

Minimization of the root of a quadratic
functional under a system of affine equality
constraints
with application in portfolio management

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February 12, 2007

Abstract

We present an explicit closed form solution of the problem of minimizing the root of a quadratic functional subject to a system of affine constraints. The result generalizes [10], where the optimization problem was solved under only one linear constraint. This is of interest for solving significant problems pertaining to financial economics as well as some classes of feasibility and optimization problems which frequently occur in tomography and other fields. The particular case when the expected return of finance portfolio is certain is discussed as well as some other examples.

2000 Mathematics Subject Classification: 90C25, 49N10, 91B28.

Key words: minimization, root of quadratic functional, linear constraints, covariance, optimal portfolio management.

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1 Introduction

In this paper we generalize the result of [10], where the problem of minimization of square root functional

$$f(\mathbf{x}) = \boldsymbol{\mu}^T \mathbf{x} + \lambda \sqrt{\mathbf{x}^T \Sigma \mathbf{x}}, \quad \lambda > 0, \quad (1)$$

subject to one linear constraint

$$\mathbf{b}^T \mathbf{x} = c, \quad c \neq 0, \quad (2)$$

was considered, to the case of a number of linear constraints. Here $\boldsymbol{\mu}$, \mathbf{b} are $n \times 1$ vectors and $\Sigma = (\sigma_{ij})_{i,j=1}^n$ is $n \times n$ positive definite matrix. More precisely, let $B = (b_{ij})_{i,j=1}^{m,n}$ be $m \times n$, $m < n$, rectangular matrix of the full rank and \mathbf{c} be some $m \times 1$ vector. In this paper we obtain the conditions under which the problem of minimization of function (1) subject to the system of affine constraints

$$B\mathbf{x} = \mathbf{c}, \quad \mathbf{c} \neq \mathbf{0}, \quad (3)$$

where $\mathbf{0}$ is a vector-column of m zeros, has the solution, and find its exact closed form. The problem of minimization of the function $f : R^n \rightarrow R$ subject to a system of linear equality constraints has many applications, among which are those related to risk management in financial economics. In this note, we obtain the conditions under which the solution of this problem exists, and for that case we show how the solution can be effectively computed. First, let us notice that function $f(\mathbf{x})$ is convex as a sum of the linear functional and a convex function. The convexity of the square root of the quadratic term follows from the observation that for any $\mathbf{u}, \mathbf{v} \in R^n$ and $t \in R$,

$$\sqrt{(\mathbf{u} + t\mathbf{v})^T \Sigma (\mathbf{u} + t\mathbf{v})} = \sqrt{\mathbf{v}^T \Sigma \mathbf{v} t^2 + 2\mathbf{v}^T \Sigma \mathbf{u} t + \mathbf{u}^T \Sigma \mathbf{u}} \quad (4)$$

is a strictly convex function of t (the square root of a quadratic univariate polynomial with a positive leading coefficient is strictly convex), and consequently $\sqrt{\mathbf{x}^T \Sigma \mathbf{x}}$ is strictly convex and so is function $f(\mathbf{x})$.

Let us notice that in the special case when matrix B is of $2 \times n$ dimension and equals

$$B = \begin{pmatrix} 1 & \cdots & 1 \\ \mu_1 & \cdots & \mu_n \end{pmatrix} \quad (5)$$

and vector $\mathbf{c}^T = (1, R)$ the solution of problem (1), (3) coincides with the solution of the problem of minimization of function

$$q(\mathbf{x}) = \mathbf{x}^T \Sigma \mathbf{x} \quad (6)$$

under constraints

$$\mathbf{1}^T \mathbf{x} = 1, \quad (7)$$

$$\boldsymbol{\mu}^T \mathbf{x} = R, \quad (8)$$

where $\mathbf{1}$ is the vector-column of n ones. This is directly related to the Markowitz optimal portfolio solution under a certain expected portfolio return, which is well documented (see, for example, [15], Section 6, [2], Section 8.2.1, [1], Section 4.4). Here vector \mathbf{x} is interpreted as a weight of the portfolio of risk returns $P = \mathbf{x}^T \mathbf{X}$, where $\mathbf{X} = (X_1, \dots, X_n)^T$ is a vector of random variables-returns with expectations $E\mathbf{X} = (EX_1, \dots, EX_n)^T = \boldsymbol{\mu}$ and covariation matrix

$$\text{cov}(\mathbf{X}) = E(\mathbf{X} - E\mathbf{X})(\mathbf{X} - E\mathbf{X})^T = \Sigma,$$

and function (6) is simply

$$q(\mathbf{x}) = \text{Var}(P), \quad (9)$$

where $\text{Var}(P)$ is variance of P , which is called the variance premium (see [6], Sect: Premium principles). Then function (1) has a special meaning in the Actuarial sciences: it is the standard deviation premium because it can be rewritten as follows

$$f(\mathbf{x}) = E(P) + \lambda \sqrt{\text{Var}(P)},$$

([6], Sect: Premium principles). We provide a closed form solution of the problem (1), (3) and show that for the special case of constraints (7) and (8) the solution coincides with the Markowitz mean-variance solution.

Let us notice that the solution of the problem of the minimization function (1) with constraints (3) provides the optimal portfolio management under all positive homogeneous and translation invariant risk measures for the class of multivariate elliptical distributions of risks (see [14], Section 6.1, [9], [7], [8]). These measures are of significant interest in financial economics. The important examples of such measures are short fall (or value-at-risk) and expected short fall (or tail conditional expectation) among others ([12]).

The details of these applications are actually beyond the scope of this paper, and are considered separately ([11]).

At the same time, we point out another interpretation and application of the presented result which are related to relative projections onto closed convex sets. We denote, as usual, $\langle \mathbf{x}, \mathbf{y} \rangle = \mathbf{x}^T \mathbf{y}$ the Euclidean inner product in R^n . Let

$$h(\mathbf{x}) = \lambda \sqrt{\mathbf{x}^T \Sigma \mathbf{x}} \quad (10)$$

and for $\boldsymbol{\xi} \in R^n$

$$h^*(\boldsymbol{\xi}) = \sup_{\mathbf{z} \in R^n} (\langle \boldsymbol{\xi}, \mathbf{z} \rangle - h(\mathbf{z}))$$

be the Fenchel conjugate of h . Then function

$$W^h(\boldsymbol{\xi}, \mathbf{x}) = h(\mathbf{x}) - \langle \boldsymbol{\xi}, \mathbf{x} \rangle + h^*(\boldsymbol{\xi})$$

is called the generalized distance in R^n . For any $\boldsymbol{\xi} \in R^n$ and for any closed convex nonempty set C in R^n , there exists a unique minimizer of the function $W^h(\boldsymbol{\xi}, \cdot)$ over C (see [4], Section 4.2). This vector is denoted by $P_C^h(\boldsymbol{\xi})$ and is called the projection of $\boldsymbol{\xi}$ on C relative to the function h (or the proximal projection of $\boldsymbol{\xi}$ relative to h). Then the purpose of the present paper is, in fact, equivalent to that of determining the minimum of $W^h(-\boldsymbol{\mu}, \cdot)$ over the closed set

$$C = \{\mathbf{x} | B\mathbf{x} = \mathbf{c}\}, \quad \mathbf{c} \neq \mathbf{0}. \quad (11)$$

Since $\boldsymbol{\mu}$ which we are considering is an arbitrary vector in R^n , solving the problem which we pose above is equivalent to **exactly solving the problem of computing** $P_C^h(\boldsymbol{\xi})$ for any $\boldsymbol{\xi}$. Notice that the function h is a norm in R^n when Σ is positive definite, which is the case here. Moreover, h satisfies the requirements of Theorem 4.8 in the [4] (when placed in the specific context of the space R^n). Therefore, Theorem 4.8 applies and gives the formula for computing the vector $P_C^h(\boldsymbol{\xi})$ with C as above. However, calculability of $P_C^h(\boldsymbol{\xi})$ by that formula depends on the calculability of the gradient of the Fenchel dual of h . Our result shows an explicit way of determining $P_C^h(\boldsymbol{\xi})$ when h and C are as above (see (10) and (11)). This is important because it may help solve numerically feasibility and optimization problems such as those discussed in the book [5]. In fact, once computation of $P_C^h(\boldsymbol{\xi})$ is numerically doable in an efficient way, many feasibility and optimization algorithms become practically implementable. This is one of the merits of the present paper: it makes some sophisticated algorithms, such as those for solving the optimization and equilibrium problems discussed in [3], applicable to a larger class of the problems than previously known.

2 Main result

Choosing the first $n - m$ variables we have the natural partition of vector $\mathbf{x}^T = (\mathbf{x}_1^T, \mathbf{x}_2^T)$, $\mathbf{x}_1 = (x_1, \dots, x_{n-m})^T$, $\mathbf{x}_2 = (x_{n-m+1}, \dots, x_n)^T$ and the corresponding partition of vectors $\boldsymbol{\mu}^T = (\boldsymbol{\mu}_1^T, \boldsymbol{\mu}_2^T)$, $\mathbf{1}^T = (\mathbf{1}_1^T, \mathbf{1}_2^T)$ ($\mathbf{1}$ is vector of n ones), matrix Σ ,

$$\Sigma = \begin{pmatrix} \Sigma_{11} & \Sigma_{12} \\ \Sigma_{21} & \Sigma_{22} \end{pmatrix} \quad (12)$$

and matrix

$$B = \begin{pmatrix} B_{21} & B_{22} \end{pmatrix}$$

where matrices B_{21} and B_{22} are of dimensions $m \times (n - m)$ and $m \times m$, respectively. As matrix B is of full rank suppose without loss of generality that matrix B_{22} is non singular. Define $m \times (n - m)$ and $(n - m) \times m$ matrices

$$D_{21} = B_{22}^{-1} B_{21}, D_{12} = D_{21}^T \quad (13)$$

and $(n - m) \times (n - m)$ matrix

$$Q = \Sigma_{11} - \Sigma_{12} D_{21} - D_{12} \Sigma_{21} + D_{12} \Sigma_{22} D_{21} = (q_{ij})_{i,j=1}^{n-m}. \quad (14)$$

Lemma 1 *As Σ is positive definite, Q is also positive definite.*

Proof. We give the probabilistic proof of the Lemma. Along with positive definite matrix Σ , one may consider an n -variate normally distributed vector \mathbf{Z} with vector - expectation $\mathbf{0}$ and covariance matrix Σ (see [16], Section 1.2.1), and so we say $\mathbf{Z} \sim N_n(\mathbf{0}, \Sigma)$. Then vector $\mathbf{Z}_1 = (Z_1, \dots, Z_{n-m})^T \sim N_{n-m}(\mathbf{0}_1, \Sigma_{11})$, vector $\mathbf{Z}_2 = (Z_{n-m+1}, \dots, Z_n)^T \sim N_m(\mathbf{0}_2, \Sigma_{22})$ and $\mathbf{Y} = \mathbf{Z}_1 - D_{12} \mathbf{Z}_2 \sim N_{n-m}(\mathbf{0}_1, Q)$, because

$$\begin{aligned} \text{cov}(\mathbf{Y}) &= E(\mathbf{Z}_1 - D_{12} \mathbf{Z}_2)(\mathbf{Z}_1 - D_{12} \mathbf{Z}_2)^T \\ &= E \mathbf{Z}_1 \mathbf{Z}_1^T - D_{12} E(\mathbf{Z}_2 \mathbf{Z}_1^T) - E(\mathbf{Z}_1 \mathbf{Z}_2^T) D_{21} + D_{12} E(\mathbf{Z}_2 \mathbf{Z}_2^T) D_{21} = Q, \end{aligned}$$

and the linear transformation of \mathbf{Z} has maximal rank. ■

Denote by

$$\boldsymbol{\Delta} = D_{12} \boldsymbol{\mu}_2 - \boldsymbol{\mu}_1. \quad (15)$$

Theorem 1 *If*

$$\lambda > \sqrt{\boldsymbol{\Delta}^T Q^{-1} \boldsymbol{\Delta}}, \quad (16)$$

the problem of the minimization of function (1) subject to (3) has the finite solution

$$\begin{aligned} \mathbf{x}^* &= \Sigma^{-1}B^T(B\Sigma^{-1}B^T)^{-1}\mathbf{c} \\ &+ \sqrt{\frac{\mathbf{c}^T(B\Sigma^{-1}B^T)^{-1}\mathbf{c}}{(\lambda^2 - \Delta^TQ^{-1}\Delta)}} (\Delta^TQ^{-1}, -\Delta^TQ^{-1}D_{12})^T \end{aligned} \quad (17)$$

Proof. Define vector $\mathbf{d}_2 = B_{22}^{-1}\mathbf{c}$. Then from the system of constraints (3) and from (13) it follows that $\mathbf{x}^T = (\mathbf{x}_1^T, \mathbf{d}_2^T - \mathbf{x}_1^T D_{12})^T$ and then straightforwardly

$$\mathbf{x}^T \Sigma \mathbf{x} = \mathbf{x}_1^T Q \mathbf{x}_1 + 2\mathbf{d}_2^T (\Sigma_{21} - \Sigma_{22} D_{21}) \mathbf{x}_1 + \mathbf{d}_2^T \Sigma_{22} \mathbf{d}_2. \quad (18)$$

Then the goal-function

$$\begin{aligned} f(\mathbf{x}) &= g(\mathbf{x}_1) = \boldsymbol{\mu}_2^T \mathbf{d}_2 + (\boldsymbol{\mu}_1 - D_{12} \boldsymbol{\mu}_2)^T \mathbf{x}_1 \\ &+ \lambda \sqrt{\mathbf{x}_1^T Q \mathbf{x}_1 + 2\mathbf{d}_2^T (\Sigma_{21} - \Sigma_{22} D_{21}) \mathbf{x}_1 + \mathbf{d}_2^T \Sigma_{22} \mathbf{d}_2} \end{aligned}$$

is a function of $n - m$ variables $\mathbf{x}_1 = (x_1, \dots, x_{n-m})^T$ and the problem reduces to the problem of finding the unconditional minimum

$$\min_{\mathbf{x}_1 \in R^{n-m}} g(\mathbf{x}_1).$$

As a corollary of the well known solution of the quadratic programming problem

$$\mathbf{x}^0 = \arg \min_{B\mathbf{x}=\mathbf{c}} \mathbf{x}^T \Sigma \mathbf{x} = \Sigma^{-1}B^T(B\Sigma^{-1}B^T)^{-1}\mathbf{c}, \quad (19)$$

and

$$\begin{aligned} \mathbf{x}_1^T Q \mathbf{x}_1 + 2\mathbf{d}_2^T (\Sigma_{21} - \Sigma_{22} D_{21}) \mathbf{x}_1 + \mathbf{d}_2^T \Sigma_{22} \mathbf{d}_2 &\geq \mathbf{x}^{0T} \Sigma \mathbf{x}^0 \\ &= \mathbf{c}^T (B\Sigma^{-1}B^T)^{-1} \mathbf{c} > 0, \\ \mathbf{x}_1 &\in R^{n-m}, \end{aligned} \quad (20)$$

as matrix $B\Sigma^{-1}B^T > 0$ and $\mathbf{c} \neq \mathbf{0}$ (see, for example, [13], Chapter 14.1). This means that function $\sqrt{\mathbf{x}_1^T Q \mathbf{x}_1 + 2\mathbf{d}_2^T (\Sigma_{21} - \Sigma_{22} D_{21}) \mathbf{x}_1 + \mathbf{d}_2^T \Sigma_{22} \mathbf{d}_2}$ is differentiable for any $\mathbf{x}_1 \in R^{n-m}$. For the same reasons as given in (4), taking into account the last inequality, one may conclude that this function, and together with it the

function $g(\mathbf{x}_1)$, is strictly convex on R^{n-m} . Denote by $\frac{d^T}{d\mathbf{x}_1} = (\frac{d}{dx_1} \dots \frac{d}{dx_{n-m}})$ -vector row of the first $n - m$ derivatives and let

$$\mathbf{x}^* = (\mathbf{x}_1^{*T}, \mathbf{d}_2^T - \mathbf{x}_1^{*T} D_{12})^T \quad (21)$$

and $\mathbf{x}^0 = (\mathbf{x}_1^{0T}, \mathbf{d}_2^T - \mathbf{x}_1^{0T} D_{12})^T$ be partitions of the vector-solution of the problem \mathbf{x}^* and vector \mathbf{x}^0 represented by (19), respectively. Then the vector \mathbf{x}_1^* is the unique solution of the vector-equations

$$\frac{d}{d\mathbf{x}_1} g(\mathbf{x}_1) = (\boldsymbol{\mu}_1 - D_{12} \boldsymbol{\mu}_2) + \frac{\lambda(Q\mathbf{x}_1 + (\Sigma_{12} - D_{12}\Sigma_{22})\mathbf{d}_2)}{\sqrt{\mathbf{x}_1^T Q \mathbf{x}_1 + 2\mathbf{d}_2^T (\Sigma_{21} - \Sigma_{22} D_{21}) \mathbf{x}_1 + \mathbf{d}_2^T \Sigma_{22} \mathbf{d}_2}} = \mathbf{0}_1,$$

where $\mathbf{0}_1$ is vector-column of $(n - m)$ zeros, which can be rewritten in the form

$$(Q\mathbf{x}_1 + (\Sigma_{12} - D_{12}\Sigma_{22})\mathbf{d}_2) = \boldsymbol{\tau} \sqrt{\mathbf{x}_1^T Q \mathbf{x}_1 + 2\mathbf{d}_2^T (\Sigma_{21} - \Sigma_{22} D_{21}) \mathbf{x}_1 + \mathbf{d}_2^T \Sigma_{22} \mathbf{d}_2},$$

where

$$\boldsymbol{\tau} = (\tau_1, \dots, \tau_{n-m})^T = (D_{12}\boldsymbol{\mu}_2 - \boldsymbol{\mu}_1)/\lambda. \quad (22)$$

Consider \mathbf{x}_1^* in the form $\mathbf{x}_1^* = \mathbf{x}_1^0 + \mathbf{y}^*$, where $\mathbf{y}^{*T} = (y_1, \dots, y_{n-m})$ is $(n - m)$ dimension vector. Then, as \mathbf{x}^0 is a solution of problem (19), it follows that

$$Q\mathbf{x}_1^0 + (\Sigma_{12} - D_{12}\Sigma_{22})\mathbf{d}_2 = \mathbf{0}_1 \quad (23)$$

and \mathbf{y}^* is the unique solution of the vector-equations

$$\mathbf{y}^* = Q^{-1} \boldsymbol{\tau} \sqrt{\mathbf{x}_1^T Q \mathbf{x}_1 + 2\mathbf{d}_2^T (\Sigma_{21} - \Sigma_{22} D_{21}) \mathbf{x}_1 + \mathbf{d}_2^T \Sigma_{22} \mathbf{d}_2}. \quad (24)$$

As $\boldsymbol{\tau} = \mathbf{0}_1$ (means $\boldsymbol{\mu}_1 = B_{12} B_{22}^{-1} \boldsymbol{\mu}_2$), it results trivially that $\mathbf{y}^* = \mathbf{0}_1$. Suppose vector $\boldsymbol{\tau} \neq \mathbf{0}$, then as matrix $Q^{-1} = (\delta_{ij})_{i,j=1}^{n-m}$ is nonsingular (positive definite) there exists row $\boldsymbol{\delta}_i^T = (\delta_{i1}, \dots, \delta_{in-m})$ of Q^{-1} such that $\boldsymbol{\delta}_i^T \boldsymbol{\tau} \neq 0$. Suppose for convenience and without loss of generality that $i = 1$. Then, using the following partition of matrix Q^{-1} into the 2 matrices Q_1^{-1} and Q_2^{-1}

$$Q^{-1} = \begin{pmatrix} Q_1^{-1} \\ Q_2^{-1} \end{pmatrix}, \quad (25)$$

where Q_1^{-1} is simply the first row of Q^{-1} (i.e., $Q_1^{-1} = \boldsymbol{\delta}_1^T$) and Q_2^{-1} consists of other $(n - m - 1)$ rows of Q^{-1} , from (24) we have

$$\mathbf{y}^* = y_1^* (\mathbf{1}, \mathbf{a}^T)^T, \quad (26)$$

where

$$\mathbf{a} = Q_2^{-1} \boldsymbol{\tau} / Q_1^{-1} \boldsymbol{\tau}. \quad (27)$$

Substituting (26) into the first equation of (24), we get straightforwardly,

$$\begin{aligned} y_1^* &= Q_1^{-1} \boldsymbol{\tau} \sqrt{\mathbf{x}^{0T} \Sigma \mathbf{x}^0 + 2(\mathbf{x}_1^{0T} Q + \mathbf{d}_2^T (\Sigma_{21} - \Sigma_{22})) \mathbf{y}^* + \mathbf{y}^{*T} Q \mathbf{y}^*} \\ &= Q_1^{-1} \boldsymbol{\tau} \sqrt{\mathbf{c}^T (B \Sigma^{-1} B^T)^{-1} \mathbf{c} + y_1^{*2} (1, \mathbf{a}^T) Q (1, \mathbf{a}^T)^T} \end{aligned} \quad (28)$$

taking into account (23), (18) and the right hand side of (20). Squaring both parts of the equation and using the partitions of matrices Q and Q^{-1} we get

$$y_1^{*2} = \frac{(Q_1^{-1} \boldsymbol{\tau})^2 \mathbf{c}^T (B \Sigma^{-1} B^T)^{-1} \mathbf{c}}{(1 - (Q_1^{-1} \boldsymbol{\tau})^2 (1, \mathbf{a}^T) Q (1, \mathbf{a}^T)^T)}. \quad (29)$$

In ([10], eq. (29)) it was shown that

$$(1, \mathbf{a}^T) Q (1, \mathbf{a}^T)^T = \frac{1}{(Q_1^{-1} \boldsymbol{\tau})^2} \boldsymbol{\tau}^T Q^{-1} \boldsymbol{\tau}. \quad (30)$$

Substituting (30) into (29) and taking into account that from (28) it follows that $\text{sign}(y_1^*) = \text{sign}(Q_1^{-1} \boldsymbol{\tau})$, we find that there exists the sole solution

$$y_1^* = Q_1^{-1} \boldsymbol{\tau} \sqrt{\frac{\mathbf{c}^T (B \Sigma^{-1} B^T)^{-1} \mathbf{c}}{1 - \boldsymbol{\tau}^T Q^{-1} \boldsymbol{\tau}}} \quad (31)$$

subject to condition (16). Substituting (31) into (26) we obtain, taking into account (27), (22) and (25),

$$\mathbf{y}^* = Q^{-1} \boldsymbol{\Delta} \sqrt{\frac{\mathbf{c}^T (B \Sigma^{-1} B^T)^{-1} \mathbf{c}}{\lambda^2 - \boldsymbol{\Delta}^T Q^{-1} \boldsymbol{\Delta}}}.$$

The theorem follows, taking into account that from (21) and (19)

$$\mathbf{x}^* = ((\mathbf{x}_1^0 + \mathbf{y}^*)^T, (\mathbf{d}_2 - D_{21} \mathbf{x}_1^0 - D_{21} \mathbf{y}^*)^T)^T = \mathbf{x}^0 + (\mathbf{y}^{*T}, -\mathbf{y}^{*T} D_{12})^T$$

■

3 Illustrations

3.1 Only one linear constraint

In the case that the problem has only one constraint (2) we have that $m = 1$, matrix B is a vector-row, $B = \mathbf{b}^T$, $B_{21} = (b_1, \dots, b_{n-1}) = \mathbf{b}_1^T$, $B_{22} = b_n$, vector \mathbf{c} is simply a real number, $\mathbf{c} = c$, $\boldsymbol{\mu}_1 = (\mu_1, \dots, \mu_{n-1})^T$, $\boldsymbol{\mu}_2 = \mu_n$. Then for $b_n \neq 0$,

$$\boldsymbol{\Delta} = \frac{\mu_n}{b_n} \mathbf{b}_1 - \boldsymbol{\mu}_1.$$

The partition of matrix Σ is of the form

$$\Sigma = \begin{pmatrix} \Sigma_{11} & \boldsymbol{\sigma} \\ \boldsymbol{\sigma}^T & \sigma_{nn} \end{pmatrix},$$

where Σ_{11} is matrix of dimension $(n-1) \times (n-1)$, and matrix

$$Q = \Sigma_{11} - \mathbf{1}_1 \boldsymbol{\sigma}^T - \boldsymbol{\sigma} \mathbf{1}_1^T + \sigma_{nn} \mathbf{1}_1 \mathbf{1}_1^T.$$

Then from Theorem 1 it follows that

$$\mathbf{x}^* = \frac{c}{(\mathbf{b}^T \Sigma^{-1} \mathbf{b})} \Sigma^{-1} \mathbf{b} + \frac{|c|}{\sqrt{(\lambda^2 - \boldsymbol{\Delta}^T Q^{-1} \boldsymbol{\Delta})(\mathbf{b}^T \Sigma^{-1} \mathbf{b})}} (\boldsymbol{\Delta}^T Q^{-1}, -\frac{1}{b_n} \mathbf{b}_1^T Q^{-1} \boldsymbol{\Delta})^T \quad (32)$$

In the case of constraint (7), $\mathbf{b} = \mathbf{1}$, $\mathbf{b}_1 = D_{12} = \mathbf{1}_1$, where $\mathbf{1}_1$ is the vector-column of $n-1$ ones, $c = 1$ and $\boldsymbol{\Delta} = (\mu_n - \mu_1, \dots, \mu_n - \mu_{n-1})^T$. Then from 32 it follows that

$$\mathbf{x}^* = \frac{1}{(\mathbf{1}^T \Sigma^{-1} \mathbf{1})} \Sigma^{-1} \mathbf{1} + \frac{1}{\sqrt{(\lambda^2 - \boldsymbol{\Delta}^T Q^{-1} \boldsymbol{\Delta})(\mathbf{1}^T \Sigma^{-1} \mathbf{1})}} (\boldsymbol{\Delta}^T Q^{-1}, -\mathbf{1}_1^T Q^{-1} \boldsymbol{\Delta})^T,$$

which conforms well with Theorem 1 [10].

3.2 Optimal portfolio under certain expected return

Suppose now that together with constraint (7) we have also constraint (8). This means that the expected portfolio return is certain and equal to R .

Then $m = 2$, matrix B has the form given in (5), and the partition of B is of the form

$$B_{\mathbf{21}} = \begin{pmatrix} 1 & \cdots & 1 \\ \mu_1 & \cdots & \mu_{n-2} \end{pmatrix}, B_{\mathbf{22}} = \begin{pmatrix} 1 & 1 \\ \mu_{n-1} & \mu_n \end{pmatrix}.$$

Vectors $\mathbf{c}^T = (1, R)$, $\mathbf{d}_2 = B_{\mathbf{22}}^{-1}\mathbf{c} = \frac{1}{\mu_n - \mu_{n-1}}((\mu_n - R), (R - \mu_{n-1}))^T$. By straightforward calculations one obtains

$$D_{\mathbf{21}} = B_{\mathbf{22}}^{-1}B_{\mathbf{21}} = \frac{1}{\mu_n - \mu_{n-1}} \begin{pmatrix} \mu_n - \mu_1 & \cdots & \mu_n - \mu_{n-2} \\ \mu_1 - \mu_{n-1} & \cdots & \mu_{n-2} - \mu_{n-1} \end{pmatrix}.$$

and

$$D_{\mathbf{12}}\boldsymbol{\mu}_2 = (\mu_1, \dots, \mu_{n-2})^T = \boldsymbol{\mu}_1$$

Then $\boldsymbol{\Delta} = D_{\mathbf{12}}\boldsymbol{\mu}_2 - \boldsymbol{\mu}_1 = 0$ and consequently

$$\mathbf{x}^* = \mathbf{x}^0 = \arg \min_{B\mathbf{x}=\mathbf{c}} \mathbf{x}^T \Sigma \mathbf{x} = \Sigma^{-1} B^T (B \Sigma^{-1} B^T)^{-1} \mathbf{c},$$

i.e., the solutions of the problem of minimization of a square root functional (1) and a quadratic functional (6) coincide under constraints (7), (8) and presented solution is the Markowitz mean-variance optimal portfolio solution under certain expected portfolio return.

Suppose now that the expected sum of last $n - k$ portfolio returns is certain. That reduces to the following system of constraints

$$\begin{cases} \mathbf{1}^T \mathbf{x} = 1, \\ \sum_{i=k+1}^n \mu_i x_i = R. \end{cases} \quad (33)$$

Then

$$B = \begin{pmatrix} 1 & \cdots & 1 & 1 & \cdots & 1 \\ 0 & \cdots & 0 & \mu_{k+1} & \cdots & \mu_n \end{pmatrix},$$

and for $k \leq n - 2$,

$$B_{\mathbf{21}} = \begin{pmatrix} 1 & \cdots & 1 & 1 & \cdots & 1 \\ 0 & \cdots & 0 & \mu_{k+1} & \cdots & \mu_{n-2} \end{pmatrix}, B_{\mathbf{22}} = \begin{pmatrix} 1 & 1 \\ \mu_{n-1} & \mu_n \end{pmatrix}.$$

Straightforward calculation shows that

$$D_{\mathbf{21}} = \frac{1}{\mu_n - \mu_{n-1}} \begin{pmatrix} \mu_n & \cdots & \mu_n & \mu_n - \mu_{k+1} & \cdots & \mu_n - \mu_{n-2} \\ -\mu_{n-1} & \cdots & -\mu_{n-1} & \mu_{k+1} - \mu_{n-1} & \cdots & \mu_{n-2} - \mu_{n-1} \end{pmatrix}. \quad (34)$$

Consider partition of vector $\boldsymbol{\mu}_1 = (\boldsymbol{\mu}_1^T, \boldsymbol{\mu}_2^T)^T$, $\boldsymbol{\mu}_{\bar{1}} = (\mu_1, \dots, \mu_k)^T$, $\boldsymbol{\mu}_{\bar{2}} = (\mu_{k+1}, \dots, \mu_{n-2})^T$ and the corresponding partition of null-vector $\mathbf{0}_1 = (\mathbf{0}_{\bar{1}}^T, \mathbf{0}_{\bar{2}}^T)^T$ and $(n-2) \times (n-2)$ matrix

$$Q^{-1} = \begin{pmatrix} Q_{\bar{1}\bar{1}}^{-1} & Q_{\bar{1}\bar{2}}^{-1} \\ Q_{\bar{2}\bar{1}}^{-1} & Q_{\bar{2}\bar{2}}^{-1} \end{pmatrix}.$$

Then from (34) it follows that

$$D_{12}\boldsymbol{\mu}_2 = (\mathbf{0}_{\bar{1}}^T, \boldsymbol{\mu}_{\bar{2}}^T)^T,$$

and from (15) Δ reduces to $\Delta = -(\boldsymbol{\mu}_{\bar{1}}^T, \mathbf{0}_{\bar{2}}^T)^T$. Finally, from Theorem 1 it follows that for $\lambda > \sqrt{\boldsymbol{\mu}_{\bar{1}}^T Q_{\bar{1}\bar{1}}^{-1} \boldsymbol{\mu}_{\bar{1}}}$,

$$\begin{aligned} \mathbf{x}^* &= \Sigma^{-1} B^T (B \Sigma^{-1} B^T)^{-1} \mathbf{c} \\ &\quad - \sqrt{\frac{\mathbf{c}^T (B \Sigma^{-1} B^T)^{-1} \mathbf{c}}{(\lambda^2 - \boldsymbol{\mu}_{\bar{1}}^T Q_{\bar{1}\bar{1}}^{-1} \boldsymbol{\mu}_{\bar{1}})}} (\boldsymbol{\mu}_{\bar{1}}^T Q_{\bar{1}\bar{1}}^{-1}, \boldsymbol{\mu}_{\bar{1}}^T Q_{\bar{1}\bar{2}}^{-1}, -\boldsymbol{\mu}_{\bar{1}}^T (Q_{\bar{1}\bar{1}}^{-1} D_{\bar{1}2} + Q_{\bar{1}\bar{2}}^{-1} D_{\bar{2}2}))^T, \end{aligned} \quad (35)$$

where $2 \times k$ and $2 \times (n-2-k)$ matrices $D_{\bar{2}\bar{1}}$ and $D_{\bar{2}\bar{2}}$ are of the form, respectively,

$$\begin{aligned} D_{\bar{2}\bar{1}} &= \frac{1}{\mu_n - \mu_{n-1}} \begin{pmatrix} \mu_n & \cdots & \mu_n \\ -\mu_{n-1} & \cdots & -\mu_{n-1} \end{pmatrix}, \\ D_{\bar{2}\bar{2}} &= \frac{1}{\mu_n - \mu_{n-1}} \begin{pmatrix} \mu_n - \mu_{k+1} & \cdots & \mu_n - \mu_{n-2} \\ \mu_{k+1} - \mu_{n-1} & \cdots & \mu_{n-2} - \mu_{n-1} \end{pmatrix}, \end{aligned}$$

and $D_{\bar{1}2} = D_{\bar{2}\bar{1}}^T$, $D_{\bar{2}2} = D_{\bar{2}\bar{2}}^T$. Since $k \geq 1$, this solution does not now coincide with that of minimization of quadratic functional.

3.3 Numerical Example

We illustrate the results in the problem of optimal portfolio selection. We consider a portfolio of 10 stocks from NASDAQ/Computers (ADOBE Sys. Inc., Compuware Corp., NVIDIA Corp., Starles Inc., Verisign Inc., Sandisk Corp., Microsoft Corp., Symantec Corp., Citrix Sys Inc., Intuit Inc.) for the year 2005, and denote by $\mathbf{X} = (X_1, \dots, X_n)^T$ with $n = 10$ stock weekly

Table 1: Expected returns

Stock	ADOBE	Compuware	NVIDIA	Staples	VeriSign
Mean	-0.0061	0.0081	0.0096	-0.0058	-0.0064
Stock	Sandisk	Microsoft	Citrix	Intuit	Symantec
Mean	0.0198	-0.0002	0.0038	0.0041	-0.0061

Table 2: Covariance matrix of returns

	ADOBE	Compuware	NVIDIA	Staples	VeriSign
ADOBE	0.006102	0.001173	0.000118	0.000513	0.000121
Compuware	0.001173	0.003310	0.001047	0.000498	0.000847
NVIDIA	0.000118	0.001047	0.002145	0.000122	0.000772
Staples	0.000513	0.000498	0.000122	0.002940	-0.000547
VeriSign	0.000121	0.000847	0.000772	-0.000547	0.003486
	Sandisk	Microsoft	Citrix	Intuit	Symantec
Sandisk	0.004013	-0.000033	0.000844	0.000131	0.000083
Microsoft	-0.000033	0.000485	0.000220	0.000167	0.000062
Citrix	0.000844	0.000220	0.001365	0.000397	0.000445
Intuit	0.000131	0.000167	0.000397	0.000876	0.000027
Symantec	0.000083	0.000062	0.000445	0.000027	0.002542

returns. The vector of means and covariance matrix weekly return are given in Tables 1 and 2.

The random return on the portfolio is $P = \sum_{j=1}^n x_j X_j$, where $\sum_{j=1}^n x_j = 1$. The loss, being the negative of this, is given by

$$L = - \sum_{j=1}^n x_j X_j.$$

Consider the problem of minimization of the standard deviation premium of L

$$s(\mathbf{x}) = E(L) + \lambda \sqrt{Var(L)} = -\boldsymbol{\mu}^T \mathbf{x} + \lambda \sqrt{\mathbf{x}^T \Sigma \mathbf{x}}, \quad (36)$$

where vector $\boldsymbol{\mu}$ is vector of expected returns and Σ is 10×10 covariance matrix of returns presented in Tables 1 and 2, respectively, under system of constraints (33) with $k = 4$. This means that the expected return of the sum of the last 6 stocks (Verisign Inc., Sandisk Corp., Microsoft Corp., Symantec Corp., Citrix Sys Inc., Intuit Inc) is certain and equaled $R = 0.2$. Then vector $\mathbf{c} = (1, 0.2)^T$, matrix

$$B = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & \mu_5 & \mu_6 & \mu_7 & \mu_8 & \mu_9 & \mu_{10} \end{pmatrix},$$

where $\mu_5 - \mu_{10}$ are taken from Table 1, matrix

$$Q_{\mathbf{1}\mathbf{1}}^{-1} = \begin{pmatrix} 182.743 & -64.677 & 22.439 & -13.337 \\ -64.677 & 424.257 & -148.572 & -35.951 \\ 22.439 & -148.572 & 611.705 & -7.247 \\ -13.337 & -35.951 & -7.247 & 386.601 \end{pmatrix}$$

and the lower boundary for λ is

$$B = \sqrt{(\mu_1, \mu_2, \mu_3, \mu_4) Q_{\mathbf{7}\mathbf{7}}^{-1} (\mu_1, \mu_2, \mu_3, \mu_4)^T} = 0.2958.$$

Theorem 1 (formula (35)) provides the explicit solution for the $s(\mathbf{x})$ -optimal (minimal) portfolio reported in Table 3. For comparison, the first row of the Table presents the solution when the expected return of the full portfolio is certain. The last column of the second part of Table 3 provides the meanings of the goal function for both solutions. One can see that the goal of the solution provided by (35) is naturally lower.

Table 3: Optimal portfolio

	ADOBE	Compuware	NVIDIA	Staples	VeriSign
\mathbf{x}_0	-0.8836	0.0994	-1.9063	-3.7878	-4.3503
\mathbf{x}^*	-1.8755	2.0312	1.4352	-5.642	-5.4436

	Sandisk	Microsoft	Citrix	Intuit	Symantec	goal function $s(\cdot)$
\mathbf{x}_0	6.4363	-0.1776	1.1217	9.2869	-0.2784	0.1168
\mathbf{x}^*	6.6337	-0.6448	0.3516	8.0707	0.2306	0.0862

In additional in Table 4 we give the solution of the problem of minimization of functional $s(\mathbf{x})$ under matrices of constraints

$$B_1 = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 3\mu_1 & \mu_2 & 2\mu_3 & 2\mu_4 & \mu_5 & \mu_6 & 2\mu_7 & \mu_8 & \mu_9 & 3\mu_{10} \end{pmatrix},$$

$$B_2 = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 3\mu_1 & \mu_2 & 2\mu_3 & 2\mu_4 & \mu_5 & \mu_6 & 2\mu_7 & \mu_8 & \mu_9 & 3\mu_{10} \\ 5\mu_1 & 4\mu_2 & 6\mu_3 & 2\mu_4 & 3\mu_5 & \mu_6 & \mu_7 & 8\mu_8 & 7\mu_9 & \mu_{10} \end{pmatrix}$$

and vectors $\mathbf{c}_1 = (1, 0.5)^T$ and $\mathbf{c}_2 = (1, 0.5, 0.9)^T$, respectively. As in the case of matrix B_2 the problem has the additional constraint that the goal of the B_2 -solution is greater than that of corresponding to B_1 .

Table 4: Solution of the minimization problem under different systems of constraints

System of constr.	ADOBE	Compuware	NVIDIA	Staples	VeriSign
B_1	-2.946	3.662	4.453	-7.124	-6.390
B_2	-2.425	4.41	4.209	-5.213	-0.829

System of constr.	Sandisk	Microsoft	Citrix	Intuit	Symant.	goal f. $s(\cdot)$
B_1	7.179	-5.626	1.082	8.111	-1.400	0.103
B_2	7.823	5.476	-3.925	-0.931	-7.596	0.194

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