$M/M/\infty$ QUEUES IN QUASI-MARKOVIAN RANDOM ENVIRONMENT

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ABSTRACT. In this paper we investigate an $M/M/\infty$ queue whose parameters depend on an external random environment that we assume to be a quasi-Markovian process with finite state space. For this model we show a recursive formula that allows to compute all the factorial moments for the number of customers in the system in steady state. The used technique is based on the calculation of the row moments of the area of a bidimensional random set. Finally some examples where it is possible to get explicit formulas are given together with comparisons with previous known results.

1. INTRODUCTION

The $M/M/\infty$ queue is one of the simplest model in queueing theory. This is due to the joint situation to have a memory-less arrival process and an infinite set of servers that allows customers to behave independently from each other. This suddenly stops to be true after introducing some correlation between customers. In this paper we achieve that by introducing an independent random environment that modulates the system parameters, i.e. the arrival rate and the server speeds. A similar study was already initiated by O'Cinneide and Purdue (1986) where the authors looked at the case when the environment is given by a finite state Markov process. For this case they showed how to compute the factorial moments for the number of customers in the system in steady state. Here we extend their analysis to the case of a quasi-Markovian random environment.

This extension is interesting both for exploiting the technique previously developed in D'Auria (2005) and for making the model more attractive for application purposes. Indeed, despite its simplicity, the $M/M/\infty$ system is often used to model pure delay systems, such as highways, satellite links or long communication cables, or to approximate the behavior of multi server systems. When these kinds of systems are subject to external influences, such as day time changing rates, it is then helpful to look at extended models, such as the one proposed in this work, in order to analyze or predict their behaviors.

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2. Model description

To start, we define the random environment $\{\Gamma(t), t \in \mathbb{R}\}$ with values in the finite state space $E = \{1, \ldots, K\}$. We assume that the sojourn time in the state $i \in E$, let T_i, is an independent positive random variable whose distribution function has Laplace transform denoted by $\tau_i(s) := \mathbb{E}[e^{-sT_i}]$. In the following we show that the Laplace transform is the only information we need to compute the moments. When the sojourn time in state $i \in E$ expires, the environment jumps to state $j \in E$ with probability p_{ij}. Denoting by $\mathbf{P} := \{p_{ij}\}_{i,j \in E}$ the routing matrix, we can define the reverse routing matrix

$$\mathbf{Q} := \mathbf{\Pi}^{-1}\, \mathbf{P}^t\, \mathbf{\Pi}, \tag{2.1}$$

where $\mathbf{\Pi} := \mathbf{diag}(\vec{\pi})$ and $\vec{\pi}$ is the stationary distribution of the Markov chain generated by $\mathbf{P}$ (see Brémaud, 1999).

We assume that when the environment is in state $i \in E$ customers arrive according to a Poisson rate $\lambda_i \geq 0$. Each of them brings an independent request of service, σ, that is exponentially distributed with rate μ. All servers work at constant speed $\beta_i = \mu_i/\mu \leq 1$. To avoid trivial cases we assume that $\mu, \lambda > 0$ where $\lambda := \max_{i \in E} \lambda_i$. By the results in D'Auria (2005) the system is stable and we are allowed to study its stationary regime.

We then look at the system at time 0 and we count the number of customers still in the system. We order them according to their arrival time $\{u_n\}_{n \in \mathbb{Z}}$ with $u_n < u_{n+1}$ and $u_{-1} < 0 \leq u_0$, and we denote by $G(\sigma)$ the common exponential distribution function of the $\{\sigma_n\}_{n \in \mathbb{Z}}$.

The n-th customer, $n < 0$, will be in the system at time 0 iff its service time, σ_n, is bigger than the work done by the server it has occupied during the time interval $[u_n, 0)$. We denote this quantity by $F_\Gamma(u_n)$ and, as the subscript shows, it is a random quantity that depends on the random environment Γ. Its value can be computed in the following way,

$$F_\Gamma(t) := \int_t^0 \beta_{\Gamma(u)} du. \tag{2.2}$$

Denoting by N the number of customers in stationary regime we have that it can be given by

$$N = \sum_{n<0} 1\{\sigma_n > F_\Gamma(u_n)\}, \tag{2.3}$$

where $1\{\cdot\}$ is the indicator function of the set $\{\cdot\}$.

An equivalent and more powerful form to write equation (2.3) is obtained by introducing the arrival point process

$$\mathcal{N}_\Gamma := \sum_{n \in \mathbb{Z}} \delta_{(u_n, \sigma_n)}, \tag{2.4}$$

that is a Random Point Process (RPP) (see Daley and Vere-Jones, 1988) which locates one Dirac delta measure to each arrival point $\{(u_n, \sigma_n)\}_{n \in \mathbb{Z}}$. The subscript Γ stays to denote that $\mathcal{N}_\Gamma$ depends on the random environment by the arrival times $\{u_n\}_{n\in\mathbb{Z}}$, such that given a realization of Γ the sequence belongs to an inhomogeneous Poisson process. More precisely, if we look at the conditional intensity rate $\lambda_\Gamma(A) := \mathbb{E}[\mathcal{N}_\Gamma(A)|\Gamma]$, it is given by

$$\lambda_\Gamma(A) = \int_A \lambda_{\Gamma(u)} du\, G(d\sigma). \tag{2.5}$$

Using $\mathcal{N}_\Gamma$, equation (2.3) reduces to the following

$$N = \mathcal{N}_\Gamma(A_\Gamma), \tag{2.6}$$

that is the measure of the random set

$$A_\Gamma := \{(u,\sigma) \in \mathbb{R}^- \times \mathbb{R}^+ : \sigma > F_\Gamma(u)\}, \tag{2.7}$$

by the random point measure $\mathcal{N}_\Gamma$. Since, given Γ, the point measure $\mathcal{N}_\Gamma$ is a Poisson measure we get that, given Γ, N is a Poisson random variable with parameter $|A|_\Gamma := \lambda_\Gamma(A_\Gamma)$. Therefore N can be finally written in the following simplified way

$$N = \mathrm{Po}(|A|_\Gamma). \tag{2.8}$$

Remark 2.1. A more correct but clumsy notation for $|A|_\Gamma$ would have been $|A_\Gamma|_\Gamma$ since both the set and the intensity measure are functions of the environment.

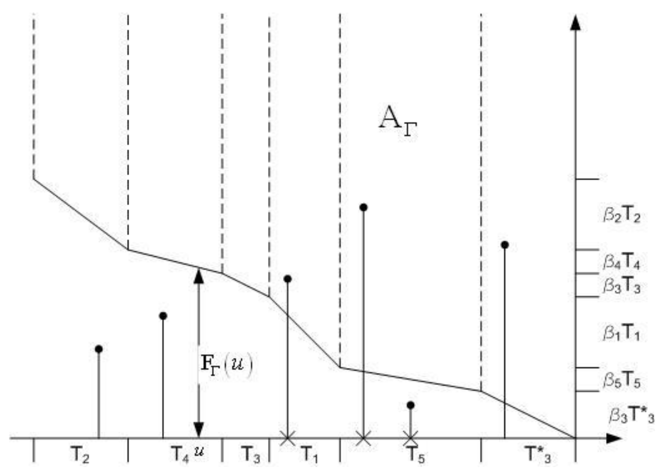

FIGURE 1. Example of realization.

Figure 1 shows an example of realization where the random environment has 5 states: the dots are the centers of the Dirac deltas of the RPP $\mathcal{N}_\Gamma$, while the piecewise linear function $F_\Gamma(u)$ denotes the lower bound of the set of integration A_Γ. The customers present in the system at time 0 are then the ones whose dots fall in the set A_Γ; in the shown example $N = 3$.

3. Computing the factorial moments

Before beginning to compute the factorial moments of the random variable N, it is worthwhile to review some basic results about the different kinds of moments and their relations with the various generating functions. A good reference about the following relations especially in connection with Random Point Processes is Daley and Vere-Jones (1988), Chapter 5.

Given a random variable X, we denote by $\psi_X(s) := \mathbb{E}[e^{sX}]$ its moment generating function and by $\phi_X(z) := \mathbb{E}[z^X]$ its probability generating function.

The factorial moment of order i of X, $f_X^{(i)}$ is defined as

$$f_X^{(i)} := \mathbb{E}\left[(X)_i\right] = \sum_{n=0}^{\infty} (n)_i p_n,$$

where $p_n = \Pr\{X = n\}$ and $(n)_i := n(n-1)\cdots(n-i+1)$ is the falling factorial. It can be directly computed by the i-th derivative of the probability generating function, i.e. $f_X^{(i)} = \lim_{z\to 1} \phi_X^{(i)}(z)$. Knowing the factorial moments it is then easy to compute the raw moments. Indeed, by taking the expectations to both sides of the following known equivalence (Abramowitz and Stegun, 1964)

$$X^n = \sum_{i=0}^{n} \mathfrak{S}_n^{(i)} (X)_i,$$

where $\mathfrak{S}_n^{(i)}$ is a Stirling Number of the Second Kind, we obtain the following relation between the n-th moment of X, $m_X^{(n)} := \mathbb{E}\left[X^n\right]$, and the factorial moments of order $i \le n$,

$$m_X^{(n)} = \sum_{i=0}^{n} \mathfrak{S}_n^{(i)} f_X^{(i)}. \tag{3.1}$$

The reverse relation is obtained by using the Stirling Numbers of the First Kind, $\mathfrak{s}_i^{(n)}$ (see Abramowitz and Stegun, 1964), that satisfy the following known relation

$$(X)_i = \sum_{n=0}^{i} \mathfrak{s}_i^{(n)} X^n,$$

so that, taking the expectations of both sides, finally we get

$$f_X^{(i)} = \sum_{n=0}^{i} \mathfrak{s}_i^{(n)} m_X^{(n)}. \tag{3.2}$$

It is interesting to notice that relation (3.1) comes directly from using the fact that $\psi_X(s) = \phi_X(e^s)$ and that $m_X^{(n)} = \lim_{s\to 0} \psi_X^{(n)}(s)$. Indeed,

$$\lim_{s\to 0} \psi_X^{(n)}(s) = \lim_{s\to 0} \frac{d^n}{ds^n} \phi_X(e^s) = \sum_{i=0}^{n} \mathfrak{S}_n^{(i)} \phi_X^{(i)}(1),$$

where in the last equation we used Faá di Bruno's formula for the expansion of derivatives of order n for composition of functions (see Abramowitz and Stegun, 1964) and the fact that $\lim_{s\to 0} \frac{d^n}{ds^n} e^s = 1$.

If X turns out to be a randomized Poisson random variable, i.e. $X = \mathrm{Po}(Y)$ with Y an additional random variable, we have that

$$\phi_X(z) = \psi_Y(z-1),$$

so that taking the derivatives of order n, we get

$$\lim_{z\to 1} \phi_X^{(n)}(z) = \lim_{z\to 1} \psi_Y^{(n)}(z-1) = \lim_{s\to 0} \psi_Y^{(n)}(s),$$

or, in other words, that the factorial moments of X are directly the raw moments of Y,

$$f_X^{(n)} = m_Y^{(n)},$$

and the latter often is easier to compute.

This is exactly what happens in our case where, as shown by relation (2.8), N is a randomized Poisson and that is why we are interested into its factorial moments rather then directly its raw moments. Indeed we have that the following relation holds

$$f_N^{(n)} = m_{|A|_\Gamma}^{(n)}, \tag{3.3}$$

and our task reduces to the computation of the raw moments of the area of the random set A_Γ.

4. Computing the row moments of $|A|_\Gamma$

In this section we compute the row moments of the measure of the set A_Γ, defined in (2.7), when measured by the random intensity measure λ_Γ, defined in (2.5). We use a fixed point technique and to this aim we look at a modified environment process, Γ_0, that is the Palm version of the process Γ, i.e. we assume that in 0 it has a transition. We denote by $n \in E$ the first state it has assumed before 0, i.e. $n := \Gamma_0(0^-)$, and by T_n its corresponding sojourn time. While, as depicted in Figure 1, for the process Γ the sojourn time in the first state before 0 would be given by a residual sojourn time, for the process Γ_0 it is distributed as any other sojourn time corresponding to the same state. We define by $A_{0n} := (A_{\Gamma_0} | \Gamma_0(0^-) = n)$, $n \in E$, the set A_{Γ_0} conditioned to the event that the first state occupied by the environment before 0 is the state n, and we call $|A_{0n}|$ its measure.

Figure 2 shows as example the set A_{0n} when $n = 3$, together with its decomposition in the set C_3 and the set $\mathcal{T}_{\beta_3 T_3} A_{05}$. To this we have defined by C_n the restriction of the set A_{0n} up the first transition of the process Γ_0, i.e.

$$C_n := A_{0n} \cap \{(x,y) \in \mathbb{R}^- \times \mathbb{R}^+ | |x| < T_n\}, \tag{4.1}$$

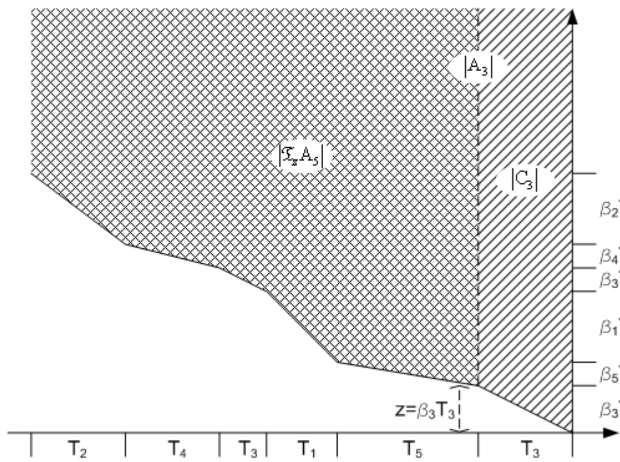

FIGURE 2. Decomposition of $|A_{03}|$ as $|C_3| + |\mathcal{T}_z A_{05}|$.

and by $\mathcal{T}_z A$ the z-translated version of the set A in the vertical direction, i.e.

$$\mathcal{T}_z A := \{(x, y) \in \mathbb{R}^2 | (x, y - z) \in A\}. \tag{4.2}$$

We denote by $j \in E$ the state of the environment before the first transition before 0, i.e. $j := \Gamma_0(-T_n^-)$, so that, $-T_n$ being a regeneration point for the process Γ_0, we have the independence of the sets C_n and $\mathcal{T}_{\beta_n T_n} A_{0j}$ conditioned to the values of the states before and after the transitions, i.e. j and n. $\beta_n T_n = F_{\Gamma_0}(-T_n)$ is the exact amount of work the non-empty servers have done during the time interval $[-T_n, 0)$ being in state n.

Thanks to this we can write down the following set of stochastic equations

$$|A_{0n}| \stackrel{\mathrm{d}}{=} |C_n| + \sum_{j=1}^{K} 1\{n \leftarrow j\} |\mathcal{T}_{\beta_n T_n} A_{0j}|, \tag{4.3}$$

where the indicator function $1\{n \leftarrow j\}$ selects the backward state transition of the environment from the state n to the state j; this would happen, according to definition (2.1), with probability q_{nj}.

Thanks to the fact that along the vertical axis the measure function is given by G that is exponential we have that the following lemma holds:

Lemma 4.1. *Given the transformation $\mathcal{T}_z$, defined in (4.2), we have that*

$$|\mathcal{T}_z A| = e^{-\mu z} |A|, \tag{4.4}$$

for any set $A \subset \mathbb{R}^- \times \mathbb{R}^+$.

Proof. By using definition (2.5) we have

$$\lambda_{\Gamma_0}(\mathcal{T}_z A) = \int_{\mathcal{T}_z A} \lambda_{\Gamma_0(u)} du\, e^{-\mu\sigma} d\sigma$$
$$= e^{-\mu z} \int_{\mathcal{T}_z A} \lambda_{\Gamma_0(u)} du\, e^{-\mu(\sigma - z)} d\sigma = e^{-\mu z} \lambda_{\Gamma_0}(A).$$

□

By using Lemma 4.1, equation (4.3) simplifies in the following

$$|A_{0n}| \overset{\mathrm{d}}{=} |C_n| + e^{-\mu\beta_n T_n} \sum_{j=1}^{K} 1\{n \leftarrow j\} |A_{0j}|, \tag{4.5}$$

that is the starting point to prove the following main result:

Theorem 4.2. *Let us define* $\vec{m}_0^{(i)} \in \mathbb{R}^K$ *as the column vector whose* j*-th coordinate is the* i*-th moment of the random variable* $|A_{0j}|$*, i.e.* $m_{0j}^{(i)} := m_{|A_{0j}|}^{(i)}$ *then the following relation holds*

$$\sum_{j=0}^{n} (-1)^j \binom{n}{j} \mathbf{R}^{n-j} \mathbf{B}_n\, \vec{m}_0^{(j)} = 0, \tag{4.6}$$

where $\mathbf{R} := \mathbf{diag}\,(\rho_i)$*,* $\rho_i := \lambda_i/\mu_i$ *and the matrix* $\mathbf{B}_n := \mathbf{diag}\left(\tau_i^{-1}(n\mu_i)\right) - \mathbf{Q}$*. It is then possible to express the* n*-th moment vector* $\vec{m}_0^{(n)}$ *in terms of the previous ones,* $\vec{m}_0^{(j)}$*,* $j = 0, \dots, n-1$*, in the following way*

$$\vec{m}_0^{(n)} = \sum_{j=0}^{n-1} (-1)^{n-1-j} \binom{n}{j} \mathbf{B}_n^{-1} \mathbf{R}^{n-j} \mathbf{B}_n\, \vec{m}_0^{(j)}. \tag{4.7}$$

Proof. We first compute the values of the variable $|C_n|$ in the following way

$$C_n = \lambda_n \int_0^{T_n} e^{-\mu_n x} dx = \rho_n (1 - e^{-\mu_n T_n}).$$

Then substituting its value in equation (4.5), it gives

$$|A_{0n}| \overset{\mathrm{d}}{=} \rho_n (1 - e^{-\mu_n T_n}) + e^{-\mu_n T_n} \sum_{j=1}^{K} 1\{n \leftarrow j\} |A_{0j}|, \tag{4.8}$$

that can be rewritten as

$$|A_{0n}| - \rho_n \overset{\mathrm{d}}{=} \sum_{j=1}^{K} 1\{n \leftarrow j\} (|A_{0j}| - \rho_n) e^{-\mu_n T_n}. \tag{4.9}$$

We denote by $\psi_{0n}(s) := \mathbb{E}\left[e^{s|A_{0n}|}\right]$ the moment generating function of $|A_{0n}|$ so that applying the exponential function to both members of equation (4.9) previously multiplied by s and then taking the expectation, we obtain

$$\begin{aligned}\psi_{0n}(s)e^{-s\rho_n} &= \mathbb{E}\left[\sum_{j=1}^{K} q_{nj} e^{s(A_{0j}-\rho_n)e^{-\mu_n T_n}}\right] \\ &= \mathbb{E}\left[\sum_{j=1}^{K} q_{nj}\psi_{0j}(se^{-\mu_n T_n})e^{-s\rho_n e^{-\mu_n T_n}}\right].\end{aligned}$$

Last expression can be written in matrix form in the following way

$$e^{-s\mathbf{R}}\vec{\psi}_0(s) = \mathbb{E}\left[e^{-s\mathbf{RT}}(\mathbf{Q}\vec{\psi}_0)(s\mathbf{T})\right], \tag{4.10}$$

where $\mathbf{T} := \mathbf{diag}(e^{-\mu_n T_n})$ and where with notation $\vec{v}(\mathbf{W})$, with $\mathbf{W}$ a diagonal matrix, we denote a vector whose j-th component is $v_j(w_{jj})$. We use then the following matrix formulas for derivatives

$$D^{(n)}[e^{-s\mathbf{W}}\vec{v}(s)] = \sum_{j=0}^{n}(-1)^{n-j}\binom{n}{j}e^{-s\mathbf{W}}\mathbf{W}^{n-j}D^{(j)}[\vec{v}(s)], \tag{4.11}$$

and

$$D^{(n)}[\vec{v}(s\mathbf{W})] = \mathbf{W}^n\vec{v}^{(n)}(s\mathbf{W}), \tag{4.12}$$

to compute the n-th derivative of both sides of equation (4.10) so that

$$\begin{aligned}&\sum_{j=0}^{n}(-1)^{n-j}\binom{n}{j}e^{-s\mathbf{R}}\mathbf{R}^{n-j}\vec{\psi}_0^{(j)}(s) = \\ &\qquad = \mathbb{E}\left[\sum_{j=0}^{n}(-1)^{n-j}\binom{n}{j}e^{-s\mathbf{RT}}\mathbf{R}^{n-j}\mathbf{T}^{n-j}D^{(j)}[\mathbf{Q}\vec{\psi}_0(s\mathbf{T})]\right] \\ &\qquad = \mathbb{E}\left[\sum_{j=0}^{n}(-1)^{n-j}\binom{n}{j}e^{-s\mathbf{RT}}\mathbf{R}^{n-j}\mathbf{T}^{n}(\mathbf{Q}\vec{\psi}_0^{(j)})(s\mathbf{T})\right].\end{aligned}$$

Remembering that $\vec{m}_0^{(n)} = \lim_{s\to 0}\psi_0^{(n)}(s)$ and taking the limit of last expression as $s \to 0$, we get

$$\sum_{j=0}^{n}(-1)^{n-j}\binom{n}{j}\mathbf{R}^{n-j}\vec{m}^{(j)} = \mathbb{E}\left[\sum_{j=0}^{n}(-1)^{n-j}\binom{n}{j}\mathbf{R}^{n-j}\mathbf{T}^{n}\mathbf{Q}\vec{m}_0^{(j)}\right]. \tag{4.13}$$

Multiplying on the left side by $(-1)^{-n}\mathbb{E}[\mathbf{T}^n]^{-1}$, the last expression can be easily rearranged in

$$\sum_{j=0}^{n}(-1)^{-j}\binom{n}{j}\mathbf{R}^{n-j}[\mathbb{E}\,[\mathbf{T}^n]^{-1} - \mathbf{Q}]\vec{m}_0^{(j)} = 0, \tag{4.14}$$

that gives the result. □

It is remarkable that it is possible to express equation (4.6) in terms of the forward transition chain $\mathbf{P}$. The result is contained in the following corollary.

Corollary 4.3. *A result similar to equation (4.6) is valid for the row vector* $\vec{m}_0^{,(i)} := \left(\vec{m}_0^{(i)}\right)^t$*, that involves the matrix* $\mathbf{P}$ *instead of the matrix* $\mathbf{Q}$*, i.e.*

$$\sum_{j=0}^{n}(-1)^j\binom{n}{j}\vec{m}_0^{,(j)}\,\mathbf{\Pi}\,\mathbf{B}'_n\,\mathbf{R}^{n-j}=0, \tag{4.15}$$

where the matrix $\mathbf{B}'_n := \mathbf{diag}\left(\tau_i^{-1}(n\mu_i)\right) - \mathbf{P}$

Proof. After taking the transposition of both sides of equation (4.14) we get

$$0=\sum_{j=0}^{n}(-1)^j\binom{n}{j}\vec{m}_0^{,(j)}\,[\mathbb{E}\,[\mathbf{T}^n]^{-1}-\mathbf{Q}^t]\mathbf{R}^{n-j}$$

and multiplying on the right by $\mathbf{\Pi}$

$$\begin{aligned}&=\sum_{j=0}^{n}(-1)^j\binom{n}{j}\vec{m}_0^{,(j)}\mathbf{\Pi}\,\mathbf{\Pi}^{-1}\,[\mathbb{E}\,[\mathbf{T}^n]^{-1}-\mathbf{Q}^t]\mathbf{\Pi}\mathbf{R}^{n-j}\\&=\sum_{j=0}^{n}(-1)^j\binom{n}{j}\vec{m}_0^{,(j)}\mathbf{\Pi}\,[\mathbb{E}\,[\mathbf{T}^n]^{-1}-\mathbf{\Pi}^{-1}\mathbf{Q}^t\mathbf{\Pi}]\mathbf{R}^{n-j}\end{aligned}$$

that gives the proof taking into account equation (2.1). □

Given the raw moments of the area $|A_{\Gamma_0}|$, we can successively compute the moments of the areas of the sets $A_j := (A_\Gamma|\Gamma(0)=j)$, $j\in E$. Following previous definitions we define $m_j^{(i)} := m_{|A_j|}^{(i)}$. Similarly to equation (4.3) we have the following equation

$$|A_n| \stackrel{\mathrm{d}}{=} |C_n^*| + \sum_{j=1}^{K} 1\{n\leftarrow j\}|\mathcal{T}_{\beta_n T_n^*}A_{0j}|, \tag{4.16}$$

with $|C_n^*| = \rho_n(1-e^{-\mu_n T_n^*})$. T_n^* refers to a residual sojourn time of the environment in state $n\in E$; this means that the Laplace transform of its distribution is given by $\tau_n^*(s)=\bar{\tau}_n(1-\tau_n(s))/s$, with $\bar{\tau}_n := \mathbb{E}[T_n]^{-1}$. For the vector of row moments $\vec{m}^{(n)}$ the following theorem holds.

Theorem 4.4. *The vector* $\vec{m}^{(n)}$ *satisfies the following relation with the vector* $\vec{m}_0^{(n)}$

$$\sum_{j=0}^{n}(-1)^j\binom{n}{j}\mathbf{R}^{n-j}[\vec{m}^{(j)}-\mathbf{E}_n\vec{m}_0^{(j)}]=0, \tag{4.17}$$

with $\mathbf{E}_n := \mathbf{diag}(\tau_i^*(n\mu_i)/\tau_i(n\mu_i))$. *Therefore the vector* $\vec{m}^{(n)}$ *can be computed from the previous moments* $\{\vec{m}^{(j)}\}_{j<n}$ *and the corresponding vectors* $\{\vec{m}_0^{(j)}\}_{j\le n}$ *in the following way*

$$\vec{m}^{(n)} = \mathbf{E}_n\vec{m}_0^{(n)} + \sum_{j=0}^{n-1}(-1)^{n-1-j}\binom{n}{j}\mathbf{R}^{n-j}[\vec{m}^{(j)} - \mathbf{E}_n\vec{m}_0^{(j)}], \tag{4.18}$$

finally

$$m^{(n)} := m_{A_\Gamma}^{(n)} = \vec{m}^{(n)}\vec{\pi}. \tag{4.19}$$

Proof. Starting by equation (4.16) and following the same calculations that brought us from equation (4.3) to equation (4.14), we get

$$\sum_{j=0}^{n}(-1)^j\binom{n}{j}\mathbf{R}^{n-j}[\mathbb{E}\,[\mathbf{T}^{*n}]^{-1}\,\vec{m}^{(j)} - \mathbf{Q}\vec{m}_0^{(j)}] = 0, \tag{4.20}$$

that after subtracting equation (4.14) gives

$$\sum_{j=0}^{n}(-1)^j\binom{n}{j}\mathbf{R}^{n-j}[\mathbb{E}\,[\mathbf{T}^{*n}]^{-1}\,\vec{m}^{(j)} - \mathbb{E}\,[\mathbf{T}^{n}]^{-1}\,\vec{m}_0^{(j)}] = 0, \tag{4.21}$$

and by multiplying on the right by $\mathbb{E}\,[\mathbf{T}^{*n}]$ we finally get the result. □

In order to check our results we compare equation (4.15) for the exponential case with results in O'Cinneide and Purdue (1986) here repeated in formula (4.22). For this case since $T_i^* \sim T_i$, we have that $\vec{m}^{(n)} = \vec{m}_0^{(n)}$.

Remark 4.5. It is worth to notice that in O'Cinneide and Purdue (1986), they actually computed the factorial moments of the random row vector $(N\,1\{\Gamma_0 = i\})_{i\in E}$ while here we compute the factorial moments of the row vector $(N|\Gamma_0 = i)_{i\in E}$. This explains the presence, in formula (4.22), of the additional factor given by matrix Π.

Corollary 4.6. *In case the sojourn times* T_j *are exponentially distributed with parameters* $\bar{\tau}_j$ *we have that with* $n > 0$

$$\vec{m}'^{(n)}\mathbf{\Pi}\,(n\mathbf{M} - \mathbf{G}) = n\,\vec{m}'^{(n-1)}\,\mathbf{\Lambda} \tag{4.22}$$

where $\mathbf{M} := \mathbf{diag}(\mu_i)$, $\mathbf{\Lambda} := \mathbf{diag}(\lambda_i)$ *and* $\mathbf{G} := \bar{\mathbf{T}}(\mathbf{I}-\mathbf{P})$, *with* $\bar{\mathbf{T}} := \mathbf{diag}(\bar{\tau}_j)$, *is the generator of the Environment that turns out to be a Markov Process.*

Proof. When the sojourn times T_j are exponentially distributed we have that

$$\mathbf{T}^{-n} = n\mathbf{M}\bar{\mathbf{T}}^{-1} + \mathbf{I}.$$

In equation (4.7) we have that

$$\mathbf{B}_n^{-1}\mathbf{R}^{n-j}\mathbf{B}_n = [\bar{\mathbf{T}}\mathbf{B}_n]^{-1}\mathbf{R}^{n-j}[\bar{\mathbf{T}}\mathbf{B}_n]$$

and by defining the generator of the reverse-time Markov process $\mathbf{H} := \bar{\mathbf{T}}(\mathbf{Q} - \mathbf{I})$

$$= [n\mathbf{M} - \mathbf{H}]^{-1}\mathbf{R}^{n-j}[n\mathbf{M} - \mathbf{H}],$$

so that we can rewrite equation (4.6) as

$$\sum_{j=0}^{n}(-1)^{n-j}\binom{n}{j}\mathbf{R}^{n-j}[n\mathbf{M} - \mathbf{H}]\,\vec{m}^{(j)} = 0. \tag{4.23}$$

Since $\vec{m}^{(0)} = \vec{1}$ we have $\mathbf{H}\vec{m}^{(0)} = 0$, and therefore equation (4.23) coincides with equation (A.3) when $\vec{v}^{(k)} := \vec{m}^{(k)}$ and $\mathbf{H}_k = \mathbf{H}$. This implies that the assumption expressed in (A.1) gives the only possible solution for the $\{\vec{m}^{(k)}\}_{k\in\mathbb{N}}$.

By taking the transposition of equation (A.1) we get

$$\vec{m}'^{(k)}(\mathbf{\Pi}\,\mathbf{\Pi}^{-1})(k\mathbf{M} - \mathbf{H}^t) = k\,\vec{m}'^{(k-1)}\mathbf{\Lambda},$$

that after multiplying on the right side by $\mathbf{\Pi}$ and simplifying reduces to

$$\vec{m}'^{(k)}\mathbf{\Pi}(k\mathbf{M} - \mathbf{\Pi}^{-1}\mathbf{H}^t\mathbf{\Pi}) = k\,\vec{m}'^{(k-1)}\mathbf{\Lambda},$$

that gives the result noticing that $\mathbf{G} = \mathbf{\Pi}^{-1}\mathbf{H}^t\mathbf{\Pi}$. $\square$

5. Some explicit formulas - Case $K = 2$

Formulas (4.7) and (4.18) show that generally to find the n-th moment of the random number of users in the system involves in a complex way the knowledge of all previous moments. Indeed, the exponential case, that was already solved in O'Cinneide and Purdue (1986), is easier as the n-th vector of moments is related only by a factor to the $(n-1)$-th one. That was anyway hidden in a non-trivial way in formula (4.7) so that there could be some other special cases where an easier expression holds. In this section we give a look to the case when the environment has only two stages, i.e. $K = 2$. That is a very special case; indeed, by assuming all sojourn times exponentially distributed, it is known how to compute the complete distribution of the number of customers in the system at steady state (see Keilson and Servi (1993), Baykal-Gursoy and Xiao (2004) and D'Auria (2005)).

By expliciting equation (4.9) for the case $K = 2$ we get

$$|A_{01}| - \rho_1 \overset{\mathrm{d}}{=} (|A_{02}| - \rho_1)e^{-\mu_1 T_1} \tag{5.1}$$

$$|A_{02}| - \rho_2 \overset{\mathrm{d}}{=} (|A_{01}| - \rho_2)e^{-\mu_2 T_2}. \tag{5.2}$$

We define $\tilde{m}_{0i}^{(k)} := \mathbb{E}[(|A_{0i}| - \rho_1)^k]$ and take the mean of the n-powers of expression (5.1) so getting

$$\tilde{m}_{01}^{(n)} = \tilde{m}_{02}^{(n)}\tau_1(n\mu_1). \tag{5.3}$$

By adding and subtracting ρ_1 to both sides of equation (5.2) we get

$$(|A_{02}| - \rho_1) - \rho_* \overset{\mathrm{d}}{=} (|A_{01}| - \rho_1)e^{-\mu_2 T_2} - \rho_* e^{-\mu_2 T_2},$$

with $\rho_* = \rho_2 - \rho_1$. Then using equation (5.1) we obtain a recursive equation involving only $|A_{02}| - \rho_1$,

$$(|A_{02}| - \rho_1) - \rho_* \stackrel{\mathrm{d}}{=} (|A_{02}| - \rho_1) e^{-\mu_1 T_1} e^{-\mu_2 T_2} - \rho_* e^{-\mu_2 T_2}.$$

Taking the n-th power and then the expectation of both sides we get

$$\sum_{k=0}^{n} (-1)^{n-k} \binom{n}{k} \rho_*^{n-k} \, \tilde{m}_{02}^{(k)} = \tau_2(n\mu_2) \, \mathbb{E}[(|A_{02}| - \rho_1) e^{-\mu_1 T_1} - \rho_*] \\ = \tau_2(n\mu_2) \sum_{k=0}^{n} (-1)^{n-k} \binom{n}{k} \rho_*^{n-k} \tau_1(k\mu_1) \, \tilde{m}_{02}^{(k)},$$

that, taking into account equation (5.3), can be rearranged to get the following

$$\sum_{k=0}^{n} (-1)^{n-k} \binom{n}{k} \rho_*^{n-k} (\tau_2^{-1}(n\mu_2) \tau_1^{-1}(k\mu_1) - 1) \, \tilde{m}_{01}^{(k)} = 0. \tag{5.4}$$

Theorem 5.1. *Assuming that the sojourn times of state 2 are exponentially distributed, i.e. $T_2 \sim Exp(\bar{\tau}_2)$, the solution of formula (5.4) is given by*

$$\tilde{m}_{01}^{(n)} = \left(\frac{\mu_2 \, \rho_*}{\bar{\tau}_2} \right)^n \prod_{k=1}^{n} \frac{k \, \tau_1^{-1}((k-1)\mu_1)}{\tau_1^{-1}(k\mu_1) \tau_2^{-1}(k\mu_2) - 1}, \tag{5.5}$$

and therefore

$$\tilde{m}_{02}^{(n)} = \left(\frac{\mu_2 \, \rho_*}{\bar{\tau}_2} \right)^n \tau_1^{-1}(n\mu_1) \prod_{k=1}^{n} \frac{k \, \tau_1^{-1}((k-1)\mu_1)}{\tau_1^{-1}(k\mu_1) \tau_2^{-1}(k\mu_2) - 1}. \tag{5.6}$$

Finally

$$m_{0i}^{(n)} = \sum_{k=0}^{n} \binom{n}{k} \rho_1^{n-k} \tilde{m}_{0i}^{(k)}. \tag{5.7}$$

Proof. Substituting $\tau_2^{-1}(s) = 1 + s/\bar{\tau}_2$ in equation (5.4) and rearranging it, we get

$$\sum_{k=0}^{n} (-1)^{n-k} \binom{n}{k} \rho_*^{n-k} \left(n \frac{\mu_2}{\bar{\tau}_2} - (\tau_1(k\mu_1) - 1) \right) \frac{\tilde{m}_{01}^{(k)}}{\tau_1(k\mu_1)} = 0. \tag{5.8}$$

Then applying Lemma A.1 in the scalar case, with $\mathbf{M} = (\mu_2/\bar{\tau}_2)$, $\mathbf{R} = (\rho_*)$, $\mathbf{H}_k = (\tau_1(k\mu_1) - 1)$ and $\vec{v}^{\,(k-1)} = (\tilde{m}_{01}^{(k)}/\tau_1(k\mu_1))$, we notice that a set of solutions is given by

$$\left(\tau_2^{-1}(k\mu_2) - \tau_1(k\mu_1)\right) \frac{\tilde{m}_{01}^{(k)}}{\tau_1(k\mu_1)} = k \, \rho_* \frac{\mu_2}{\bar{\tau}_2} \, \frac{\tilde{m}_{01}^{(k-1)}}{\tau_1((k-1)\mu_1)}, \tag{5.9}$$

that is then uniquely defined given that $\tilde{m}_{01}^{(0)} = 1$. Therefore equation (5.5) holds. Equation (5.6) results by applying (5.3) to (5.5) and finally equation (5.7) comes from the fact that $m_{0i}^{(n)} = \mathbb{E}[((|A_{0i}| - \rho_1) + \rho_1)^k]$. □

Example 5.2. **Case $T_1 \sim$ Exp$(\bar{\gamma}_1)$.** In this special case equation (5.5) simplifies in

$$\tilde{m}_{01}^{(n)} = \rho_*^n \frac{(\tau_1/\mu_1)_n}{(\tau_1/\mu_1 + \tau_2/\mu_2)_n},$$

with $(\cdot)_n$ being the falling factorial. Therefore the moment generating function of $|A_{01}| - \rho_1$ is given by the Kummer function $M(\tau_1/\mu_1,\, \tau_1/\mu_1 + \tau_2/\mu_2,\, \rho_* s)$ (see Abramowitz and Stegun (1964)), in accordance to what is shown in Baykal-Gursoy and Xiao (2004) and D'Auria (2005).

Example 5.3. **Case $T_1 \sim$ Gamma$(\kappa, 1/\bar{\gamma}_1)$.** For this case we have that $\tau_1^{-1}(k\mu_1) = (1 + k\mu_1/\bar{\tau}_1)^\kappa$. Therefore equation (5.5) simplifies in

$$\tilde{m}_{01}^{(n)} = \frac{\rho_*^n [(\tau_1/\mu_1)_n]^\kappa}{\prod_{k=1}^n \left[(\tau_1/\mu_1 + k)^\kappa (\tau_2/\mu_2 + k) - (\tau_2/\mu_2)(\tau_1/\mu_1)^\kappa\right]}.$$

6. Conclusions

In this paper we showed that using a matrix-geometric approach it is possible to solve the problem to find the factorial moments of the random number of customers in an M/M/∞ system when its parameters are modulated by a quasi-markovian random environment. We showed that this is possible by looking at this random variable as the random measure of a bidimensional random set by a modulated Poisson random measure. Finally the case when the environment has only 2 states is more deeply investigated and it is shown that explicit formulas are obtainable given that one state has exponential sojourn times. It is then plausible to believe that for this last case it would be possible to get an explicit expression for the complete characteristic function.

Appendix A. Technical Lemmas

Lemma A.1. *If for $k > 0$*

$$(k\mathbf{M} - \mathbf{H}_k)\vec{v}^{(k)} = k\,\mathbf{R}\mathbf{M}\,\vec{v}^{(k-1)}, \tag{A.1}$$

where $\mathbf{M}, \mathbf{R}, \{\mathbf{H}_k\}_{k\geq 0}$ are general matrices with $\mathbf{H}_0 v^{(0)} = 0$, than the following relation holds for $k \leq n$

$$\sum_{j=0}^{k-1} (-1)^{n-1-j} \binom{n}{j} \mathbf{R}^{n-j} [n\mathbf{M} - \mathbf{H}_j]\vec{v}^{(j)} = (-1)^{n-k} \binom{n}{k} \mathbf{R}^{n-k} \left[k\mathbf{M} - \mathbf{H}_k\right]\vec{v}^{(k)}. \tag{A.2}$$

In particular for $k = n$ we have

$$\sum_{j=0}^{n} (-1)^{n-j} \binom{n}{j} \mathbf{R}^{n-j} [n\mathbf{M} - \mathbf{H}_j]\vec{v}^{(j)} = 0. \tag{A.3}$$

Proof. As base of induction assume $k = 1$ then we have

$$\begin{aligned}(-1)^{n-1}\mathbf{R}^n[n\mathbf{M} - \mathbf{H}_0]\vec{v}^{(0)} &= (-1)^{n-1}\mathbf{R}^n[n\mathbf{M}]\vec{v}^{(0)} \\ &= (-1)^{n-1}n\mathbf{R}^{n-1}\mathbf{R}\mathbf{M}\vec{v}^{(0)} \\ &= (-1)^{n-1}\binom{n}{1}\mathbf{R}^{n-1}(\mathbf{M} - \mathbf{H}_1)\vec{v}^{(1)}\end{aligned}$$

Now assuming the relation valid for $k < n$ we have for $k + 1$ that

$$\begin{aligned}&\sum_{j=0}^{k}(-1)^{n-1-j}\binom{n}{j}\mathbf{R}^{n-j}[n\mathbf{M} - \mathbf{H}_j]\vec{v}^{(j)} \\ &\quad= (-1)^{n-k}\binom{n}{k}\mathbf{R}^{n-k}\,[k\mathbf{M} - \mathbf{H}_k]\vec{v}^{(k)} + (-1)^{n-1-k}\binom{n}{k}\mathbf{R}^{n-k}[n\mathbf{M} - \mathbf{H}_k]\vec{v}^{(k)} \\ &\quad= (-1)^{n-1-k}\binom{n}{k}\mathbf{R}^{n-k}(n-k)\mathbf{M}\vec{v}^{(k)} \\ &\quad= (-1)^{n-1-k}\binom{n}{k}\mathbf{R}^{n-k-1}(n-k)\mathbf{R}\mathbf{M}\vec{v}^{(k)}\end{aligned}$$

and by equation (A.1),

$$\begin{aligned}&= (-1)^{n-1-k}\binom{n}{k}\frac{n-k}{k+1}\mathbf{R}^{n-k-1}((k+1)\mathbf{M} - \mathbf{H}_{k+1})\vec{v}^{(k+1)} \\ &= (-1)^{n-(k+1)}\binom{n}{k+1}\mathbf{R}^{n-(k+1)}((k+1)\mathbf{M} - \mathbf{H}_{k+1})\vec{v}^{(k+1)}\end{aligned}$$

□

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