Asymptotic entropy and Green speed for random walks on countable groups

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Abstract

We study asymptotic properties of the Green metric associated to transient random walks on countable groups. We prove that the rate of escape of the random walk computed in the Green metric equals its asymptotic entropy. The proof relies on integral representations of both quantities with the extended Martin kernel. In the case of finitely generated groups, where this result is known (Benjamini & Peres [3]), we give an alternative proof relying on a version of the so-called fundamental inequality (relating the rate of escape, the entropy and the logarithmic volume growth) extended to random walks with unbounded support.

1 Introduction

Let Γ be an infinite countable group and let (Z_n) be a transient random walk on Γ.

In order to study asymptotic properties of the random walk, we define the Green (or hitting) metric:

$$d_G(x,y) = -\ln \mathbb{P}^x[\tau_y < \infty]\,,$$

where τ_y is the hitting time of the element y by the random walk started at x.

Looking at the random walk through the Green metric leads to nice geometrical interpretation of probabilistic quantities describing the long time behaviour of the walk. We illustrate this claim by showing that the rate of escape computed in d_G coincides with the asymptotic entropy of the random walk, see Theorem 1.1. As another example of interest in the Green metric, we also explain how the Martin compactification of Γ can be interpreted as the Busemann compactification of Γ equiped with d_G. In a forthcoming paper [5] we use the Green metric to study fine geometric properties of the harmonic measure on boundaries of hyperbolic groups.

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Before stating our theorem, let us first recall some definitions. The rate of escape of the random walk computed in the Green metric (in short the Green speed) is defined by the almost sure limit

$$\ell_G \stackrel{\text{def.}}{=} \lim_{n\to\infty} \frac{d_G(e, Z_n)}{n}\,.$$

The asymptotic entropy of the random walk is defined by

$$h \stackrel{\text{def.}}{=} \lim_{n\to\infty} \frac{-\ln \mu^n(Z_n)}{n}$$

where μ is the law of the increment of the random walk (i.e. the law of Z_1) and μ^n is the n-th convolution power of μ (i.e. the law of Z_n). This limit almost surely exists and is finite if the entropy of μ

$$H(\mu) \stackrel{\text{def.}}{=} -\sum_{x\in\Gamma} \mu(x)\ln \mu(x)$$

is finite. The asymptotic entropy h plays a very important role in the description of the large time behaviour of the random walk as illustrated in Derriennic [9, 10], Guivarc'h [15], Kaimanovich [17], Kaimanovich & Vershik [18], or Vershik [24] among others. For instance it is known that $h = 0$ if and only if the Poisson boundary of the random walk is trivial.

Our main result is the following

Theorem 1.1 *For any transient random walk on a countable group such that $H(\mu) < \infty$, the asymptotic entropy h and the Green speed ℓ_G are equal.*

In part 2 we prove this result using an integral representation of h on the Martin boundary of Γ (Lemma 2.6) and interpreting the Green speed of the random walk as a limit of a Martin kernel (Proposition 2.4). This proof does not use any quantitative bound on the transition probabilities of the random walk and therefore applies to transient random walks on any countable groups even non-finitely generated ones.

In part 3 we consider the case of a finitely generated group Γ and we discuss the connection of Theorem 1.1 with the so-called 'fundamental inequality' $h \leq \ell \cdot v$ where ℓ and v denote the rate of escape and the logarithmic volume growth in some left invariant metric on the group with a finite first moment. We first derive a new general version of the fundamental inequality for any random walk (with bounded or unbounded support) and any (geodesic or non-geodesic) left invariant metric on the group with a finite first moment, see Proposition 3.4. Then we use heat kernel estimates to get bounds on the logarithmic volume growth in the Green metric, see Proposition 3.1. Thus we finally obtain another proof of Theorem 1.1, valid for finitely generated groups of superpolynomial volume growth. In the case of groups with polynomial volume growth, h and ℓ_G are both zero.

For finitely generated groups, Benjamini & Peres [3] gave a different proof of the equality $h = \ell$. Even if their proof is written for finitely supported random walks, their method also works for random walks with infinite support (see the proof of Proposition 3.1).

2 Countable groups

2.1 The Green metric

We will give the definition of the Green metric associated to transient random walks and recall some of its properties from Blachère & Brofferio [4].

Let μ be a probability measure on Γ whose support generates the whole group Γ. (We will always make this generating hypothesis). We do not assume that μ is symmetric nor that it is finitely supported. Let (X_k) be a sequence of i.i.d. random variables whose common law is μ. The process

$$Z_k \overset{\text{def.}}{=} xX_1X_2\cdots X_k,$$

with $Z_0 = x \in \Gamma$, is an irreducible random walk on Γ starting at x with law μ. We denote $\mathbb{P}^x$ and $\mathbb{E}^x$, respectively, the probability and expectation related to a random walk starting at x. When $x = e$ (the identity of the group), the exponent will be omitted.

From now on, we will always assume the random walk to be transient i.e., with positive probability, it never returns to its starting point. This assumption is always satisfied if Γ is not a finite extension of $\mathbb{Z}$ or $\mathbb{Z}^2$ (see Woess [25, Sect. I.3.B]). On a finite extension of $\mathbb{Z}$ or $\mathbb{Z}^2$, there exists a canonical projection φ onto an Abelian subgroup $(\{e\}, \mathbb{Z}$ or $\mathbb{Z}^2)$, see Alexopoulos [1]. We define the first moment of the canonical projection of the random walk:

$$M_1(\mu) \overset{\text{def.}}{=} \sum_{x\in\Gamma} \|\varphi(x)\|\,\mu(x)\,,$$

where $\|\varphi(x)\|$ is the norm of $\varphi(x)$. When $M_1(\mu) < \infty$, the random walk is transient if and only if it has a non-zero drift ($\sum_{x\in\Gamma}\varphi(x)\mu(x) \neq 0$). But there are examples of recurrent and transient random walks with $M_1(\mu) = \infty$. There are even examples of transient symmetric random walks on $\mathbb{Z}$. For these results and examples, see Spitzer [23].

The **Green function** $G(x, y)$ is defined as the expected number of visits at y

for a random walk starting at x:

$$G(x,y) \stackrel{\text{def.}}{=} \mathbb{E}^x\left[\sum_{n=0}^{\infty} \mathbb{1}_{\{Z_n=y\}}\right] = \sum_{n=0}^{\infty} \mathbb{P}^x[Z_n = y]\,.$$

Since the random walk is chosen to be transient, the Green function is finite for every x and y.

Let τ_y be the first hitting time of y by the random walk:

$$\tau_y \stackrel{\text{def.}}{=} \inf\{k \geq 0 \;:\; Z_k = y\}\,.$$

When y is never attained, let $\tau_y = \infty$. The **hitting probability** of y starting at x is

$$F(x,y) \stackrel{\text{def.}}{=} \mathbb{P}^x[\tau_y < \infty]\,.$$

Note that $F(x,y)$ is positive since the support of μ generates Γ, and that F and G are invariant by left diagonal multiplication. In particular, $G(y,y) = G(e,e)$. A straightforward computation (using the strong Markov property) shows that the functions F and G are proportional:

$$G(x,y) = G(y,y)F(x,y) = G(e,e)F(x,y)\,. \tag{1}$$

The metric we will use is the **Green metric** (or Hitting metric, defined in [4]):

$$d_G(x,y) \stackrel{\text{def.}}{=} -\ln F(x,y) = \ln G(e,e) - \ln G(x,y)\,.$$

Throughout the article, we will call (with some abuse of notation) **metric** *any non-negative real function* $d(\cdot,\cdot)$ *on* $\Gamma \times \Gamma$ *which satisfies the triangle inequality, vanishes on the diagonal and satisfies*

$$d(x,y) = 0 = d(y,x) \Longrightarrow x = y\,. \tag{2}$$

Lemma 2.1 ([4] Lemma 2.1) *The function* $d_G(\cdot,\cdot)$ *is a left invariant metric on* Γ.

Proof

As $F(x,y)$ is bounded by 1, then $d_G(\cdot,\cdot)$ is non-negative. It is also clear that $F(x,x) = 1$ and therefore $d_G(x,x) = 0$ for any $x \in \Gamma$.

The invariance of $F(\cdot,\cdot)$ by left diagonal multiplication implies the same property for $d_G(\cdot,\cdot)$. Also note that since the random walk is transient we have:

$$\forall x \neq y,\; 1 > \mathbb{P}^x[\tau_x' < \infty] \geq \mathbb{P}^x[\tau_y < \infty]\mathbb{P}^y[\tau_x < \infty] = F(x,y)F(y,x)\,,$$

where $\tau'_x \stackrel{\text{def.}}{=} \inf\{k \geq 1 \ : \ Z_k = x\}$. Thus

$$d_G(x,y) = d_G(y,x) = 0 \Longleftrightarrow F(x,y) = F(y,x) = 1 \Longleftrightarrow x = y\,.$$

Finally,

$$\mathbb{P}^x[\tau_z < \infty] \geq \mathbb{P}^x[\tau_y < \infty]\mathbb{P}^y[\tau_z < \infty]$$

leads to the triangular inequality: $d_G(x,z) \leq d_G(x,y) + d_G(y,z)$. □

Remark 2.2 *The converse implication in (2) is true for the Green metric as soon as Γ is not isomorphic to $\mathbb{Z}$. Indeed, we can show that if there exist $x \neq y \in \Gamma$ such that $d_G(x,y) = 0$ then Γ is isomorphic to $\mathbb{Z}$. (one first proves that $\#\{z \in \mathrm{Supp}(\mu)\backslash\{e\} \text{ s.t. } d_G(e,z) = 0\} \leq 1$).*

Observe that, if μ is symmetric ($\mu(x) = \mu(x^{-1})$ for all $x \in \Gamma$), then the Green function $G(\cdot,\cdot)$ and the Green function d_G are also symmetric and therefore d_G becomes a genuine distance on Γ.

2.2 Entropy and Green speed

The measure μ is now supposed to have finite **entropy**:

$$H(\mu) \stackrel{\text{def.}}{=} -\sum_{x\in\Gamma} \mu(x) \ln \mu(x) < \infty\,.$$

The first moment of μ in the Green metric is, by definition, the expected Green distance between e and Z_1, which is also the expected Green distance between Z_n and Z_{n+1} for any n and has the following analytic expression:

$$\mathbb{E}[d_G(e,Z_1)] = \sum_{x\in\Gamma} \mu(x) \cdot d_G(e,x)\,.$$

Lemma 2.3 *The finiteness of the entropy $H(\mu)$ implies the finiteness of the first moment of μ with respect to the Green metric.*

Proof
By construction, the law of $Z_1 = X_1$ under $\mathbb{P}$ is μ. Since $\mathbb{P}[\tau_x < \infty] \geq \mathbb{P}[Z_1 = x] = \mu(x)$ holds, we have

$$\sum_{x\in\Gamma} \mu(x) \cdot d_G(e,x) = -\sum_{x\in\Gamma} \mu(x) \cdot \ln(\mathbb{P}[\tau_x < \infty]) \leq -\sum_{x\in\Gamma} \mu(x) \cdot \ln(\mu(x)) = H(\mu)\,.$$

So that: $H(\mu) < \infty \Longrightarrow \mathbb{E}[d_G(e,X_1)] < \infty$.

□

Let ℓ_G be the rate of escape of the random walk Z_n in the Green metric $d_G(e,.)$:

$$\ell_G = \ell_G(\mu) \stackrel{\text{def.}}{=} \lim_{n\to\infty} \frac{d_G(e,Z_n)}{n} = \lim_{n\to\infty} \frac{-\ln F(e,Z_n)}{n} = \lim_{n\to\infty} \frac{-\ln G(e,Z_n)}{n}\,,$$

since the functions $F(\cdot,\cdot)$ and $G(\cdot,\cdot)$ differ only by a multiplicative constant. We call ℓ_G the **Green speed**. Under the hypothesis that μ has finite entropy, by the sub-additive ergodic Theorem (Kingman [22], Derriennic [9]), this limit exists almost surely and in L^1.

The sub-additive ergodic Theorem of Kingman also allows one to define the asymptotic entropy as the almost sure and L^1 limit:

$$h \stackrel{\text{def.}}{=} \lim_{n\to\infty} \frac{-\ln \mu^n(Z_n)}{n}\,,$$

where μ^n is the n^{th} convolution power of the measure μ.

Taking expectations, we deduce that h also satisfies

$$h = \lim_n \frac{H(\mu^n)}{n}\,.$$

The properties of the asymptotic entropy are studied in great generality in the articles mentioned in the introduction. In particular, it turns out that h can also be interpreted as a Fisher information. We shall use this fact to conclude the proof of our Theorem, see Lemma 2.6.

2.3 Martin boundary and proof of Theorem 1.1

The **Martin kernel** is defined (using (1)) for all $(x,y)\in\Gamma\times\Gamma$ by

$$K(x,y) \stackrel{\text{def.}}{=} \frac{G(x,y)}{G(e,y)} = \frac{F(x,y)}{F(e,y)}\,.$$

The Martin kernel continuously extends in a compactification of Γ called the Martin compactification $\Gamma\cup\partial_M\Gamma$ where $\partial_M\Gamma$ is the **Martin boundary**. Let us briefly recall the construction of $\partial_M\Gamma$: let $\Psi:\Gamma\to C(\Gamma)$ be defined by $y\longmapsto K(\cdot,y)$. Here $C(\Gamma)$ is the space of real valued functions defined on Γ endowed with the topology of pointwise convergence. It turns out that Ψ is injective and thus we may identify Γ with its image. The closure of $\Psi(\Gamma)$ is compact in $C(\Gamma)$ and, by definition, $\partial_M\Gamma = \overline{\Psi(\Gamma)}\setminus\Psi(\Gamma)$ is the Martin boundary. In the compact space $\Gamma\cup\partial_M\Gamma$, for any initial point x, the random walk Z_n almost surely converges to some random variable $Z_\infty\in\partial_M\Gamma$ (see for instance Dynkin [12] or Woess [25]).

We note that, by means of the Green metric, one can also consider the Martin compactification as a special example of a **Busemann compactification**. We recall that the Busemann compactification of a proper metric space (X,d) is obtained through the embedding $\Phi : X \to C(X)$ defined by $y \longmapsto d(\cdot,y)-d(e,y)$. (Here e denotes an arbitrary base point.) In general, $C(X)$ should be endowed with the topology of uniform convergence on compact sets. The Busemann compactification of X is the closure of the image $\Phi(X)$ in $C(X)$. We refer to Ballmann, Gromov & Schroeder [2] and to Karlsson & Ladrappier [20] and the references therein for further details.

If one now chooses as X the group Γ itself and for the distance d the Green metric, both constructions of the Martin and Busemann compactifications coincide as it is straightforward from the relation:

$$d_G(\cdot,y)-d_G(e,y) = -\ln K(\cdot,y)\,.$$

We first prove that the Green speed can be expressed in terms of the extended Martin kernel. Theorem 1.1 will then be a direct consequence of the formulas in Proposition 2.4 and Lemma 2.6. For that purpose we need to define the reversed law $\tilde{\mu}$:

$$\forall x\in\Gamma\,,\quad \tilde{\mu}(x) \overset{\text{def.}}{=} \mu(x^{-1})\,.$$

Note that $H(\tilde{\mu})=H(\mu)$.

Proposition 2.4 *Let μ be a probability measure on Γ with finite entropy $H(\mu)$ and whose support generates Γ. Let (Z_n) be a random walk on Γ of law μ (starting at e) and let $\tilde{X}_1$ be an independent random variable of law $\tilde{\mu}$. Then*

$$\ell_G = \mathbb{E}\tilde{\mathbb{E}}[-\ln K(\tilde{X}_1,Z_\infty)]\,,$$

where $\tilde{\mathbb{E}}$ refers to the integration with respect to the random variable $\tilde{X}_1$ and $\mathbb{E}$ refers to the integration with respect to the random walk (Z_n).

Proof

As μ is supposed to have finite entropy, ℓ_G is well defined as an almost sure and L^1 limit. We will prove that the sequence

$$\mathbb{E}[d_G(e,Z_{n+1})-d_G(e,Z_n)] = \mathbb{E}[-\ln G(e,Z_{n+1})+\ln G(e,Z_n)]\,,$$

converges to $\mathbb{E}\tilde{\mathbb{E}}[-\ln K(\tilde{X}_1,Z_\infty)]$. Since its limit in the Cesaro sense is ℓ_G, it implies the formula in Proposition 2.4.

By definition of the reversed law $\tilde{\mu}$, $\tilde{X}_1^{-1}$ has the same law as X_1 the first increment of the random walk (Z_n). Note also that $X_2\cdots X_{n+1}$ has the same law as $Z_n = X_1\cdots X_n$. Since we have assumed that $\tilde{X}_1$ is independent of the sequence (Z_n), $Z_{n+1}=X_1\cdot X_2\cdots X_{n+1}$ has the same law as $\tilde{X}_1^{-1}\cdot Z_n$ and therefore, using the translation invariance, $G(e,Z_{n+1})$ has the same law as $G(\tilde{X}_1,Z_n)$. Thus

$$\begin{aligned}\mathbb{E}[-\ln G(e,Z_{n+1})+\ln G(e,Z_n)] &= \mathbb{E}\tilde{\mathbb{E}}[-\ln G(\tilde{X}_1,Z_n)+\ln G(e,Z_n)]\\ &= \mathbb{E}\tilde{\mathbb{E}}[-\ln K(\tilde{X}_1,Z_n)]\,.\end{aligned}$$

By continuity of the Martin kernel up to the Martin boundary, for every $x \in \Gamma$, the sequence $K(x, Z_n)$ almost surely converges to $K(x, Z_\infty)$. We need an integrable bound for $-\ln K(\tilde{X}_1, Z_n)$ (uniformly in n) to justify the convergence of the expectation.

To prove that $-\ln K(\tilde{X}_1, Z_n)$ cannot go too far in the negative direction, we first prove a maximal inequality for the sequence $(K(\tilde{X}_1, Z_n))_n$ following Dynkin [12].

Lemma 2.5 *For any $a > 0$,*

$$\mathbb{P}\tilde{\mathbb{P}}[\sup_n K(\tilde{X}_1, Z_n) \geq a] \leq \frac{1}{a}.$$

where $\tilde{\mathbb{P}}$ refers to the measure associated to the random variable $\tilde{X}_1$ and $\mathbb{P}$ refers to the measure associated to the random walk (Z_n).

Proof
We fix an integer R. Let σ_R be the time of the last visit to the ball $B_G(e, R)$ for the random walk (Z_n). (We will only consider this random time for starting points within $B_G(e, R)$. Since the random walk is transient, σ_R is well defined and almost surely finite.) Let us define the sequence $(Z_{\sigma_R - k})$ $(k \in \mathbb{N})$. As this sequence (in Γ) is only defined for $k \leq \sigma_R$, we take the following convention for negative indices:

$$\{k > \sigma_R\} \Longrightarrow \{Z_{\sigma_R - k} \stackrel{\text{def.}}{=} \star\},$$

so that the sequence $(Z_{\sigma_R - k})_{k \in \mathbb{N}}$ is well defined and takes its values in $\Gamma \cup \{\star\}$. Note that Z_{σ_R} takes its value in $B_G(e, R)$.

Let us call $\mathcal{F}_k$ the σ-algebra generated by $(Z_{\sigma_R}, \ldots, Z_{\sigma_R - k})$ and observe that

$$\mathbb{1}_{\{k \leq \sigma_R\}} \in \mathcal{F}_k,$$

since $\{k \leq \sigma_R\}$ means that none of $Z_{\sigma_R}, \ldots, Z_{\sigma_R - k}$ equals $\star$. With the convention that, for any $x \in \Gamma$, $K(x, \star) = 0$, we can define, for any x in Γ, the non-negative sequence $(K(x, Z_{\sigma_R - k}))$ $(k \in \mathbb{N})$. This sequence is adapted to the filtration $(\mathcal{F}_k)$ and we will prove, following Dynkin [12, §6,7], that it is a supermartingale with respect to $(\mathcal{F}_k)$.

For this purpose, let us check that for any positive integer k and any sequence $z_0, z_1, \ldots, z_{k-1}$ in $\Gamma \cup \{\star\}$ (with $z_0 \in B_G(e, R)$),

$$\mathbb{E}\left[K(x, Z_{\sigma_R - k}) \prod_{j=0}^{k-1} \mathbb{1}_{\{Z_{\sigma_R - j} = z_j\}}\right]$$
$$= \left(K(x, z_{k-1}) - \delta_x(z_{k-1}) G(e, x)^{-1}\right) \cdot \mathbb{E}\left[\prod_{j=0}^{k-1} \mathbb{1}_{\{Z_{\sigma_R - j} = z_j\}}\right]. \quad (3)$$

We first compute the left-hand side of (3) in the case where none of $z_0, z_1, \dots, z_{k-1}$ equals $\star$. Using first that $K(x,\star)=0$,

$$\begin{aligned}
\sum_{z_k\in\Gamma\cup\{\star\}} & \mathbb{P}[Z_{\sigma_R}=z_0,\dots,Z_{\sigma_R-(k-1)}=z_{k-1},Z_{\sigma_R-k}=z_k]\cdot K(x,z_k)\\
&= \sum_{z_k\in\Gamma}\mathbb{P}[Z_{\sigma_R}=z_0,\dots,Z_{\sigma_R-k}=z_k]\cdot K(x,z_k)\\
&= \sum_{z_k\in\Gamma}\mathbb{P}[k\le\sigma_R, Z_{\sigma_R}=z_0,\dots,Z_{\sigma_R-k}=z_k]\cdot K(x,z_k)\,,
\end{aligned}$$

since the fact that none of $z_0,\dots,z_k$ equals $\star$ means in particular

$$\bigcap_{j=0}^{k}\{Z_{\sigma_R-j}=z_j\}\subset\{k\le\sigma_R\}\,.$$

Then

$$\begin{aligned}
\sum_{z_k\in\Gamma\cup\{\star\}} & \mathbb{P}[Z_{\sigma_R}=z_0,\dots,Z_{\sigma_R-(k-1)}=z_{k-1},Z_{\sigma_R-k}=z_k]\cdot K(x,z_k)\\
&= \sum_{z_k\in\Gamma}\sum_{m=k}^{\infty}\mathbb{P}[\sigma_R=m, Z_m=z_0,\dots,Z_{m-k}=z_k]\cdot K(x,z_k)\\
&= \sum_{z_k\in\Gamma}\sum_{m=k}^{\infty}\mathbb{P}[Z_{m-k}=z_k]\mu(z_k^{-1}z_{k-1})\cdots\mu(z_1^{-1}z_0)\mathbb{P}^{z_0}[\sigma_R=0]\cdot K(x,z_k)\\
&= \mu(z_{k-1}^{-1}z_{k-2})\cdots\mu(z_1^{-1}z_0)\mathbb{P}^{z_0}[\sigma_R=0]\sum_{z_k\in\Gamma}G(e,z_k)\mu(z_k^{-1}z_{k-1})\cdot K(x,z_k)\\
&= \mu(z_{k-1}^{-1}z_{k-2})\cdots\mu(z_1^{-1}z_0)\mathbb{P}^{z_0}[\sigma_R=0]\sum_{z_k\in\Gamma}G(x,z_k)\mu(z_k^{-1}z_{k-1})\\
&= \mu(z_{k-1}^{-1}z_{k-2})\cdots\mu(z_1^{-1}z_0)\mathbb{P}^{z_0}[\sigma_R=0]\left(G(x,z_{k-1})-\delta_x(z_{k-1})\right)\,.
\end{aligned}$$

Using the same kind of computation we get that the right-hand side of (3) equals

$$\begin{aligned}
\sum_{m=k-1}^{\infty} & \mathbb{P}[\sigma_R=m, Z_m=z_0,\dots,Z_{m-(k-1)}=z_{k-1}]\left(K(x,z_{k-1})-\delta_x(z_{k-1})G(e,x)^{-1}\right)\\
&= \sum_{m=k-1}^{\infty}\mathbb{P}[Z_{m-(k-1)}=z_{k-1}]\mu(z_{k-1}^{-1}z_{k-2})\cdots\mu(z_1^{-1}z_0)\mathbb{P}^{z_0}[\sigma_R=0]\\
&\qquad\times\left(K(x,z_{k-1})-\delta_x(z_{k-1})G(e,x)^{-1}\right)\\
&= \mu(z_{k-1}^{-1}z_{k-2})\cdots\mu(z_1^{-1}z_0)\mathbb{P}^{z_0}[\sigma_R=0]\left(G(x,z_{k-1})-\delta_x(z_{k-1})\right)\,.
\end{aligned}$$

So (3) is true as soon as $z_0, \dots, z_{k-1}$ take values in Γ. Now suppose that $z_j = \star$ for some $j \leq k-1$, then

$$\{Z_{\sigma_R - j} = z_j\} \Longrightarrow \{Z_{\sigma_R-(k-1)} = \star\} \Longrightarrow \{Z_{\sigma_R-k} = \star\} .$$

Since $K(x,\star) = 0$, the left-hand side of (3) is zero. To check that the right-hand side is also zero, observe that

$$z_{k-1} \neq \star \Longrightarrow \ \mathbb{1}_{\{Z_{\sigma_R-j}=z_j\}} \cdot \ \mathbb{1}_{\{Z_{\sigma_R-(k-1)}=z_{(k-1)}\}} = 0 \Longrightarrow \mathbb{E}\left[\prod_{j=0}^{k-1} \mathbb{1}_{\{Z_{\sigma_R-j}=z_j\}}\right] = 0 ,$$

and, as $x \in \Gamma$,

$$z_{k-1} = \star \Longrightarrow K(x, z_{k-1}) = 0 \quad \text{and} \quad \delta_x(z_{k-1}) = 0 .$$

The proof of (3) is now complete. Since the Green function is positive, we deduce from (3)

$$\mathbb{E}\left[K(x, Z_{\sigma_R-k})\prod_{j=0}^{k-1} \mathbb{1}_{\{Z_{\sigma_R-j}=z_j\}}\right] \leq K(x, z_{k-1}) \cdot \mathbb{E}\left[\prod_{j=0}^{k-1} \mathbb{1}_{\{Z_{\sigma_R-j}=z_j\}}\right] ,$$

thus proving the supermartingale property of the sequence $(K(x, Z_{\sigma_R-k}))$ $(k \in \mathbb{N})$.

We use similar arguments to compute the expectation of the value of the supermartingale at time $k=0$: $\mathbb{E}[K(x, Z_{\sigma_R})]$ which turns out not to depend on R.

$$\begin{aligned}
\mathbb{E}[K(x, Z_{\sigma_R})] &= \sum_{m=0}^{\infty} \sum_{z \in B_G(e,R)} \mathbb{P}[\sigma_R = m, Z_m = z] \cdot K(x,z) \\
&= \sum_{m=0}^{\infty} \sum_{z \in B_G(e,R)} \mathbb{P}^z[\sigma_R = 0] \cdot \mathbb{P}[Z_m = z] \cdot K(x,z) \\
&= \sum_{z \in B_G(e,R)} \mathbb{P}^z[\sigma_R = 0] \cdot G(x,z) \\
&= \sum_{z \in B_G(e,R)} \mathbb{P}^z[\sigma_R = 0] \sum_{m=0}^{\infty} \mathbb{P}^x[Z_m = z] \\
&= \sum_{z \in B_G(e,R)} \sum_{m=0}^{\infty} \mathbb{P}^x[\sigma_R = m, Z_{\sigma_R} = z] \\
&= \mathbb{P}^x[\sigma_R < \infty] = 1 .
\end{aligned}$$

We can now use Doob's maximal inequality for non-negative supermartingales, see for instance Breiman [6, Prop. 5.13], to get that:

$$\forall x \in \Gamma , \quad \mathbb{P}[\sup_k K(x, Z_{\sigma_R-k}) \geq a] \leq \frac{1}{a} .$$

So $\mathbb{P}\tilde{\mathbb{P}}[\sup_k K(\tilde{X}_1, Z_{\sigma_R - k}) \geq a] \leq \frac{1}{a}$, and, letting R tend to infinity,

$$\mathbb{P}\tilde{\mathbb{P}}[\sup_n K(\tilde{X}_1, Z_n) \geq a] \leq \frac{1}{a}\,.$$

$\square$

Let us go back to the proof of Proposition 2.4: Lemma 2.5 implies that, for any $b > 0$,

$$\mathbb{P}\tilde{\mathbb{P}}[\sup_n \ln K(\tilde{X}_1, Z_n) \geq b] \leq \mathrm{e}^{-b}\,,$$

and therefore $\mathbb{E}\tilde{\mathbb{E}}[\sup_n \ln K(\tilde{X}_1, Z_n)\ 1\!\!1_{K(\tilde{X}_1, Z_n)\geq 1}] < \infty$.

On the other hand, we have

$$K(x, Z_n) = \frac{\mathbb{P}^x[\tau_{Z_n} < \infty]}{\mathbb{P}^e[\tau_{Z_n} < \infty]} \geq \frac{\mathbb{P}^x[\tau_e < \infty]\cdot \mathbb{P}^e[\tau_{Z_n} < \infty]}{\mathbb{P}^e[\tau_{Z_n} < \infty]} = \mathbb{P}^e[\tau_{x^{-1}} < \infty] \geq \tilde{\mu}(x)\,,$$

and

$$\tilde{\mathbb{E}}[-\ln \tilde{\mu}(\tilde{X}_1)] = H(\tilde{\mu}) = H(\mu) < \infty\,.$$

Writing that

$$\begin{aligned} |\ln K(\tilde{X}_1, Z_n)| &= \ln K(\tilde{X}_1, Z_n)\ 1\!\!1_{K(\tilde{X}_1, Z_n)\geq 1} - \ln K(\tilde{X}_1, Z_n)\ 1\!\!1_{K(\tilde{X}_1, Z_n)\leq 1} \\ &\leq \ln K(\tilde{X}_1, Z_n)\ 1\!\!1_{K(\tilde{X}_1, Z_n)\geq 1} - \ln \tilde{\mu}(\tilde{X}_1)\,, \end{aligned}$$

we conclude that the random variable $\sup_n |\ln K(\tilde{X}_1, Z_n)|$ is integrable. We can therefore apply the dominated convergence theorem to deduce that the sequence $\mathbb{E}[-\ln G(e, Z_{n+1}) + \ln G(e, Z_n)]$ converges to

$$\mathbb{E}\tilde{\mathbb{E}}[-\ln K(\tilde{X}_1, Z_\infty)]\,.$$

$\square$

Lemma 2.6 *Let Γ be a countable group and μ be a probability measure on Γ whose support generates Γ, with finite entropy $H(\mu)$. Then*

$$h = \mathbb{E}\tilde{\mathbb{E}}[-\ln K(\tilde{X}_1, Z_\infty)]\,.$$

Proof
Recall that $\tilde{\mu}$ is the law of $\tilde{X}_1$. We have

$$\mathbb{E}\tilde{\mathbb{E}}[-\ln K(\tilde{X}_1, Z_\infty)] = \int_\Gamma\int_{\partial_M\Gamma} -\ln(K(x,\xi))\, d\nu(\xi)\, d\tilde{\mu}(x)\,,$$

where $\nu_y(\cdot)$ is the harmonic measure on the Martin boundary $\partial_M\Gamma$ for a random walk (of law μ) starting at y and $\nu(\cdot) = \nu_e(\cdot)$. By the Martin boundary convergence Theorem, see Hunt [16] or Woess [25, Th. 24.10], the Martin kernel $K(x, \xi)$ is the Radon-Nikodym derivative of ν_x by ν at ξ. Therefore

$$\mathbb{E}\tilde{\mathbb{E}}[-\ln K(\tilde{X}_1, Z_\infty)] = \int_\Gamma\int_{\partial_M\Gamma} -\ln\left(\frac{d\nu_x(\xi)}{d\nu(\xi)}\right) d\nu(\xi)\, d\mu(x^{-1})\,.$$

We will make the following changes of variables. As $\partial_M\Gamma$ is stable by left multiplication, the change of variables $\xi \longmapsto x^{-1}\xi$ gives $\nu_x(\xi) \longmapsto \nu(\xi)$ and $\nu(\xi) \longmapsto \nu_{x^{-1}}(\xi)$. Hence, changing also x into x^{-1}, gives

$$\begin{aligned}\mathbb{E}\tilde{\mathbb{E}}[-\ln K(\tilde{X}_1, Z_\infty)] &= \int_\Gamma\int_{\partial_M\Gamma} -\ln\left(\frac{d\nu(\xi)}{d\nu_x(\xi)}\right) d\nu_x(\xi)\, d\mu(x) \\ &= \int_\Gamma\int_{\partial_M\Gamma} \ln\left(\frac{d\nu_x(\xi)}{d\nu(\xi)}\right) d\nu_x(\xi)\, d\mu(x)\,. \qquad (4)\end{aligned}$$

Observe that $d\nu_x(\xi)/d\nu(\xi)$ is the Radon-Nikodym derivative of the joint law of $(\tilde{X}_1^{-1}, Z_\infty)$ with respect to the product measure $\mu(\cdot) \otimes \nu(\cdot)$. Therefore (4) means that $\mathbb{E}\tilde{\mathbb{E}}[-\ln K(\tilde{X}_1, Z_\infty)]$ is the relative entropy of the joint law of $(\tilde{X}_1^{-1}, Z_\infty)$ with respect to $\mu(\cdot) \otimes \nu(\cdot)$, which equals the asymptotic entropy h (see Derriennic [11] who actually takes the latter as the definition of the asymptotic entropy and proves that both definitions coincide.) □

3 Finitely generated groups

We now restrict ourselves to a finitely generated group Γ.

3.1 Volume growth in the Green metric

For a given finite generating set $\mathcal{S}$, we define the associated **word metric**:

$$d_w(x, y) \stackrel{\text{def.}}{=} \min\{n \text{ s.t. } x^{-1}y = g_1 g_2 \cdots g_n \text{ with } g_i \in \mathcal{S}\}\,.$$

This distance is the geodesic graph distance of the Cayley graph of Γ defined by $\mathcal{S}$. Different choices of the generating set lead to different word distances in the same quasi-isometry class. When μ is symmetric and finitely supported, the two metrics $d_G(\cdot,\cdot)$ and $d_w(\cdot,\cdot)$ can be compared (see [4, Lemma 2.2]). These two metrics are equivalent for any non-amenable group and also for some amenable groups, e.g. the Lamplighter group $\mathbb{Z} \wr \mathbb{Z}_2$.

Throughout the article, the notion of growth of the group Γ always refers to the function $V_w(n) \stackrel{\text{def.}}{=} \#\{x \in \Gamma \text{ s.t. } d_w(e, x) \leq n\}$ for some (equivalently any) symmetric finite generating set. The group will be said to have

- polynomial growth when $V_w(n) = O(n^D)$ for some constant D (the largest integer D satisfying this condition is called the **degree** of the group);
- superpolynomial growth when $V_w(n)/n^D$ tends to infinity for every D;
- subexponential growth when $V_w(n) = o(e^{Cn})$ for every constant $C > 0$;
- exponential growth when $V_w(n)/e^{Cn}$ tends to infinity for some $C > 0$.

We are now interested in the asymptotic behaviour of the volume of the balls for the Green metric. Let us define $B_G(e,n) \stackrel{\text{def.}}{=} \{x \in \Gamma \text{ s.t. } d_G(e,x) \leq n\}$, $V_G(n) \stackrel{\text{def.}}{=} \#B_G(e,n)$ and the corresponding logarithmic volume growth:

$$v_G \stackrel{\text{def.}}{=} \limsup_{n\to\infty} \frac{\ln(V_G(n))}{n}\,.$$

Proposition 3.1 *Let us suppose that Γ is not a finite extension of $\mathbb{Z}$ or $\mathbb{Z}^2$. For any random walk on Γ,*

i. If Γ has superpolynomial growth, then $v_G \leq 1$;

ii. If Γ has polynomial growth of degree D, then $v_G \leq \frac{D}{D-2}$.

Proof
Observe that Proposition 2.3 in [4] proves (i) when μ has finite support and is symmetric.

We recall a classical result (e.g. see Woess [25]): let μ be a symmetric measure with finite support and let Γ having at least polynomial growth of degree D ($D \geq 3$), then

$$\exists C_e > 1 \text{ s.t. } \forall x, y \in \Gamma \text{ and } k \in \mathbb{N} \quad \mathbb{P}^x[Z_k = y] \leq C_e k^{-D/2}\,. \tag{5}$$

The above estimate remains valid even without the symmetry and the finite support hypotheses. Indeed Coulhon's result [7, Prop. IV.4] (see also Coulhon & Saloff-Coste [8]) allows one to extend upper bounds of the n^{th} convolution power of a symmetric probability measure μ_1 to the n^{th} convolution power of another probability measure μ_2 under the following condition:

$$\exists c > 0 \quad \text{s.t.} \quad \forall x, \quad \mu_1(x) \leq c\mu_2(x)\,. \tag{6}$$

For a general probability measure μ whose support generates Γ, there exists K such that the support of μ^K contains any finite symmetric generating set $\mathcal{S}$ of Γ.

Hence, choosing $\mu_2 = \mu^K$, $c = (\min_{x\in\mathcal{S}} \mu_2(x))^{-1}$ and $\mu_1 = (1/\#\mathcal{S}) \times \delta_{\mathcal{S}}(x)$, the uniform distribution on $\mathcal{S}$, we see that the measures μ_1 and μ_2 satisfy condition (6). Therefore the estimate (5) remains valid for μ, with a possible different constant C_e.

The same argument as in [4] shows that (5) implies

$$V_G(n) \leq C\exp\left(\frac{D}{D-2} \cdot n\right) ,$$

for some constant C. Thus $v_G \leq \frac{D}{D-2}$. For groups with superpolynomial growth, letting D going to infinity gives $v_G \leq 1$. □

Remark 3.2 *If the measure μ has a finite support, then it is already known that $v_G \geq 1$ [4, Prop. 2.3]. From Lemma 3.3 and Proposition 3.4, we will also get that $v_G \geq 1$ when μ has finite entropy and $h > 0$, but μ may have an infinite support. It implies that $v_G = 1$ for groups with superpolynomial growth and measures of finite entropy such that $h > 0$.*

3.2 The "fundamental" inequality

We now present a different proof of Theorem 1.1 in the case of finitely generated groups. The interest of this proof comes from an extended version of the "fundamental" inequality relating the asymptotic entropy, the logarithmic volume growth and the rate of escape.

There is a general obvious link between the Green speed and the asymptotic entropy:

Lemma 3.3 *For any random walk with finite entropy $H(\mu)$, we have $\ell_G \leq h$.*

Proof
The sequence $\frac{1}{n}d_G(e, Z_n)$ converges to ℓ_G in L^1. Therefore

$$\ell_G = \lim_{n\to\infty} \frac{-\sum_{x\in\Gamma} \mu^n(x) \ln\left(\sum_{k=0}^{\infty} \mu^k(x)\right)}{n} \leq \lim_{n\to\infty} \frac{-\sum_{x\in\Gamma} \mu^n(x) \ln \mu^n(x)}{n} = h\,.$$

□

Our aim is to prove the other inequality and deduce that $h = \ell_G$.

Groups with polynomial volume growth. For groups with polynomial growth, Lemma 3.3 gives the (trivial) equality since any random walk has a zero asymptotic entropy. Indeed, these groups have a trivial Poisson boundary (Dynkin & Malyutov [13]) which is equivalent to $h = 0$ for measures with finite entropy, Derriennic [10] and Kaimanovich & Vershik [18], see also Kaimanovich [17, Th. 1.6.7].

Groups with superpolynomial volume growth. We rely on the so-called fundamental inequality:

$$h \le \ell_G \cdot v_G \,, \tag{7}$$

which holds when μ has finite entropy. For groups with superpolynomial growth, Proposition 3.1 gives $v_G \le 1$ and therefore inequality (7) implies that $h \le \ell_G$ and we conclude that $h = \ell_G$. Thus all that remains to be done in order to complete the proof of Theorem 1.1 in the case of groups with superpolynomial growth is justify (7). This is the content of the next Proposition.

A version of inequality (7), when the speed and volume growth are computed in a word metric, is proved by Guivarc'h [15] and is discussed in great details by Vershik [24]. The same proofs as in [15] or [24] would apply to any invariant metric on Γ, for instance the Green metric, provided μ has finite support. The fundamental inequality is also known to hold for measures with unbounded support and a finite first moment in a word metric. See for instance Erschler [14, Lem. 6] or Karlsson & Ledrappier [21] but note that their argument seems to apply only to word metrics and observe that the Green metric is not a word metric in general: as a matter of fact it need not even be a geodesic metric. We shall derive the fundamental inequality in the Green metric, under the mere assumption that the entropy of μ is finite.

We present our result in a general setting (for any invariant metric and group) since it has its own interest.

Proposition 3.4 *Let μ be the law of the increment of a random walk on a countable group Γ, starting at a point e, and let $d(\cdot,\cdot)$ be a left invariant metric. Under the following hypothesis*

- *The measure μ has finite entropy,*
- *The measure μ has finite first moment with respect to the metric d,*
- *The logarithmic volume growth $v \overset{def.}{=} \limsup_{n\to\infty} \frac{\ln(\#B(e,n))}{n}$ is finite,*

the asymptotic entropy h, the rate of escape $\ell \overset{def.}{=} \lim_n \frac{d(e,Z_n)}{n}$ (limit both in L^1 and almost surely) and the logarithmic volume growth v satisfy the following inequality:

$$h \le \ell \cdot v \,.$$

Proof
The proof relies on the idea of Guivarc'h [15, Prop. C.2]. Fix $\varepsilon > 0$ and, for all integer n, let $B^n_\varepsilon \overset{\text{def.}}{=} B(e, (\ell+\varepsilon)n)$ (here the balls are defined for the metric $d(e,\cdot)$). We split $\Gamma \backslash B^n_\varepsilon$ into a sequence of annuli: choose $K > \ell + \varepsilon$ and define

$$\begin{aligned} \mathcal{C}^{n,K}_\varepsilon &\overset{\text{def.}}{=} B(e,Kn)\backslash B^n_\varepsilon \\ \forall i \ge 1\,,\ \mathcal{C}^{n,K}_i &\overset{\text{def.}}{=} B(e,2^iKn)\backslash B(e,2^{i-1}Kn)\,. \end{aligned}$$

Define the conditional entropy

$$H(\mu \,|\, A) \stackrel{\text{def.}}{=} -\sum_{x \in A} \frac{\mu(x)}{\mu(A)} \ln \frac{\mu(x)}{\mu(A)} .$$

The entropy of μ^n can then be written as

$$\begin{aligned} H(\mu^n) = \mu^n(B^n_\varepsilon) \cdot H(\mu^n \,|\, B^n_\varepsilon) + \mu^n(\mathcal{C}^{n,K}_\varepsilon) \cdot H(\mu^n \,|\, \mathcal{C}^{n,K}_\varepsilon) \\ + \sum_{i=1}^{\infty} \mu^n(\mathcal{C}^{n,K}_i) \cdot H(\mu^n \,|\, \mathcal{C}^{n,K}_i) + H'_n , \end{aligned} \quad (8)$$

where

$$H'_n \stackrel{\text{def.}}{=} -\mu^n(B^n_\varepsilon) \cdot \ln(\mu^n(B^n_\varepsilon)) - \mu^n(\mathcal{C}^{n,K}_\varepsilon) \cdot \ln(\mu^n(\mathcal{C}^{n,K}_\varepsilon)) - \sum_{i=1}^{\infty} \mu^n(\mathcal{C}^{n,K}_i) \cdot \ln(\mu^n(\mathcal{C}^{n,K}_i)) . \quad (9)$$

We will repeatedly use the fact that the entropy of any probability measure supported by a finite set is maximal for the uniform measure and then equals the logarithm of the volume. First observe that

$$H(\mu^n \,|\, B^n_\varepsilon) \leq \ln(\# B^n_\varepsilon) \leq (\ell + \varepsilon) \cdot v \cdot n + o(n) ,$$

and thus the first term in (8) satisfies

$$\lim_n \frac{\mu^n(B^n_\varepsilon) \cdot H(\mu^n \,|\, B^n_\varepsilon)}{n} \leq (\ell + \varepsilon) \cdot v .$$

For the second term in (8), we get that

$$H(\mu^n \,|\, \mathcal{C}^{n,K}_\varepsilon) \leq \ln(\# \mathcal{C}^{n,K}_\varepsilon) \leq K \cdot v \cdot n + o(n) .$$

On the other hand, ℓ is also the limit in probability of $d(e, Z_n)/n$, hence $\forall \varepsilon > 0$, $\lim_n \mu^n(B^n_\varepsilon) = 1$. Therefore $\lim_n \mu^n(\mathcal{C}^{n,K}_\varepsilon) = 0$ and the second term in (8) satisfies

$$\lim_n \frac{\mu^n(\mathcal{C}^{n,K}_\varepsilon) \cdot H(\mu^n \,|\, \mathcal{C}^{n,K}_\varepsilon)}{n} = 0 .$$

For the third term in (8), as before, we have

$$H(\mu^n \,|\, \mathcal{C}^{n,K}_i) \leq \ln(\# \mathcal{C}^{n,K}_i) \leq 2^i K \cdot v \cdot n + o(n) ,$$

and, by the definition of $\mathcal{C}^{n,K}_i$,

$$\mu^n(\mathcal{C}^{n,K}_i) = \mathbb{E}\left[\, \mathbb{1}_{\{Z_n \in \mathcal{C}^{n,K}_i\}} \right] \leq \mathbb{E}\left[\frac{d(e, Z_n)}{2^{i-1} K n} \cdot \; \mathbb{1}_{\{Z_n \in \mathcal{C}^{n,K}_i\}} \right] . \quad (10)$$

So,

$$\begin{aligned}\frac{1}{n}\sum_{i=1}^{\infty}\mu^n(\mathcal{C}_i^{n,K})\cdot H(\mu^n\,|\,\mathcal{C}_i^{n,K}) &\leq \left(\frac{2v}{n}+o\left(\frac{1}{n}\right)\right)\mathbb{E}\left[d(e,Z_n)\sum_{i=1}^{\infty}\ \mathbb{1}_{\{Z_n\in\mathcal{C}_i^{n,K}\}}\right]\\ &= \left(\frac{2v}{n}+o\left(\frac{1}{n}\right)\right)\mathbb{E}\left[d(e,Z_n)\cdot\ \mathbb{1}_{\{d(e,Z_n)>Kn\}}\right]\,.\end{aligned}$$

As $d(e,Z_n)\leq\sum_{k=1}^n d(e,X_k)$,

$$\begin{aligned}\frac{1}{n}\sum_{i=1}^{\infty}\mu^n(\mathcal{C}_i^{n,K})\cdot H(\mu^n\,|\,\mathcal{C}_i^{n,K}) &\leq \left(\frac{2v}{n}+o\left(\frac{1}{n}\right)\right)\\ &\qquad\times\sum_{j=1}^{n}\mathbb{E}\left[d(e,X_j)\cdot\ \mathbb{1}_{\{\sum_{k=1}^n d(e,X_k)>Kn\}}\right]\\ &= (2v+o(1))\mathbb{E}\left[d(e,X_1)\cdot\ \mathbb{1}_{\{\sum_{k=1}^n d(e,X_k)>Kn\}}\right]\,,\end{aligned}$$

since $X_1,\dots,X_n$ are i.i.d., so that the random variables

$$Y_j\stackrel{\text{def.}}{=}d(e,X_j)\cdot\ \mathbb{1}_{\{\sum_{k=1}^n d(e,X_k)>Kn\}}\,,$$

have the same distribution.

By the strong law of large numbers, the sequence $\frac{1}{n}\sum_{k=1}^n d(e,X_k)$ almost surely converges to $\mathbb{E}[d(e,X_1)]\stackrel{\text{def.}}{=}m<\infty$. As a consequence, for any $K>m$, we have

$$d(e,X_1)\cdot\ \mathbb{1}_{\{\sum_{k=1}^n d(e,X_k)>Kn\}}\xrightarrow{\text{a.s.}}0\,. \tag{11}$$

Moreover, as

$$d(e,X_1)\cdot\ \mathbb{1}_{\{\sum_{k=1}^n d(e,X_k)>Kn\}}\leq d(e,X_1)\,,$$

which is integrable, the limit in (11) occurs also in L^1. Then

$$\lim_n\frac{1}{n}\sum_{i=1}^{\infty}\mu^n(\mathcal{C}_i^{n,K})\cdot H(\mu^n\,|\,\mathcal{C}_i^{n,K})=0\,.$$

We are left with H_n'. As $\lim_n\mu^n(B_\varepsilon^n)=1$ and $\lim_n\mu^n(\mathcal{C}_\varepsilon^{n,K})=0$,

$$\lim_n[-\mu^n(B_\varepsilon^n)\cdot\ln(\mu^n(B_\varepsilon^n))-\mu^n(\mathcal{C}_\varepsilon^{n,K})\cdot\ln(\mu^n(\mathcal{C}_\varepsilon^{n,K}))]=0\,.$$

For the last term in (9), note that (10) gives

$$\mu^n(\mathcal{C}_i^{n,K})\leq\frac{1}{2^{i-1}Kn}\sum_{k=1}^{n}\mathbb{E}[d(e,X_k)]\leq\frac{m}{2^{i-1}K}\,.$$

Together with the inequality $-a\ln(a) \le 2\mathrm{e}^{-1}\sqrt{a}$, we get

$$-\sum_{i=1}^{\infty} \mu^n(\mathcal{C}_i^{n,K}) \cdot \ln(\mu^n(\mathcal{C}_i^{n,K})) \le 2\mathrm{e}^{-1} \sum_{i=1}^{\infty} \sqrt{\mu^n(\mathcal{C}_i^{n,K})} < \infty \,.$$

So $\lim_n H'_n/n = 0$.

Finally, taking the limit $n \to \infty$, we deduce from (8) that $h \le (\ell + \varepsilon) \cdot v$ for any ε, so $h \le \ell \cdot v$. □

We conclude by a last remark.

Remark 3.5 *The proof of Theorem 1.1 using the Martin boundary relies on the translation invariance of* Γ*, but the hypothesis that the graph is a Cayley graph of a countable group seems too strong. It would be interesting to extend this proof to the case of space homogeneous Markov chains (see Kaimanovich & Woess [19]).*

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References

[1] G. Alexopoulos, *Random walks on discrete groups of polynomial volume growth*, Ann. Probab. **30** (2002), no. 2, 723–801.

[2] W. Ballmann, M. Gromov and V. Schroeder, *Manifolds of nonpositive curvature*, Birkhäuser Boston Inc., Boston, MA, 1985, Progress in Mathematics, Vol. 61.

[3] I. Benjamini and Y. Peres, *Tree-indexed random walks on groups and first passage percolation*, Probab. Th. Rel. Fields. **98** (1994), no. 1, 91–112.

[4] S. Blachère and S. Brofferio, *Internal diffusion limited aggregation on discrete groups having exponential growth*, Probab. Th. Rel. Fields. **137** (2007), no. 3-4, 323–343.

[5] S. Blachère, P. Haïssinsky, and P. Mathieu, *Harmonic measures versus quasiconformal measures for hyperbolic groups*, (in preparation).

[6] L. Breiman, *Probability*, Addison-Wesley Publishing Company, Reading, Mass., 1968.

[7] T. Coulhon, *Ultracontractivity and Nash type inequalities*, J. Funct. Anal. **141** (1996), no. 2, 510–539.

[8] T. Coulhon and L. Saloff-Coste, *Marches aléatoires non symétriques sur les groupes unimodulaires*, C. R. Acad. Sci. Paris Sér. I Math. **310** (1990), no. 8, 627–630.

[9] Y. Derriennic, *Sur le théorème ergodique sous-additif*, C. R. Acad. Sci. Paris Sér. A-B **281** (1975), no. 22, Aii, A985–A988.

[10] ______, *Quelques applications du théorème ergodique sous-additif*, Conference on Random Walks (Kleebach, 1979) (French), Astérisque, vol. 74, Soc. Math. France, Paris, 1980, pp. 183–201, 4.

[11] ______, *Entropie, théorèmes limite et marches aléatoires*, Probability measures on groups, VIII (Oberwolfach, 1985), Lecture Notes in Math., vol. 1210, Springer, Berlin, 1986, pp. 241–284.

[12] E. Dynkin, *The boundary theory of Markov processes (discrete case)*, Uspehi Mat. Nauk **24** (1969), no. 2, 3–42.

[13] E. Dynkin and M. Malyutov, *Random walks on groups with a finite number of generators*, Soviet Math. Doklady **2** (1961), 399–402.

[14] A. Erschler, *On drift and entropy growth for random walks on groups*, Ann. Probab. **31** (2003), no. 3, 1193–1204.

[15] Y. Guivarc'h, *Sur la loi des grands nombres et le rayon spectral d'une marche aléatoire*, Conference on Random Walks (Kleebach, 1979) (French), Astérisque, vol. 74, Soc. Math. France, Paris, 1980, pp. 47–98, 3.

[16] G. A. Hunt, *Markoff chains and Martin boundaries*, Illinois J. Math. **4** (1960), 313–340.

[17] V. Kaimanovich, *Poisson boundary of discrete groups*, preprint.

[18] V. Kaimanovich and A. Vershik, *Random walks on discrete groups: boundary and entropy*, Ann. Probab. **11** (1983), no. 3, 457-490.

[19] V. Kaimanovich and W. Woess, *Boundary and entropy of space homogeneous Markov chains*, Ann. Probab. **30** (2002), no. 1, 323–363.

[20] A. Karlsson and F. Ledrappier, *On laws of large numbers for random walks*, Ann. Probab. **34** (2006), no. 5, 1693–1706.

[21] ______, *Linear drift and entropy for random walks*, preprint.

[22] J. F. C. Kingman, *The ergodic theory of subadditive stochastic processes*, J. Roy. Statist. Soc. Ser. B **30** (1968), 499–510.

[23] F. Spitzer, *Principles of random walks*, second ed., Springer-Verlag, New York, 1976, Graduate Texts in Mathematics, Vol. 34.

[24] A. M. Vershik, *Dynamic theory of growth in groups: entropy, boundaries, examples*, Uspekhi Mat. Nauk **55** (2000), no. 4(334), 59–128, (Translation in Russian Math. Surveys 55 (2000), no. 4, 667–733).

[25] W. Woess, *Random walks on infinite graphs and groups*, Cambridge University Press, Cambridge, 2000.

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