times are equal to 1, and the server is always free at the beginning of any time slot. It is easy to show that for this system the maximum throughput is equal to e^{-1}. Further, the Markov chain $\{W_n\}$ is positive recurrent if $\mathbf{E}\xi_1 < e^{-1}$ and transient if $\mathbf{E}\xi_1 > e^{-1}$.

When information on the numbers W_n of users is unavailable, various decentralised adaptive algorithms have been introduced and studied. Algorithms of this type use information on what occurred in the previous time slot: either conflict or successful service or an empty session. More precisely, let B_n be the number of users trying to transmit at time n. In decentralised algorithms, there are only values of $\min\{B_n, 2\}$ available at time $n+1$. For such a system, under independence and exponential moment assumptions for ξ_n, Hajek [11] proved that $\mathbf{E}\xi_n < e^{-1} \approx 0.37$ is necessary and sufficient for the existence of a stable algorithm. Mikhailov [14] generalised this result by weakening the exponential moment assumption to the requirement that only the second moment needs to exist, while Foss [9] generalised it further by dropping this as well as the independence assumption. We also refer to Ephremides and Hajek [6] for a survey which includes, in particular, results in this direction.

These analysis ignore the network's spatial diversity and, in particular, the fact that there may be only *partial* interaction between users, depending on the distance between them. A development of random access protocols for wireless networks has created a new need of theoretical results on the stability and performance of such protocols when spatial interaction between the sources is taken into account.

In this work, a new model is presented, which captures the main feature of wireless networks: the spatial reuse of a common communication channel. This feature brings a new conceptual difficulty into the analysis of the stability of random access protocols. Here we consider only spatial centralised schemes, the study of decentralised ones is a subject of our future research.

The remainder of this paper is organised as follows. The end of this introduction is devoted to the description of the model and the statement of our main stability result. In Section 2 we prove that fluid

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limits of the workload in the system satisfy a differential equation. Section 3 is devoted to the behavior of the fluid limits on the boundary of the positive orthant. In Section 4 we present the proof of our main stability result and formulate one of its possible generalisations. In Section 5 we present a result on rates of convergence to the stable regime of the system. Sections 6 and 7 contain some interesting results on behaviour of the solutions to the differential equation satisfied by fluid limits. Finally, in Section 8 we conclude with some extensions of our model, which are in a certain sense more applicable to real systems. These extensions include, in particular, the system that captures the fact that various changes in environment conditions may result in changes in the radius and/or direction of interference between the message transmissions.

1.2 Model Description

Let $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ be a non-directed graph with a finite set of vertices, say $\mathcal{V} = \{1, ..., K\}$. We suppose that G is connected. For the graph $\mathcal{G}$ we use the standard notion of the graph distance. Denote by D the maximum graph distance in $\mathcal{G}$ (the diameter of $\mathcal{G}$). For $i \in \mathcal{V}$, let $V_i = \{i\} \cup \{j \in \mathcal{V} : \text{ such that } (i,j) \in \mathcal{E}\}$, that is the set of vertices at a maximum distance of 1 from the point i in the graph.

We introduce the following service system with spatial (neighborhood) interactions associated with the graph $\mathcal{G}$. We assume that time is slotted, i.e., arrivals and services may occur only at times $n = 1, 2, \ldots$. Suppose that there are service stations at each point of $\mathcal{G}$. The arrival process is denoted by $A = (A(n))_{n\in\mathbb{N}}$, where $A(n) \in \mathbb{N}^K$ is the number of users arriving at time n at each vertex. For $t > s$, denote by $A(t, s) = \sum_{n=\lceil t\rceil}^{\lceil s\rceil-1} A(n)$ the number of users arriving between time t and $s-$. We suppose that $(A(n))$ is an i.i.d. sequence. We also suppose that $EA_i(n) = \lambda_i > 0$ for every $i = 1, \ldots, K$.

Let $W(n) \in \mathbb{R}_+^K$ be the workload at time n in the system, that is, $W_i(n)$ is the number of users at vertex i at time n. At time n, a user at vertex i requires service independently of the others with probability $1/\sum_{j\in V_i} W_j(n)$. This user receives service if he is the only user requiring service in V_i at time n. We suppose that all service times are equal to 1 and that any user leaves the system immediately upon service completion. Let $N_i(n)$ be the number of users requiring service at time n at vertex i. $N_i(n)$ is a binomial random variable with parameters $\left(W_i(n), \dfrac{1}{\sum_{j\in V_i} W_j(n)}\right)$ and $(N_i(n)), 1 \le i \le K$ are independent variables conditioned on $W(n)$. W is clearly an irreducible Markov chain on $\mathbb{N}^K$. We have the following relation on the values of the workload at subsequent time instances:

$$W_i(n) = W_i(n-1) + A_i(n) - \mathbb{1}(N_i(n-1) = 1) \prod_{j\in V_i\setminus\{i\}} \mathbb{1}(N_j(n-1) = 0). \tag{1}$$

To explicitly show the dependence of $W(n)$ on the initial condition $W(0) = x$, we may sometimes write $W^x(n)$.

If $x_i > 0$, the i-th component of drift vector is given by the following expression:

$$E\left[\, W_i(1) - W_i(0)|W(0) = x \,\right] = \lambda_i - \frac{x_i}{\sum_{k\in V_i} x_k}\left(1 - \frac{1}{\sum_{k\in V_i} x_k}\right)^{x_i-1} \prod_{j\in V_i\setminus\{i\}}\left(1 - \frac{1}{\sum_{k\in V_j} x_k}\right)^{x_j}, \tag{2}$$

and if $x_i = 0$, then $E\left[\, W_i(1) - W_i(0)|W(0) = x \,\right] = \lambda_i$.

We re-write the expression for the drift vector in the following way:

$$E\left[\, W(1) - W(0)|W(0) = x \,\right] = \underline{\lambda} - G(x).$$

Here $\underline{\lambda}$ is the K-dimensional vector with it's i-th component equal to λ_i and G is a function from $\mathbb{R}^K$ to $\mathbb{R}^K$ defined by

$$G_i(x) = \begin{cases} \frac{x_i}{\sum_{k\in V_i} x_k}\left(1 - \frac{1}{\sum_{k\in V_i} x_k}\right)^{x_i-1} \prod_{j\in V_i\setminus\{i\}}\left(1 - \frac{1}{\sum_{k\in V_j} x_k}\right)^{x_j}, & \text{if } \ x_i > 0, \\ 0, \quad \text{if} \quad x_i = 0. \end{cases}$$

For $x \in \mathbb{R}^K$, we define $\phi_i(x) = \frac{x_i}{\sum_{j\in V_j} x_j}$. Let $\phi(x) = (\phi_1(x), ..., \phi_K(x))'$. Note that G_i is bounded by 1 and if $\sum_{k\in V_i} x_k > 0$ then

$$\lim_{t\to+\infty} G_i(tx) = \tilde{G}_i(x) = \phi_i(x) e^{-\sum_{j\in V_i} \phi_i(x)}.$$

In particular $\tilde{G}_i$ is homogeneous of order 0, i.e. $\tilde{G}_i(cx) = \tilde{G}_i(x)$ for any $c > 0$.

We now make some comments on the model. In this paper, we mostly consider the so-called *symmetric* case, i.e. when $\lambda_i = \lambda$ for all $i = 1, \dots, K$ and the graph $\mathcal{G}$ is $V-1$ regular: the cardinal of V_i is equal to V for all i. First notice that even in this case, the graph $\mathcal{G}$ is not necessarily completely symmetric. Figure 1 shows an example of a 3-regular graph which is not completely symmetric.

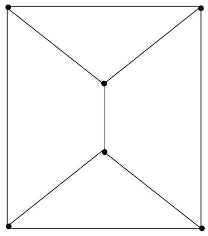

Figure 1: A regular graph which is not completely symmetric.

Note also that the system is not monotone. Indeed, $x \le y$ (component-wise) does not imply that $W^x(1)$ is dominated stochastically by $W^y(1)$ (check this by coupling). The system is neither monotone with respect to the graph structure. If G_1 is embedded into G_2, this does not imply that the workload process built on graph G_1 is dominated stochastically by the workload built on graph G_2.

In this work, we also present some results on the non-symmetric case. In particular, using methods suggested recently in [18], in Remark 4.2 we give sufficient conditions for the stability of the system with space-inhomogeneous input (not necessarily identical λ's). Some other generalisations of the model are described in Section 8.

1.3 Stability Result

We first explain the intuition hidden behind the result.

The access protocol favours an equilibrium of the workload in the network: assume that the workload at node i is much larger than the workload in its neighbouring nodes, V_i. Then $\phi_i(x)$ will be close to 1, whereas for all the nodes j in V_i, $\phi_j(x)$ will be close to 0. Thus the workload at node j in V_i will tend to get closer to the workload at node i. This balance mechanism hints that the diagonal $\Delta = \{x \in \mathbb{R}^K : x_1 = x_2 ... = x_K\}$ is attractive.

If the workload is on the diagonal: $W(0) = c\mathbb{1}$ where $c \in \mathbb{N}^*$, we obtain:

$$E(W(1) - W(0)|W(0) = c\mathbb{1}) = \left(\lambda - \frac{1}{V}\left(1 - \frac{1}{Vc}\right)^{Vc-1}\right)\mathbb{1}.$$

Hence, as c tends to infinity, the drift vector converges to $(\lambda - e^{-1}/V)\mathbb{1}$.

So finally, we end up with the conjecture that if $\lambda < e^{-1}/V$, the Markov chain W is ergodic. This conjecture is clearly true for the fully isolated graph and the complete graphs.

The reasons that led to this conjecture appear to be wrong (as will follow from the results of Sections 6 and 7, in general the diagonal is not attractive). However, the conjecture itself is true and we can formulate our main stability result that will be proved in Section 4.

Theorem 1.1. *If $\lambda < e^{-1}/V$, the Markov chain W is ergodic.*

A classical strategy to analyse the positive recurrence of this type of Markov chain is via the fluid approximation. We will prove that the fluid approximation satisfies an ordinary differential equation.

Our heuristics suggest also that if $\lambda > e^{-1}/V$ then W is transient. Corollary 6.3 in Section 6 is a partial result which corroborates this intuition. In this paper, the transience of W is stated as a conjecture.

2 Fluid Approximation Method

This section deals with a general (not necessarily symmetric) case.

2.1 General Properties

In what follows, we endow $\mathbb{R}^K$ with the L^1-norm: $|x| = \sum_{k=1}^K |x_k|$. Let $(x^n), n \in \mathbb{N}$, be a sequence in $\mathbb{N}^K$ such that $\lim_n |x^n| = \infty$. For $t \in [0, T]$, we define:

$$X^n(t) = \frac{W^{x^n}(\lceil |x^n| t \rceil)}{|x^n|}.$$

To simplify the notation, for $t \in \mathbb{R}_+$, we set $W(t) = W(\lceil t \rceil)$.

Let $\mathbb{D}([0,T], \mathbb{R}^K)$ denote the space of càdlàg functions from $[0,T]$ to $\mathbb{R}^K$ endowed with the usual Skorokhod topology, i.e. the distance between the functions f_1 and f_2 is given by the following expression:

$$d_T(f_1, f_2) = \inf \sup_{t \in [0,T]} \{|g(t) - t| + \rho(f_1(g(t)), f_2(t))\},$$

where ρ is the L^1-metric in R^K and the outer infimum is taken over all monotone continuous functions $g : [0,T] \to [0,T]$ such that $g(0) = 0$ and $g(T) = T$. Denote by $\mathbb{D}([0,\infty))$ the space of R^K–valued càdlàg functions on $[0,\infty)$ with the metric

$$d(f_1, f_2) = \sum_1^\infty 2^{-T} \frac{d_T(f_1, f_2)}{1 + d_T(f_1, f_2}.$$

Note that $X^n \in \mathbb{D}([0,T], \mathbb{R}^K)$ for all n.

Lemma 2.1. *(i) For any sequence x^n such that $|x^n| \to \infty$, a.s. the family $\{(X^n), n \in \mathbb{N}\}$ has a compact closure in the Skorokhod topology, and any accumulation point z of A is almost surely continuous.*

(ii) Function z is Lipschitz with the constant $K \max\{\overline{\lambda}, 1\}$ where $\overline{\lambda} = \frac{\sum_{i=1}^K \lambda_i}{K}$.

Proof of Lemma 2.1

(i) One can obtain the proof of this assertion by following the lines of the proof of [3], Theorem 4.1 or [17], Theorem 7.1. Formally, the proofs of the mentioned theorems are given for multi-class networks. However, as pointed out in [10], the tightness of such families holds under weaker conditions (see [10], Assumption 2.19).

(ii) Since G_i is bounded by 1:

$$\begin{aligned} |X^n(t) - X_i^n(s)| &\leq \max\left\{ \frac{|A(s|x^n|, t|x^n|)|}{|x^n|}, \frac{K|x^n|(t-s)}{|x^n|} \right\} \\ &\leq \max\left\{ \frac{1}{|x^n|} \sum_{k=\lceil |x^n| s \rceil}^{\lfloor |x^n| t \rfloor} V_k, K(t-s) \right\}, \end{aligned}$$

where V_k is the total number of arrivals at time k. Sequence $\{V_k\}_{k \in \mathbb{N}}$ consists of i.i.d. random variables with $EV_k = K\overline{\lambda}$. By the law of large numbers, the result now follows if we let $n \to \infty$.

□

Definition 2.2. *Any accumulation (in the Skorokhod topology) point z of the sequence X^n is called fluid limit. The collection of all fluid limits is called the fluid model.*

Note that it follows from the definition of X^n and z that $z_i(t) \geq 0$ for all $i = 1, \ldots, K$ and for all t.

Corollary 2.3. *The trajectories of fluid limits are self-similar. More precisely, for any fluid limit z and for any $u > 0$ such that $\mathsf{P}(|z(u)| > 0) > 0$, the random process $\{\tilde{z}(t), t \geq 0\}$ with conditional distribution*

$$\mathsf{P}(\tilde{z}(t) \in \cdot\,) = \mathsf{P}\left(\frac{z(u+t)}{|z(u)|} \in \cdot \,\big|\, z(u) \right)$$

is also a fluid limit on the set $|z(u)| > 0$.

This result may be obtained by following the lines of the proof of Stolyar [17], Lemma 6.1. However, the same remark as the one given in the proof of Lemma 2.1, (i) applies here.

Definition 2.4. *We say that the fluid model is* stable *if there exists a deterministic time t_0 and $\epsilon \geq 0$, such that for all fluid limits z satisfying $|z(0)| = 1$, $|z(t)| \leq \varepsilon$ for $t \geq t_0$ a.s.*

The definition of fluid stability is standard and appears in most papers dealing with the fluid approximation method.

2.2 Fluid Model Criterion for Stability

Theorem 1.1 can be restated via the fluid approximation method.

Lemma 2.5. *If the fluid model is stable then W is ergodic.*

Proof. (i) One can again obtain a proof of this assertion by following the lines of the proofs of Dai [3] or Stolyar [17] which are given for multi-class networks.

(ii) Theorem 3.2 of Meyn [12] contains the statement (ii) for multi-class queueing networks. Here again, Meyn's proof also applies to our framework without major changes.

□

By Lemma 2.5, Theorem 1.1 can be restated as:

Theorem 2.6. *If $\lambda < e^{-1}/V$, the fluid model is stable.*

2.3 Fluid Limit Evolution Equation

In what follows we write $\varphi_i(t) = \phi_i(z(t)) = \dfrac{z_i(t)}{\sum\limits_{j\in V_i} z_j(t)}$.

Theorem 2.7. *Take any fluid limit z. Assume for all i, $\sum_{j\in V_i} z_j(t) > 0$. If $t > 0$, z_i has a derivative at point t and a right derivative at 0 if $t = 0$. Moreover, for $t > 0$:*

$$z_i^{'}(t) = \lambda_i - \varphi_i(t) e^{-\sum\limits_{j\in V_i} \varphi_j(t)} = \lambda_i - \tilde{G}_i(z_i(t)). \tag{3}$$

For $t = 0$ this equation holds with the right derivative.

Under the assumptions of the Theorem, this differential equation admits a unique solution, thus all the converging subsequences of (X^n) converge toward the same deterministic limit.

When the assumption: for all i, $\sum_{j\in V_i} z_j(0) > 0$ is not fulfilled, some boundary effects arise. These boundary conditions are discussed in Section 3.

Proof of Theorem 2.7

(i) We first suppose that $z_i(t) > 0$. To treat this case, we need the following technical result.

Lemma 2.8. *There exists $C > 0$ such that $|G_i(x) - \tilde{G}_i(x)| \le \min(1, C/x_i)$ if $x_i \ge 2$.*

Proof of Lemma 2.8.

Using that $|e^{-y_1} - e^{y_2}| \le |y_1 - y_2|$ for all $y_1, y_2 \ge 0$, we obtain the following:

$$|G_i(x) - \tilde{G}_i(x)| \le \left| \ln\left(1 - \frac{1}{\sum_{k\in V_i} x_k}\right)\right| + \left| \sum_{j\in V_i} \left(x_j \ln\left(1 - \frac{1}{\sum_{k\in V_j} x_k}\right) + \frac{x_j}{\sum_{k\in V_j} x_k}\right)\right|. \tag{4}$$

For every j, denote $y_j = \dfrac{1}{\sum_{k\in V_j} x_k}$. Then, using that $|\ln(1-y) + y| \le \dfrac{y^2}{2(1-y)^2}$ for $y \in (0,1)$, we obtain that

$$|G_i(x) - \tilde{G}_i(x)| \le y_i + \frac{y_i^2}{2(1-y_i)^2} + \sum_{j\in V_i} \frac{x_j y_j^2}{2(1-y_j)^2}.$$

The required bound now follows from the facts that

$$y_j \le 1/x_i, \quad x_j y_j \le 1 \quad \text{and} \quad y_j \le 1/2$$

for all $j \in V_i$.

□

Assume now that $t = 0$ (the result for an arbitrary t follows from the self-similarity of fluid limits). Let $x = z(0)$. Suppose that $s < x_i$. Let $k \le |x^n| s$, then $W_i^{x^n}(k) \ge x_i^n - k \ge |x^n|(x_i^n/|x^n| - s)$. Hence, $W_i^{x^n}(k) \ge 2$ for $k \le |x^n| s$ for large enough n.

We need to show that $\lim_{s\to 0} \dfrac{z_i(s)-z_i(0)}{s} = \lambda_i - \tilde{G}_i(z(0))$. Consider the following expression:

$$\begin{aligned}
X_i^n(s) - X_i^n(0) &= \frac{1}{|x^n|}\sum_{k=0}^{\lfloor |x^n|s\rfloor -1}\Big(W_i^{x^n}(k+1) - W_i^{x^n}(k)\Big) \\
&= \frac{1}{|x^n|}\sum_{k=0}^{\lfloor |x^n|s\rfloor -1} E\Big[W_i^{x^n}(k+1) - W_i^{x^n}(k)|W^{x^n}(k)\Big] \\
&\quad + \frac{1}{|x^n|}\sum_{k=0}^{\lfloor |x^n|s\rfloor -1}\Big(W_i^{x^n}(k+1) - E[W_i^{x^n}(k+1)|W_i^{x^n}(k)]\Big) \\
&= \frac{1}{|x^n|}\sum_{k=0}^{\lfloor |x^n|s\rfloor -1}\Big(\lambda_i - G_i(W^{x^n}(k))\Big) + \frac{1}{|x^n|}\sum_{k=1}^{\lfloor |x^n|s\rfloor} D_k^n, \qquad (5)
\end{aligned}$$

where

$$D_k^n = W_i^{x^n}(k) - E\Big(W_i^{x^n}(k)|W^{x^n}(k-1)\Big) = A_i(k) - \lambda_i + q_i(k) - E\Big(q_i(k)|W^{x^n}(k-1)\Big)$$

with $q_i(k) = I(N_i(k-1)=1)\prod_{j\in V_i\setminus\{i\}} I(N_j(k-1)=0)$. We have $\dfrac{1}{|x^n|}\sum_{k=1}^{\lfloor |x^n|s\rfloor}(A_i(k)-\lambda_i) \to 0$ a.s. when $n\to\infty$ and we can use Theorem VII.3 of Feller [8] (applied to $b_k = 1/k$) to deduce that almost surely

$$\frac{1}{|x^n|}\sum_{k=1}^{\lfloor |x^n|s\rfloor}\Big(q_i(k) - E\Big(q_i(k)|W^{x^n}(k-1)\Big)\Big) \to 0 \qquad (6)$$

as $n\to\infty$.

It remains to find the limit of the first term in equation (5). We decompose this term as follows:

$$\frac{1}{|x^n|}\sum_{k=0}^{\lfloor |x^n|s\rfloor -1}\Big(\lambda_i - G_i(W^{x^n}(k))\Big) = \frac{1}{|x^n|}\sum_{k=0}^{\lfloor |x^n|s\rfloor -1}\left(\lambda_i - \tilde{G}_i\left(X^n\left(\frac{k}{|x^n|}\right)\right)\right) + \epsilon(s,n),$$

where by Lemma 2.8

$$|\epsilon(s,n)| \le C\frac{1}{|x^n|}\sum_{k=0}^{\lfloor |x^n|s\rfloor -1}\frac{1}{W_i^{x^n}(k)} \le C\frac{1}{|x^n|}\sum_{k=0}^{\lfloor |x^n|s\rfloor -1}\frac{1}{x_i^n - k} \to 0$$

as $n\to\infty$ uniformly in $s\le x_i$. Further, from the uniform convergence of X^n to z and the continuity of $\tilde{G}$ we deduce that

$$\frac{z_i(s)-z_i(0)}{s} = \lambda_i - \lim_{n\to\infty}\frac{\sum_{k=0}^{\lfloor |x^n|s\rfloor -1}\tilde{G}_i\left(z\left(\frac{k}{|x^n|}\right)\right)}{|x^n|s}.$$

Since $\frac{1}{|x^n|}\sum_{k=1}^{\lfloor |x^n|s\rfloor}\tilde{G}_i(z(\frac{k-1}{|x^n|}))$ is a Riemann sum of a continuous bounded function, it converges to $\int_0^s \tilde{G}_i(z(u))du$ and we have

$$\lim_{s\to 0}\frac{z_i(s)-z_i(0)}{s} = \lambda_i - \lim_{s\to 0}\frac{\int_0^s \tilde{G}_i(z(u))du}{s} = \lambda_i - \tilde{G}_i(z(0)). \qquad (7)$$

(ii) it remains to treat the following case: $z_i(0)=0$ and $\sum_{j\in V_i} z_j(0) > 0$. Notice that $\tilde{G}_i(z_i(0)) = 0$. In view of equations (5) and (6) it suffices to show that:

$$\lim_{s\to 0^+}\lim_{n\to\infty}\frac{1}{|x^n|s}\sum_{k=0}^{\lfloor |x^n|s\rfloor -1} G_i(W^{x^n}(k)) = 0. \qquad (8)$$

By assumption, there exists $j\in V_i$ such that $z_j(0) = \lim_{n\to\infty} x_j^n/|x^n| > \alpha > 0$. Let $\epsilon > 0$, in particular, there exists n_0 such that for all $n\ge n_0$, $x_j^n/|x^n| > \alpha$ and $x_i^n/|x^n| < \epsilon$.

Let $n \geq n_0$, pick $0 < s < \alpha$ and fix $\epsilon < \alpha$, then for n large enough, $W_i^{x^n}(k) \leq \epsilon|x^n| + A_i(0,k)$, $W_j^{x^n}(k) \geq \alpha|x^n| - k$ and:

$$\begin{aligned} G_i(W^{x^n}(k)) &\leq \frac{W_i^{x^n}(k)}{W_i^{x^n}(k) + W_j^{x^n}(k)} \\ &\leq \frac{\epsilon|x^n| + A_i(0,k)}{(\alpha+\epsilon)|x^n| - k} \end{aligned}$$

By the strong law of large numbers, a.s. $\lim_{t\to+\infty} A_i(0,t)/t = \lambda_i$. Let $\widetilde{\lambda} > \lambda_i$. It is clear that a.s. we may find k_0 such that for $k_0 \leq k \leq s|x^n|$ (we may suppose that $|x^n|$ is large enough to be larger than k_0/s):

$$G_i(W^{x^n}(k)) \leq \frac{\epsilon|x^n| + \widetilde{\lambda}k}{(\alpha+\epsilon)|x^n| - k},$$

and

$$\frac{1}{|x^n|}\sum_{k=0}^{\lfloor |x^n|s\rfloor - 1} G_i(W^{x^n}(k)) \leq \frac{k_0}{|x^n|} + \frac{\epsilon}{\alpha+\epsilon-s} + \frac{1}{|x^n|}\sum_{k=0}^{\lfloor |x^n|s\rfloor - 1} \frac{k\widetilde{\lambda}}{\alpha|x^n| - k}$$

A direct computation shows that:

$$\lim_{n\to\infty} \frac{1}{|x^n|}\sum_{k=0}^{\lfloor |x^n|s\rfloor - 1} \frac{k\widetilde{\lambda}}{\alpha|x^n| - k} = -\widetilde{\lambda}(s + \alpha\ln(1-\frac{s}{\alpha})).$$

We obtain, almost surely:

$$\limsup_n \frac{1}{|x^n|}\sum_{k=0}^{\lfloor |x^n|s\rfloor - 1} G_i(W^{x^n}(k)) \leq \frac{\epsilon}{\alpha+\epsilon-s} - \widetilde{\lambda}(s + \alpha\ln(1-\frac{s}{\alpha}))$$

Since this last inequality holds for all $\epsilon > 0$ and $\widetilde{\lambda} > \lambda_i$, we have:

$$\limsup_n \frac{1}{|x^n|}\sum_{k=0}^{\lfloor |x^n|s\rfloor - 1} G_i(W^{x^n}(k)) \leq -\lambda_i(s + \alpha\ln(1-\frac{s}{\alpha})).$$

It then follows immediately that:

$$\lim_{s\to 0^+}\limsup_n \frac{1}{|x^n|s}\sum_{k=0}^{\lfloor |x^n|s\rfloor - 1} G_i(W^{x^n}(k)) = 0.$$

□

3 Properties of the Fluid Limit on the Boundary

Conjecture 3.1. *We think that any fluid limit z has a right derivative at point 0 in all coordinates for any vector $z(0)$ (even if there exists i such that $x_j = z_j(0) = 0$ for all $j \in V_i$). We also believe that $z'(0)$ does not depend on the sequence x^n and only depends on $x = \lim_n x^n/|x^n|$. If it is so then all fluid limits are deterministic functions.*

In this Section, we will prove a weaker statement that will be sufficient to prove that the boundary of the positive orthant does not play any role in the stability of the fluid model. Denote

$$\tau_h = \inf\{t \geq 0 : |z(t)| < h\}.$$

Denote also $\lambda_* = \min\{\lambda_1, \ldots, \lambda_K\} > 0$. Assume that $|z(0)| = 1$ then $\max_i z_i(0) \geq 1/K$. The fact that $z_i'(t) \geq \lambda_* - 1$ for all i and t also implies that

$$\tau_{1-\varepsilon} \geq \frac{\varepsilon}{K(1-\lambda_*)}. \tag{9}$$

Theorem 3.2. *Assume that* $|z(0)| = 1$. *Then there exist positive constants* b *and* ϵ_0 *such that for any* $\epsilon \in (0, \epsilon_0)$, $\min_i z_i(t) \geq b\epsilon$ *for any* $t \in [c\epsilon, \tau_{1-\epsilon})$ *where* $c = 1/K(1-\lambda_*)$.

The following corollary is immediate:

Corollary 3.3. *For any* $0 < h < 1$, $z_i(t) > 0$ *for all* $0 < t < \tau_h$.

The forthcoming Lemma 6.1 and Corollary 3.3 imply:

Corollary 3.4. *Assume that* $|z(0)| > 0$, *then either:*

- *there exists* c *such that* $z(c) = 0$ *and* $z(t)$ *remains in* H *for all* $t \in (0, c)$ *or*

- $z(t)$ *remains in* H *for all* $t > 0$.

The rest of this paragraph is devoted to the proof of Theorem 3.2. We begin with some technical lemmas.

Lemma 3.5. *There exist positive constants* $K_1 > 1$ *and* K_2 *such that, for any fluid limit* z, *if* $z_i(t) > K_1 z_j(t)$ *for two neighboring nodes* i *and* j *then* $z_j'(t) > K_2$.

Proof of Lemma 3.5

Note that the existence of $z_j'(t)$ is guaranteed by Theorem 2.7. Indeed $z_i(t) > 0$ and therefore $\sum_{k \in V_j} z_k(t) > 0$. To prove Lemma 3.5, note that

$$z_j'(t) > \lambda_* - \frac{z_j(t)}{\sum_{k \in V_j} z_k(t)} \geq \lambda_* - \frac{z_j(t)}{z_i(t) + z_j(t)} > \lambda_* - \frac{1}{1 + K_1}$$

and we may take $K_1 = 2/\lambda_* - 1$ and $K_2 = \lambda_*/2$.

□

Lemma 3.6. *There exist constants* $C_1 \geq C_2 > 0$ *such that for any* $h > 0$ *if* $|z(0)| \geq h_1$ *and* $\min_i z_i(0) \geq C_1 h$ *then* $\min_i z_i(t) \geq C_2 h_1$ *for all* $t \leq \tau_h$.

Proof of Lemma 3.6

Let D be the maximum graph distance of $\mathcal{G}$. Put $C_1 = \dfrac{1}{K K_1^{D+1}}$ and put $C_2 = \dfrac{C_1}{K_1^{D-1}}$. We may prove Lemma 3.6 for $h = 1$. The result for arbitrary h follows from the self-similarity of fluid limits.

It is sufficient to show that for any $t < \tau_1$ if $\min_i z_i(t) \geq C_1$ then there exists $0 < s < \infty$ such that

$$\min_i z_i(t+s) \geq C_1 \tag{10}$$

and

$$\min_i z_i(u) \geq C_2 \quad \text{for all} \quad t \leq u \leq t+s. \tag{11}$$

Indeed, assume that the last statement holds and Lemma 3.6 is not valid. Then there exists $t \leq \tau_1$ such that $\min_i z_i(t) < C_2$. It follows from the continuity of fluid limit that there exists the last moment $v < t$ when $\min_i z_i(v) \geq C_1$. However, our last statement implies that there exists $s > 0$ such that $\min_i z_i(v+s) \geq C_1$ and $\min_i z_i(u) \geq C_2$ for all $v \leq u \leq v+s$. Clearly, $v + s < t$, which contradicts our assumption on v being the last moment before t when $\min_i z_i(v) \geq C_1$.

Let now t be any time such that $t < \tau_1$ and $\min_i z_i(t) \geq C_1$. Note that $\max_i z_i(t) \geq 1/N = C_1 K_1^{D+1}$ since $t < \tau_1$. To simplify the notation, assume that $z_1(t) = \max_i z_i(t)$. Let T be such that $z_1(t+u) \geq C_1 K_1^D$ for all $0 \leq u \leq T$. Note that $z_i'(u) \geq \lambda_* - 1$ for all i and u. This implies that

$$T \geq \frac{C_1(K_1^{D+1} - K_1^D)}{1 - \lambda_*} = \frac{C_1 K_1^D (K_1 - 1)}{1 - \lambda_*}. \tag{12}$$

Let now d be the maximum distance in $\mathcal{G}$ from node 1. Clearly, $d \leq D$. For $j = 1, \ldots, d$, denote by A_j the set of nodes at distance j from node 1.

We show that there exists $0 < s < T$ such that (10) and (11) hold. First, we show that $\min z_i(u) \geq C_2$ for all $t \leq u \leq t + T$. Note that $z_i(u) \geq C_1$ for all $i \in A_1$ and $t \leq u < t + T$. Indeed, assume that there exist $i \in A_1$ and $t \leq u < t + T$ such that $z_i(u) < C_1$. Then, by continuity, there exists a last moment $t \leq u_1 < u$ such that $z_i(u_1) \geq C_1$. Lemma 3.5 implies that $z_i'(u_1) \geq K_2 > 0$ and hence, there exists time $u_2 > u_1$ such that $z_i(u_2) \geq C_1$ that contradicts our assumption on u_1. Using induction and following the same arguments, we can show that $z_i(u) \geq C_1/K_1^{j-1}$ for all $i \in A_j$ and $t \leq u \leq t+T$ for any $j = 1, \ldots, d$. Hence, $\min_i z_i(u) \geq C_1/K_1^{d-1} \geq C_1/K_1^{D-1} = C_2$ for all $t \leq u \leq t+T$.

Let us now show that there exists $0 < s < T$ such that (10) holds. For every $j = 1, \ldots, d$, denote by t_j the time needed to achieve the level $C_1 K_1^{d-j}$ starting from the level C_1/K_1^{j-1} and moving with the speed K_2. Clearly, $t_j = \dfrac{C_1(K_1^{d-1}-1)}{K_2 K_1^{j-1}}$. Note that (10) and (11) hold with $s = \sum_{j=1}^{d} t_j$ if $T \geq \sum_{j=1}^{d} t_j$. Indeed, $\min_{j \in A_1} z_j$ will achieve the level $C_1 K_1^{d-1}$ no later than $t + t_1$ and will not become smaller than this level before time $t+T$, since all nodes in A_1 are neighbours of node 1 and $z_1(u) \geq K_1^D$ for all $t \leq u \leq t+T$. Note also that $\min_{j \in A_2} z_j$ will become greater than $C_1 K_1^{d-2}$ no later than $t + t_1 + t_2$ since it cannot become smaller than C_1/K_1 before the time $t + t_1$, and after this time it is either greater than $C_1 K_1^{d-2}$ or it grows with at least the speed K_2 (this follows from Lemma 3.5 and the fact that any node in A_2 has a neighbour in A_1). We can continue these arguments to prove that $\min_{j \in A_d} z_j$ will become greater than C_1 no later than $t + \sum_{i=1}^{d} t_i$ if $T \geq \sum_{i=1}^{d} t_i$.

Note that

$$\begin{aligned}\sum_{i=1}^{d} t_i &= \frac{C_1(K_1^{d-1}-1)(1+K_1+\ldots+K_1^{d-1})}{K_2 K_1^{d-1}} = \frac{C_1(K_1^{d-1}-1)(K_1^d-1)}{K_2 K_1^{d-1}(K_1-1)} \\ &\leq \frac{C_1(K_1^d-1)}{K_2(K_1-1)} \leq \frac{C_1(K_1^D-1)}{K_2(K_1-1)} \end{aligned} \tag{13}$$

If we take $K_2 = \lambda_*/2$ and $K_1 = 2/\lambda_* - 1$ then $(1-\lambda_*)/K_2 = K_1 - 1$. Note also that in this case $K_1 \geq 2$. It now follows from (12) and (13) that $T \geq \sum_{i=1}^{d} t_i$.

□

One can see from the proof of Lemma 3.6 that the following (stronger) result holds.

Lemma 3.7. *For any $h_1 > 0$ there exists $\widehat{h_2} > 0$ such that for any $h_2 \leq \widehat{h_2}$ there exists $0 < h_3 \leq h_2$ such that if $|z(0)| \geq h_1$ and $\min_i z_i(0) \geq h_2$ then $\min_i z_i(t) \geq h_3$ for all $t \leq \tau_{h_1}$.*

Remark 3.8. *Lemma 3.7 is valid with $\widehat{h_2} = \dfrac{h_1}{K K_1^{D+1}}$.*

Proof of Theorem 3.2

The proof of Theorem 3.2 is similar to that of Lemma 3.6.

Take ϵ_0 such that $\dfrac{K_2(K_1-1)\epsilon}{(K_1^D-1)} \leq \dfrac{1-\epsilon}{K K_1^{D+1}}$ for all $\epsilon \leq \epsilon_0$ and take $a = \dfrac{K_2(K_1-1)}{(K_1^D-1)}$. In this case $a\epsilon \leq \dfrac{1-\epsilon}{K K_1^{D+1}}$, and in view of Lemma 3.7 and Remark 3.8, it is enough to prove that $\min_i z_i(c\epsilon) \geq a\epsilon$.

Let D be the graph distance of $\mathcal{G}$. Note that $\max_i z_i(0) \geq 1/K$. Assume that $z_1(0) = \max_i z_i(0)$.

Let T be such that $z_1(u) \geq a\epsilon K_1^D$ for all $0 \leq u \leq T$. Note that $z_i'(t) \geq \lambda_* - 1$ for all i and t. This implies that

$$T \geq \frac{1/K - a\epsilon K_1^D}{1-\lambda_*} = \frac{1 - K a\epsilon K_1^D}{K(1-\lambda_*)}. \tag{14}$$

Now let d be the maximum distance in $\mathcal{G}$ from node 1. Clearly, $d \leq D$. For $j = 1, \ldots, d$, denote by A_j the set of nodes at distance j from node 1. For every $j = 1, \ldots, d$, denote by t_j the time needed to achieve the level $a\epsilon K_1^{d-j}$ starting from the level 0 and moving with the speed K_2. Clearly, $t_j = \dfrac{a\epsilon K_1^{d-j}}{K_2}$. Denote $T_1 = \sum_{j=1}^{D} t_j$. Note that

$$T_1 = \frac{a\epsilon(K_1^D-1)}{K_2(K_1-1)} = \frac{\epsilon}{K(1-\lambda_*)} = c\epsilon. \tag{15}$$

Following the same arguments as in the proof of Lemma 3.6, we can show that $\min_i z_i(c\varepsilon) = \min_i z_i(T_1) \geq a\varepsilon$ if $T_1 \leq T$.

It remains to prove that $T_1 \leq T$. This is so due to (14), (15) and the fact that $a\epsilon \leq \dfrac{1-\varepsilon}{K K_1^{D+1}}$.

Remark 3.9. *Denote by* $\nu(z,h,b) = \inf\{t \geq z : |z(t)| < h \quad or \quad \min_i z_i(t) < b\}$ *the time (after moment* z*) of the first exit from the set* $\{|z| \geq h\} \cap \{\min_i z_i \geq b\}$*. Theorem 3.2 implies that there exist* $b > 0$ *and* $z \geq 0$ *such that* $\tau_{1-\varepsilon} = \nu(z\epsilon, 1-\varepsilon, b\epsilon)$ *for any initial condition* $z(0)$ *with* $|z(0)| = 1$*.*

4 Proof of Theorem 1.1

In this Section we first present the proof of our main stability result and then formulate its generalisation. Recall that here we deal with the symmetric case. We start with the proof of the stability. Due to Theorem 2.6 it is enough to prove that there exists a deterministic time t_0 such that for all fluid limits z satisfying $|z(0)| = 1$, $z(t) = 0$ for $t \geq t_0$ a.s.

Lemma 4.1. *If* $z_i(t) > 0$ *for all* $i = 1, \ldots, K$ *then*

$$\left(\sum_i z_i^2(t)\right)' \leq \left(\lambda - \frac{e^{-1}}{V}\right)\sum_i z_i(t)$$

and hence, if $\lambda < \frac{e^{-1}}{V}$*,*

$$\left(\sum_i z_i^2(t)\right)' \leq -\varepsilon \sum_i z_i(t)$$

for some $\varepsilon > 0$*.*

Proof of Lemma 4.1.
Clearly, it is sufficient to prove the following inequality

$$\frac{\sum\limits_i z_i \varphi_i \exp\left\{-\sum\limits_{j\in V_i} \varphi_j\right\}}{\sum\limits_k z_k} \geq \frac{e^{-1}}{V} \tag{16}$$

where we slightly abuse the notation by writing z_i instead of $z_i(t)$. We can write the LHS of the previous inequality in the following form:

$$\sum_i p_i f(y_i)$$

where $p_i = \frac{z_i}{\sum\limits_k z_k}$, $\quad y_i = -\sum\limits_{j\in V_i} \varphi_j - \ln\frac{1}{\varphi_i}$ $\quad$ and $\quad f(z) = e^z$. Function f is convex and $\sum\limits_i p_i = 1$, hence, $\sum\limits_i p_i f(y_i) \geq f(\sum\limits_i p_i y_i)$ and

$$\frac{\sum\limits_i z_i \varphi_i \exp\left\{-\sum\limits_{j\in V_i} \varphi_j\right\}}{\sum\limits_k z_k} \geq \exp\left\{-\sum_i \frac{z_i}{\sum\limits_k z_k}\sum_{j\in V_i}\varphi_j - \sum_i \frac{z_i}{\sum\limits_k z_k}\ln\frac{1}{\varphi_i}\right\}. \tag{17}$$

Now consider $\sum\limits_i \frac{z_i}{\sum\limits_k z_k}\sum\limits_{j\in V_i}\varphi_j$ and $\sum\limits_i \frac{z_i}{\sum\limits_k z_k}\ln\frac{1}{\varphi_i}$ separately:

$$\sum_i \frac{z_i}{\sum\limits_k z_k}\sum_{j\in V_i}\varphi_j = \frac{\sum\limits_i z_i \sum\limits_{j\in V_i}\varphi_j}{\sum\limits_k z_k} = \frac{\sum\limits_j \varphi_j \sum\limits_{i\in V_j} z_i}{\sum\limits_k z_k} = \frac{\sum\limits_j z_j}{\sum\limits_k z_k} = 1. \tag{18}$$

(we used the facts that $j \in V_i$ if and only if $i \in V_j$ and that $\varphi_j \sum\limits_{i\in V_j} z_i = z_j$.)

Note that the function ln is concave, hence,

$$\sum_i \frac{z_i}{\sum\limits_k z_k}\ln\frac{1}{\varphi_i} \leq \ln\left(\sum_i \frac{z_i}{\sum\limits_k z_k}\frac{1}{\varphi_i}\right) = \ln\left(\frac{\sum\limits_i \frac{z_i}{\varphi_i}}{\sum\limits_k z_k}\right) = \ln\left(\frac{\sum\limits_i\sum\limits_{j\in V_i} z_j}{\sum\limits_k z_k}\right) = \ln V. \tag{19}$$

Inequality (16) follows now from (17), (18) and (19).

□

Proof of Theorem 1.1.
Corollary 3.3 implies that if $|z(0)| = 1$, then $z_i(t) > 0$ for all $i = 1, \dots, K$ and all $t > 0$. Then we can use Lemma 4.1. Note also that for any positive values of $\{x_i\}$ it holds that $\sum_i x_i \geq \sqrt{\sum_i x_i^2}$. Hence, Lemma 4.1 implies that

$$\left(\sum_i z_i^2(t)\right)' \leq -\varepsilon\sqrt{\sum_i z_i^2(t)}$$

and hence,

$$\left(\sqrt{\sum_i z_i^2(t)}\right)' \leq -\varepsilon/2$$

and the result follows.

□

Remark 4.2. *By applying methods used in [18], we can get a similar (but less explicit) stability result in a more general situation. Assume now that the system is not symmetric, in general, i.e. that values of λ_i may differ for different i and the graph $\mathcal{G}$ may be irregular.*
Let

$$M = \{\mu : \mu_i = p_i e^{-\sum_{j\in V_i} p_j}, i = 1, \dots, K, \quad \text{for some} \quad \overline{p} = (p_1, \dots, p_K) \quad \text{with} \quad p_i \geq 0\}.$$

One can show that the vector $(\varphi_1, \dots, \varphi_K)$ with $\varphi_i = \dfrac{z_i}{\sum_{j\in V_i} z_j}$ maximises the function $\sum_{i=1}^{K} z_i \ln \mu_i$ over all vectors $\mu \in M$. Based on that, one can obtain the following.

Theorem 4.3. *Assume that there exists a vector $\mu \in M$ such that $\lambda < \mu$ component-wise. Then the Markov chain W_n is recurrent.*

A proof of Theorem 4.3 follows the lines of the proof of Theorem 4 in [18].

5 Rates of Convergence

In this section, we again consider the symmetric case. We will obtain rates of convergence of W_n to its stationary distribution in the total variation norm.

Define the total variation distance between measures π_1 and π_2 by

$$||\pi_1(\cdot) - \pi_2(\cdot)|| = \sup_{|g|\leq 1}\left|\int g(y)\pi_1(dy) - \int g(y)\pi_2(dy)\right|.$$

Theorem 5.1. *Assume that $\lambda < e^{-1}/V$ and $EA_i(n)^{p+1} < \infty$ for some $p \geq 1$ and for all $i = 1, \dots, K$ and n. Then*

$$\lim_{n\to\infty} n^p ||\mathsf{P}^n(x, \cdot) - \pi(\cdot)|| = 0, \quad x \in \mathbb{N}^K,$$

where $\mathsf{P}^n(x, \cdot)$ — distribution of $W^x(n)$ and $\pi(\cdot)$ — stationary measure for W.

Proof of Theorem 5.1
The proof of Theorem 5.1 is based on the following lemma which is an analogue of Proposition 5.3 of Dai and Meyn [4].

Lemma 5.2. *Assume that the conditions of Theorem 5.1 are satisfied. Then, for some constants $c < \infty$, $\delta > 0$ and a finite set C,*

$$E\left(\sum_{n=0}^{\tau_C(\delta)} |W^x(n)|^p\right) \leq c|x|^{p+1}$$

for any $x \in \mathbb{N}^K$, where $\tau_C(\delta) = \min(n \geq \delta : W(n) \in C)$.

Proof of Lemma 5.2

The proof of Lemma 5.2 follows the lines of the proof of Proposition 5.3 of [4].

It follows from Theorem 2.6 that there exists t_0 such that

$$\lim_{|x|\to\infty} \frac{W^x(|x|t_0)}{|x|} = 0$$

a.s. Note also that the family of random variables $\left\{ \frac{|W^x(|x|t_0)|^{p+1}}{|x|^{p+1}} \right\}$ is uniformly integrable, since

$$\frac{|W^x(|x|t_0)|^{p+1}}{|x|^{p+1}} \le \frac{\left(\sum_{m=0}^{|x|t_0}\sum_{i=1}^K A_i(m)\right)^{p+1}}{|x|^{p+1}} \le t_0^{p+1} \frac{\sum_{m=0}^{|x|t_0}\left(\sum_{i=1}^K A_i(m)\right)^{p+1}}{|x|t_0}$$

and the family $\left\{ \frac{\sum_{m=0}^{|x|t_0}\left(\sum_{i=1}^K A_i(m)\right)^{p+1}}{|x|t_0} \right\}$ is uniformly integrable. The latter is guaranteed by the existence of $EA_i(m)^{p+1}$ for all $i = 1, \ldots, K$ and for all m. Hence,

$$\lim_{|x|\to\infty} \frac{E\left[|W^x(|x|t_0)|^{p+1}\right]}{|x|^{p+1}} = 0.$$

Choose L such that

$$E\left[|W^x(|x|t_0)|^{p+1}\right] \le \frac{1}{2}|x|^{p+1} \tag{20}$$

for $|x| \ge L$. Define, as in the proof of Proposition 5.3 of [4], the sequence of stopping times $\sigma_0 = 0, \sigma_1 = t(x)$, and $\sigma_{k+1} = \sigma_k + \theta_{\sigma_k}\sigma_1, k \ge 1$, where $t(x) = t_0 \max(L, |x|)$, θ — shift operator on the sample space. We assume that t_0 is integer. The stochastic process $\hat{W}_k = W(\sigma_k)$ is a Markov chain with the transition kernel

$$\hat{P}(x, A) = \mathsf{P}(W^x(t(x)) \in A).$$

Now (20) implies that

$$E\left\{|\hat{W}_1|^{p+1} - |\hat{W}_0|^{p+1} | \hat{W}_0 = x\right\} \le -\frac{1}{2}|x|^{p+1} + b\mathbf{I}_C(x),$$

where set $C = \{x : |x| \le L\}$ and b is some constant. The Comparison Theorem (Meyn and Tweedie [13], p. 337) yields that

$$E\left[\sum_{n=0}^{k_*-1} |W^x(\sigma_k)|^{p+1}\right] = E\left[\sum_{n=0}^{k_*-1} |\hat{W}(k)|^{p+1}\right] \le 2\left\{|x|^{p+1} + b\mathbf{I}_C(x)\right\} \tag{21}$$

where $k_* = \min(k \ge 1 : \hat{W}(k) \in C\}$. To prove Lemma 5.2 , we first show that for some constant c_0

$$E\left[\sum_{n=\sigma_k}^{\sigma_{k+1}} |W^x(n)|^p | \mathcal{F}_{\sigma_k}\right] \le c_0 W^x(\sigma_k)^{p+1} \tag{22}$$

which by the strong Markov property amounts to

$$E\sum_{n=0}^{t(x)} |W^x(n)|^p \le c_0|x|^{p+1}$$

This follows from the fact that

$$\sum_{n=0}^{t(x)} |W^x(n)|^p \le \sum_{n=0}^{t(x)} \left(\sum_{m=0}^{n}\sum_{i=1}^{K} A_i(m)\right)^p \le \sum_{n=0}^{t(x)} \left(\sum_{m=0}^{t(x)}\sum_{i=1}^{K} A_i(m)\right)^p$$

a.s. and from our assumption that $EA_i(m) < \infty$ for all $i = 1, \ldots, K$ and for all m. Substituting (22) into (21), we have

$$E\left[\sum_{k=0}^{\infty} E\left[\sum_{n=\sigma_k}^{\sigma_{k+1}} |W^x(n)|^p | \mathcal{F}_{\sigma_k}\right] \mathbf{I}k < k_*\right] \le c|x|^{p+1}.$$

By the Fubini theorem and the smoothing property of the conditional expectation, the LHS is precisely $E\left[\sum_{n=0}^{\sigma_{k_*}}(1+|W^x(n)|^p)\right]$. The proposition now follows from the fact that $\sigma_{k_*} \geq \tau_C(t_0 L)$.

□

We now use Proposition 5.4 of [4] with $t = 1$. Applied to our case, it gives the following bound:

$$E\left\{V(W(1)) - V(W(0))|W(0) = x\right\} \leq -f(x) + \kappa \tag{23}$$

with $V(x) = E\left(\sum_{n=0}^{\tau_C(\delta)} |W^x(n)|^p\right)$ and $f(x) = |x|^p$.

Note that Lemma 5.2 implies that $V(x) \leq c|x|^{p+1}$. Now (23) yields that

$$E\left\{V(W(1)) - V(W(0))|W(0) = x\right\} \leq V(x)^{\frac{p}{p+1}} + b\mathbf{I}_C$$

for the set $C = \{x : |x| \leq L\}$ and for some constant b. The result now follows from Theorem 2.5 of Douc et al. [5].

□

6 Local Stability of Fluid Limits on the Positive Orthant

In this Section we investigate the behaviour of the solution to the differential equation satisfied by fluid limits. In particular, we show that if the input rate λ is sufficiently small, then the diagonal is locally unstable.

6.1 Orbits of the Fluid Limits

Denote $H = \{x \in \mathbb{R}^K : x_i > 0 \quad \text{for all} \quad i = 1, \ldots, K\}$ and $\mathbb{1} = (1, ..., 1)'$. For $z(t)$ in H, the differential equation (3) is restated in closed form as:

$$\dot{z}(t) = F(\phi(z(t))), \tag{24}$$

with $F(x)_i = \lambda - x_i e^{-\sum_{j\in V_i} x_j}$. Let $\Delta = \{x \in H : x_1 = x_2 ... = x_K\}$ and $C_u = \{x \in H : |x/|x| - \mathbb{1}/K| \leq u\}$, $u > 0$, C_u is a cone with direction Δ. We note that the diagonal is an orbit of the differential equation: $F(\phi(c\mathbb{1})) = (\lambda - e^{-1}/V)\mathbb{1}$. We are going to prove that the diagonal is also locally attractive.

Lemma 6.1. *Assume that $z(0) \in H$, then:*

- *there either exists c such that $z(c) = 0$ and $z(t)$ remains in H for all $t \in (0, c)$ or*
- *$z(t)$ remains in H for all $t > 0$.*

Proof of Lemma 6.1.

Restricted on the open set H, $F \circ \phi$ is $\mathcal{C}^\infty(\mathbb{R}^n)$. Therefore, the solutions of equation (24) are locally uniquely defined as long as $z(t)$ remains in H. Now, suppose the contrary: that $t \mapsto z(t)$ leaves H at time c at $y = \lim_{t\to c-} z(t) \in \partial H \backslash \{0\}$.

Let $a_i = \limsup_{t\to c^-} \phi(z(t))$, $a_i \in [0, 1]$. Since $y \neq 0$, there exist i_1 and i_2 such that $y_{i_1} = 0$ and $y_{i_2} > 0$. Note also that G is connected, this implies that there exists k such that $y_k = 0$ and $\sum_{j\in V_k} y_k > 0$ (consider the path from i_1 to i_2). Hence, $a_k = 0$ and $\lim_{t\to c^-} F_k(\phi(z(t))) = \lambda > 0$, this implies that $t \mapsto x_k(t)$ increases on a left neighbourhood of c, which is contradictory with $y_k = \lim_{t\to c-} z(t) = 0$. □

Lemma 6.1 implies that for an initial condition in H the fluid limit $z(t)$ remains in H or finally reaches 0 at time c. By convention, we set that $\phi(0) = \mathbb{1}/V$, thus after time c, the orbit of z remains on the diagonal: for $t \geq 0$, $z(t + c) = (\lambda - e^{-1}/V)\mathbb{1}t$. Notice also that if $x \in H$, then $F(\phi(z)) \leq \lambda\mathbb{1}$ (component-wise). In view of Lemma 6.1 this immediately implies that if $z(0)$ is in H then the maximal solution of equation (24) is defined on $\mathbb{R}_+$. Lemma 6.1 also implies that if $z(0) = \lim_n x^n/|x^n| \in H$ then the fluid limit is deterministic.

Let A be the adjacency matrix of G and $\{\nu_1, ..., \nu_K\}$ its eigenvalues with $\nu_i \leq \nu_{i+1}$. The spectral gap γ is defined by:

$$\gamma = \min_{i<K}(\nu_K - \nu_i) = \nu_K - \nu_{K-1}.$$

Note that since G is $V - 1$ regular, $\nu_K = V$. The main result of this section is the following.

Theorem 6.2. *If $\lambda > \frac{e^{-1}}{V}(1 - \frac{\gamma^2}{V^2})$, there exists $u > 0$ such that for all solutions $t \to z(t)$ of equation (24) with the initial condition in C_u,*

$$\lim_{t \to +\infty} \phi(z(t)) = \mathbb{1}/V.$$

If $\lambda < \frac{e^{-1}}{V}(1 - \frac{\gamma^2}{V^2})$, the diagonal is locally unstable.

Corollary 6.3. *If $\lambda > \frac{e^{-1}}{V}(1 - \frac{\gamma^2}{V^2})$, there exists $u > 0$ such that if $z(t)$ is a solution of equation (24) with an initial condition $z(0)$ in C_u,*

- *if $\lambda < e^{-1}/V$, then there exists $c > 0$ such that $z(c) = 0$.*
- *if $\lambda > e^{-1}/V$, then $z(t) \sim (\lambda - e^{-1}/V)t$.*

Proof of Corollary 6.3

Let $z(t)$ be the maximal solution with $z(0) \in H$. From Theorem 6.2, $\lim \phi(z(t)) = \mathbb{1}/V$. Since F is $\mathcal{C}^\infty(\mathbb{R}^n)$ on a neighborhood of $\mathbb{1}/V$, $\lim_{t \to +\infty} \dot{z}(t) = (\lambda - e^{-1}/V)\mathbb{1}$. If $\lambda \neq e^{-1}/V$, this implies that, as t tends toward infinity:

$$z(t) \sim (\lambda - e^{-1}/V)t\mathbb{1}. \tag{25}$$

Suppose first that $\lambda - e^{-1}/V < 0$ then from equation (25), $z(t)$ leaves H in finite time. Lemma 6.1 implies in turn that there exists $c > 0$ such that $z(c) = 0$. The first assertion of Corollary 6.3 is proved. □

Remark 6.4. The first statement of Corollary 6.3 can be strengthened in the following way:

there exists $\delta > 0$ and $v > 0$ such that for all $z(0) \in C_v$, $z(\delta|z(0)|) = 0$.

Indeed, let δ such that $0 < \delta^{-1} < e^{-1}/V - \lambda$. There exists $\epsilon > 0$ such that for all i and $z \in C_\epsilon$, $\phi(z)_i < -\delta$. We then define $v = \min(u, \epsilon)$.

6.2 Proof of Theorem 6.2

The proof of this theorem is an application of the stability theory for differential equations.

6.2.1 Spectral Analysis

We need to consider the eigenvalues of $D(F \circ \phi)(x)$ for $x \in \Delta$, where $Df(x)$ is the differential of f at x. $F \circ \phi$ is homogeneous of order 0: for all $c > 0$, $F(\phi(cx)) = F(\phi(x))$. Hence:

$$D(F \circ \phi)(c\mathbb{1}) = c^{-1} D(F \circ \phi)(\mathbb{1}).$$

Since Δ is an orbit of equation (24), $\mathbb{1}$ is an eigenvector of $D(F \circ \phi)(\mathbb{1})$ associated with the eigenvalue 0.

Lemma 6.5. *The eigenvalues of $D(F \circ \phi)(\mathbb{1})$ are $(0, \eta_1, \cdots, \eta_{K-1})$ with $\eta_i = -\frac{e^{-1}}{V^3}(V - \nu_{K-i})^2$. In particular, for all $i \geq 1$, $\eta_i < 0$.*

Proof of Lemma 6.1.

A direct computation leads to:

$$(D(F \circ \phi)(\mathbb{1}))_{ij} = \begin{cases} -\frac{e^{-1}(V-1)}{V^2} & \text{if } \; j = i, \\ \frac{e^{-1}}{V^3}|V_i \cup V_j| & \text{if } \; j \in V_i \backslash i \\ -\frac{e^{-1}}{V^3}|V_i \cap V_j| & \text{if } \; j \notin V_i \end{cases},$$

Not surprisingly, $D(F \circ \phi)(\mathbb{1}).\mathbb{1} = 0$. Indeed, let $M = -eV^3 D(F \circ \phi)(\mathbb{1})$. Using the fact that $|V_i \cup V_j| = 2V - |V_i \cap V_j|$, we deduce that:

$$(M\mathbf{1})_i = V(V-1) - 2V(V-1) + \sum_{j \neq i} |V_i \cap V_j| = \sum_{j=1}^{K} |V_i \cap V_j| - V^2 = 0.$$

Let E denote the identity matrix and A the adjacency matrix of $\mathcal{G}$, since $(A^2)_{ij} = |V_i \cap V_j|$, we have the following decomposition:

$$M = V^2 E - 2VA + A^2 = (A - VE)^2.$$

The matrix A is irreducible by hypothesis ($\mathcal{G}$ is connected), thus $(A - VE)$ is an ML-Matrix (refer to Seneta [16]). In graph theory, this matrix is referred as the Laplacian matrix of G. From Corollary 1 of Theorem 1 in Seneta [16], the spectral radius of A is V, Theorem 2.6 (d) of [16] implies that $\dim Ker(A - VE) = 1$ and all the eigenvalues of $(A - E)^2$ different from 0 are positive reals (remind that the spectrum of A is real). $\square$

6.2.2 Orbit of $\psi \circ z$

We define:

$$\Sigma = \{x \in H \,:\, \sum_{i=1}^{K} x_i = 1\} = H \cap \langle \mathbb{1}, \cdot \rangle^{-1}(\{1\}) = \psi(H),$$

where $\psi(x) = x/|x|$. Σ is clearly a C^∞-convex manifold of codimension 1. We define the following differential equation on Σ:

$$\dot{y} = D\psi(y)F(\phi(y)) = \alpha(y) \tag{26}$$

with an initial condition $y(0)$ in Σ. α is a $C^\infty(\Sigma)$ function and $\alpha(y) \in T_y(\Sigma)$ the tangent space of Σ at y. The next step is to compare the orbits of equation (26) and equation (24). The next lemma asserts that the orbits of the solution of $\dot{y} = \alpha(y)$ and $\psi \circ x$ where $t \mapsto z(t)$ is a solution of equation (24) are indeed equal.

Lemma 6.6. *Let $z(0)$ be in H and let $z(t)$ be the maximal solution of equation (24). Let $y(t)$ be the maximal solution of $\dot{y} = G(y)$, with the initial condition $y(0) = \psi(z(0))$. Then it is defined on $\mathbb{R}_+$, and there exists an increasing continuous bijective function $\mu : \mathbb{R}_+ \to \mathbb{R}_+$ such that:*

$$y \circ \mu = \psi \circ z$$

Proof of Lemma 6.6.

This lemma is a classical result. For an initial condition in H, we have $F(z(t)) \leq \lambda \mathbb{1}$. Indeed, while $z(t) \in H$ it is clear. If $z(t) \notin H$, from Lemma 6.1, $z(t) \in \Delta \cap -\overline{H}$, thus $F(z(t)) = (\lambda - 1/Ve^{-1})\mathbb{1} \leq \lambda \mathbb{1})$. It follows that $|z(t)| = \sum_{j=1}^{n} z_j(t) \leq K\lambda t + \sum_{j=1}^{n} z_j(0)$.

Suppose now that for all t, $z(t) \in H$, then $\int_0^{+\infty} \frac{ds}{\sum_{j=1}^{n} z_j(s)}$ diverges. By the intermediate value theorem, we deduce that there exists an increasing continuous function ν such that:

$$\text{for all } t \geq 0, \quad \int_0^{\nu(t)} \frac{ds}{\sum_{j=1}^{K} z_j(s)} = t. \tag{27}$$

In particular:

$$\dot{\nu}(t) = \sum_{j=1}^{K} z_j(\nu(t)).$$

Let $w = \psi \circ z \circ \nu$, $w(0) = \psi(w(0)) = y(0)$. We have:

$$\begin{aligned} \dot{w}(t) &= \dot{\nu}(t) \frac{d}{ds} \psi(z(s)) \Big|_{s=\nu(t)} \\ &= \big(\sum_{j=1}^{K} z_j(\nu(t))\big) D\psi(z(\nu(t))).F(w(t)). \end{aligned}$$

The function ψ is homogeneous of order 0 and thus $D\psi(cz) = c^{-1}D\psi(z)$ for all $c > 0$. It follows that:

$$\begin{aligned} \dot{w}(t) &= D\psi(\frac{z(\nu(t))}{\sum_{j=1}^{n} z_j(\nu(t))}).F(w(t)) \\ &= G(w(t)). \end{aligned}$$

The solution of the differential equation is unique, therefore $w(t) = y(t)$. The lemma is proved with $\mu = \nu^{-1}$.

If $z(t)$ leaves H, then due to Lemma 6.1 there exists c such that $z(c) = 0$ and $z(t) = (\lambda - e^{-1}/V)(t - c)\mathbb{1}$ for $t \geq c$. Then the mapping ν is built on $[0, c]$, as we did previously, and $\nu(t) = \nu(c) + t - c$ for $t \geq c$. Then the same proof holds. $\square$

6.2.3 Local Stability of $\psi \circ z$.

Clearly, $y_0 = \mathbb{1}/K$ is an equilibrium point of equation (26). In the next lemma we prove that this equilibrium is locally stable.

Lemma 6.7. *If $\lambda > \frac{e^{-1}}{V}(1 - \frac{\gamma^2}{V^2})$, there exists $u > 0$ such that for all solutions $t \mapsto y(t)$ of equation (26) with $|y(0) - y_0| < u$,*

$$\lim_{t \to +\infty} \sup_{y(0) \in \Sigma : |y(0) - y_0| < u} |y(t) - \mathbb{1}/V| = 0.$$

Proof of Lemma 6.7.

We denote by $D\alpha(y)|_{T_y(\Sigma)}$ the differential of α at y restricted to the $K-1$ dimensional subspace $T_y(\Sigma)$. We examine if all the eigenvalues of $D\alpha(y_0)|_{T_y(\Sigma)}$ have a negative real part, this will imply the local stability (refer for example to Coddington and Levinson [2]). Let $D^2\psi(y)(\cdot,\cdot)$ denote the second differential of ψ at y, seen as a bilinear mapping. We have:

$$D\alpha(y) = D^2\psi(y)(F(\phi(y)), \cdot) + D\psi(y)D(F \circ \phi)(y). \tag{28}$$

The first term in this last equation is a matrix and its entry (i, j) is equal to:

$$\sum_{k=1}^{K} \frac{\partial^2 \psi(y)_i}{\partial y_j \partial y_k} F(\phi(y))_k.$$

For $y = y_0$, $F(\phi(y_0)) = (\lambda - e^{-1}/V)\mathbb{1}$, and a straightforward computation gives:

$$D^2\psi(y_0)(F(\phi(y_0)), \cdot) = (\lambda - e^{-1}/V)(J - KE),$$

where E is the identity matrix and J is the matrix with all its entries equal to 1. We also have $D\psi(y_0) = (KE - J)/K$. Finally, equation (28) can be rewritten as:

$$D\alpha(y_0) = 1/K(KE - J)\left(D(F \circ \phi)(y_0) - (\lambda - e^{-1}/V)E \right).$$

$(KE - J)$ commutes with all symmetric matrices and $(KE - J)$ has two eigenvalues K (with multiplicity $K-1$) and 0 (with multiplicity 1, associated to the eigenvector $\mathbb{1}$). By Lemma 6.5, the eigenvalues of $D(F \circ \phi)(y_0) - (\lambda - e^{-1}/V)E$ are

$$0 \le i \le K-1 : \ \mu_i = -e^{-1}(V - \nu_{K-i})^2/V^3 - \lambda + e^{-1}/V.$$

The eigenvector associated to $\mu_0 = \lambda - e^{-1}/V$ is $\mathbb{1}$. Thus we have proved that $\lambda - e^{-1}/V$ is an eigenvalue of multiplicity 1 for $D\alpha(y_0)$ and that the other eigenvalues are $(\mu_i)_{i \ge 1}$. These eigenvalues have negative real parts if and only if $\mu_1 = -e^{-1}\gamma^2/V^3 - \lambda + e^{-1}/V < 0$, that is $\lambda > e^{-1}(1 - \gamma^2/V^2)/V$. The vector space generated by the associated eigenvectors is precisely the tangent hyperplane $T_{y_0}(\Sigma) = \mathbb{1}^{\perp}$, the hyperplane orthogonal to $\mathbb{1}$. □

We can then prove Theorem 6.2. Let $|z(0)| \in C_u$ and $y(0) = z(0)/|z(0)|$, by Lemmas 6.6 and 6.7:

$$\lim_{t \to +\infty} \psi(z(t)) = \lim_{t \to +\infty} y(\mu(t)) = \mathbb{1}/K.$$

In particular, $\phi(z(t))$ tends towards $\mathbb{1}/V$ as t tends toward infinity.

7 Absence of Attraction to the Diagonal in One Particular Case

As it has already been pointed out in Section 1.3 and in the previous Section, the diagonal may not be locally stable for small enough values of λ. In this Section, we present an example of a graph for which there are locally stable sets of parameters away from the main diagonal if λ is sufficiently small.

Consider a graph $\mathcal{G}$ with 4 vertices placed on a circle. Number the vertices 1, 2, 3, 4 clockwise and assume that each vertex is linked with its 2 neighbours (so that, for example, vertex 1 has links with 2 and 4). In this case, $K = 4$ and $V = 3$ (see Figure 7).

For a service system associated with this graph, consider the equation (26). It is clear that we can write it in the form

$$\dot{y}_i(t) = \left(\lambda - \varphi_i(t) e^{-\sum_{j\in V_i} \varphi_j(t)}\right) - y_i \sum_{k=1}^{K} \left(\lambda - \varphi_k(t) e^{-\sum_{j\in V_k} \varphi_j(t)}\right), \quad i = 1, \dots, K.$$

We are interested in the so-called stable points of the latter system of differential equations, i.e. points for which all the RHS's in the system above are identically 0 and so if $y(0)$ is such a point, $y(t)$ stays at this point for all $t \geq 0$. Clearly, one stable point is $(1/K, \dots, 1/K)$, which corresponds to the diagonal. However, if $\lambda < \frac{e^{-1}}{V}\left(1 - \frac{\gamma^2}{V^2}\right)$ $(= 5/27e^{-1}$ in our case), then there exist other stable points.

Take $y_1(0) = y_2(0)$ and $y_3(0) = y_4(0)$. Since $y_1(0) + y_2(0) + y_3(0) + y_4(0) = 1$, the equality $y_3(0) = (1 - 2y_1(0))/2$ holds, and the system of differential equations at time $t = 0$ reduces to just one (i.e. any) of them. One can show that, for any $\lambda < 5/27e^{-1}$, the RHS of this equation equals 0 at three different points: at $\overline{y^{(1)}} = (1/4, 1/4, 1/4, 1/4)$ and at two others, say $\overline{y^{(2)}}$ and $\overline{y^{(3)}}$. One can find approximate values of these points numerically. For instance, if $\lambda = 0.001$, then $\overline{y^{(2)}} \approx (0.01, 0.01, 0.49, 0.49)$ and $\overline{y^{(3)}} \approx (0.49, 0.49, 0.01, 0.01)$. Numerical results also show that these points are locally stable.

8 Extensions of the Model

8.1 Random Neighbourhood

In this Subsection we consider a possible extension of our model. Assume there is a fixed number of points $1, \dots, K$ and a set of non-directed graphs $\{\mathcal{G}^j\}_{j=1}^{L}$ each having points $1, \dots, K$ as its vertices. Assume that at each time n the neighbourhood relations are given by the graph $\mathcal{G}^{\eta_n}$ where η_n are independent identically distributed random variables taking the value j with probability p_j. The need to consider such a variability of neigbourhood relations may be justified by, for instance, the fact that a change of environment conditions may lead to a change of the radius and/or direction of interactions.

Denote by $\mathcal{V}_i^j$ the neighbourhood of the point i in the graph $\mathcal{G}^j$ and by V_i^j its cardinal. We assume that the system is "regular" in some sense: $\mathbf{E}V_i^{\eta_1} = V$ for all i.

Following the proof of Theorem 2.7, one can show that fluid limits of the model described above satisfy the following differential equation

$$z_i^{'}(t) = \lambda - \sum_{k=1}^{L} p_k \varphi_i^k(t) e^{-\sum\limits_{j\in V_i^k} \varphi_j^k(t)}$$

where $\varphi_i^k(t)$ are defined in an obvious way. Using the same methods as those used in the proof of Theorem 1.1, it can be shown that the system with random neighbourhood is stable if $\lambda < \frac{e^{-1}}{V}$.

8.2 Non-Regular Graphs with Space-Inhomogeneous Input

Although Remark 4.2 provides sufficient conditions for stability in this case, they are not easy to verify. Here we give some other conditions that are also sufficient for the stability of the system. Assume now that $\mathbf{E}V_i^{\eta_1} = V_i$, and V_i are not necessarily equal. Put $V = \max_i V_i$. Assume also that $\mathbf{E}\xi_i^n = \lambda_i$, so that the input is "space-inhomogeneous". Put $\lambda = \max_i \lambda_i$. Clearly, all the results concerning fluid limits also hold in this case, and it is easy to see that one can prove the following result.

Theorem 8.1. *The system described above is stable provided* $\lambda < \frac{e^{-1}}{V}$.

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