An expansion for self-interacting random walks

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December 19, 2006

Abstract

We derive a perturbation expansion for general interacting random walks, where steps are made on the basis of the history of the path. Examples of models where this expansion applies are reinforced random walk, excited random walk, the true (weakly) self-avoiding walk and loop-erased random walk. We use the expansion to prove a law of large numbers and central limit theorem for two models: (i) A directed version of once-reinforced random walk on $\mathbb{Z}^d$ for sufficiently small reinforcement parameters. This model is such that if the reinforcement parameter is set to zero, then the resulting random walk has independent increments with a non-zero drift; and (ii) Excited random walk in dimension $d > 8$ when the excitement parameter is sufficiently small.

1 Introduction

Recently, many models have been introduced of random walks with a certain self-interaction. A few examples are self-reinforced random walks [8, 24, 25], excited random walks [2, 20, 21, 29, 30], true-self avoiding walks and loop-erased random walks. Proofs in these models often rely on martingale methods, or explicit comparisons to random walk properties. In some of the examples, laws of large numbers are derived. The difficulty is that the limiting parameters are rather implicit, so that it is hard to derive analytical properties of them. For example, it is quite reasonable to assume that the drift for excited random walk is monotone increasing in the excitement parameter, but a proof of this fact is currently missing (see [2, 20, 21, 29, 30]). Similarly, it has not been proved that the speed for once-reinforced random walk on the tree is monotone decreasing in the reinforcement parameter (see [8]).

In the past decades, the lace expansion has proved to be an extremely useful technique to investigate a variety of models above their upper-critical dimension, where Gaussian limits are expected. Examples are self-avoiding walks above 4 dimensions [3, 11, 26, 27, 28], lattice trees above 8 dimensions [6, 7, 10, 19], the contact process above 4 dimensions [16], oriented percolation above 4 dimensions [18, 22, 23], and percolation above 6 dimensions [9, 12, 13]. An essential ingredient in the proofs is the fact that the above models are *self-repellent.* There are many more models where a Gaussian limit is expected above a certain upper critical dimension, but using the

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lace expansion for these models is hard as they are not strictly self-repellent. In this paper, we perform a first step for a successful application of the lace expansion methodology, namely, we derive the expansion for general interacting random walks. The goal is to use this expansion for some of the simpler interacting stochastic processes available. We will study two examples in detail, namely (i) *once-reinforced random walk with drift*; and (ii) excited random walk, as we explain in more detail below. The advantage is that this method gives a formula for the speed in the model in terms of the expansion coefficients, which may make it possible to prove monotonicity properties of the speed in the relevant parameters. The problem of deriving monotonicity properties is deferred to a future publication.

We will study a particular version of once-reinforced random walk, where the initial weights are such that the corresponding random walk has a *non-zero drift*. A similar situation was investigated in [8], where once-reinforced random walk was investigated on the tree. In particular, our results can be used to reprove some results in [8], when the reinforcement parameter in the tree is sufficiently small. We stress that our method is quite general, and also applies to the case where the reinforcement parameter is small, but negative, which one could call *directed once-self-repelled random walk*.

We also study *excited random walk*, where the random walker has a drift in the direction of the first component each time when the walker visits a new site. It was shown that this process has ballistic behaviour when $d \geq 2$ in [2, 20, 21], while there is no ballistic behaviour in one dimension. However, in each of these cases, a central limit theorem is unknown, and will be proved in this paper when the excitation parameter is sufficiently small and $d > 8$. We also derive a law of large numbers when $d > 5$. The advantage of this method compared to the methodology in [2, 20, 21, 29, 30] is that we obtain a formula for the speed in terms of the expansion coefficients.

2 The main results

We start by introducing some notation. A path ω is a sequence $\{\omega_i\}_{i=0}^{\infty}$ for which $\omega_i \in \mathbb{Z}^d$ for all $i \geq 0$. For *random walk*, the random vector $\{\omega_{i+1} - \omega_i\}_{i=0}^{\infty}$ is an i.i.d. sequence. We let $\mathbb{P}$ be the random walk law, where the random walk starts at the origin. We write $\vec{\omega}_n$ for the vector

$$\vec{\omega}_n = (\omega_0, \dots, \omega_n), \tag{2.1}$$

that is, for the first n positions of the walk and its starting point. Let

$$D(x) = \mathbb{P}(\omega_1 = x) \tag{2.2}$$

be the random walk transition probability, so that

$$\mathbb{P}(\vec{\omega}_n = (x_0, x_1, \dots, x_n)) = \prod_{i=0}^{n-1} D(x_{i+1} - x_i). \tag{2.3}$$

We restrict our attention to D with finite range L so that $\sum_{x:|x|>L} D(x) = 0$ and all moments of D exist. For interacting random walks, a similar expression to (2.3) is valid, but the term appearing in the product may depend on the history of the path. For this, and for a general path $\vec{\omega}_i$, we write $p^{\vec{\omega}_i}(x_i, x_{i+1})$ for the conditional probability that the walk steps from $\omega_i = x_i$ to x_{i+1}, given the history of the path $\vec{\omega}_i = (\omega_0, \ldots, \omega_i)$. We write $\mathbb{Q}$ for the law of the interacting random walk, i.e.,

$$\mathbb{Q}(\vec{\omega}_n = (x_0, x_1, \ldots, x_n)) = \prod_{i=0}^{n-1} p^{\vec{\omega}_i}(x_i, x_{i+1}). \tag{2.4}$$

The goal of this paper is to investigate the *two-point function*

$$c_n(x) = \mathbb{Q}(\omega_n = x). \tag{2.5}$$

The definition in (2.4) is quite general. In this paper, we will derive an expansion for the two-point function in full generality. However, for the analytical results we will focus on directed once-edge-reinforced random walks and excited random walks. In Sections 2.1 and 2.2 below, we will define the models and state the results.

2.1 Once edge-reinforced random walk with drift

In this section, we introduce an example of a once edge-reinforced random walk with drift. In Section 2.2, we introduce excited random walk.

For a directed edge b, let $w_s(b)$ to be the *weight of the edge* b, which is given recursively by

$$w_t(b) = w_{t-1}(b) + I[(\omega_{t-1}, \omega_t) = b]\beta_{\ell_t(b)}, \tag{2.6}$$

where $w_0(b)$ will be specified shortly, and

$$\ell_t(b) = \sum_{i=1}^{t} I[(\omega_{i-1}, \omega_i) = b] \tag{2.7}$$

is the number of times the edge b is traversed and $t \mapsto \beta_t$ is a sequence of $\mathbb{R}$-valued reinforcement parameters. Then,

$$p^{\vec{\omega}_i}(x_i, x_{i+1}) = \frac{w_i(x_i, x_{i+1})}{\sum_y w_i(x_i, y)}. \tag{2.8}$$

We have defined a directed version of ERRW. Undirected ERRW is defined in a similar fashion, where $\ell_t(b)$ in (2.6) is replaced by

$$\hat{\ell}_t(b) = \sum_{i=1}^{t} \Big(I[(\omega_{t-1}, \omega_t) = b] + I[(\omega_t, \omega_{t-1}) = b] \Big). \tag{2.9}$$

We will deal with *once-reinforced random walks*, where

$$\beta_t = \beta \delta_{t,1}, \tag{2.10}$$

i.e., the step along an edge is reinforced the first time it is traversed, and remains reinforced forever afterwards. However, our technique extends to *boundedly-reinforced random walks*, where we assume that

$$\beta = \sum_{t=0}^{\infty} |\beta_t| < \infty. \tag{2.11}$$

Note that the parameters β_s are allowed to be negative (provided $w_0(b)+\sum_{t=1}^{m}\beta_t$ remains bounded away from 0). We denote by $\mathbb{Q}_\beta$ the distribution of the above once-reinforced random walk with drift, and we let $\mathbb{E}_\beta$ denote expectation with respect to $\mathbb{Q}_\beta$. We denote by $\mathrm{Var}_\beta(\omega_n)$ the covariance matrix of the random vector ω_n under the measure $\mathbb{Q}_\beta$. We also denote convergence in distribution by $\stackrel{d}{\Longrightarrow}$.

The main assumption for our reinforced random walk is that $w_0(b)$ is translation invariant, and that

$$\sum_x x w_0(0,x) \neq 0, \tag{2.12}$$

i.e., the random walk distribution arising for $\beta = 0$ has *non-zero drift*. We write $\mathcal{N}(0,\Sigma)$ for the multivariate normal distribution with mean the zero vector and covariance matrix Σ.

Theorem 2.1 (A CLT for finitely reinforced random walk with drift)**.** *Fix $d \geq 1$ and assume* (2.12)*. There exist $\beta_0 = \beta_0(d, w_0) > 0$, $\theta = \theta(\beta, w_0, d) \in [-1,1]^d$ and finite $\Sigma = \Sigma(\beta, w_0, d)$ such that, for all $\beta \leq \beta_0$,*
(a)

$$\mathbb{E}_\beta[\omega_n] = \theta n[1 + O(\frac{1}{n})]. \tag{2.13}$$

(b)

$$\mathrm{Var}_\beta(\omega_n) = \Sigma n[1 + O(\frac{1}{n})]. \tag{2.14}$$

(c) the end-point ω_n satisfies a central limit theorem under $\mathbb{Q}_\beta$, that is,

$$\frac{\omega_n - \theta n}{\sqrt{n}} \stackrel{d}{\Longrightarrow} \mathcal{N}(0,\Sigma). \tag{2.15}$$

The method used in this paper can also be applied to once-edge-reinforced random walk on a tree, where x in (2.13)–(2.15) should be replaced by the height of the point x in the tree. Therefore, our methods can be used to reprove part of the results in [8], where a central limit theorem was proved for *all* β. We defer the application of our methods to once-edge reinforced random walks on a tree to the future.

We remark that parts of our methods apply to once-reinforced random walk where the initial weights are *undirected*. However, we cannot yet prove the bounds on the expansion coefficients, one of the crucial steps in the analysis.

2.2 Excited random walk

In this section, we introduce excited random walk, which is the second model to which we shall apply our expansion method. Excited random walk (ERW) is obtained by taking

$$p^{\vec{\omega}_i}(x_i, x_{i+1}) = p_0(x_{i+1} - x_i)\delta_{x_i,\vec{\omega}_{i-1}} + p_\beta(x_{i+1} - x_i)[1 - \delta_{x_i,\vec{\omega}_{i-1}}], \tag{2.16}$$

where $\delta_{x_i,\vec{\omega}_{i-1}}$ denotes the indicator that $x_i = \omega_j$ for some $0 \leq j \leq i-1$, and where

$$p_0(x) = \frac{1}{2d} I[|x| = 1] \tag{2.17}$$

is the nearest-neighbour step distribution, while, for $\beta \in [0,1]$,

$$p_\beta(x) = \frac{1+\beta e_1 \cdot x}{2d} I[|x|=1], \tag{2.18}$$

where $e_1 = (1,0,\ldots,0)$ and $x \cdot y$ is the inner-product between x and y. In words, the random walker gets excited each time he/she visits a new site, and, when the random walk is excited, it has a positive drift in the direction of the first coordinate. For a description in terms of cookies, see [29].

The question is whether excited random walk has a positive drift. This question has been answered affirmatively for ERW in $d \geq 4$ in [2], for $d = 3$ in [20], and for $d = 2$ in [21]. For $d = 1$, it is known that ERW is recurrent and diffusive [4]. Additional results on one dimensional (multi)-excited random walks can be found in [1].

We denote by $\mathbb{Q}_\beta$ the distribution of the above excited random walk, and we let $\mathbb{E}_\beta$ denote expectation with respect to $\mathbb{Q}_\beta$. We denote by $\mathrm{Var}_\beta(\omega_n)$ the covariance matrix of the random vector ω_n under the measure $\mathbb{Q}_\beta$.

Our main result for excited random walk is the following theorem:

Theorem 2.2 (A CLT for ERW above 8 dimensions)**.** *Fix $d > 8$. Then, there exists $\beta_0 = \beta_0(d) > 0$, $\theta = (\theta_1(\beta,d), 0, \ldots, 0)$ and finite $\Sigma = \Sigma(\beta,d)$ such that, for all $\beta \leq \beta_0$,*
(a)

$$\mathbb{E}_\beta[\omega_n] = \theta n[1 + O(\frac{1}{n})]. \tag{2.19}$$

(b)

$$\mathrm{Var}_\beta(\omega_n) = \Sigma n[1 + O(\frac{\log n}{n^{1 \wedge \frac{d-7}{2}}})]. \tag{2.20}$$

(c) the end-point ω_n satisfies a central limit theorem under $\mathbb{Q}_\beta$, that is,

$$\frac{\omega_n - \theta n}{\sqrt{n}} \overset{d}{\Longrightarrow} \mathcal{N}(0,\Sigma). \tag{2.21}$$

We believe that the methods in [2, 29, 30] can be used to prove the law of large numbers in Theorem 2.2. The CLT and error terms in Theorem 2.2 (and Theorem 2.1) are probably new. Furthermore, the advantage of our method is that it gives a description for the speed, which may allow one to prove that the speed is monotonically increasing in the excitement parameter β, for d sufficiently large. It may be possible to prove that the CLT in Theorem 2.2 holds for every $\beta \in [0,1]$ when the dimension d is sufficiently large. It would be of interest to prove that the speed in very high dimensions is increasing for all β. We plan to return to this question in a future publication.

Unfortunately, our methods do not apply to general $d \geq 2$. However, when $d > 5$, we *can* prove a weak law of large numbers:

Theorem 2.3 (A LLN for ERW above 5 dimensions)**.** *Fix $d > 5$. Then, there exists $\beta_0 = \beta_0(d) > 0$ and $\theta = (\theta_1(\beta,d), 0, \ldots, 0)$ such that, for all $\beta \leq \beta_0$,*
(a)

$$\mathbb{E}_\beta[\omega_n] = \theta n[1 + O(\frac{\log n}{n^{1 \wedge (d-5)/2}})]. \tag{2.22}$$

(b) the end-point ω_n satisfies a law of large numbers under $\mathbb{Q}_\beta$, that is,

$$\frac{\omega_n}{n} \stackrel{d}{\Longrightarrow} \theta. \tag{2.23}$$

2.3 Overview of the proof

The main tool used is a *perturbation expansion for the two-point function.* Such an expansion is often called a *lace expansion*, and takes the form of a recurrence relation

$$c_{n+1}(x) = \sum_y D(y)c_n(x-y) + \sum_y \sum_{m=2}^{\infty} \pi_m(y)c_{n+1-m}(x-y) \tag{2.24}$$

for certain expansion coefficients $\{\pi_m\}_{m=2}^{\infty}$, and where

$$D(x) = p^{\varnothing}(0,x) \tag{2.25}$$

is the transition probability function for the first step. A recurrence relation such as (2.24) is derived for the oriented percolation and self-avoiding walk two-point functions, and plays an essential part in the proofs that these models are Gaussian above the upper-critical dimension. For self-avoiding walk, $c_n(x)$ equals the number of n-step self-avoiding walks starting at 0 and ending at x, and $\sum_x c_n(x)$ equals the total number of self-avoiding walks, which grows exponentially at a certain rate that needs to be determined in the course of the proof. For interacting random walks, $\sum_x c_n(x) = 1$. This essential difference gives rise to a difference in the strategy for proofs.

In any lace expansion analysis, there are three main steps. The first is the expansion in (2.24), which, for general self-interacting random walks, will be derived in Section 3. The second step is to derive bounds on the lace expansion coefficients. These bounds will be derived in Section 6. The final step is the analysis of the recurrence relation, using the bounds on the lace expansion coefficients. For this analysis, we will make use of induction. This inductive analysis is similar to the one in [14], where a lace expansion was used to prove ballistic behaviour and a central limit theorem for general one-dimensional weakly self-avoiding walk models. In turn, this induction was inspired by the analyses in [15, 17]. In the induction argument, we shall make use of the characteristic function of the end-point of the self-interacting random walk, which is the Fourier transform

$$\hat{c}_n(k) = \sum_{x\in\mathbb{Z}^d} e^{ik\cdot x} c_n(x). \tag{2.26}$$

Taking the Fourier transform of (2.24) yields

$$\hat{c}_{n+1}(k) = \hat{D}(k)c_n(k) + \sum_{m=2}^{\infty} \hat{\pi}_m(k)\hat{c}_{n+1-m}(k) \tag{2.27}$$

We shall present two separate induction arguments. The first proves a law of large numbers as in Theorem 2.3 under weak assumptions on the expansion coefficients, the second is a more involved induction argument proving the central limit theorem as in Theorem 2.1 and Theorem 2.2 under stronger assumptions on the expansion coefficients.

The remainder of the paper is organised as follows. In Section 3, we present the expansion for self-interacting random walks, which applies in the general context described in Section 2. In Sections 4 and 5, we describe the induction arguments for the law of large numbers and central limit theorem respectively. Finally, in Section 6, we prove the bounds on the lace expansion coefficients for the two models under consideration.

3 The expansion for interacting random walks

Before we can start to prove (3.2), we need some more notation.

We will make use of the convolution of functions, which is defined for absolutely summable functions f, g on $\mathbb{Z}^d$ by

$$(f * g)(x) = \sum_y f(y)g(x-y), \tag{3.1}$$

so that we can rewrite (2.24) as

$$c_{n+1}(x) = (D * c_n)(x) + \sum_{m=2}^{\infty} (\pi_m * c_{n+1-m})(x). \tag{3.2}$$

If η and ω are two paths of length at least j and m respectively and such that $\eta_j = \omega_0$, then the concatenation $\vec{\eta}_j \circ \vec{\omega}_m$ is defined by

$$(\vec{\eta}_j \circ \vec{\omega}_m)(i) = \begin{cases} \eta_i & \text{when } 0 \le i \le j, \\ \omega_{i-j} & \text{when } j \le i \le m+j. \end{cases} \tag{3.3}$$

Given $\vec{\eta}_m$, we define a probability measure $\mathbb{Q}^{\vec{\eta}_m}$ on walks path starting from η_m, by specifying its value on particular cylinder sets (in a consistent manner) as follows

$$\mathbb{Q}^{\vec{\eta}_m}(\vec{\omega}_n = (x_0, x_1, \ldots, x_n)) \equiv \prod_{i=0}^{n-1} p^{\vec{\eta}_m \circ \vec{\omega}_i}(x_i, x_{i+1}), \tag{3.4}$$

and extending the measure to all finite-dimensional cylinder sets in the natural (consistent) way. We write $\mathbb{E}^{\vec{\eta}_m}$ for the expected value with respect to $\mathbb{Q}^{\vec{\eta}_m}$. Then, we have that

$$\prod_{i=0}^{j-1} p^{\vec{\omega}_1^{(0)} \circ \vec{\eta}_i}(\eta_i, \eta_{i+1}) = \mathbb{Q}^{\vec{\omega}_1^{(0)}}(\vec{\omega}_j^{(1)} = \vec{\eta}_j). \tag{3.5}$$

We further write

$$c_n^{\vec{\eta}_m}(\eta_m, x) = \mathbb{Q}^{\vec{\eta}_m}(\omega_n = x), \tag{3.6}$$

Any path of length $n+1$ is a path of length 1 concatenated with a path of length n, and in terms of the above notation, we can rewrite

$$c_{n+1}(x) = \sum_{\vec{\omega}_1^{(0)}} D(\omega_1^{(0)}) \sum_{\vec{\omega}_n^{(1)} : \omega_1^{(0)} \to x} \prod_{i=0}^{n-1} p^{\vec{\omega}_1^{(0)} \circ \vec{\omega}_i^{(1)}}(\omega_i^{(1)}, \omega_{i+1}^{(1)}). \tag{3.7}$$

If we would have that

$$p^{\vec{\omega}_1^{(0)} \circ \vec{\omega}_i^{(1)}} = p^{\vec{\omega}_i^{(1)}}, \tag{3.8}$$

then we would be back in the random walk case, since we would arrive at

$$c_{n+1}(x) = \sum_{\vec{\omega}_1^{(0)}} D(\omega_1^{(0)}) \sum_{\vec{\omega}_n^{(1)} : \omega_1^{(0)} \to x} \prod_{i=0}^{n-1} p^{\vec{\omega}_i^{(1)}}(\omega_i^{(1)}, \omega_{i+1}^{(1)}) = (D * c_n)(x). \tag{3.9}$$

For interacting random walks, $p^{\vec\omega_1^{(0)}\circ\vec\omega_i^{(1)}}$ does not equal $p^{\vec\omega_i^{(1)}}$ in general, and we are left to deal with the difference between the two. For given m and i we write

$$p^{\vec\eta_m\circ\vec\omega_i}(\omega_i,\omega_{i+1}) = p^{\vec\omega_i}(\omega_i,\omega_{i+1}) + \Big(p^{\vec\eta_m\circ\vec\omega_i} - p^{\vec\omega_i}\Big)(\omega_i,\omega_{i+1}). \tag{3.10}$$

With this substitution, we have that

$$\prod_{i=0}^{n-1} p^{\vec\omega_1^{(0)}\circ\vec\omega_i^{(1)}}(\omega_i^{(1)},\omega_{i+1}^{(1)}) = \prod_{i=0}^{n-1}\Big[p^{\vec\omega_i^{(1)}}(\omega_i^{(1)},\omega_{i+1}^{(1)}) + \Big(p^{\vec\omega_1^{(0)}\circ\vec\omega_i^{(1)}}(\omega_i^{(1)},\omega_{i+1}^{(1)}) - p^{\vec\omega_i^{(1)}}(\omega_i^{(1)},\omega_{i+1}^{(1)})\Big)\Big]. \tag{3.11}$$

In (3.11), the first term has 'forgotten' the first step, while the second term makes up for this mistake. We would like to expand out the product in (3.11). Note that for all $\{a_i\}_{i=0}^{n-1}$ and $\{b_i\}_{i=0}^{n-1}$,

$$\prod_{i=0}^{n-1}(a_i+b_i) = \prod_{i=0}^{n-1} a_i + \sum_{j=0}^{n-1}\Big(\prod_{i=0}^{j-1}(a_i+b_i)\Big)b_j\Big(\prod_{i=j+1}^{n-1} a_i\Big), \tag{3.12}$$

where the empty product arising in $\prod_{i=0}^{j-1}(a_i+b_i)$ when $j=0$ and in $\prod_{i=j+1}^{n-1} a_i$ when $j=n-1$, is defined to be equal to 1.

Applying this to (3.7) with $a_i = p^{\vec\omega_i^{(1)}}(\omega_i^{(1)},\omega_{i+1}^{(1)})$, $b_i = \Big(p^{\vec\omega_1^{(0)}\circ\vec\omega_i^{(1)}} - p^{\vec\omega_i^{(1)}}\Big)(\omega_i^{(1)},\omega_{i+1}^{(1)})$, we arrive at

$$\begin{aligned} c_{n+1}(x) = {} & \sum_{\vec\omega_1^{(0)}} D(\omega_1^{(0)}) \sum_{\vec\omega_n^{(1)}:\omega_1^{(0)}\to x} \prod_{i=0}^{n-1} p^{\vec\omega_i^{(1)}}(\omega_i^{(1)},\omega_{i+1}^{(1)}) \\ & + \sum_{j=0}^{n-1}\sum_{\vec\omega_1^{(0)}} D(\omega_1^{(0)}) \sum_{\vec\omega_n^{(1)}:\omega_1^{(0)}\to x} \Big[\prod_{i=0}^{j-1} p^{\vec\omega_1^{(0)}\circ\vec\omega_i^{(1)}}(\omega_i^{(1)},\omega_{i+1}^{(1)})\Big]\Big(p^{\vec\omega_1^{(0)}\circ\vec\omega_j^{(1)}} - p^{\vec\omega_j^{(1)}}\Big)(\omega_j^{(1)},\omega_{j+1}^{(1)}) \\ & \times\Big[\prod_{i=j+1}^{n-1} p^{\vec\omega_i^{(1)}}(\omega_i^{(1)},\omega_{i+1}^{(1)})\Big]. \end{aligned} \tag{3.13}$$

The first term equals $(D*c_n)(x)$ by (3.9). To rewrite the second term, we need some more notation. We abbreviate

$$\Delta_{j+1}^{(1)} = \Big(p^{\vec\omega_1^{(0)}\circ\vec\omega_j^{(1)}} - p^{\vec\omega_j^{(1)}}\Big)(\omega_j^{(1)},\omega_{j+1}^{(1)}), \tag{3.14}$$

so that (3.13) becomes

$$\begin{aligned} c_{n+1}(x) &= (D*c_n)(x) + \sum_{j=0}^{n-1}\sum_{\vec\omega_1^{(0)}} D(\omega_1^{(0)}) \sum_{\vec\omega_n^{(1)}:\omega_1^{(0)}\to x}\Big[\prod_{i=0}^{j-1} p^{\vec\omega_1^{(0)}\circ\vec\omega_i^{(1)}}(\omega_i^{(1)},\omega_{i+1}^{(1)})\Big]\Delta_{j+1}^{(1)}\Big[\prod_{i=j+1}^{n-1} p^{\vec\omega_i^{(1)}}(\omega_i^{(1)},\omega_{i+1}^{(1)})\Big] \\ &= (D*c_n)(x) + \sum_{j=0}^{n-1}\sum_{\vec\omega_1^{(0)}} D(\omega_1^{(0)}) \sum_{\vec\omega_{j+1}^{(1)}:\omega_0^{(1)}=\omega_1^{(0)}}\Big[\prod_{i=0}^{j-1} p^{\vec\omega_1^{(0)}\circ\vec\omega_i^{(1)}}(\omega_i^{(1)},\omega_{i+1}^{(1)})\Big]\Delta_{j+1}^{(1)} \\ &\quad\times \sum_{(\omega_{j+2}^{(1)},\dots,\omega_n^{(1)}):\omega_n^{(1)}=x}\Big[\prod_{i=j+1}^{n-1} p^{\vec\omega_i^{(1)}}(\omega_i^{(1)},\omega_{i+1}^{(1)})\Big]. \end{aligned} \tag{3.15}$$

From (3.6) we have that

$$\sum_{(\omega_{j+2}^{(1)},\dots,\omega_n^{(1)}):\omega_n^{(1)}=x}\Big[\prod_{i=j+1}^{n-1} p^{\vec{\omega}_i^{(1)}}(\omega_i^{(1)},\omega_{i+1}^{(1)})\Big] = c_{n-j-1}^{\vec{\omega}_{j+1}^{(1)}}(\omega_{j+1}^{(1)},x). \tag{3.16}$$

Therefore, (3.15) is equal to

$$\begin{aligned} c_{n+1}(x) &= (D*c_n)(x) + \sum_{j=0}^{n-1}\sum_{\vec{\omega}_1^{(0)}} D(\omega_1^{(0)}) \sum_{\vec{\omega}_{j+1}^{(1)}} \mathbb{Q}^{\vec{\omega}_1^{(0)}}(\vec{\omega}_j^{(1)})\Delta_{j+1}^{(1)}\ c_{n-j-1}^{\vec{\omega}_{j+1}^{(1)}}(\omega_{j+1}^{(1)},x) \\ &= (D*c_n)(x) + \sum_{j=0}^{n-1}\sum_{\vec{\omega}_1^{(0)}} D(\omega_1^{(0)})\mathbb{E}^{\vec{\omega}_1^{(0)}}\Big[\Delta_{j+1}^{(1)}\ c_{n-j-1}^{\vec{\omega}_{j+1}^{(1)}}(\omega_{j+1}^{(1)},x)\Big]. \end{aligned} \tag{3.17}$$

Note that both $\Delta_{j+1}^{(1)}$ and $c_{n-j-1}^{\vec{\omega}_{j+1}^{(1)}}(\omega_{j+1}^{(1)},x)$ are random variables in this expression. This completes the first step of the expansion.

For the second step of the expansion, we note that a type of two-point function $c_{n-j-1}^{\vec{\omega}_{j+1}^{(1)}}(\omega_{j+1}^{(1)},x)$ appears on the right side of (3.17). The second step of the expansion involves expanding out the dependence of this two-point function on the history $\vec{\omega}_{j+1}^{(1)}$. Given $\vec{\omega}_{j+1}^{(1)}$ we write

$$c_{n-j-1}^{\vec{\omega}_{j+1}^{(1)}}(\omega_{j+1}^{(1)},x) = c_{n-j-1}(\omega_{j+1}^{(1)},x) + \left(c_{n-j-1}^{\vec{\omega}_{j+1}^{(1)}}(\omega_{j+1}^{(1)},x) - c_{n-j-1}(\omega_{j+1}^{(1)},x)\right). \tag{3.18}$$

The contribution to (3.17) from the first term on the right of (3.18) is

$$\sum_{j=0}^{n-1}\sum_y\sum_{\vec{\omega}_1^{(0)}} D(\omega_1^{(0)})\mathbb{E}^{\vec{\omega}_1^{(0)}}\Big[\Delta_{j+1}^{(1)} I_{\{\omega_{j+1}^{(1)}=y\}}\Big]c_{n-j-1}(x-y) \equiv \sum_{m=2}^{n+1}\pi_m^{(1)}(y)c_{n+1-m}(x-y), \tag{3.19}$$

where, for $m \geq 2$,

$$\begin{aligned} \pi_m^{(1)}(y) &= \sum_{\vec{\omega}_1^{(0)}} D(\omega_1^{(0)})\mathbb{E}^{\vec{\omega}_1^{(0)}}\Big[\Delta_{m-1}^{(1)}\ I_{\{\omega_{m-1}^{(1)}=y\}}\Big] \\ &= \sum_{\vec{\omega}_1^{(0)}} D(\omega_1^{(0)})\mathbb{E}^{\vec{\omega}_1^{(0)}}\Big[\Big(p^{\vec{\omega}_1^{(0)}\circ\vec{\omega}_{m-2}^{(1)}} - p^{\vec{\omega}_{m-2}^{(1)}}\Big)(\omega_{m-2}^{(1)},\omega_{m-2}^{(1)})\ I_{\{\omega_{m-1}^{(1)}=y\}}\Big], \end{aligned} \tag{3.20}$$

To investigate the contribution to (3.17) from the term in brackets on the right of (3.18), we consider the difference between $c_n^{\vec{\eta}_m}(\eta_m,x)$ and $c_n(\eta_m,x)$ for general $\vec{\eta}_m$, n and x. We first write

$$c_n^{\vec{\eta}_m}(\eta_m,x) = \sum_{\vec{\omega}_n:\eta_m\to x}\prod_{i=0}^{n-1} p^{\vec{\eta}_m\circ\vec{\omega}_i}(\omega_i,\omega_{i+1}), \tag{3.21}$$

and then use (3.10) and (3.12) to end up with

$$c_n^{\vec{\eta}_m}(\eta_m,x) = c_n(\eta_m,x)+\sum_{j=0}^{n-1}\sum_{\vec{\omega}_n:\eta_m\to x}\Big[\prod_{i=0}^{j-1} p^{\vec{\eta}_m\circ\vec{\omega}_i}(\omega_i,\omega_{i+1})\Big]\Big(p^{\vec{\eta}_m\circ\vec{\omega}_j}-p^{\vec{\omega}_j}\Big)(\omega_j,\omega_{j+1})\prod_{i=j+1}^{n-1} p^{\vec{\omega}_i}(\omega_i,\omega_{i+1})\Big]. \tag{3.22}$$

Therefore, we obtain

$$c_n^{\vec{\eta}_m}(\eta_m,x)=c_n(\eta_m,x)+\sum_{j=0}^{n-1}\mathbb{E}^{\vec{\eta}_m}\Big[\Big(p^{\vec{\eta}_m\circ\vec{\omega}_j}-p^{\vec{\omega}_j}\Big)(\omega_j,\omega_{j+1})c_{n-j-1}^{\vec{\omega}_{j+1}}(\omega_{j+1},x)\Big]. \tag{3.23}$$

In (3.23), the first term is a regular two-point function, i.e., it does not depend on the history $\vec{\eta}_m$). In the correction term a history-dependent two point function $c_{n-j-1}^{\vec{\omega}_{j+1}}$ appears to which we can iteratively use (3.23). Thus,

$$\begin{aligned}c_{n+1}(x)=&(D*c_n)(x)+\sum_{j=0}^{n}(\pi_j^{(1)}*c_{n-j-1})(x)\\&+\sum_{j_1,j_2}\sum_{\vec{\omega}_1^{(0)}}D(\omega_1^{(0)})\mathbb{E}^{\vec{\omega}_1^{(0)}}\Big[\Delta_{j_1+1}^{(1)}\;\mathbb{E}^{\vec{\omega}_{j_1+1}^{(1)}}\Big[\Delta_{j_2+1}^{(2)}\;c_{n-j_1-j_2-2}^{\vec{\omega}_{j_2+1}^{(2)}}(\omega_{j_2+1}^{(2)},x)\Big]\Big],\end{aligned} \tag{3.24}$$

where we write, for $N\geq 1$,

$$\Delta_{j_N+1}^{(N)}=\Big(p^{\vec{\omega}_{j_{N-1}+1}^{(N-1)}\circ\vec{\omega}_{j_N}^{(N)}}-p^{\vec{\omega}_{j_N}^{(N)}}\Big)(\omega_{j_N}^{(N)},\omega_{j_N+1}^{(N)}), \tag{3.25}$$

with $j_0\equiv 0$.

For $N\geq 1$, we further define

$$\pi_m^{(N)}(y)=\sum_{j_1+\cdots+j_N=m-N-1}\sum_{\vec{\omega}_1^{(0)}}D(\omega_1^{(0)})\mathbb{E}_1^{\vec{\omega}_1^{(0)}}\Big[\Delta_{j_1+1}^{(1)}\;\mathbb{E}_2^{\vec{\omega}_{j_1+1}^{(1)}}\Big[\Delta_{j_2+1}^{(2)}\;\cdots\mathbb{E}_N^{\vec{\omega}_{j_{N-1}+1}^{(N-1)}}\Big[\Delta_{j_N+1}^{(N)}I_{\{\omega_{j_N+1}^{(N)}=y\}}\Big]\cdots\Big], \tag{3.26}$$

which is zero when $N+1>m$. Note that (3.26) reduces to (3.20) in the case $N=1$. We emphasize that, conditionally on $\vec{\omega}_{j_M+1}^{(M)}$, the probability measure $\mathbb{P}_{M+1}^{\vec{\omega}_{j_M+1}^{(M)}}$ is the law of $\vec{\omega}_{j_{M+1}+1}^{(M+1)}$. That is to say that $\vec{\omega}_{j_M+1}^{(M)}$ acts as the history for $\vec{\omega}_{j_{M+1}+1}^{(M+1)}$.

Equation (2.24) follows by iteratively replacing the two-point function in the expectation in (3.23) by using the equality (3.23), until the second term on the right of (3.23) vanishes. This must happen when $N=n+1$, and we define

$$\pi_m(y)=\sum_{N=1}^{\infty}\pi_m^{(N)}(y), \tag{3.27}$$

and $\pi_m^{(N)}(y)$ is the N^{th} term arising in the expansion. This completes the derivation of the expansion. In Section 3.1 below, we discuss the expansion and derive some consequences of it.

3.1 Discussion of the expansion

In this section, we discuss the consequences of the expansion in (2.24).

The lace expansion coefficients. The lace expansion coefficients involve the factors

$$\Delta_{j_N+1}^{(N)}=\Big(p^{\vec{\omega}_{j_{N-1}+1}^{(N-1)}\circ\vec{\omega}_{j_N}^{(N)}}-p^{\vec{\omega}_{j_N}^{(N)}}\Big)(\omega_{j_N}^{(N)},\omega_{j_N+1}^{(N)}) \tag{3.28}$$

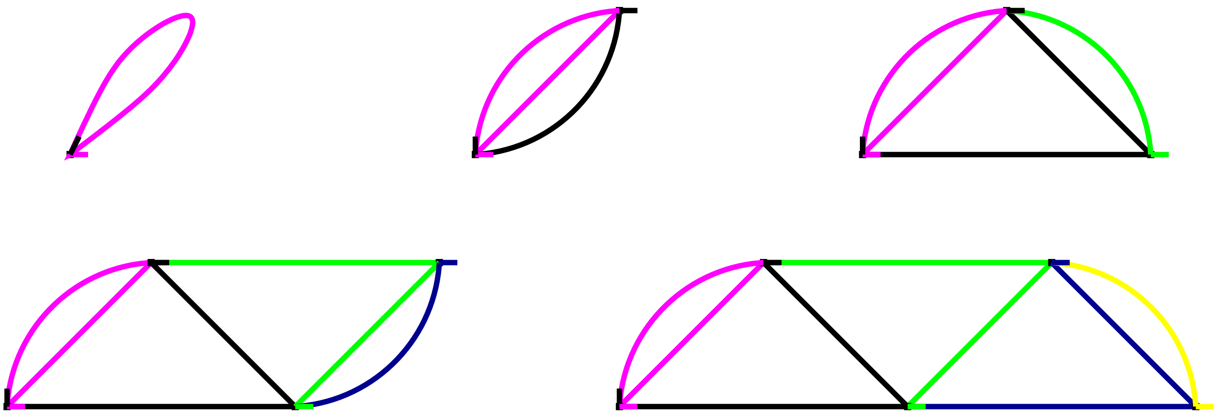

Figure 1: The diagrams for $\pi_m^{(N)}$, $N = 1, \ldots, 5$, arising from the expansion and the bound (3.30) for both models. The subwalks (indicated by different shades) in the diagrams have the previous subwalk as their history. An intersection of two subwalks and a small factor β appears at each vertex.

in (3.25). This difference is identically zero when the histories $\vec{\omega}^{(N-1)}_{j_{N-1}+1} \circ \vec{\omega}^{(N)}_{j_N}$ and $\vec{\omega}^{(N)}_{j_N}$ give the same transition probabilities to go from $\omega^{(N)}_{j_N}$ to $\omega^{(N)}_{j_N+1}$. For excited random walk, $\Delta^{(N)}_{j_N+1}$ is non-zero precisely when $\omega^{(N)}_{j_N}$ has already been visited by $\vec{\omega}^{(N-1)}_{j_{N-1}+1}$ but not by $\vec{\omega}^{(N)}_{j_N-1}$ so that

$$\begin{aligned} |\Delta^{(N)}_{j_N+1}| \leq& |\Delta^{(N)}_{j_N+1}| I[\omega^{(N)}_{j_N} \in \vec{\omega}^{(N-1)}_{j_{N-1}+1}] I[\omega^{(N)}_{j_N} \notin \vec{\omega}^{(N)}_{j_N-1}] \\ \leq& C I[\omega^{(N)}_{j_N} \in \vec{\omega}^{(N-1)}_{j_{N-1}+1}] I[\omega^{(N)}_{j_N} \notin \vec{\omega}^{(N)}_{j_N-1}] \leq C\beta I[\omega^{(N)}_{j_N} \in \vec{\omega}^{(N-1)}_{j_{N-1}}]. \end{aligned} \tag{3.29}$$

For once-edge-reinforced random walk, the difference (3.25) is nonzero exactly when the vertex $\omega^{(N)}_{j_N}$ has already been visited by $\vec{\omega}^{(N-1)}_{j_{N-1}+1}$ via an edge that was not traversed by $\vec{\omega}^{(N)}_{j_N}$. Therefore, we also have for once-edge-reinforced random walk that

$$|\Delta^{(N)}_{j_N+1}| \leq C\beta I[\omega^{(N)}_{j_N} \in \vec{\omega}^{(N-1)}_{j_{N-1}}] \tag{3.30}$$

We conclude that for both models, each factor $|\Delta^{(N)}_{j_N+1}|$:

1. enforces an intersection between the path and its previous history;
2. gives rise to a factor β, making $\pi^{(N)}_m(y)$ small when β is sufficiently small and N is large.

The quantities $\pi^{(N)}_m(y)$ combined with the bound (3.30) for both models, can be represented by diagrams of the form displayed in Figure 1 for $N = 1, \ldots, 5$. The first step is special, as it has no history. Thereafter, each subwalk $\vec{\omega}^{(i)}_{j_i+1}$ (indicated by shading in Figure 1) has the previous subwalk $\vec{\omega}^{(i-1)}_{j_{i-1}+1}$ as its history. The apparent similarity with the self-avoiding walk diagrams (see for example [11]) is natural due to the intersections enforced by the factors $\Delta^{(i)}_{j_i+1}$ as described above. A small factor β arises from each intersection (represented by vertices in Figure 1), and the number of intersections increases with the complexity of the diagram.

The speed and variance. By convention our vectors are considered to be column vectors. Thus if $\theta \in \mathbb{R}^d$, then $\theta\theta^t$ is a $d \times d$ matrix with real entries.

The limiting speed $\theta = \theta(\beta)$ and covariance matrix $\Sigma = \Sigma(\beta)$ appearing in Theorems 2.1–2.3 are given by

$$\theta(\beta) \;=\; \theta_\varnothing - i \sum_{m=2}^{\infty} \nabla \hat{\pi}_m(0), \tag{3.31}$$

$$\Sigma(\beta) \;=\; \Sigma_\varnothing - \theta\theta^t - \sum_{m=2}^{\infty} \nabla^2 \Big[e^{-i\theta\cdot k(m-1)} \hat{\pi}_m(k) \Big]_{k=0}, \tag{3.32}$$

where $\theta_\varnothing$ is the expected drift of the transition probability $D = p^\varnothing$, i.e.,

$$\theta_\varnothing = \sum_{x\in\mathbb{Z}^d} xD(x), \tag{3.33}$$

while $\Sigma_\varnothing$ is the covariance matrix of $D = p^\varnothing$ given by

$$(\Sigma_\varnothing)_{i,j} = \sum_{x\in\mathbb{Z}^d} x_i x_j D(x), \tag{3.34}$$

and $\nabla^2 f(k)$ is the matrix consisting of the double derivatives of $k \mapsto f(k)$.

These formulas can be heuristically derived from the recurrence relation (2.24). Indeed, take the Fourier transform to obtain

$$\hat{c}_{n+1}(k) = \hat{D}(k)\hat{c}_n(k) + \sum_{m=2}^{\infty} \hat{\pi}_m(k)\hat{c}_{n+1-m}(k). \tag{3.35}$$

Now replace $\hat{c}_l(k)$ throughout the recurrence relation by $e^{i\theta\cdot kl - \frac{1}{2}k^t\Sigma kl}$, in accordance with Theorem 2.1(c)–2.2(c). Then, dividing by $e^{i\theta\cdot kn-\frac{1}{2}k^t\Sigma kn}$, we obtain

$$e^{i\theta\cdot k-\frac{1}{2}k^t\Sigma k} \approx \hat{D}(k) + \sum_{m=2}^{\infty} \hat{\pi}_m(k) e^{-i\theta\cdot k(m-1)+\frac{1}{2}k^t\Sigma k(m-1)}. \tag{3.36}$$

Expanding to linear order in k yields (3.31) and expanding to second order in k yields (3.32), when we note that Σ (as defined in (2.14) and (2.20)) must be symmetric, and

$$\hat{\pi}_m(0) = 0 \quad \text{and} \quad \hat{D}(k) = 1 + ik\cdot\theta_\varnothing - \frac{1}{2}k^t\Sigma_\varnothing k + O(|k|^3). \tag{3.37}$$

The results in this paper, as well as the proofs, follow part of the ideas in [14], where it was shown that certain weakly self-avoiding walk models in $d = 1$ behave ballistically.

4 Induction for the weak law of large numbers

In this section we prove a law of large numbers from the recurrence relation (2.24), or, more precisely, its Fourier transform (3.36), assuming certain bounds on the coefficients $\hat{\pi}_m(k)$. The bounds roughly correspond to upper bounds on the accuracy of the Taylor approximation of $\hat{\pi}_m(k)$ up to *first* order.

We start by formulating a general assumption, which must be verified for a specific model and prove the main result, Theorem 4.1, under this assumption.

Assumption (LLN). *There exists a sequence* $\{b_m\}_{m\geq 1}$*, independent of* β *and with* $b_1 \geq 1$*, and a constant* C_β *satisfying* $\lim_{\beta\downarrow 0} C_\beta = 0$ *such that*

$$\hat{\pi}_m(0) = 0, \qquad |\nabla\hat{\pi}_m(0)| \leq C_\beta b_m, \qquad |\nabla^2\hat{\pi}_m(0)| \leq C_\beta m b_m, \tag{4.1}$$

and uniformly in $k \in [-\pi,\pi]^d$,

$$|\hat{\pi}_m(k) - k\cdot\nabla\hat{\pi}_m(0)| \leq C_\beta |k|^2 m b_m, \tag{4.2}$$

where

$$B \equiv \sum_{m=1}^{\infty} b_m < \infty. \tag{4.3}$$

Theorem 4.1. *(Weak law of large numbers) When Assumption (LLN) holds, there exist* $\beta_0 = \beta_0(d) > 0$ *and* $\theta = \theta(\beta)$ *such that for all* $\beta \leq \beta_0$,

$$\mathbb{E}_\beta[\omega_n] = \theta n\Big[1 + O\Big(\frac{1}{n}\sum_{m=1}^{\infty}(n\wedge m)b_m\Big)\Big]. \tag{4.4}$$

Furthermore, for every $U > 0$ *and every* $k \in \mathbb{R}^d$ *such that* $|k| \leq U$,

$$\mathbb{E}_\beta[e^{ik\cdot\omega_n/n}] = e^{ik\cdot\theta}\left(1 + O\left(\frac{|k|}{n}\sum_{m=1}^{\infty}(n\wedge m)b_m\right) + O\left(\frac{|k|^2}{n}\sum_{m=1}^{n} m b_m\right)\right), \tag{4.5}$$

where the constants in the O *terms depend on* U*, and the constant* θ *given by* (3.31) *is model dependent.*

Observe that $n^{-1}\sum_{m=1}^{\infty}(n\wedge m)b_m = o(1)$ and $n^{-1}\sum_{m=1}^{n} m b_m = o(1)$ when (4.3) holds. Thus (4.5) implies that $\lim_{n\to\infty}\mathbb{E}_\beta[e^{ik\cdot\omega_n/n}] = e^{ik\cdot\theta}$, which is equivalent to the statement of convergence in probability, $\omega_n/n \xrightarrow{P} \theta$.

Note that since D has finite range, there exists a constant $C_1 \geq 1$ independent of β such that

$$|\hat{D}(k) - 1 - ik\cdot\theta_\varnothing| \leq C_1|k|^2, \tag{4.6}$$

and let $K_1 = 2C_1$, which is independent of β.

Set $\theta_1 = \theta_\varnothing$, and for $n \geq 2$ we define the following approximation to θ:

$$\theta_n = \theta_\varnothing - i\sum_{m=2}^{n}\nabla\hat{\pi}_m(0). \tag{4.7}$$

Our induction hypothesis for the law of large numbers in Theorem 4.1 is that the following bound holds for all $\beta \leq \beta_0$, all $0 \leq j \leq n$ and for each fixed $U > 0$ and some δ_1 independent of β:

For all $|k| \leq \frac{U}{n}\wedge\delta_1$ and some $K \geq 1$ independent of β we can write,

$$\hat{c}_j(k) = \exp\left[\sum_{l=1}^{j}\Big(ik\cdot\theta_l + e_l(k)\Big)\right] \quad \text{where} \quad |e_j(k)| \leq K|k|^2\sum_{l=1}^{j} l b_l, \tag{4.8}$$

and where the empty sum, arising when $j = 0$, is defined to be 0.

The initialisation of the induction (the $n = 0$ case) holds trivially since $1 = e^0$. In Section 4.1 we will advance the induction hypothesis. In Section 4.2 we will use it to prove Theorem 4.1

4.1 The LLN induction advanced

The induction step will be achieved as soon as we are able to write

$$\frac{\hat{c}_{n+1}(k)}{\hat{c}_n(k)} = \exp\Big[ik\cdot\theta_{n+1} + e_{n+1}(k)\Big], \tag{4.9}$$

for $e_{n+1}(k)$ satisfying the required bound. For this, we write

$$\frac{\hat{c}_{n+1}(k)}{\hat{c}_n(k)} = 1 + ik\theta_{n+1} + e'_{n+1}(k) \tag{4.10}$$

and then set

$$e_{n+1}(k) = \log\Big[1 + ik\theta_{n+1} + e'_{n+1}(k)\Big] - ik\theta_{n+1}. \tag{4.11}$$

The following lemma is a trivial consequence of (3.33) and (4.7):

Lemma 4.1. *We have $|\theta_\varnothing| \le L$ and when Assumption (LLN) holds we have $|\theta_n| \le L + C_\beta B$ for every n.*

Choose $\beta_0 > 0$ so that $C_\beta \le 1$ for all $\beta \le \beta_0$, and suppose that the required bound (4.8) holds for $e'_{n+1}(k)$ with constant K_1. By Lemma 4.1, $|k||\theta_{n+1}| + |e'_{n+1}(k)| \le \frac{1}{2}$ for k satisfying the bound specified in the preamble of the induction hypothesis (4.8) with $\delta_1 = \frac{1}{8K_1UB} \wedge \frac{1}{4(L+B)}$. Therefore we may apply Taylor's Theorem $|\log(1+x) - x| \le 4|x|^2$ for $|x| \le \frac{1}{2}$, to (4.11). This implies that when the required bound holds for $e'_{n+1}(k)$ with constant K_1, it also holds for $e_{n+1}(k)$ for some K independent of β, since the terms of order k in (4.11) cancel. Specifically, if $|e'_{n+1}(k)| \le K_1|k|^2\sum_{l=1}^{n+1} lb_l$, then

$$\begin{aligned}|e_{n+1}(k)| \le&\Big(|k|^2\sum_{l=1}^{n+1} lb_l\Big)K_1\Big(1 + 4|k|^2\sum_{l=1}^{n+1} lb_l + 8|k|(|\theta_\varnothing| + \sum_{m=2}^{\infty} C_{\beta_0}b_m)\Big) + 4|k|^2\Big(|\theta_\varnothing| + \sum_{m=2}^{\infty} C_{\beta_0}b_m\Big)^2\\ \le&\Big(|k|^2\sum_{l=1}^{n+1} lb_l\Big)K_1\Big(1 + 8UB + 8U(L+B)\Big) + 4|k|^2\Big(L+B\Big)^2 \le K\Big(|k|^2\sum_{l=1}^{n+1} lb_l\Big),\end{aligned} \tag{4.12}$$

for $K \ge K_1(1 + 8U + 8U(L+B)) + (L+B)^2$, which is independent of β. Note that we have used the facts that $|\theta_\varnothing| \le L$ and $b_1 \ge 1$.

The rest of this section will be devoted to the proof of the following lemma:

Lemma 4.2. *If $e_j(k)$ satisfies the bound in (4.8) for all $j \le n$, then*

$$|e'_{n+1}(k)| \;\le\; K_1|k|^2\sum_{l=1}^{n+1} lb_l. \tag{4.13}$$

To prove (4.13), we divide the recursion relation (2.27) by $\hat{c}_n(k)$ and use the equality $\hat{\pi}_m(0) = 0$ of (4.1) to obtain

$$\frac{\hat{c}_{n+1}(k)}{\hat{c}_n(k)} \;=\; \hat{D}(k) + \sum_{m=2}^{n+1}[\hat{\pi}_m(k) - \hat{\pi}_m(0)]\frac{\hat{c}_{n+1-m}(k)}{\hat{c}_n(k)}. \tag{4.14}$$

We can rewrite (4.14) as

$$\frac{\hat{c}_{n+1}(k)}{\hat{c}_n(k)} \;=\; 1 \;+ik\cdot\theta_{n+1}+e'_{n+1}(k),$$

where

$$e'_{n+1}(k) \;=\; [\hat{D}(k)-1-ik\cdot\theta_\varnothing]+\sum_{m=2}^{n+1}\Big[\hat{\pi}_m(k)-k\cdot\nabla\hat{\pi}_m(0)\Big]+\sum_{m=2}^{n+1}\hat{\pi}_m(k)\Big[\frac{\hat{c}_{n+1-m}(k)}{\hat{c}_n(k)}-1\Big].$$

The first term is taken care of by (5.8). Furthermore, by (4.2), we have that

$$|\hat{\pi}_m(k)-k\cdot\nabla\hat{\pi}_m(0)|\le C_\beta|k|^2 m b_m. \tag{4.15}$$

Finally, using the induction hypothesis (4.8) for $e_l(k)$ with $l\le n$,

$$\begin{aligned}\Big|\frac{\hat{c}_{n+1-m}(k)}{\hat{c}_n(k)}-1\Big| \;&=\; \Big|\exp\Big[-\sum_{l=n+1-m}^{n}\Big(ik\cdot\theta_l+e_l(k)\Big)\Big]-1\Big| \\ &\le\; Cm|k|+CK|k|^2\sum_{l=n+1-m}^{n}\sum_{s=1}^{l} sb_s. \end{aligned}\tag{4.16}$$

When $|k|n\le U$, we have that for any $l\le n$,

$$|k|\sum_{s=1}^{l} sb_s\le UB. \tag{4.17}$$

Since $K\ge 1$, there exists a C independent of K such that

$$\Big|\frac{\hat{c}_{n+1-m}(k)}{\hat{c}_n(k)}-1\Big|\le CKm|k|. \tag{4.18}$$

Using (4.1) and (4.2), it follows that

$$\Big|\sum_{m=2}^{n+1}\hat{\pi}_m(k)\Big[\frac{\hat{c}_{n+1-m}(k)}{\hat{c}_n(k)}-1\Big]\Big|\le CK\sum_{m=2}^{n+1}(C_\beta|k|b_m)(m|k|)\le CKC_\beta|k|^2\sum_{m=2}^{n+1} mb_m. \tag{4.19}$$

Summarising (4.15)-(4.19) we have

$$|e'_{n+1}(k)|\le C_1|k|^2+CKC_\beta|k|^2\sum_{m=2}^{n+1} mb_m\le (C_1+CC_\beta K)|k|^2\sum_{m=2}^{n+1} mb_m, \tag{4.20}$$

for C sufficiently large but independent of β and K. Recalling that $K_1=2C_1$ and choosing β_0 sufficiently small so that $CC_\beta K\le\frac{K_1}{2}$ (and $C_\beta\le 1$) for all $\beta\le\beta_0$, we conclude that $C_1+CC_\beta K\le K_1$ and therefore (4.13) holds as required for all $\beta\le\beta_0$. This completes the proof of Lemma 4.2.

4.2 Proof of Theorem 4.1

To prove (4.4), we note that from (4.8), which is now known to be valid for all n,

$$\mathbb{E}_\beta[\omega_n] = -i\nabla\hat{c}_n(0) = -i\sum_{j=1}^{n}\Big[i\theta_j + \nabla e_j(0)\Big]. \tag{4.21}$$

Since $|e_j(k)| = O(|k|^2)$, we have that $\nabla e_j(0) = 0$. Therefore,

$$\mathbb{E}_\beta[\omega_n] = \sum_{j=1}^{n}\theta_j = n\theta + \sum_{j=1}^{n}[\theta_j - \theta]. \tag{4.22}$$

By (3.31), (4.7) and (4.1), we have that

$$\sum_{j=1}^{n}[\theta_j - \theta] = i\sum_{j=1}^{n}\sum_{s=j+1}^{\infty}\nabla\hat{\pi}_s(0) = i\sum_{s=2}^{\infty}(n\wedge(s-1))\nabla\hat{\pi}_s(0) = O\left(\sum_{s=1}^{\infty}(n\wedge s)b_s\right). \tag{4.23}$$

For (4.5), fix $U > 0$ and let $|k| \le U$. Then there exists n_U such that $\frac{|k|}{n} \le \frac{U}{n} \wedge \frac{1}{8K_1UB} \wedge \frac{1}{4(L+B)}$ for all $n \ge n_U$ and for such n we use (4.8) in the form

$$\hat{c}_n(k) = e^{ik\cdot\theta n}\exp\Big[\sum_{l=1}^{n}[ik\cdot(\theta_l - \theta) + e_l(k)]\Big], \quad \text{with} \quad |e_j(k)| \le K|k|^2\sum_{l=1}^{j} lb_l. \tag{4.24}$$

By (4.23),

$$\sum_{l=1}^{n} ik\cdot(\theta_l - \theta) = O\left(|k|\sum_{s=1}^{\infty}(n\wedge s)b_s\right). \tag{4.25}$$

Similarly,

$$\sum_{j=1}^{n}|e_j(k)| \le K|k|^2\sum_{j=1}^{n}\sum_{l=1}^{j} lb_l \le K|k|^2\sum_{l=1}^{n}(n-l+1)lb_l \le K|k|^2 n\sum_{l=1}^{n} lb_l. \tag{4.26}$$

Together (4.25) and (4.26) prove (4.5) for $n \ge n_U$, where the constant K appearing in (4.26) depends on U. For $n \le n_U$, the result (4.5) (with constant depending on n_U and hence on U) is trivial by applying the Taylor approximation $|e^{iy} - 1| \le |y|$ and using the fact that $|\omega_n| \le Ln$. □

5 Induction for the central limit theorem

In this section we prove a central limit theorem from the recurrence relation (2.24), or more precisely its Fourier transform (3.36), assuming certain bounds on the coefficients $\hat{\pi}_m(k)$. The bounds roughly correspond to upper bounds on the accuracy of the Taylor approximation of $\hat{\pi}_m(k)$ up to *second* order, and the argument is an extension of the one in Section 4. In this section we make use of the L^1 norm of a $d \times d$ matrix Σ defined by

$$|\Sigma| = \sum_{i,j=1}^{d}|[\Sigma]_{ij}|. \tag{5.1}$$

We start by formulating a general assumption, and prove the main result, Theorem 5.1, under this assumption.

Assumption (CLT). *There exists a non-increasing sequence* $\{b_m\}_{m\geq 1}$ *independent of* β *with* $b_1 \geq 1$, *and a constant* C_β *with* $\lim_{\beta\downarrow 0} C_\beta = 0$, *such that*
(i)

$$\hat{\pi}_m(0) = 0, \qquad |\nabla\hat{\pi}_m(0)| \leq C_\beta b_m, \qquad |\nabla^2\hat{\pi}_m(0)| \leq C_\beta m b_m. \tag{5.2}$$

(ii) for all $k \in [-\pi,\pi]^d$,

$$\Big|\hat{\pi}_m(k) - k\cdot\nabla\hat{\pi}_m(0) - \frac{1}{2}k^t\nabla^2\hat{\pi}_m(0)k\Big| \leq C_\beta |k|^3 m^2 b_m. \tag{5.3}$$

Moreover $B^* \equiv \sum_m m b_m < \infty$,

$$d_n \equiv \sum_{m=2}^{n} m b_m \sum_{l=n+1-m}^{n} b_{l+1} = o(1), \quad \text{as } n\to\infty, \text{ and}$$

$$a_n \equiv \max\left\{\sum_{m=1}^{n} m^2 b_m, 1\right\} \leq C_a\sqrt{n}, \quad \text{for all } n \text{ and some } C_a \geq 1. \tag{5.4}$$

We will prove a generalised version of Theorems 2.1 and 2.2, which is formulated below:

Theorem 5.1. *(Central limit theorem) When Assumption (CLT) holds, there exist* $\beta_0 = \beta_0(d) > 0$, *and* $\theta = \theta(\beta)$, *and* $\Sigma = \Sigma(\beta)$ *such that, for all* $\beta \leq \beta_0$,
(a)

$$\mathbb{E}_\beta[\omega_n] = \theta n\left[1 + O\left(\frac{1}{n}\right)\right]. \tag{5.5}$$

(b)

$$\mathrm{Var}_\beta(\omega_n) = \Sigma n + O\left(1 \vee \sum_{m=1}^{\infty}(m\wedge n) m b_m\right) + O\left(\sum_{j=1}^{n} d_j\right). \tag{5.6}$$

(c) for every $U > 0$ *and every* $k \in \mathbb{R}^d$ *such that* $|k| \leq U$,

$$\mathbb{E}_\beta[e^{ik\cdot\frac{(\omega_n-\theta n)}{\sqrt{n}}}] = e^{-\frac{1}{2}k^t\Sigma k}\Big[1+O(|k|n^{-1/2})+O\Big(\frac{|k|^2}{n}\sum_{m=1}^{\infty}(m\wedge n)mb_m\Big)+O\Big(\frac{|k|^2}{n}\sum_{j=1}^{n}d_j\Big)+O\Big(\frac{|k|^3}{n^{3/2}}\sum_{j=1}^{n}a_j\Big)\Big], \tag{5.7}$$

where the error estimate in (c) depends on U. *The constants* θ *and* Σ *(given by* (3.31), (3.32)*) are model dependent.*

It is not hard to see that each of the O terms in (5.7) is indeed an error term when we assume that Assumption (CLT) holds. However, in the general set-up in Assumption (CLT), it is not clear to us which term on the right-hand side of (5.7) is typically the largest.

Note that since D has finite range, there exists a constant $C_2 \geq 1$ independent of β such that

$$|\hat{D}(k) - 1 - ik\cdot\theta_\varnothing - \frac{1}{2}k^t\Sigma_\varnothing k| \leq C_2|k|^3, \tag{5.8}$$

and let $K_2 = 2C_2$, which is independent of β.

Recall (4.7) and define the following approximation to Σ:

$$\Sigma_n = \Sigma_\varnothing - \theta_n\theta_n^t - \sum_{m=2}^{n}\nabla^2\Big[e^{i\theta_n\cdot k(m-1)}\hat{\pi}_m(k)\Big]_{k=0}. \tag{5.9}$$

Let $B \equiv \sum_m b_m$ and $d^* \equiv \sup_n d_n$.

Our induction hypothesis for the central limit theorem is that the following bound holds for all $\beta \leq \beta_0$, all $0 \leq j \leq n$, each fixed $U > 0$ and some δ_2, independent of β:

For $|k| \leq U/(C_a\sqrt{n}) \wedge \delta_2$, and some K independent of β we can write,

$$\hat{c}_j(k) = \exp\Big[\sum_{l=1}^{j} ik \cdot \theta_l - \frac{1}{2}k^t\Sigma_l + r_l(k)\Big] \quad \text{with } |r_j(k)| \leq K(|k|^2 d_n + |k|^3 a_n), \tag{5.10}$$

where the empty sum appearing when $j = 0$ is defined to be zero.

The initialisation of the induction ($n = 0$ case) holds trivially as $1 = e^0$.

5.1 The CLT induction advanced

We follow the same strategy as in Section 4.1, now expanding the Fourier transform one order further.

The induction step will be achieved as soon as we are able to write

$$\frac{\hat{c}_{n+1}(k)}{\hat{c}_n(k)} = \exp\Big[ik \cdot \theta_{n+1} - \frac{1}{2}k^t\Sigma_{n+1}k + r_{n+1}(k)\Big], \tag{5.11}$$

for $r_{n+1}(k)$ satisfying the required bound. For this, we write

$$\frac{\hat{c}_{n+1}(k)}{\hat{c}_n(k)} = 1 + ik \cdot \theta_{n+1} - \frac{1}{2}k^t(\Sigma_{n+1} + \theta_{n+1}\theta_{n+1}^t)k + r'_{n+1}(k) \tag{5.12}$$

and then set

$$r_{n+1}(k) = \log\Big[1 + ik \cdot \theta_{n+1} - \frac{1}{2}k^t(\Sigma_{n+1} + \theta_{n+1}\theta_{n+1}^t)k + r'_{n+1}(k)\Big] - ik \cdot \theta_{n+1} + \frac{1}{2}k^t\Sigma_{n+1}k. \tag{5.13}$$

The following lemma is an easy consequence of (3.34) and (5.9):

Lemma 5.1. *We have $|\theta_\varnothing| \leq L$ and $|\Sigma_\varnothing| \leq d^2L^2$, and when Assumption (CLT) holds we have, for all n, $|\theta_n| \leq L + C_\beta B$, and*

$$\begin{aligned} |\Sigma_n| \leq & d^2L^2 + (L + C_\beta B)^2 + 2d^2(L + C_\beta B)B^* + C_\beta B^*, \quad \textit{and} \\ |\Sigma_n + \theta_n\theta_n^t| \leq & d^2L^2 + 2d^2(L + C_\beta B)B^* + C_\beta B^*. \end{aligned} \tag{5.14}$$

Choose β_0 so that $C_\beta \leq 1$ for all $\beta \leq \beta_0$, and suppose that the required bound (5.10) holds for $r'_{n+1}(k)$ with constant K_2. By Lemma 5.1, $|k||\theta_{n+1}| + \frac{1}{2}|k|^2|\Sigma_{n+1} - \theta_{n+1}\theta_{n+1}^t| + |r'_{n+1}(k)| \leq \frac{1}{2}$ for k satisfying the bound specified in the preamble of the induction hypothesis (5.10) with

$$\delta_2 = \Big(6(L + B) + 3(d^2L^2 + 2d^2(L + B)B^* + B^*) + 6K_2(2 + d^*)\Big)^{-1}. \tag{5.15}$$

Therefore we may apply Taylor's Theorem $|\log(1 + x) - x + \frac{x^2}{2}| \leq 8|x|^3$ for $|x| \leq \frac{1}{2}$ to (5.13). This implies that when the required bound holds for $r'_{n+1}(k)$ with constant K_2, it also holds for $r_{n+1}(k)$ for some K independent of β, since the terms of order k and $|k|^2$ in (5.13) cancel. Specifically, if

$|r'_{n+1}(k)| \le K_2(|k|^2 d_{n+1} + |k|^3 a_{n+1})$ then using Taylor's Theorem, followed by the assumed bound on $r'_{n+1}(k)$ and the facts that $a_{n+1} \ge 1$ and $|k| \le UC_a^{-1}$, we obtain

$$\begin{aligned}
|r_{n+1}(k)| \le & |r'_{n+1}(k)| + |k||\theta_{n+1}| \left(\frac{1}{2}|k|^2|\Sigma_{n+1} + \theta_{n+1}\theta_{n+1}^t| + |r'_{n+1}(k)|\right) \\
& + \frac{1}{2}\left(\frac{1}{2}|k|^2|\Sigma_{n+1} + \theta_{n+1}\theta_{n+1}^t| + |r'_{n+1}(k)|\right)^2 \\
& + 8\left(|k||\theta_{n+1}| + \frac{1}{2}|k|^2|\Sigma_{n+1} + \theta_{n+1}\theta_{n+1}^t| + |r'_{n+1}(k)|\right)^3 \\
\le & CK_2(|k|^2 d_{n+1} + |k|^3 a_{n+1}) \le K(|k|^2 d_{n+1} + |k|^3 a_{n+1}),
\end{aligned} \tag{5.16}$$

when $K \ge CK_2$. Here $C \ge 1$ is a constant that depends on $U, C_a, B, B^*, d^*, L, d$, but is independent of β. Most of this section will be devoted to the proof of the following lemma:

Lemma 5.2. *If* (5.10) *holds for all $j \le n$, then*

$$|r'_{n+1}(k)| \quad \le K_2(|k|^2 d_{n+1} + |k|^3 a_{n+1}). \tag{5.17}$$

In the course of the proof of Lemma 5.2, we will frequently use the following lemma, whose proof follows easily by applying Taylor's Theorem at $t = 0$ to the map from $\mathbb{R} \to \mathbb{C}$ given by $t \mapsto e^{tx}$:

Lemma 5.3. *For all $x \in \mathbb{C}, j \in \mathbb{N}$,*

$$\left| e^x - \sum_{l=0}^{j} \frac{x^l}{l!} \right| \le \frac{|x|^{j+1}}{(j+1)!} e^{|\mathrm{Re}(x)|},$$

where $\mathrm{Re}(x)$ *is the real part of x.*

5.1.1 Proof of Lemma 5.2

The proof involves expressing $r'_{n+1}(k)$ as a sum of three terms and showing that each term is bounded in absolute value by the right hand side of (5.17).

Recall (5.12), then

$$r'_{n+1}(k) \quad = I + II,$$

where

$$\begin{aligned}
I &= [\hat{D}(k) - 1 - ik\cdot\theta_{\varnothing} + \frac{1}{2}k^t\Sigma_{\varnothing}k] + \sum_{m=2}^{n+1}\left[\hat{\pi}_m(k) - k\nabla\hat{\pi}_m(0) - \frac{1}{2}k^t\nabla^2\hat{\pi}_m(0)k\right], \\
II &= \sum_{m=2}^{n+1}\left[\hat{\pi}_m(k)\Big[\frac{\hat{c}_{n+1-m}(k)}{\hat{c}_n(k)} - 1\Big] + k\cdot\nabla\hat{\pi}_m(0)i(m-1)k\cdot\theta_{n+1}\right].
\end{aligned}$$

We will bound $|I|$ and $|II|$, and then choose β_0 sufficiently small so that $|I| + |II|$ satisfies the bound on the right hand side of (5.17). By (5.8) and (5.3) in Assumption (CLT), and the fact that $a_{n+1} \ge 1$ we have

$$|I| \quad \le C_2|k|^3 + \textstyle\sum_{m=2}^{n+1} C_\beta |k|^3 m^2 b_m \le (C_2 + C_\beta)|k|^3 a_{n+1}. \tag{5.18}$$

To bound II, we first split $II = II_1 + II_2$, with

$$II_1 = \sum_{m=2}^{n+1} [\hat{\pi}_m(k) - k\cdot\nabla\hat{\pi}_m(0)] \left[\frac{\hat{c}_{n+1-m}(k)}{\hat{c}_n(k)} - 1\right],$$

$$II_2 = \sum_{m=2}^{n+1} k\cdot\nabla\hat{\pi}_m(0) \left[\frac{\hat{c}_{n+1-m}(k)}{\hat{c}_n(k)} - 1 + i(m-1)k\cdot\theta_{n+1}\right].$$

For II_1, we use (5.3) in Assumption (CLT) and Lemma 5.3 for $j=0$, to get

$$\begin{aligned}
|II_1| &\leq \sum_{m=2}^{n+1} |\hat{\pi}_m(k) - k\cdot\nabla\hat{\pi}_m(0)| \left|\frac{\hat{c}_{n+1-m}(k)}{\hat{c}_n(k)} - 1\right| \\
&\leq \sum_{m=2}^{n+1} CC_\beta|k|^2 m b_m \left|\exp\left[-\sum_{l=n+2-m}^{n}\left[ik\cdot\theta_l - \frac{1}{2}k^t\Sigma_l k + r_l(k)\right]\right] - 1\right|, \\
&\leq CC_\beta|k|^2 \sum_{m=2}^{n+1} m b_m e^{\chi_{m,n}(k)} \sum_{l=n+2-m}^{n}\left[|k||\theta_l| + \frac{1}{2}|k|^2|\Sigma_l| + |r_l(k)|\right],
\end{aligned}$$

with

$$\chi_{m,n}(k) = \sum_{l=n+2-m}^{n}\left[\frac{|k|^2}{2}|\Sigma_l| + |r_l(k)|\right], \tag{5.19}$$

Since a_n is increasing, for $|k| \leq U/(C_a\sqrt{n})$ it follows from Lemma 5.1 and (5.10) that

$$e^{\chi_{m,n}(k)} \leq e^{m|k|^2(C+d^*+a_n|k|)} \leq C, \tag{5.20}$$

and by Lemma 5.1 and (5.10),

$$|k||\theta_l| + \frac{|k|^2}{2}|\Sigma_l| + |r_l(k)| \leq C(1+C_\beta K)|k|. \tag{5.21}$$

Therefore,

$$|II_1| \leq CC_\beta(1+C_\beta K)|k|^3 \sum_{m=2}^{n+1} m^2 b_m = CC_\beta(1+C_\beta K)|k|^3 a_{n+1}. \tag{5.22}$$

For II_2 we use (5.10), Lemma 5.3 for $j=1$ and (5.20), to obtain

$$\begin{aligned}
|II_2| &\leq |k|\sum_{m=3}^{n+1} C_\beta b_m \left|\exp\left[-\sum_{l=n+2-m}^{n}\left[ik\cdot\theta_l - \frac{1}{2}k^t\Sigma_l k + r_l(k)\right]\right] - 1 + ik\cdot\theta_{n+1}(m-1)\right| \\
&\leq C_\beta|k|\sum_{m=2}^{n+1} b_m \sum_{l=n+2-m}^{n}\left[|k||\theta_{n+1}-\theta_l| + \frac{|k|^2}{2}|\Sigma_l| + |r_l(k)|\right] \\
&\quad + C_\beta|k|\sum_{m=2}^{n+1} b_m \left[\sum_{l=n+2-m}^{n}\left[|k||\theta_l| + \frac{|k|^2}{2}|\Sigma_l| + |r_l(k)|\right]\right]^2 e^{\chi_{m,n}(k)}.
\end{aligned}$$

The second sum can be bounded, using (5.21) and (5.20), as

$$CC_\beta(1+C_\beta K)^2|k|^3 \sum_{m=2}^{n+1} m^2 b_m \leq CC_\beta(1+C_\beta K)^2|k|^3 a_{n+1}. \tag{5.23}$$

By a similar argument as in (5.20), we have for $|k| \leq U/(C_a\sqrt{n})$,

$$\frac{|k|^2}{2}|\Sigma_l| + |r_l(k)| \leq C|k|^2. \tag{5.24}$$

Therefore,

$$C_\beta|k|\sum_{m=2}^{n+1} b_m \sum_{l=n+2-m}^{n}\left[\frac{|k|^2}{2}|\Sigma_l| + |r_l(k)|\right] \leq CC_\beta|k|^3\sum_{m=2}^{n+1} mb_m \leq CC_\beta|k|^3 a_{n+1}. \tag{5.25}$$

We continue with the last contribution to II_2. For this, we note that, since $\{b_m\}_{m\geq 1}$ is a decreasing sequence,

$$|\theta_{n+1} - \theta_l| \leq \sum_{s=l+1}^{n+1} b_s \leq (n-l+1)b_{l+1}. \tag{5.26}$$

Thus,

$$\sum_{m=2}^{n+1} b_m \sum_{l=n+2-m}^{n} |\theta_{n+1} - \theta_l| \leq \sum_{m=2}^{n+1} mb_m \sum_{l=n+2-m}^{n} b_{l+1} = d_{n+1}. \tag{5.27}$$

We conclude that

$$|II_2| \leq CC_\beta(1 + CC_\beta K)^2(|k|^2 d_{n+1} + |k|^3 a_{n+1}). \tag{5.28}$$

We have shown that

$$|I| + |II| \leq \Big(C_2 + C_\beta + CC_\beta(1 + CC_\beta K)^2\Big)(|k|^2 d_{n+1} + |k|^3 a_{n+1}). \tag{5.29}$$

Choose β_0 sufficiently small so that for all $\beta \leq \beta_0$, $C_\beta + CC_\beta(1 + CC_\beta K)^2 \leq \frac{1}{2}K_2$. Recall that $K_2 = 2C_2$. Then for $\beta \leq \beta_0$, $|I| + |II| \leq (C_2 + \frac{1}{2}K_2)(|k|^2 d_{n+1} + |k|^3 a_{n+1}) \leq K_2|k|^2 d_{n+1} + |k|^3 a_{n+1})$ as required. This completes the proof of Lemma 5.2. □

5.2 Proof of Theorem 5.1

We will make use of the following lemma:

Lemma 5.4. *For all $\beta \leq \beta_0$, and all $j \in \mathbb{N}$,*

(i) $\nabla r_j(0) = 0$, *and*

(ii) $|\nabla^2 r_j(0)| \leq 2Kd^2 d_j$.

Proof. The induction hypothesis (5.10), now verified for all j, states that $|r_j(k)| \leq K(|k|^2 d_j + |k|^3 a_j)$. Therefore, letting $[\nabla r_j(0)]_i$ denote the i^{th} coordinate of the vector $\nabla r_j(0)$, we have

$$|[\nabla r_j(0)]_i| = \lim_{k_i\to 0}\frac{|r_j(0,\ldots,0,k_i,0,\ldots,0)|}{|k_i|} \leq \lim_{k_i\to 0}\frac{K|k_i|^2 d_j + |k_i|^3 a_j)}{|k_i|} = 0 \tag{5.30}$$

Since all partial derivatives of $\hat{c}_n(k)$ up to second order exist and are continuous, and $\hat{c}_n(0) = 1$, we have from (5.12) and (5.13) that all partial derivatives of $r_j(k)$ up to second order exist in a neighbourhood of 0 and are continuous. Let $[\nabla^2 r_j(0)]_{lm}$ denote the $(l,m)^{\text{th}}$ entry of the matrix $\nabla^2 r_j(0)$ and suppose that $|r_j(k)| \le J_1|k|^2 + J_2|k|^3$. We claim that this implies that $|[\nabla^2 r_j(0)]_{lm}| \le 2J_1$ for each m, l, from which part *(ii)* of the lemma follows immediately. Without loss of generality we suppose that $l, m \in \{1, 2\}$.

Let $h(k_1, k_2) = r_j(k_1, k_2, 0, \ldots, 0)$. By the second order mean value theorem, $f_{u_1,u_2}(t) \equiv h(tu_1, tu_2)$ satisfies

$$f_{u_1,u_2}(t) = f_{u_1,u_2}(0) + f'_{u_1,u_2}(0)t + f''_{u_1,u_2}(t^*)\frac{t^2}{2} \tag{5.31}$$

for some $t^* \equiv t^*(t, u_1, u_2) \in (0, t)$.

Now $f_{u_1,u_2}(0) = h(0,0) = 0$ and

$$|f'_{u_1,u_2}(0)| = \lim_{t\to 0}\left|\frac{h(tu_1, tu_2) - h(0,0)}{t}\right| = \lim_{t\to 0}\left|\frac{h(tu_1, tu_2)}{t}\right| \le \lim_{t\to 0}\left|\frac{C_{u_1,u_2}(t^2 + t^3)}{t}\right| = 0 \tag{5.32}$$

where we have used the bound on $|r_j(k)|$ in the last inequality. Thus (5.31) reduces to

$$f_{u_1,u_2}(t) = f''_{u_1,u_2}(t^*)\frac{t^2}{2}, \tag{5.33}$$

and by hypothesis the left hand side is bounded in absolute value by $J_1 t^2(u_1^2+u_2^2) + J_2 t^3(u_1^2+u_2^2)^{3/2}$.

We now set $t_n = 1/n$ and let $t_n^* = t^*(t_n, u_1, u_2)$. Then for each n, $|f''_{u_1,u_2}(t_n^*)| \le J_1(u_1^2 + u_2^2) + n^{-1}J_2(u_1^2 + u_2^2)^{3/2}$. By the multivariate chain rule $\frac{d}{dt}h(\vec{g}(t)) = \nabla h \cdot \vec{g}'(t)$ we have

$$f''_{u_1,u_2}(t_n^*) = u_1^2 h_{11}(t_n^* u_1, t_n^* u_2) + u_2^2 h_{22}(t_n^* u_1, t_n^* u_2) + 2u_1 u_2 h_{12}(t_n^* u_1, t_n^* u_2), \tag{5.34}$$

and thus

$$|u_1^2 h_{11}(t_n^* u_1, t_n^* u_2) + u_2^2 h_{22}(t_n^* u_1, t_n^* u_2) + 2u_1 u_2 h_{12}(t_n^* u_1, t_n^* u_2)| \le J_1(u_1^2+u_2^2) + \frac{J_2}{n}(u_1^2+u_2^2)^{3/2}. \tag{5.35}$$

Putting $u_1 = 1, u_2 = 0$ in (5.35) gives $|h_{11}(t_n^*, 0)| \le J_1 + n^{-1}J_2$, and similarly $|h_{22}(t_n^*, 0)| \le J_1 + n^{-1}J_2$. Letting $n \to \infty$ and using the fact that $t_n^* \in (0, t_n)$ (so that $t_n = 1/n \to 0$ implies that $t_n^* \to 0$ as $n \to \infty$) we have $|h_{11}(0,0)| \le J_1$ by continuity of the partial derivatives. Similarly, by taking $u_1 = 0, u_2 = 1$, we obtain $|h_{22}(0,0)| \le J_1$. Next, set $u_1 = u_2 = 1$ in (5.35) and use $|a + b| \le d \Rightarrow |a| \le d + |b|$ to see that

$$2|h_{12}(t_n^* u_1, t_n^* u_2)| \le |h_{11}(t_n^* u_1, t_n^* u_2) + h_{22}(t_n^* u_1, t_n^* u_2)| + J_1(u_1^2 + u_2^2) + \frac{J_2}{n}(u_1^2 + u_2^2)^{3/2}. \tag{5.36}$$

Now use the triangle inequality and let $n \to \infty$ to get $|h_{12}(0,0)| \le \frac{3}{2}J_1$. □

We are now ready to prove the statements in Theorem 5.1(a)–(c) one by one.

Proof of Theorem 5.1(a): Using (5.10) and Lemma 5.4(i), we have

$$\sum_{x\in\mathbb{Z}^d} x c_n(x) \;=\; -i\nabla\hat{c}_n(0) = -i\sum_{j=1}^{n}\Big[i\theta_j + \nabla r_j(0)\Big] = n\theta + \sum_{j=1}^{n}[\theta_j - \theta], \tag{5.37}$$

so that it suffices to prove that

$$\sum_{j=1}^{n}[\theta_j - \theta] = O(1). \tag{5.38}$$

For this, we use (3.31), (4.7) as well as the second bound in (5.2) and to note that

$$\sum_{j=1}^{n}|\theta_j-\theta| \le \sum_{j=1}^{n}\sum_{m=j+1}^{\infty}|\nabla\hat{\pi}_m(0)| \le C_\beta\sum_{j=1}^{\infty}\sum_{m=j+1}^{\infty} b_m = C_\beta\sum_{m=1}^{\infty} m b_m = O(1), \tag{5.39}$$

by (5.4). □

Proof of Theorem 5.1(b): Recall that $\mathrm{Var}_\beta(\omega_n)$ is the covariance matrix of ω_n. Then

$$[\mathrm{Var}_\beta(\omega_n)]_{lm} = \sum_x x_l x_m c_n(x) - \left(\sum_x x_l c_n(x)\right)\left(\sum_x x_m c_n(x)\right). \tag{5.40}$$

By (5.10) and Lemma 5.4(i-ii), and writing $[\theta_p]_l$ for the l^{th} component of θ_p,

$$\begin{aligned}
\sum_{x\in\mathbb{Z}^d} x_l x_m c_n(x) &= -[\nabla^2\hat{c}_n(0)]_{lm} \\
&= \sum_{p=1}^{n}\Big([\Sigma_p]_{lm} - [\nabla^2 r_p(0)]_{lm}\Big) - \sum_{p,q=1}^{n}\left(i[\theta_p]_l + [\nabla r_p(0)]_l\right)\left(i[\theta_q]_m + [\nabla r_q(0)]_m\right) \\
&= \sum_{p=1}^{n}\left([\Sigma_p]_{lm} + O(d_p)\right) + \sum_{p=1}^{n}[\theta_p]_l\sum_{q=1}^{n}[\theta_q]_m.
\end{aligned} \tag{5.41}$$

It follows from (5.37) that $\sum_x x_l c_n(x) = \sum_{p=1}^n [\theta_p]_l$ and from (5.40) and (5.41) that

$$\begin{aligned}
[\mathrm{Var}_\beta(\omega_n)]_{lm} &= \sum_{p=1}^{n}\left([\Sigma_p]_{lm} + O(d_p)\right) \\
&= n[\Sigma]_{lm} + O\left(\sum_{p=1}^{n} d_p\right) + \sum_{p=1}^{n}([\Sigma_p]_{lm} - [\Sigma]_{lm}).
\end{aligned}$$

Therefore to complete the proof, it is sufficient to show that for $p\le n$,

$$|[\Sigma_p]_{lm} - [\Sigma]_{lm}| = O\left(1\vee(p\wedge n)pb_p\right). \tag{5.42}$$

By (3.32) and (5.9), the left hand side of (5.42) is bounded by

$$\begin{aligned}
&|[\theta]_l[\theta]_m - [\theta_p]_l[\theta_p]_m| + \sum_{r=p+1}^{\infty}\left|\Big[[\nabla^2 e^{i(r-1)k\cdot\theta_p}\hat{\pi}_r(k)]_{lm}\Big]_{k=0}\right| \\
&\quad\le |[\theta]_l||[\theta]_m - [\theta_p]_m| + |[\theta_p]_m||[\theta]_l - [\theta_p]_l| \\
&\qquad + \sum_{r=p+1}^{\infty}\Big((r-1)\Big(|[\nabla\hat{\pi}_r(0)]_l| + |[\nabla\hat{\pi}_r(0)]_m|\Big) + |[\nabla^2\hat{\pi}_r(0)]_{lm}|\Big),
\end{aligned} \tag{5.43}$$

since $\hat{\pi}_r(0)=0$ by (5.2). The first two terms are $O(1)$ using the fact that $|\theta|$ is finite and the $|\theta_p|$ are uniformly bounded together with (5.39). By (5.2), the last term is bounded by

$$C_\beta\sum_{p=1}^{n}\sum_{r=p+1}^{\infty} rb_r \le C_\beta\sum_{r=1}^{\infty}(r\wedge n)rb_r,$$

which completes the proof. □

Proof of Theorem 5.1(c): Fix $U > 0$ and let $k \leq UC_a^{-1}$ and let $k_n = \frac{k}{\sqrt{n}}$. There exists n_U such that for all $n \geq n_U$ we have $k_n \leq \frac{U}{C_a\sqrt{n}} \wedge (6(L+B) + 3(d^2L^2 + 2d^2(L+B)B^* + B^*) + 6K_2(2+d^*))^{-1}$. For such n we use (5.10) to write

$$\begin{aligned}\hat{c}_n(k_n) &= \exp\Big[\sum_{j=1}^n \Big(ik_n\cdot\theta_j - \frac{1}{2}k_n^t\Sigma_j k_n + r_j(k_n)\Big)\Big]\\ &= \exp\Big[ik\cdot\theta\sqrt{n} - \frac{1}{2}k^t\Sigma k\Big]\\ &\qquad\times\exp\Big[\sum_{j=1}^n ik_n\cdot[\theta_j-\theta] - \frac{1}{2}k_n^t\sum_{j=1}^n[\Sigma_j-\Sigma]k_n\Big]\exp\sum_{j=1}^n r_j(k_n).\end{aligned}\tag{5.44}$$

From (5.39) we have

$$\Big|\sum_{j=1}^n ik_n\cdot[\theta_j-\theta]\Big| \leq |k_n|\sum_{j=1}^n|\theta_j-\theta| = O(|k_n|),\tag{5.45}$$

and using (5.42) we obtain

$$\Big|\frac{k_n}{2}\sum_{j=1}^n[\Sigma_j-\Sigma]k_n^t\Big| \leq \frac{|k_n|^2}{2}\sum_{j=1}^n|\Sigma_j-\Sigma| = O\Big(|k_n|^2\sum_{m=1}^\infty(m\wedge n)mb_m\Big).\tag{5.46}$$

Finally we use (5.10) to get

$$\sum_{j=1}^n|r_j(k_n)| \leq O\Big(\frac{|k|^2}{n}\sum_{j=1}^n d_j\Big) + O\Big(\frac{|k|^3}{n^{3/2}}\sum_{j=1}^n a_j\Big).\tag{5.47}$$

This proves the bound in Theorem 5.1(c) for $n \geq n_U$. The bound holds trivially for $n \leq n_U$ by allowing the constants to depend on n_U (and hence on U). □

6 Bounds on the lace expansion

In this section, we give bounds on the lace expansion coefficients, and verify that these bounds imply Theorems 2.1, 2.2 and 2.3. We start in Section 6.1 by formulating some general bounds on $\hat{\pi}_m(0), \nabla\hat{\pi}_m(0)$ and $\nabla^2\hat{\pi}_m(0)$ that will reduce the bounds on the derivatives of $\hat{\pi}_m(k)$ to a single bound, which we will prove separately for each model. In Section 6.2, we prove the bounds on the lace expansion coefficients for once edge-reinforced random walk with drift, and complete the proof of Theorem 2.1. In Section 6.3, we prove the bounds on the lace expansion coefficients for excited random walk, and complete the proof of Theorems 2.2–2.3.

6.1 Reduction to a single bound

Recall (3.26) and define $\mathcal{A}_{m,N} = \{(j_1,\dots,j_N)\in\mathbb{Z}_+^N : \sum_{l=1}^N j_l = m-N-1\}$, and

$$\pi_m^{(N)}(x,y) = \sum_{\vec{j}\in\mathcal{A}_{m,N}}\sum_{\vec{\omega}_1^{(0)}} D(\omega_1^{(0)})\mathbb{E}_1^{\vec{\omega}_1^{(0)}}\Big[\Delta_{j_1+1}^{(1)}\,\mathbb{E}_2^{\vec{\omega}_{j_1+1}^{(1)}}\Big[\Delta_{j_2+1}^{(2)}\cdots\mathbb{E}_N^{\vec{\omega}_{j_{N-1}+1}^{(N-1)}}\Big[\Delta_{j_N+1}^{(N)}I_{\{\omega_{j_N}^{(N)}=x,\omega_{j_N+1}^{(N)}=y\}}\Big]\cdots\Big],\tag{6.1}$$

so that

$$\pi_m^{(N)}(y) = \sum_x \pi_m^{(N)}(x,y). \tag{6.2}$$

We also let

$$\pi_m(x,y) = \sum_{N=1}^{\infty} \pi_m^{(N)}(x,y). \tag{6.3}$$

The starting point for the bounds on the lace expansion coefficients for self-interacting random walks is the following proposition:

Proposition 6.1 (Reduction of the bounds on the expansion coefficients)**.**
For a self-interacting stochastic process with range $L < \infty$, where $\pi_m^{(N)}(y)$ is given by (3.26)*, the following bounds hold:*

$$\hat{\pi}_m(0) = 0, \tag{6.4}$$

$$|\nabla \hat{\pi}_m(0)| \leq \sqrt{d} L \sum_{x,y} |\pi_m(x,y)|, \tag{6.5}$$

$$|\nabla^2 \hat{\pi}_m(0)| \leq (dL)^2 (2m-1) \sum_{x,y} |\pi_m(x,y)|, \tag{6.6}$$

$$\Big|\hat{\pi}_m(k) - k \cdot \nabla \hat{\pi}_m(0)\Big| \leq |k|^2 m L^2 \sum_{x,y} |\pi_m(x,y)|, \tag{6.7}$$

$$\Big|\hat{\pi}_m(k) - k \cdot \nabla \hat{\pi}_m(0) - k \nabla^2 \hat{\pi}_m(0) k^t\Big| \leq |k|^3 m^2 L^3 \sum_{x,y} |\pi_m(x,y)|. \tag{6.8}$$

Proof. We note that for every $x \in \mathbb{Z}^d$ and $N \geq 1$,

$$\sum_y \Delta_{j_N+1}^{(N)} I_{\{\omega_{j_N}^{(N)}=x, \omega_{j_N+1}^{(N)}=y\}} = \sum_y \Big(p^{\vec{\omega}_{j_{N-1}+1}^{(N-1)} \circ \vec{\omega}_{j_N}^{(N)}}(x,y) - p^{\vec{\omega}_{j_N}^{(N)}}(x,y) \Big) = 1 - 1 = 0$$

from which it follows immediately that, for every $x \in \mathbb{Z}^d$,

$$\sum_y \pi_m(x,y) = 0. \tag{6.9}$$

Summing (6.9) over x establishes the first claim of Proposition 6.1 and the first property of $\hat{\pi}_m(k)$ in (4.1) and (5.2).

Furthermore, again by (6.9), we have that

$$\begin{aligned} [\nabla \hat{\pi}_m(0)]_l &= i \sum_y y_l \pi_m(y) = i \sum_{x,y} y_l \pi_m(x,y) = i \sum_{x,y} x_l \pi_m(x,y) + i \sum_{x,y} [y_l - x_l] \pi_m(x,y) \\ &= i \sum_{x,y} [y_l - x_l] \pi_m(x,y). \end{aligned} \tag{6.10}$$

For walks with range L, we have that $|y_j - x_j| \leq L$, so that

$$|[\nabla \hat{\pi}_m(0)]_l| \leq L \sum_{x,y} |\pi_m(x,y)|, \tag{6.11}$$

which establishes the second claim of Proposition 6.1 since $\sum_{l=1}^d u_l^2 \le d \max_l |u_l|^2$. Similarly,

$$\begin{aligned}
-[\nabla^2 \hat\pi_m(0)]_{st} &= \sum_y y_s y_t \pi_m(y) = \sum_{x,y} y_s y_t \pi_m(x,y) \\
&= \sum_{x,y} x_s x_t \pi_m(x,y) + \sum_{x,y}[y_s - x_s] x_t \pi_m(x,y) \\
&\quad + \sum_{x,y}[y_t - x_t] x_s \pi_m(x,y) + \sum_{x,y}[y_s - x_s][y_t - x_t]\pi_m(x,y) \\
&= \sum_{x,y}[y_s - x_s] x_t \pi_m(x,y) + \sum_{x,y}[y_t - x_t] x_s \pi_m(x,y) + \sum_{x,y}[y_s - x_s][y_t - x_t]\pi_m(x,y).
\end{aligned}$$

We use that $|y_j - x_j| \le L$ and $|x_j| \le L(m-1)$ to obtain

$$|[\nabla^2 \hat\pi_m(0)]_{st}| \le (2m-1)L^2 \sum_{x,y} |\pi_m(x,y)|.$$

This establishes the third claim of Proposition 6.1 by (5.1).

By (6.9),

$$\hat\pi_m(k) = \sum_{x,y} e^{ik\cdot y}\pi_m(x,y) = \sum_{x,y} e^{ik\cdot x}[e^{ik\cdot(y-x)} - 1]\pi_m(x,y).$$

Together with (6.10) this gives

$$\begin{aligned}
\hat\pi_m(k) - k\cdot\nabla\hat\pi_m(0) &= \sum_{x,y}\Big[[1 + (e^{ik\cdot x} - 1)][e^{ik\cdot(y-x)} - 1] - ik\cdot(y-x)\Big]\pi_m(x,y), \\
&= \sum_{x,y}[e^{ik\cdot x} - 1][e^{ik\cdot(y-x)} - 1]\pi_m(x,y) + \sum_{x,y}[e^{ik\cdot(y-x)} - 1 - ik\cdot(y-x)]\pi_m(x,y).
\end{aligned}$$

The Taylor estimates $|e^{iu}-1| \le |u|$ and $|e^{iu}-1-iu| \le \frac{1}{2}u^2$, together with the finite range properties of the walk proves the fourth claim of Proposition 6.1. The final claim is proved similarly by first showing that

$$\begin{aligned}
\hat\pi_m(k) - k\cdot\nabla\hat\pi_m(0) - \frac{1}{2}k\nabla^2\hat\pi_m(0)k^t = \sum_{x,y}\Big[&[e^{ik\cdot(y-x)} - 1 - ik\cdot(y-x) + \frac{1}{2}k(y-x)^t(y-x)k^t] \\
&+ [e^{ik\cdot x} - 1 - ik\cdot x][e^{ik\cdot(y-x)} - 1] \\
&+ ik\cdot x[e^{ik\cdot(y-x)} - 1 - ik\cdot(y-x)]\Big]\pi_m(x,y),
\end{aligned}$$

and then using $|e^{iu} - 1 - iu + \frac{1}{2}u^2| \le \frac{1}{6}|u|^3$ together with the previous estimates. □

We conclude that the bounds in (4.1), (5.2), (4.2) and (5.3) follow if we can show that

$$\sum_{x,y} |\pi_m(x,y)| \le C_\beta b_m, \tag{6.12}$$

for some sequence $\{b_m\}_{m\ge1}$ satisfying $\sum_m b_m < \infty$ for the law of large numbers or $\sum_m m b_m < \infty$ for the central limit theorem. We will prove (6.12) for once-edge-reinforced random walk in Section 6.2 and for excited random walk in Section 6.3 below.

6.2 Bounds for once-edge-reinforced random walk

In this section we show that for OERRWD, there exist $\beta_0 > 0$, $\mathcal{J} > 0$ and a constant C independent of β, such that (6.12) holds with

$$b_m = e^{-\mathcal{J}m}, \qquad C_\beta = C\beta^{\frac{1}{2}}. \tag{6.13}$$

Proposition 6.1 then shows that Assumptions (LLN) and (CLT) also hold, and we conclude this section with a short proof of Theorem 2.1.

The bounds in this section are based on the following large deviations estimates.

Lemma 6.2 (Large deviations)**.** *Whenever $\theta_\varnothing \neq 0$, there exist $\beta_0 = \beta_0(D(\cdot), w_0(\cdot)) > 0$ and $\mathcal{I} = \mathcal{I}(D(\cdot), w_0(\cdot)) > 0$ such that for all $|\beta| \leq \beta_0$,*

$$\sup_{\vec{\eta}} \mathbb{Q}_\beta^{\vec{\eta}}(\omega_n = \omega_0) \leq e^{-\mathcal{I}n}, \quad \textit{and} \tag{6.14}$$

$$\sup_{\vec{\omega}^{(i-2)}_{j_{i-2}+1}} \mathbb{E}^{\vec{\omega}^{(i-2)}_{j_{i-2}+1}}\left[\mathbb{Q}_\beta^{\vec{\omega}^{(i-1)}_{j_{i-1}+1}}(\omega^{(i)}_{j_i} = \omega^{(i-1)}_l)\right] \leq e^{-\mathcal{I}(j_{i-1}+1-l+j_i)}, \tag{6.15}$$

where the supremum is over all $j_{i-2}+1$-step random walk paths $\vec{\omega}^{(i-2)}_{j_{i-2}+1}$, and the law of walk i depends on walk $i-1$ but not walk $i-2$.

Proof. By Cramér's Theorem (e.g. see Theorem 2.2.30 of [5]) there exists $J = J(D(\cdot), w_0(\cdot)) > 0$ such that $\mathbb{Q}_0(\omega_n = \omega_0) \leq e^{-Jn}$ for all n. Let Ω denote the support of $D(x)$. It is easy to show that for every $\beta \in [0, \beta_0]$ and $\vec{\nu}$,

$$p_\beta^{\vec{\nu}}(x,y) \leq (1 + C\beta_0)\, p^\varnothing(x,y) \tag{6.16}$$

when $C \geq 1/w_0(x,y)$ (similarly for $\beta \in [-\beta_0, 0]$ when $C \geq (|\Omega|-1)/(\sum_{u\sim x} w_0(x,u) - \beta_0(|\Omega|-1))$.

By translation invariance, $w_0(\cdot) \geq W$ is uniformly bounded from below as a function on Ω. We fix

$$\begin{aligned} C &\geq \max\left\{\frac{|\Omega|-1}{\frac{1}{2}\sum_{u\sim 0} w_0(0,u)}, \sup_{y\sim 0}\frac{1}{w_0(0,y)}\right\}, \quad \text{and} \\ \beta_0 &\leq \min\left\{\frac{\sum_{u\sim 0} w_0(0,u)}{2(|\Omega|-1)}, J/2\right\}, \end{aligned} \tag{6.17}$$

and recall that $\mathbb{Q}_\beta^{\vec{\eta}}(\omega_n = x) = \sum_{\vec{\omega}_n}\prod_{i=0}^{n-1} p^{\vec{\omega}_i\circ\vec{\eta}}(\vec{\omega}_{i+1} - \vec{\omega}_i)$. The bound (6.14) with $\mathcal{I} = J/2$ follows immediately from this by (6.16) by choosing β_0 sufficiently small so that $\log(1 + C\beta_0) \leq J/2$. Similarly, for the second bound we write

$$\begin{aligned} &\mathbb{E}^{\vec{\omega}^{(i-2)}_{j_{i-2}+1}}\left[\mathbb{Q}_\beta^{\vec{\omega}^{(i-1)}_{j_{i-1}+1}}(\omega^{(i)}_{j_i} = \omega^{(i-1)}_l)\right] \\ =&\mathbb{E}^{\vec{\omega}^{(i-2)}_{j_{i-2}+1}}\Big[\sum_{\vec{\omega}^{(i-1)}_{j_{i-1}+1}}\prod_{k=0}^{j_{i-1}} p^{\vec{\omega}^{(i-2)}_{j_{i-2}+1}\circ\vec{\omega}^{(i-1)}_k}(\omega^{(i-1)}_k, \omega^{(i-1)}_{k+1})\sum_{\vec{\omega}^{(i)}_{j_i}}\prod_{m=0}^{j_i-1} p^{\vec{\omega}^{(i-1)}_{j_{i-1}+1}\circ\vec{\omega}^{(i)}_m}(\omega^{(i)}_m, \omega^{(i)}_{m+1}) I_{\{\omega^{(i)}_{j_i} = \omega^{(i-1)}_l\}}\Big]. \end{aligned}$$

Using (6.16) after the l^{th} step of $\vec{\omega}^{(N-1)}$ gives an upper bound of

$$
\begin{aligned}
&(1+C\beta_0)^{j_i+j_{i-1}+1-l}\mathbb{E}^{\vec{\omega}^{(i-2)}_{j_{i-2}+1}}\Big[\sum_{\vec{\omega}^{(i-1)}_l}\prod_{k=0}^{l-1} p^{\vec{\omega}^{(i-2)}_{j_{i-2}+1}\circ\vec{\omega}^{(i-1)}_k}(\omega^{(i-1)}_k,\omega^{(i-1)}_{k+1})\mathbb{Q}_0(\omega_{j_i+j_{i-1}+1-l}=\omega_0)\Big]\\
&\quad\le (1+C\beta_0)^{j_i+j_{i-1}+1-l}e^{-J(j_i+j_{i-1}+1-l)}\mathbb{E}^{\vec{\omega}^{(i-2)}_{j_{i-2}+1}}\Big[\sum_{\vec{\omega}^{(i-1)}_l}\prod_{k=0}^{l-1} p^{\vec{\omega}^{(i-2)}_{j_{i-2}+1}\circ\vec{\omega}^{(i-1)}_k}(\omega^{(i-1)}_k,\omega^{(i-1)}_{k+1})\Big]\\
&\quad\le e^{-(J-C\beta_0)(j_i+j_{i-1}+1-l)}\cdot 1\le e^{-\frac{1}{2}J(j_i+j_{i-1}+1-l)},
\end{aligned}
\tag{6.18}
$$

for $\beta_0\le J/(2C)$. This proves (6.15) with $\mathcal{I}=J/2$. □

Proposition 6.3 (Bounds on the OERRW expansion coefficients)**.** *There exist $\beta_0>0$ and $\mathcal{J}>0$ such that for all $|\beta|\le\beta_0$,*

$$\sum_{x,y}|\pi_m(x,y)|\le (C\beta)^{\frac{1}{2}}e^{-\mathcal{J}m}, \tag{6.19}$$

where $\mathcal{J}$ depends on β_0,w_0,d but is independent of β.

Proof. We bound $\sum_{x,y}|\pi^{(N)}_m(x,y)|$ and sum the resulting bound over N. For $N=1$, $m\ge 2$, (3.30) gives

$$
\begin{aligned}
\sum_{x,y}|\pi^{(1)}_m(x,y)| \quad&\le \sum_{x,y}\sum_{\vec{\omega}^{(0)}_1} D(\omega^{(0)}_1)\mathbb{E}^{\vec{\omega}^{(0)}_1}\Big[|\Delta^{(1)}_{m-1}|\ I_{\{\omega^{(1)}_{m-1}=y\}}I_{\{\omega^{(1)}_{m-2}=x\}}\Big]=\sum_{\vec{\omega}^{(0)}_1} D(\omega^{(0)}_1)\mathbb{E}^{\vec{\omega}^{(0)}_1}\Big[|\Delta^{(1)}_{m-1}|\Big]\\
&\le C\beta\sum_{\vec{\omega}^{(0)}_1} D(\omega^{(0)}_1)\mathbb{E}^{\vec{\omega}^{(0)}_1}\Big[I_{\{\omega^{(1)}_{m-2}=\omega^{(0)}_0\}}\Big]=C\beta\mathbb{Q}_\beta(\omega_{m-1}=\omega_0)\\
&\le C\beta e^{-\mathcal{I}(m-1)}\le C\beta e^{-\mathcal{I}m},
\end{aligned}
\tag{6.20}
$$

where we have applied the first bound of Lemma 6.2 in the last line, and the value of C changes from place to place.

For general N, we have that

$$\sum_{x,y}|\pi^{(N)}_m(x,y)|\le\sum_{\vec{j}\in\mathcal{A}_{m,N}}\sum_{\vec{\omega}^{(0)}_1} D(\omega^{(0)}_1)\mathbb{E}^{\vec{\omega}^{(0)}_1}\Big[|\Delta^{(1)}_{j_1+1}|\ \mathbb{E}^{\vec{\omega}^{(1)}_{j_1+1}}\Big[|\Delta^{(2)}_{j_2+1}|\ \cdots\mathbb{E}^{\vec{\omega}^{(N-1)}_{j_{N-1}+1}}\Big[|\Delta^{(N)}_{j_N+1}|\Big]\cdots\Big], \tag{6.21}$$

where, by (3.30),

$$|\Delta^{(i)}_{j_i+1}|\le C\beta\sum_{l_{i-1}=0}^{j_{i-1}} I_{\{\omega^{(i)}_{j_i}=\omega^{(i-1)}_{l_{i-1}}\}}. \tag{6.22}$$

Let $N\ge 2$, and for $q\in\{0,1\}$ let $A_q=\{i\le N:(N-i)\mod 2=q\}$ and B_q be the set of $\vec{j}\in\mathcal{A}_{m,N}$ so that $\sum_{i\in A_q}(j_i+1)\ge m/2$. For $r=0,\dots,N-1$, let $l_r\le j_r$ denote the number of steps in walk r up to the intersection point as in (6.22) (in particular, $l_0=0$). Then, combining (6.21) and (6.22),

$$
\begin{aligned}
\sum_{x,y}|\pi^{(N)}_m(x,y)|\le&(C\beta)^N\sum_{\vec{j}\in\mathcal{A}_{m,N},\vec{l}}\ \sum_{\vec{\omega}^{(0)}_1} D(\omega^{(0)}_1)\\
&\times\mathbb{E}^{\vec{\omega}^{(0)}_1}\Big[I_{\{\omega^{(1)}_{j_1}=\omega^{(0)}_{l_0}\}}\mathbb{E}^{\vec{\omega}^{(1)}_{j_1+1}}\Big[I_{\{\omega^{(2)}_{j_2}=\omega^{(1)}_{l_1}\}}\ \cdots\mathbb{E}^{\vec{\omega}^{(N-1)}_{j_{N-1}+1}}\Big[I_{\{\omega^{(N)}_{j_N}=\omega^{(N-1)}_{l_{N-1}}\}}\Big]\cdots\Big].
\end{aligned}
\tag{6.23}
$$

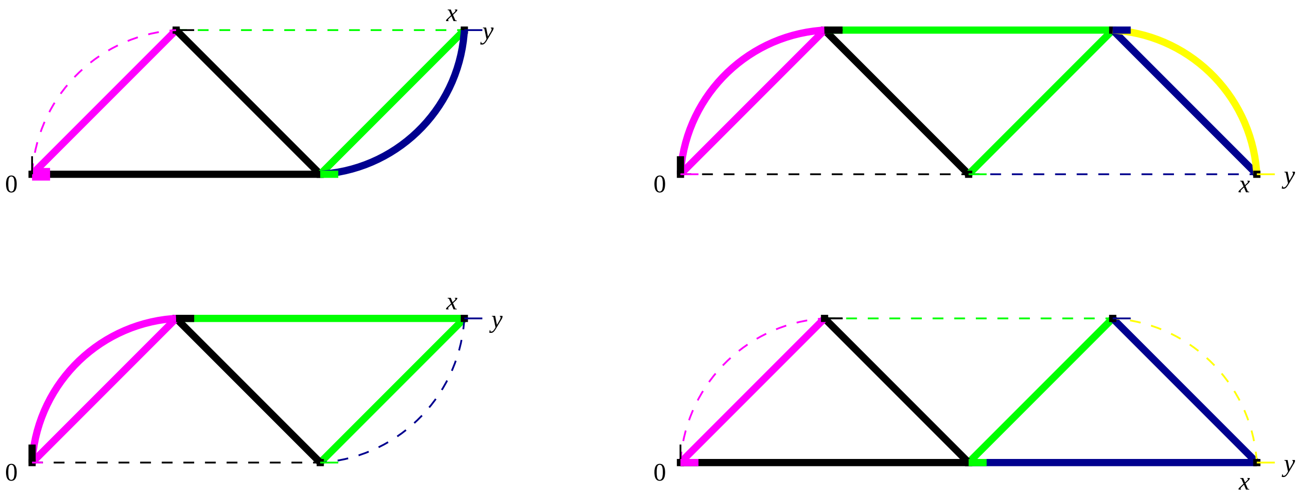

Figure 2: Illustration of the four cases of the diagrammatic bounds for OERRWD. On the left N is even ($N = 4$) and on the right N is odd ($N = 5$). In the first row $\vec{j} \in B_0$, while in the second, $\vec{j} \in B_1 \setminus B_0$. In each case the thick lines indicate the loops (whose total length is of order half of the total length m of the diagram) that give an exponentially small bound.

The bound is now split into four cases, depending on whether N is even or odd, and on whether $\vec{j} \in B_0$ or $\vec{j} \in B_1 \setminus B_0$. See Figure 2.

(a) The bound for N even and $\vec{j} \in B_0$. When N is even we bound the contribution to (6.21) from $\vec{j} \in B_0$ by using the following two facts, the first of which follows immediately from the second bound of Lemma 6.2, while the last holds with equality trivially.

The first fact is that for each even $i \in [2, N]$, uniformly in $\vec{\omega}^{(i-2)}_{j_{i-2}+1}$,

$$\mathbb{E}^{\vec{\omega}^{(i-2)}_{j_{i-2}+1}}\Big[I_{\{\omega^{(i-1)}_{j_{i-1}}=\omega^{(i-2)}_{l_{i-2}}\}}\mathbb{E}^{\vec{\omega}^{(i-1)}_{j_{i-1}+1}}\Big[I_{\{\omega^{(i)}_{j_i}=\omega^{(i-1)}_{l_{i-1}}\}}\Big]\Big] \le e^{-\mathcal{I}(j_i+(j_{i-1}+1-l_{i-1}))}. \tag{6.24}$$

The second fact is that,

$$\sum_{\vec{\omega}^{(0)}_1} D(\omega^{(0)}_1) \le 1. \tag{6.25}$$

By successive applications of (6.24) and lastly (6.25), when N is even we obtain a bound on the contribution to (6.21) from $\vec{j} \in B_0$ (whence $\sum_{i\le N, even} j_i \ge m/2 - N$), of

$$(C\beta)^N \sum_{\vec{j}\in B_0} \prod_{2\le i\le N,\ even} \sum_{\vec{l}} e^{-\mathcal{I}(j_i+(j_{i-1}+1-l_{i-1}))} \le (C\beta)^N e^{-\frac{\mathcal{I}}{2}m} \sum_{\vec{j}\in B_0} \sum_{\vec{l}} \prod_{2\le i\le N,\ even} e^{-\mathcal{I}(j_{i-1}+1-l_{i-1})}, \tag{6.26}$$

where the constant has changed (to accommodate a factor $e^{\mathcal{I}N}$). Using the fact that there are at most $j_i + 1$ possible values $\{0, 1, \ldots, j_i\}$ for l_i, this is bounded above by

$$(C\beta)^N e^{-\frac{\mathcal{I}}{2}m} \sum_{\vec{j}} \prod_{i=1}^{N} (j_i + 1), \tag{6.27}$$

which in turn can be bounded by the integral

$$e^{-\frac{\mathcal{I}}{2}m}(C\beta)^N \int_0^{m+3} x_1 \int_0^{m+3-x_1} x_2 \cdots \int_0^{m+3-(x_1+\cdots+x_{N-1})} x_N dx_N \cdots dx_1. \tag{6.28}$$

It is an easy exercise in integration by parts that

$$\int_0^{a-\sum_{i=1}^{j-1} x_i} \frac{x_j}{(2(N-j))!}\left(a-\sum_{i=1}^{j} x_i\right)^{2(N-j)} dx_j = \frac{1}{(2(N-(j-1)))!}\left(a-\sum_{i=1}^{j-1} x_i\right)^{2(N-(j-1))}. \tag{6.29}$$

Applying (6.29) N times, we bound (6.28) by

$$e^{-\frac{\mathcal{I}}{2}m}(C\beta)^N \frac{(m+3)^{2N}}{(2N)!} \le e^{-\frac{\mathcal{I}}{2}m}(C\beta)^{1/2}(C\beta)^{N/2}\frac{(m+3)^{2N}}{(2N)!}. \tag{6.30}$$

(b) The bound for N even and $\vec{j} \in B_1 \setminus B_0$. When N is even we bound the contribution to (6.21) from $\vec{j} \in B_1 \setminus B_0$ by using the following three facts, the first of which is trivial, while the second and third follow immediately from Lemma 6.2.

The first fact is that uniformly in $\vec{\omega}^{(N-1)}_{j_{N-1}+1}$,

$$\mathbb{E}^{\vec{\omega}^{(N-1)}_{j_{N-1}+1}}\left[I_{\{\omega^{(N)}_{j_N}=\omega^{(N-1)}_{l_{N-1}}\}}\right] \le 1. \tag{6.31}$$

The second fact is that for each odd $i \in [3, N-1]$, uniformly in $\vec{\omega}^{(i-2)}_{j_{i-2}+1}$, (6.24) holds. The third fact is that

$$\sum_{\vec{\omega}^{(0)}_1} D(\omega^{(0)}_1)\mathbb{E}^{\vec{\omega}^{(0)}_1}\left[I_{\{\omega^{(1)}_{j_1}=\omega^{(0)}_{l_0}\}}\right] \le e^{-\mathcal{I}j_1} \tag{6.32}$$

By first (6.31), followed by successive applications of (6.24) and lastly (6.32), when N is even we obtain a bound on the contribution to (6.21) from $\vec{j} \in B_1$, of

$$\begin{aligned}(C\beta)^N \prod_{1\le i\le N-1,\ odd}\sum_{\vec{l}} e^{-\mathcal{I}(j_i+1+(j_{i-1}+1-l_{i-1}))} &\le (C\beta)^N e^{-\frac{\mathcal{I}}{2}m}\sum_{\vec{j}\in B_1}\sum_{\vec{l}}\prod_{1\le i\le N-1,\ odd} e^{-\mathcal{I}(j_{i-1}+1-l_{i-1})}\\ &\le (C\beta)^N e^{-\frac{\mathcal{I}}{2}m}\sum_{\vec{j}}\prod_{i=1}^{N}(j_i+1),\end{aligned} \tag{6.33}$$

which is bounded by (6.30) just as in the previous case.

(c),(d) The bounds for N odd. The bounds for N odd are similar to the bounds described above, and we will omit the details. When N is odd, we bound the the contribution from $\vec{j} \in B_0$ by using the bound (6.24) successively for each $i \in [3, N]$, and finally (6.32). For $\vec{j} \in B_1 \setminus B_0$, we use (6.31), (6.24) and finally (6.25). In both cases we obtain the same bound (6.30).

To complete the proof of Proposition 6.3, we sum (6.30) over N, giving at most

$$(C\beta)^{1/2}e^{-(\mathcal{I}/2-(C\beta)^{1/4})m}. \tag{6.34}$$

Choosing β_0 sufficiently small so that $(C\beta)^{1/4} \leq \mathcal{I}/4$ for all $|\beta| \leq \beta_0$, we have that (6.34) is bounded by $(C\beta)^{1/2}e^{-\mathcal{J}m}$, where $\mathcal{J} = \mathcal{I}/4$ is independent of β. □

Proof of Theorem 2.1. We use Propositions 6.3 and 6.1 as well as Theorem 5.1 to complete the proof of Theorem 2.1. As noted at the beginning of this section, when $b_m = e^{-\mathcal{J}m}$, Assumption (CLT) is satisfied. Also, (2.13) is directly implied by (5.5). Furthermore, the error terms in (5.6) can all be seen to be $O(1)$, which proves (2.14). Finally, all the sums in (5.7) are $O(1)$, so we obtain that for $|k|$ bounded,

$$\mathbb{E}_\beta[e^{ik\cdot(\omega_n-\theta n)/\sqrt{n}}] = e^{-\frac{1}{2}k\Sigma k^t}\Big[1 + O(|k|n^{-1/2})\Big]. \tag{6.35}$$

Clearly, this implies (2.15). □

6.3 Bounds for excited random walk

In this section we show that for ERW, Assumptions (LLN) and (CLT) hold (in dimensions $d > 5$ and $d > 8$ respectively) with

$$b_m = (m+1)^{-(d-3)/2}, \qquad C_\beta = C\beta, \tag{6.36}$$

Note that in order for (4.3) to hold and for the $O(\cdot)$ terms in (4.4) and (4.5) to be error terms for $b_m = (m+1)^{-(d-3)/2}$ we require that $d > 5$. Similarly in order for $\sum_{m=1}^n m^2 b_m = O(\sqrt{n})$ in (5.4) to hold for $b_m = (m+1)^{-(d-3)/2}$, we need that $d \geq 8$. Moreover, for $n^{-3/2}\sum_{j=1}^n a_j$ to be an error term in (5.7) we require that $d > 8$.

In bounding the diagrams arising from the expansion applied to excited random walk, we will make use of the following lemma, in which $\mathbb{Q}^{\vec{\eta}}$ denotes the law of an excited random walk with history $\vec{\eta}$, where $\vec{\eta}$ is a path ending at u:

Lemma 6.4. *For excited random walk in $d > 2$ dimensions,*

$$\sup_{u,\vec{\eta}} \mathbb{Q}^{\vec{\eta}}(\omega_m = 0) \leq \frac{C}{(m+1)^{\frac{d-1}{2}}}. \tag{6.37}$$

Proof. Let $Y_m = \#\{k \leq m : \omega_k \notin \{\omega_{k-1} \pm e_1\}\}$ denote the number of steps taken in the dimensions $2, \ldots, d$ by the excited random walk up to time m. Note that for excited random walk and simple random walk, Y_n has the same distribution. Then $Y_m \sim Bin(m, q)$ where $q = (d-1)/d > \frac{1}{2}$ for $d > 2$, and standard large deviations estimates give $\mathbb{P}(Y_m < m/2) \leq e^{-mI}$ for some $I > 0$.

Now for each u, $\vec{\eta}$,

$$\mathbb{Q}^{\vec{\eta}}(\omega_m = 0) \leq \mathbb{Q}^{\vec{\eta}}(\omega_m^{[2,\ldots,d]} = 0^{[2,\ldots,d]}) = \mathbb{P}_u(\omega_m^{[2,\ldots,d]} = 0^{[2,\ldots,d]}), \tag{6.38}$$

where $\mathbb{P}_u$ denotes the law of a simple random walk starting at u. For m even, this is bounded by

$$\begin{aligned}
\mathbb{P}(\omega_m^{[2,\ldots,d]} = 0^{[2,\ldots,d]}) &\leq \sum_{k=m/2}^{m} \mathbb{P}_0(\omega_m^{[2,\ldots,d]} = 0^{[2,\ldots,d]} | Y_m = k)\mathbb{P}(Y_m = k) + \mathbb{P}(Y_m < m/2) \\
&\leq \sum_{k=m/2}^{m} \frac{C}{(k+1)^{\frac{d-1}{2}}}\mathbb{P}(Y_m = k) + e^{-\frac{1}{2}Im} \\
&\leq \frac{C}{(m+1)^{\frac{d-1}{2}}} \sum_{k=m/2}^{m} \mathbb{P}(Y_m = k) + e^{-\frac{1}{2}Im} \leq \frac{C}{(m+1)^{\frac{d-1}{2}}}.
\end{aligned} \tag{6.39}$$

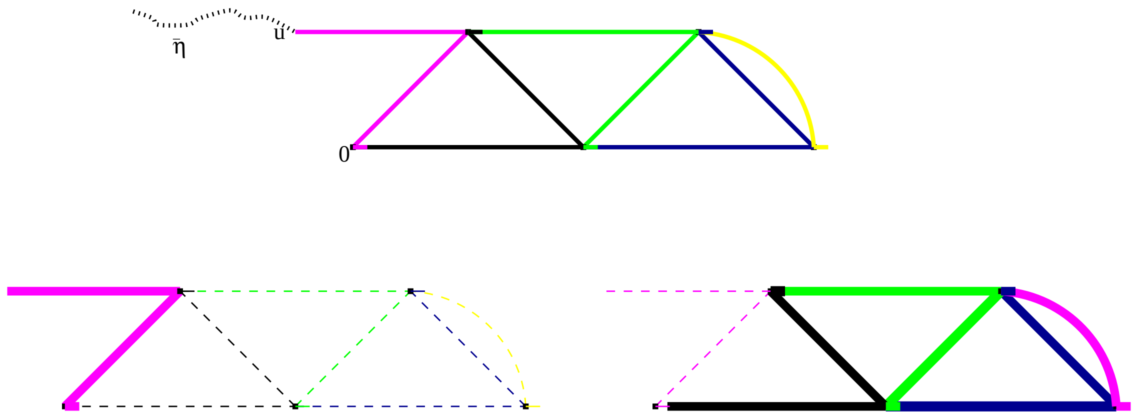

Figure 3: A diagrammatic representation of $\Pi_m^{(N),\vec{\eta}}(u)$ for $N=5$, followed by the decomposition of the diagram when $j_1 > m/2$ and when $j_1 \le m/2$ respectively. In each case, the induction hypothesis is applied to the subdiagram of length $m-(j_1+1)$ that excludes the first walk, and the required decay comes from the part of the diagram with thick lines.

For m odd, (6.38) is bounded by $2d\mathbb{P}_0(\omega_{m+1}^{[2,\dots,d]} = 0^{[2,\dots,d]})$ and we proceed as in (6.39). □

Recall that $\mathcal{A}_{m,N} \equiv \{\vec{j} \in \mathbb{Z}_+^N : \sum j_i = m-N-1\}$. For $N \ge 1$,

$$\begin{aligned}\sum_{x,y} |\pi_m^{(N)}(x,y)| &= \sum_{\vec{j}\in\mathcal{A}_{m,N}} \sum_{\vec{\omega}_1^{(0)}} D(\omega_1^{(0)}) \mathbb{E}^{\vec{\omega}_1^{(0)}}\Big[|\Delta_{j_1+1}^{(1)}|\ \mathbb{E}^{\vec{\omega}_{j_1+1}^{(1)}}\Big[|\Delta_{j_2+1}^{(2)}|\ \cdots \mathbb{E}^{\vec{\omega}_{j_{N-1}+1}^{(N-1)}}\Big[|\Delta_{j_N+1}^{(N)}|\Big]\cdots\Big] \qquad (6.40)\\ &\le (C\beta)^N \sup_{u,\vec{\eta}} \sum_{\vec{j}\in\mathcal{A}_{m,N}} \mathbb{E}^{\vec{\eta}}\Big[I_{\{\omega_{j_1}^{(1)}=0\}} \mathbb{E}^{\vec{\omega}_{j_1+1}^{(1)}}\Big[I_{\{\omega_{j_2}^{(2)}\in\vec{\omega}_{j_1}^{(1)}\}} \cdots \mathbb{E}^{\vec{\omega}_{j_{N-1}+1}^{(N-1)}}\Big[I_{\{\omega_{j_N}^{(N)}\in\vec{\omega}_{j_{N-1}}^{(N-1)}\}}\Big]\cdots\Big],\end{aligned}$$

where $\vec{\eta}$ is any random walk path of any length ending at u.

Proposition 6.5 (Bounds on the expansion coefficients for ERW)**.** *For excited random walk with $d>5$, the following bound holds:*

$$\sum_{x,y} |\pi_m^{(N)}(x,y)| \le \frac{(C\beta)^N}{(m+1)^{\frac{d-3}{2}}}. \qquad (6.41)$$

Define

$$\Pi_m^{(N),\vec{\eta}}(u) = \sum_{\vec{j}\in\mathcal{A}_{m,N}} \mathbb{E}^{\vec{\eta}}\Big[I_{\{\omega_{j_1}^{(1)}=0\}} \mathbb{E}^{\vec{\omega}_{j_1+1}^{(1)}}\Big[I_{\{\omega_{j_2}^{(2)}\in\vec{\omega}_{j_1}^{(1)}\}} \cdots \mathbb{E}^{\vec{\omega}_{j_{N-1}+1}^{(N-1)}}\Big[I_{\{\omega_{j_N}^{(N)}\in\vec{\omega}_{j_{N-1}}^{(N-1)}\}}\Big]\cdots\Big]. \qquad (6.42)$$

See the top diagram in Figure 3. Since (6.40) and (6.42) imply that

$$\sum_{x,y} |\pi_m^{(N)}(x,y)| \le (C\beta)^N \sup_{u,\vec{\eta}} \Pi_m^{(N),\vec{\eta}}(u), \qquad (6.43)$$

the conclusion of Proposition 6.5 follows immediately from the following lemma:

Lemma 6.6. *For $d > 5$,*

$$\sup_{u,\vec{\eta}} \Pi_m^{(N),\vec{\eta}}(u) \le \frac{C^N}{(m+1)^{\frac{d-3}{2}}}. \tag{6.44}$$

Proof. By induction on N.

For $N = 1$, we have

$$\Pi_m^{(1),\vec{\eta}}(u) = \mathbb{E}^{\vec{\eta}}\Big[I_{\{\omega_{j_1}^{(1)}=0\}}\Big] \le \frac{C}{(m+1)^{\frac{d-1}{2}}}, \tag{6.45}$$

uniformly in $u, \vec{\eta}$.

For $N \ge 2$, and for any u, $\vec{\eta}$,

$$\begin{aligned}\Pi_m^{(N),\vec{\eta}}(u) \le & \sum_{j_1+1\le m/2}\sum_{l_1\le j_1} \mathbb{E}_u^{\vec{\eta}}\Big[I_{\{\omega_{j_1}^{(1)}=0\}} \sum_{\vec{j}'\in\mathcal{A}_{m-(j_1+1),N-1}} \mathbb{E}^{\vec{\omega}_{j_1+1}^{(1)}}\Big[I_{\{\omega_{j_2}^{(2)}=\vec{\omega}_{l_1}^{(1)}\}} \cdots \mathbb{E}^{\vec{\omega}_{j_{N-1}+1}^{(N-1)}}\Big[I_{\{\omega_{j_N}^{(N)}\in\vec{\omega}_{j_{N-1}}^{(N-1)}\}}\Big]\cdots\Big] \\ & + \sum_{l\le m/2}\sum_{l_1\le m-l-1} \mathbb{E}_u^{\vec{\eta}}\Big[I_{\{\omega_{m-l-1}^{(1)}=0\}} \sum_{\vec{j}'\in\mathcal{A}_{l,N-1}} \mathbb{E}^{\vec{\omega}_{m-l}^{(1)}}\Big[I_{\{\omega_{j_2}^{(2)}\in\vec{\omega}_{j_1}^{(1)}\}} \cdots \mathbb{E}^{\vec{\omega}_{j_{N-1}+1}^{(N-1)}}\Big[I_{\{\omega_{j_N}^{(N)}\in\vec{\omega}_{j_{N-1}}^{(N-1)}\}}\Big]\cdots\Big]\end{aligned} \tag{6.46}$$

The first term on the right of (6.46) corresponds to the case where the first walk is short (compared to m) and is bounded by

$$\begin{aligned} & \sum_{j_1\le m/2}\sum_{l_1\le j_1} \mathbb{E}^{\vec{\eta}}\Big[I_{\{\omega_{j_1}^{(1)}=0\}} \sum_y I_{\{\vec{\omega}_{l_1}^{(1)}=y\}} \sum_{\vec{j}'\in\mathcal{A}_{m-j_1-1,N-1}} \mathbb{E}^{\vec{\omega}_{j_1+1}^{(1)}}\Big[I_{\{\omega_{j_2}^{(2)}=y\}} \cdots \mathbb{E}^{\vec{\omega}_{j_{N-1}+1}^{(N-1)}}\Big[I_{\{\omega_{j_N}^{(N)}\in\vec{\omega}_{j_{N-1}}^{(N-1)}\}}\Big]\cdots\Big] \\ \le & \sum_{j_1\le m/2}\sum_{l_1\le j_1} \mathbb{E}^{\vec{\eta}}\Big[I_{\{\omega_{j_1}^{(1)}=0\}} \sum_y I_{\{\vec{\omega}_{l_1}^{(1)}=y\}} \sup_{y'} \sum_{\vec{j}'\in\mathcal{A}_{m-j_1-1,N-1}} \mathbb{E}^{\vec{\omega}_{j_1+1}^{(1)}-y'}\Big[I_{\{\omega_{j_2}^{(2)}=0\}} \cdots \mathbb{E}^{\vec{\omega}_{j_{N-1}+1}^{(N-1)}}\Big[I_{\{\omega_{j_N}^{(N)}\in\vec{\omega}_{j_{N-1}}^{(N-1)}\}}\Big]\cdots\Big] \\ \le & \sum_{j_1\le m/2}\sum_{l_1\le j_1} \mathbb{E}^{\vec{\eta}}\Big[I_{\{\omega_{j_1}^{(1)}=0\}} \sum_y I_{\{\vec{\omega}_{l_1}^{(1)}=y\}} \sup_{u',\vec{\eta}'} \sum_{\vec{j}'\in\mathcal{A}_{m-j_1-1,N-1}} \mathbb{E}^{\vec{\eta}'}\Big[I_{\{\omega_{j_2}^{(2)}=0\}} \cdots \mathbb{E}^{\vec{\omega}_{j_{N-1}+1}^{(N-1)}}\Big[I_{\{\omega_{j_N}^{(N)}\in\vec{\omega}_{j_{N-1}}^{(N-1)}\}}\Big]\cdots\Big] \\ = & \sum_{j_1\le m/2}\sum_{l_1\le j_1} \mathbb{E}^{\vec{\eta}}\Big[I_{\{\omega_{j_1}^{(1)}=0\}} \sup_{u',\vec{\eta}'} \Pi_{m-j_1-1}^{(N-1),\vec{\eta}'}(u')\Big],\end{aligned} \tag{6.47}$$

where we have first taken the supremum over y' and then used translation invariance to obtain the second line. The internal expectations are now not dependent on $\vec{\omega}^{(1)}$. We use the induction hypothesis and the fact that $m - j_1 \ge m/2$ to bound (6.47) by

$$\frac{C^{N-1}}{(m+1)^{\frac{d-3}{2}}} \sum_{j_1\le m/2}\sum_{l_1\le j_1} \mathbb{E}_u^{\vec{\eta}}\Big[I_{\{\omega_{j_1}^{(1)}=0\}}\Big] \le \frac{C^N}{(m+1)^{\frac{d-3}{2}}} \sum_{j_1\le m/2}\sum_{l_1\le j_1} \frac{1}{(j_1+1)^{\frac{d-1}{2}}} \le \frac{C^N}{(m+1)^{\frac{d-3}{2}}}, \tag{6.48}$$

where the final inequality holds since $d > 5$.

The second term on the right of (6.46) corresponds to the case where the first walk is long (compared to m). Applying the same steps as in (6.47) and then the induction hypothesis and the fact that $m-l\geq m/2$, we bound the second term on the right of (6.46) by

$$\begin{aligned}\sum_{l\leq m/2}\sum_{l_1\leq m-l-1}\mathbb{E}^{\vec{\eta}}\Big[I_{\{\omega^{(1)}_{m-l-1}=0\}}\sup_{u',\vec{\eta}'}\Pi^{(N-1),\vec{\eta}'}_{m-l}(u')\Big]\leq&\frac{C^N}{(m+1)^{\frac{d-1}{2}}}\sum_{l\leq m/2}\sum_{l_1\leq m-l-1}\frac{1}{(m-l+1)^{\frac{d-3}{2}}}\\ \leq&\frac{C^N}{(m+1)^{\frac{d-1}{2}}}\sum_{l\leq m/2}\frac{1}{(m-l+1)^{\frac{d-5}{2}}}\\ \leq&\frac{C^N}{(m+1)^{\frac{d-3}{2}}},\quad\text{when } d>5.\end{aligned}\tag{6.49}$$

This completes the proof of Lemma 6.6, and hence also Proposition 6.5. □

Proof of Theorems 2.2 and 2.3. By Propositions 6.5 and 6.1, Assumption (LLN) holds with $C_\beta=C\beta$ and $b_m=(m+1)^{-(d-3)/2}$ when $d>5$. Thus, Theorem 4.1 applies, and it is an easy exercise to see that when $b_m=(m+1)^{-(d-3)/2}$ and $d>5$, the error terms given in (4.4) and (4.5) are sufficient to prove Theorem 2.3. Similarly, Propositions 6.5 and 6.1 show that Assumption (CLT) holds with $C_\beta=C\beta$ and $b_m=(m+1)^{-(d-3)/2}$ when $d\geq 8$. Thus, Theorem 5.1 applies and it is an easy exercise to see that when $b_m=(m+1)^{-(d-3)/2}$ and $d>8$, the error terms given in (4.4) and (4.5) are sufficient to prove Theorem 2.2. □

6.4 Discussion of the bounds

In both examples given in this paper, an estimate of the form

$$\sup_{\vec{\eta},x}\mathbb{Q}^{\vec{\eta}}(\omega_m=x)\leq A(m)\tag{6.50}$$

is crucially used in bounding the diagrams, where $A(m)$ is decreasing sufficiently rapidly in m. In the case of the reinforced random walk with drift, Cramér's Theorem enabled such a result with $A(m)$ exponentially small in m. For excited random walk the simple random walk behaviour in all but the first dimension gave such a result with $A(m)=(m+1)^{-(d-1)/2}$. In both of our examples, we ignore considerable information contained in the expansion in order to bound certain quantities arising from the expansion in terms of diagrams. In the case of excited random walk, we bounded these diagrams using very simple, but non-optimal estimates. The diagrammatic estimates are used to verify a set of non-optimal assumptions under which the central limit theorem holds. Improvements in any of these areas could lead to a reduction in the dimension above which our methods imply a central limit theorem for excited random walk.

Initially it was hoped that this expansion would apply to once (or boundedly) edge-reinforced random walk with sufficiently small reinforcement parameter and in high dimensions with no underlying drift. Although we believe that for this model (and in fact also for excited random walk) (6.50) holds with $A(m)=C(m+1)^{-d/2}$, we are unable to prove any bound which enables us to deal with the diagrams arising from the expansion. In order to prove a central limit theorem for this model using our methods, it would be sufficient to first prove a bound of the form (6.50) with $A(m)\leq C(m+1)^{-a}$, for $a>7/2$, which would be used to bound the diagrams inductively as we did for excited random walk. Unfortunately we don't see how to prove this sufficient bound (which would also imply transience of the walk) in any dimension at present. Perhaps with an improved analysis of the recursion equation (2.24), we may be able to prove the necessary bounds inductively.

Our expansion certainly applies to the partial reinforcement model in $d_R + d$ dimensions, where edges traversed in the first d_R dimensions are reinforced, while edges traversed in the other d dimensions are not reinforced. In much the same way as for excited random walk, we can bound the diagrams for this model for every d_R when $d > 7$, provided the reinforcement is bounded by some sufficiently small β.

Finally, we wish to reiterate that the expansion itself is exact, and is valid in a very general setting. We are aware that our results could also be proved using more probabilistic techniques. Using renewal-type arguments with the fact that there exist cut-times (separating the dependence of the walk before and after these times), the LLN and CLT would follow from the existence of sufficiently many moments of these cut-times. One advantage of our method is that the formulae for θ and Σ are explicit, and this should enable us to investigate monotonicity issues of $\beta \mapsto \theta(\beta)$ in a future publication. It would be of great interest to see whether a combination of the expansion with the general methodology of regeneration times could improve the results in this paper.

Acknowledgements

The work of RvdH and MH was supported in part by Netherlands Organisation for Scientific Research (NWO). RvdH thanks Vlada Limic for various discussions and encouragements at the start of this project.

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