

Lévy processes with adaptable exponent

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Abstract

In this paper we consider Lévy processes without negative jumps, reflected at the origin. Feedback information about the level of the Lévy process (“workload level”) may lead to adaptation of the Lévy exponent. Examples of such models are queueing models in which the service speed or customer arrival rate changes depending on the workload level, and dam models in which the release rate depends on the buffer content. We first consider a class of models where information about the workload level is continuously available. In particular, we consider dam processes with a two-step release rule and M/G/1 queues in which the arrival rate, service speed, and/or jump size distribution may be adapted depending on whether the workload is above or below some level K . Secondly, we consider a class of models in which the workload can only be observed at Poisson instants. At these Poisson instants, the Lévy exponent may be adapted based on the amount of work present. For both classes of models we determine the steady-state workload distribution.

Keywords: Reflected Lévy process; adaptable exponent; workload distribution; scale functions; Laplace inversion; M/G/1 queue; storage process.

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1 Introduction

The basic process under consideration in this paper is a spectrally one-sided Lévy process $\{X(t), t \geq 0\}$ [6], i.e., a Lévy process which has either no positive or no negative jumps. Such processes are often studied in the applied probability literature, with applications to, e.g., queues, dams, storage processes, and finance. It is well-known that one can write

$$\mathbb{E}[e^{-\omega X(t)}] = e^{t\phi(\omega)};$$

$\phi(\omega)$ is called the Lévy exponent. In particular, we shall mainly consider Lévy processes without negative jumps, reflected at the origin. The key feature of the paper is that the Lévy exponent may change, depending on the level of the Lévy process (“workload level”). For example, consider the classical M/G/1 queue with service speed r , arrival rate λ and service requirement distribution $B(\cdot)$ with Laplace-Stieltjes Transform (LST) $\beta(\cdot)$. The input process to the queue is a compound Poisson process, which is a special case of a Lévy process, having Lévy exponent $-\lambda(1 - \beta(\omega))$. Subtracting a deterministic drift of rate r , to take the service capacity into account, yields a Lévy process with Lévy exponent $r\omega - \lambda(1 - \beta(\omega))$. Information about the workload level may lead one to change the service speed r , arrival rate λ , and/or service requirement LST $\beta(\cdot)$, and hence the Lévy exponent. There is a large literature about queueing models in which the server may work at *different speeds*, depending on either the number of customers in the system or the amount of work in the system. Models with service speed depending on queue length or workload arise naturally as representation of congestion phenomena in, e.g., manufacturing and healthcare processes. Another bulk of papers is devoted to the – related – case in which the *arrival rate* of customers depends on the state of the system. For example, feedback information signals are being used in various communication systems to regulate the offered traffic volume in accordance with the actual level of congestion. A prime example is the Transmission Control Protocol (TCP) in the Internet. The case of *service requirement distributions* depending on the workload level has received much less attention; see, e.g., [9, 25]. We refer to Dshalalow [16] for an extensive survey, with 277 references, on queueing models with state-dependent parameters.

Another application area of Lévy processes without negative jumps and state-dependent exponent concerns the analysis of dam models, in which the level of the buffer content increases (or decreases) *gradually* during stochastic amounts of time. Dams form a historically important area of systems with state-dependent rates. The input process of a large class of dam processes studied in the literature ([11, 14, 26], or [24, Chapter 3]) is a non-decreasing Lévy process. Information about the level of the Lévy process (which we call the workload) may lead to adaptation of the release rate or, more generally, the Lévy exponent.

In the present study we restrict ourselves to two classes of models in which the determining factor is whether the workload is above or below a certain level K . First we briefly consider a class of models where information about the level of congestion (workload) is continuously available. In particular, we consider dam processes with a two-step release rule (corresponding to service speed) and M/G/1 queues with adaptable arrival rate, service speed, and/or service requirement distribution, depending on whether the workload is above or below level K . The drawback of these models is that frequent changes in the adaptable parameters are not excluded. In the second class of models, we assume that the workload can only be observed at Poisson instants. At these Poisson instants, the adaptable parameters can be changed based on the amount of work present. This is the main model class of the paper and we refer to it as Poisson observer models. The two classes are described in more detail in Section 2.

The main goal of the paper is to derive the steady-state distribution of the workload of the reflected Lévy process. We develop a solution procedure that can be used to determine such steady-state behavior for a large class of reflected Lévy processes with two Lévy exponents. The machinery used includes Laplace transforms (building upon a technique developed in [17], see also [4]), martingales, and various properties of Lévy processes.

This paper is organized as follows. Section 2 contains descriptions of the dam (M/G/1-type) and the model with a Poisson observer, and some preliminary results on reflected Lévy processes without negative jumps. In Section 3 we give an outline of a 5-step procedure to determine the steady-state behavior of various queueing systems with two service rates or, more generally, reflected Lévy processes with two Lévy exponents. In Sections 4–6 we apply this procedure to several models. Section 4 presents the steady-state workload analysis of the M/G/1 system that has different arrival rates, service speeds and service requirements when the workload is above or below K , respectively. As an example, we derive the steady-state workload distribution in a dam with two release rates (service speeds) in Subsection 4.1. In Subsection 4.2, we consider the general M/G/1 queue where, in addition to the service speed, the arrival rate and service requirements can also be adapted based on the workload. The model of a reflected Lévy process with a Poisson observer is analyzed in Sections 5 and 6. In this analysis a key role is played by so-called alternating Lévy processes reflected at the origin. These processes are studied in the appendix.

2 Models and preliminaries

In this section, we introduce the two models considered in the paper. Moreover, we give some preliminary results on reflected Lévy processes without negative jumps.

Let us first introduce some notation. For some process, denote the value of that process at time t by $X^{(t)}$ and let X correspond to its steady-state version (if it exists). We specifically consider Lévy processes without negative jumps $\{X_i^{(t)}, t \geq 0\}$, $i = 1, 2$. For convenience, we exclude cases in which the processes $\{X_i^{(t)}, t \geq 0\}$, $i = 1, 2$, have monotone paths. (A similar approach can however be applied in case $X_1^{(t)}$ is a subordinator, or $X_2^{(t)}$ is the negative of a subordinator.) Denote by $\phi_i(\cdot)$ the Lévy exponent of the process $\{X_i^{(t)}, t \geq 0\}$, i.e.,

$$\mathbb{E}[e^{-\omega X_i^{(t)}}] = e^{t\phi_i(\omega)}.$$

For $i = 1, 2$, let $Z_i^{(t)} = Z_i^{(0)} + X_i^{(t)} + L_i^{(t)}$, $t \geq 0$, starting at $Z_i^{(0)} \geq 0$, where $L_i^{(t)} = -\inf_{0 \leq s \leq t} [Z_i^{(0)} + X_i^{(s)}]^-$, i.e., $Z_i^{(t)}$ is the reflected Lévy process and $L_i^{(t)}$ is the local time in 0. In terms of dams or queues with Lévy input, $Z_i^{(t)}$ constitutes the content or workload process. Throughout, we assume that the system is stable. For the models discussed below this means that $\phi_2'(0) > 0$.

Model I: Dams and M/G/1 queues

The study of queueing models with continuously adaptable service speed $r(Z^{(t)})$ goes back to the M/G/1 dam. In such a dam, the output equals r_1 when the content of the dam is smaller than or equal to K (but positive) and r_2 when the content of the dam exceeds level K . In terms of queueing models, this corresponds to a queue with workload-dependent service speed. In this paper, we consider an extension of the classical M/G/1 dam. In addition to the service speed (output rate), the arrival rate and service requirement distribution may also depend on the amount of work present. In terms of Lévy processes, this corresponds to a reflected process where the Lévy exponent depends on the amount of work present. In particular, the Lévy exponent equals $\phi_1(\cdot)$ when the workload is smaller than K (with reflection in zero) and it is $\phi_2(\cdot)$ otherwise. For the general M/G/1 queue, the Lévy process is a compound Poisson process with an additional

negative drift. For this special case, we denote the arrival rate by λ_i , the mean and LST of the service requirement by β_i and $\beta_i(\cdot)$, respectively, and the service speed by r_i , $i = 1, 2$. Hence, the Lévy exponent reads $\phi_i(\omega) = r_i\omega - \lambda_i + \lambda_i\beta_i(\omega)$.

Model II: Poisson observer

Again, we consider a reflected Lévy process without negative jumps and with two different Lévy exponents. At Poisson instants an observer arrives at the system to observe the amount of work present. We assume that the interarrival times of the observer are exponentially distributed with mean $1/\xi$ and are independent of the processes $\{X_i^{(t)}, t \geq 0\}$, $i = 1, 2$. The observer regulates the workload process according to a two-step rule in the following way: When the workload at the observer arrival instant is larger than some fixed value $K > 0$, then the process with Lévy exponent $\phi_2(\cdot)$ is chosen until the subsequent observer arrival instant. If the observer finds a workload smaller than or equal to K , then the process with Lévy exponent $\phi_1(\cdot)$ is taken during its next interarrival time.

More precisely, define $t_0 := 0$ and let $t_n, n = 1, \dots$, be the n th arrival epoch of the Poisson observer. Let $X_i(n) := \{X_i^{(t)}(n), t \geq 0\}$, $i = 1, 2$ and $n \geq 0$, be the Lévy input process with exponent $\phi_i(\cdot)$ during the interval between the n th and $(n+1)$ st arrival instant of the observer. For fixed $i = 1, 2$, we assume that $X_i(n), n \geq 0$, are independent and identically distributed Lévy processes without negative jumps. The workload process $\{Z^{(t)}, t \geq 0\}$ is now defined recursively by $Z^{(0)} = 0$ and

$$Z^{(t)} = \begin{cases} Z^{(t_n)} + X_1^{(t-t_n)}(n) + L_1^{(t-t_n)}(n), & \text{if } t \in (t_n, t_{n+1}] \text{ and } Z^{(t_n)} \leq K, \\ Z^{(t_n)} + X_2^{(t-t_n)}(n) + L_2^{(t-t_n)}(n), & \text{if } t \in (t_n, t_{n+1}] \text{ and } Z^{(t_n)} > K, \end{cases}$$

where $L_i^{(t)}(n) = -\inf_{0 \leq s \leq t} [Z^{(t_n)} + X_i^{(s)}(n)]^-$ for $i = 1, 2, n \geq 0$, and $t \in (t_n, t_{n+1}]$. Similarly, the local time in 0 is defined recursively by $L^{(0)} = 0$ and

$$L^{(t)} = \begin{cases} L^{(t_n)} + L_1^{(t-t_n)}(n), & \text{if } t \in (t_n, t_{n+1}] \text{ and } Z^{(t_n)} \leq K, \\ L^{(t_n)} + L_2^{(t-t_n)}(n), & \text{if } t \in (t_n, t_{n+1}] \text{ and } Z^{(t_n)} > K. \end{cases}$$

We also consider the process embedded at arrival instants of the observer. Let V_n denote the workload at time $t_n, n \geq 0$. Using PASTA, it follows that the steady-state distribution of V_n equals the steady-state distribution of the workload at an arbitrary instant. Let Z denote this steady-state random variable. To describe the one-step transition probabilities of V_n , we need to determine the distribution and LST of the workload of a Lévy process with exponent $\phi_i(\cdot), i = 1, 2$, after an exponential time (see Theorem 2.2 below).

Let T denote a generic (exponential) interarrival time of the observer. We now have the following recursion relation:

$$V_{n+1} = \begin{cases} Z_1^{(T)} \mid Z_1^{(0)} = V_n, & \text{for } 0 \leq V_n \leq K, \\ Z_2^{(T)} \mid Z_2^{(0)} = V_n, & \text{for } V_n > K. \end{cases} \quad (1)$$

Preliminaries on Lévy processes

In this subsection, we present some results on reflected Lévy processes with only positive jumps and Lévy exponent $\phi(\cdot)$. As mentioned, we exclude the case that the process has monotone paths.

We first introduce the family of so-called *scale functions*. In the literature, scale functions often appear in the study of first-exit times and exit positions, see e.g. [2, 6, 7, 8, 20, 21, 22]. However, in this paper we frequently use scale functions to describe the (steady-state) workload behavior. Define, for $s \in \mathbb{R}$, $\eta(s) := \sup\{\omega \geq 0 : \phi(\omega) = s\}$ as the largest root of the equation $\phi(\omega) = s$.

Definition 2.1. *For $q \geq 0$, the q -scale function $W^{(q)} : (-\infty, \infty) \rightarrow [0, \infty)$ is the unique function whose restriction to $[0, \infty)$ is continuous and has Laplace transform*

$$\int_0^\infty e^{-\omega x} W^{(q)}(x) dx = \frac{1}{\phi(\omega) - q}, \quad \text{for } \omega > \eta(q),$$

and $W^{(q)}(x) = 0$ for $x < 0$.

In this paper, we frequently restrict ourselves to the case $q = 0$. In that case $W(\cdot) := W^{(0)}(\cdot)$, which is also often referred to as the scale function. In some special cases, the scale function can be explicitly determined. For instance, if the Lévy process is a compound Poisson process with negative drift, then $W(\cdot)$ can be related to the waiting time distribution in the M/G/1 queue. Also, in case of Brownian motion, $W(\cdot)$ has a tractable form. These examples are further discussed in Section 5. We also refer to [21] for some examples. In the remainder, the subscript i is added to the scale function when it is associated with exponent $\phi_i(\cdot)$, $i = 1, 2$, i.e., we write $W_i^{(q)}(\cdot)$ and $W_i(\cdot)$.

Now, consider the steady-state workload (denoted by V) of the reflected Lévy process. The formula for its LST is also known as the generalized Pollaczek-Khinchine formula and is presented in the following theorem, see e.g. [1, Corollary IX.3.4] or [8, 18].

Theorem 2.1. *Consider a Lévy process without negative jumps and with negative drift, i.e., $0 < \phi'(0) < \infty$. Then, for $\omega \geq 0$,*

$$\mathbb{E}e^{-\omega V} = \phi'(0) \frac{\omega}{\phi(\omega)}.$$

The distribution of the steady-state amount of work may be expressed in terms of the scale function. Using Definition 2.1 and partial integration, we obtain

$$\frac{\omega}{\phi(\omega)} = \omega \int_0^\infty e^{-\omega x} W(x) dx = W(0) + \int_{0+}^\infty e^{-\omega x} dW(x).$$

Hence, the transform in Theorem 2.1 may be readily inverted, providing

$$\mathbb{P}(V < x) = \phi'(0)W(x). \tag{2}$$

In fact, a similar result holds in case there is reflection at both 0 and some level $K > 0$. In the M/G/1 setting this model is often referred to as the finite dam. Note that reflection at K implies that ‘customers’ are admitted according to partial rejection; customers arriving at the system that cause an overflow over K are only partly accepted such that the workload equals K . Denote by V^K the steady-state workload. In [22] it was shown that, in case of a Lévy process without negative jumps, a similar proportionality result holds as for the finite dam, that is, for $x \in [0, K]$,

$$\mathbb{P}(V^K < x) = \frac{W(x)}{W(K)}. \tag{3}$$

Finally, in the model with a Poisson observer, we also need to determine the level of the process after an exponential time, see e.g. (1). Let T again be a generic exponential time with expectation $1/\xi$. Starting at v at time 0, the LST of the workload after an exponential time can be found in, e.g., [8, Theorem 4b] or [10, Theorem 2]:

Theorem 2.2. *For $\omega \geq 0$, we have*

$$\mathbb{E}[e^{-\omega Z_i^{(T)}} \mid Z_i^{(0)} = v] = \frac{\xi}{\xi - \phi_i(\omega)} \left(e^{-\omega v} - \omega \frac{e^{-\eta_i(\xi)v}}{\eta_i(\xi)} \right),$$

where $\eta_i(\xi)$ is the unique positive zero of $\xi - \phi_i(\omega)$.

Remark 2.1. The result of [8] is in fact in terms of the LST of the transient behavior of a reflected process, which is however directly related to the LST of the workload after an exponential time. This follows from the observation that $\mathbb{E}[e^{-\omega Z_i^{(T)}}]$ corresponds to ξ times the double transform of the transient workload behavior, where we take ξ for the parameter of the transform with respect to time. Thus, in the special case of the M/G/1 queue, the above result can also be obtained from, e.g., [23], Formula (2.62). Similar to [23, Section 2.3], we also have that $\mathbb{P}(Z_i^{(T)} = 0 \mid Z_i^{(0)} = v) = \xi e^{-\eta_i(\xi)v} / (\eta_i(\xi)r_i)$. \diamond

A useful relation between the steady-state workload during an exponential interval and the amount of work after an exponential time follows from PASTA, see also [10, Equation (3.4)]. In particular, for $i = 1, 2$,

$$\mathbb{E}[e^{-\omega Z_i^{(T)}} \mid Z_i^{(0)} = v] = \frac{1}{\mathbb{E}T} \mathbb{E} \left[\int_{s=0}^T e^{-\omega Z_i^{(s)}} ds \mid Z_i^{(0)} = v \right]. \quad (4)$$

The distribution of the workload after an exponential time starting from $v \geq 0$ can also be expressed in terms of scale functions. Denote $W_i^{(T)}(x; v) = \mathbb{P}(Z_i^{(T)} \leq x \mid Z_i^{(0)} = v)$ and define $\bar{W}_i^{(q)}(x) := \int_0^x W_i^{(q)}(y) dy$, with $i = 1, 2$. Applying Laplace inversion, we obtain from Theorem 2.2 that, for $x \geq 0$,

$$W_i^{(T)}(x; v) = W_i^{(\xi)}(x) \xi \frac{e^{-\eta_i(\xi)v}}{\eta_i(\xi)} - \xi \bar{W}_i^{(\xi)}(x - v) I(x \geq v), \quad (5)$$

where $I(\cdot)$ is the indicator function.

3 Solution procedure

In this section, we outline the solution procedure that can be generally used to determine the steady-state behavior of systems with two service rates or Lévy exponents. We apply this procedure to various models in Sections 4–6.

Denote by Z the random variable of interest (in the examples of the present paper corresponding to the workload), and define the LST of Z by

$$\zeta(\omega) := \int_0^\infty e^{-\omega x} d\mathbb{P}(Z < x). \quad (6)$$

The procedure for determining the distribution of Z builds upon techniques applied in [13, 17] and [15], p. 556, where the workload in an M/G/1 queue with continuously

adaptable service speed is studied. The formal outline presented below is based on [4]. The initial step is to determine two sets of equations for $\zeta(\omega)$. This step strongly depends on the specific model, but the general form of the equations can be found below. Since the procedure focuses on solving a set of two equations with special structure, we label the derivation of the proper equations as Step 0.

The basic algorithm to obtain $\mathbb{P}(Z < x)$ is as follows:

Step 0 Determine two sets of equations for $\zeta(\omega)$; derive (i) an equation involving $\zeta(\omega)$ and the incomplete LST $\int_K^\infty e^{-\omega x} d\mathbb{P}(Z < x)$:

$$\zeta(\omega) = F_1(\omega) + G_1(\omega) \int_K^\infty e^{-\omega x} d\mathbb{P}(Z < x), \quad (7)$$

and (ii) an equation involving $\zeta(\omega)$ and the incomplete LST $\int_0^K e^{-\omega x} d\mathbb{P}(Z < x)$:

$$\zeta(\omega) = F_2(\omega) + G_2(\omega) \int_0^K e^{-\omega x} d\mathbb{P}(Z < x), \quad (8)$$

for some functions $F_i(\cdot)$ and $G_i(\cdot)$, $i = 1, 2$.

Step 1 Rewrite Equation (7) such that $G_1(\omega) \int_K^\infty e^{-\omega x} d\mathbb{P}(Z < x)$ can be written as the sum of an LST with mass only on $[K, \infty)$ and an LST that does only depend on $\mathbb{P}(Z < x)$ for $x > K$ through a constant.

Step 2 Apply Laplace inversion to the reformulated Equation (7) resulting from Step 1, to determine $\mathbb{P}(Z < x)$ for $x \in (0, K]$.

Step 3 By Step 2, we may now calculate $\int_0^K e^{-\omega x} d\mathbb{P}(Z < x)$. Substitution in (8) then directly provides $\zeta(\omega)$. Applying Laplace inversion again, we determine $\mathbb{P}(Z < x)$ for $x > K$.

Step 4 The remaining constants may be found by normalization.

The easiest application of this solution procedure is to the M/G/1 dam where the service speed is continuously adapted based on the workload. This model was analyzed in, for instance, [13, 17] and [15, Section III.5.10]. We rederive their result in Subsection 4.1 to demonstrate the basic features of each step in the solution procedure. In fact, the results remain valid for dams with nondecreasing Lévy input and two output rates, see Remark 4.1 below.

The solution procedure as described in this section is specifically formulated to determine the steady-state behavior of a reflected Lévy process with two Lévy exponents and only positive jumps. However, we believe that a similar procedure can also be applied to derive steady-state results and first-exit probabilities for spectrally one-sided Lévy processes with two exponents. Whereas Lévy processes with only positive jumps are of importance in the study of queueing systems, Lévy processes with only negative jumps are of particular interest in connection with applications in finance, see e.g. [1, Chapter XIV] or [2, 24] and references therein.

For the class of Lévy processes with only negative jumps, the solution procedure has to be slightly adapted. Intuitively, it should be clear that the process on the interval (K, ∞) is

only affected by the behavior of the process on $(0, K]$ through a constant. Hence, in Step 2 we should first apply Laplace inversion on (K, ∞) and then use that result in Step 3 to determine the behavior on $(0, K]$, i.e., Steps 2 and 3 are reversed. Accordingly, in Step 1, $G_2(\omega) \int_0^K e^{-\omega x} d\mathbb{P}(Z < x)$ should be rewritten as the sum of an LST with only mass on $(0, K]$ and an LST that does only depend on $\mathbb{P}(Z < x)$ for $x \leq K$ through a constant.

4 Dam processes and M/G/1 queues

In this section, we consider (a generalization of) the classical M/G/1 dam. First, we rederive the steady-state workload distribution for the M/G/1 queue with two service speeds. Because it is the easiest application of the solution procedure, giving insight into the fundamentals of each step, we have included the derivation of this special case. Second, we consider a generalization of the M/G/1 queue, where the arrival rate and service requirement distribution may also depend on the amount of work present. Since we restrict ourselves to M/G/1-type models in this section, the Lévy exponent reduces to $\phi_i(\omega) = r_i\omega - \lambda_i + \lambda_i\beta_i(\omega)$, for $i = 1, 2$. We interchangeably use the expressions $\phi_i(\omega)$ and $r_i\omega - \lambda_i + \lambda_i\beta_i(\omega)$ depending on the most convenient representation. Define $\rho_i := \lambda_i\beta_i/r_i$. The stability condition for this case reads $\rho_2 < 1$. Before considering the two separate models, we start by deriving two sets of equations for the general M/G/1 model.

Step 0: Determining equations

For the M/G/1-type models, we consider downcrossings of level K as regeneration epochs. Let $Z^{(0)} = K$. Defining $\tau_1 := \inf\{t > 0 : Z^{(t)} \geq K\}$ as the first upcrossing of level K and $\tau_2 := \inf\{t \geq \tau_1 : Z^{(t)} = K\}$ as the subsequent downcrossing, it is easily seen that this model can be interpreted as a regenerative alternating Lévy process as described in Appendix A.

Since there is no reflection during the second part of the cycle, we have $\mathbb{E}L_2^{(\tau_2 - \tau_1)} = 0$. Note that $\mathbb{E}e^{-\omega Z^{(\tau_2)}} = \mathbb{E}e^{-\omega Z^{(0)}} = e^{-\omega K}$. Using the theory of regenerative processes, we may also write

$$\begin{aligned} \int_0^K e^{-\omega x} d\mathbb{P}(Z < x) &= \frac{1}{\mathbb{E}\tau_2} \mathbb{E}\left[\int_{s=0}^{\tau_1} e^{-\omega Z^{(s)}} ds\right], \\ \int_K^\infty e^{-\omega x} d\mathbb{P}(Z < x) &= \frac{1}{\mathbb{E}\tau_2} \mathbb{E}\left[\int_{s=\tau_1}^{\tau_2} e^{-\omega Z^{(s)}} ds\right]. \end{aligned} \quad (9)$$

Combining the above, we directly obtain two equations for $\zeta(\cdot)$ from Lemma A.1 (see Appendix A):

$$\zeta(\omega) = \frac{\mathbb{E}L_1^{(\tau_1)}}{\mathbb{E}\tau_2} \frac{\omega}{\phi_2(\omega)} + \frac{\phi_2(\omega) - \phi_1(\omega)}{\phi_2(\omega)} \int_0^K e^{-\omega x} d\mathbb{P}(Z < x), \quad (10)$$

and

$$\zeta(\omega) = \frac{\mathbb{E}L_1^{(\tau_1)}}{\mathbb{E}\tau_2} \frac{\omega}{\phi_1(\omega)} + \frac{\phi_1(\omega) - \phi_2(\omega)}{\phi_1(\omega)} \int_K^\infty e^{-\omega x} d\mathbb{P}(Z < x). \quad (11)$$

Hence, (10) and (11) correspond to Equations (8) and (7), respectively, where $F_i(\omega) = \mathbb{E}L_1^{(\tau_1)}\omega/(\phi_i(\omega)\mathbb{E}\tau_2)$ and $G_i(\omega) = (\phi_i(\omega) - \phi_{3-i}(\omega))/\phi_i(\omega)$, $i = 1, 2$.

4.1 M/G/1 dam: Change of drift

To obtain insight into the solution procedure, we start by assuming that $\lambda_1 = \lambda_2 = \lambda$ and $\beta_1(\omega) = \beta_2(\omega) = \beta(\omega)$. This concerns the classical dam model or M/G/1 queue where only the service speed is adapted, see also, e.g., [13, 17] and [15, Section III.5.10]. We also briefly consider more general dams with a nondecreasing Lévy input process (a subordinator) in Remark 4.1.

Step 1: Rewriting (11)

In fact, this step can be omitted using the fact that the term $(\phi_1(\omega) - \phi_2(\omega))/\phi_1(\omega)$ in (11) can be reduced to $(r_1 - r_2)\omega/\phi_1(\omega)$. Hence, Equation (11) reads

$$\zeta(\omega) = \frac{\mathbb{E}L_1^{(\tau_1)}}{\mathbb{E}\tau_2} \frac{\omega}{\phi_1(\omega)} + (r_1 - r_2) \frac{\omega}{\phi_1(\omega)} \int_K^\infty e^{-\omega x} d\mathbb{P}(Z < x). \quad (12)$$

Step 2: Workload distribution on $(0, K]$

The first term on the rhs of (12) can be readily inverted using Theorem 2.1 and Definition 2.1, yielding the scale function $W_1(\cdot)$. For details on the exact form of $W_1(\cdot)$ in the M/G/1 case, we refer to Subsection 5.2.

For the second term, we note that it involves a convolution, corresponding to the sum of two functions. Since the Laplace inverse of $\omega/\phi_1(\omega)$ has mass on $[0, \infty)$ and $\int_K^\infty e^{-\omega x} d\mathbb{P}(Z < x)$ is (up to a constant) the LST of a function with mass on $[K, \infty)$, its convolution has only mass on $[K, \infty)$. Hence, for $x \in (0, K]$, the Laplace inverse of $\zeta(\omega)$ is given by

$$\mathbb{P}(Z < x) = \frac{\mathbb{E}L_1^{(\tau_1)}}{\mathbb{E}\tau_2} W_1(x). \quad (13)$$

Step 3: Workload distribution on (K, ∞)

In this step, we apply Laplace inversion to (10) to obtain the distribution of Z on (K, ∞) . The inverse of the first transform on the rhs of (10) is directly given by the scale function $W_2(\cdot)$. For the second term, we use that the term $(\phi_2(\omega) - \phi_1(\omega))/\phi_2(\omega)$ can be simplified to $(r_2 - r_1)\omega/\phi_2(\omega)$, which is the LST of the scale function $W_2(\cdot)$ (times the constant $r_2 - r_1$). After this simplification, we note that this second term involves an incomplete convolution. Using the result for the distribution of Z on $(0, K]$, i.e. (13), we obtain, for $x > K$,

$$\mathbb{P}(Z < x) = \frac{\mathbb{E}L_1^{(\tau_1)}}{\mathbb{E}\tau_2} \left(W_2(x) + (r_2 - r_1) \int_0^K W_2(x - y) dW_1(y) \right). \quad (14)$$

Step 4: Determination of the constant

Finally, we determine the constant using normalization and (13). More specifically, letting $\omega \downarrow 0$ in (10) and applying (13) with $x = K$, we obtain

$$\frac{\mathbb{E}L_1^{(\tau_1)}}{\mathbb{E}\tau_2} = \frac{r_2(1 - \rho_2)}{1 + (r_2 - r_1)W_1(K)}. \quad (15)$$

Using the specific form of $W_i(\cdot)$ given in Subsection 5.2, it may be readily checked that the results coincide with the results in [13, 17] and [15], p. 556.

Remark 4.1. The above results also hold for more general dam processes, see e.g. [11, 14, 26] or [24, Chapter 3]. Assume that the input of the dam is a nondecreasing Lévy process (a subordinator) and the output rate depends on the content of the dam as described above. We exclude the degenerate case that a possible deterministic drift of the input process is larger than or equal to r_1 . It then follows directly from [26] that $0 < \tau_1 < \tau_2$ (almost surely) and the content process can be considered as a regenerative process as we did in Step 0 of the solution procedure. Hence, the steady-state distribution as given by (13), (14), and (15) remains valid. As indicated, in the M/G/1 case of a finite jump rate the scale functions $W_i(\cdot)$, $i = 1, 2$, are given in Subsection 5.2. The form of the scale function in case of an infinite jump rate is given in [14].

Finally, we note that the steady-state solution of general dam processes with release rate function $r(\cdot)$ has been given in [11]. However, their results are in terms of infinite sums of iterated kernels, which does not provide the tractable form described above. \diamond

4.2 M/G/1 dam: General case

We now consider the general M/G/1 queue with arrival rates λ_i , service requirement LSTs $\beta_i(\cdot)$, and service speeds r_i , $i = 1, 2$, and apply the procedure to derive the steady-state workload distribution.

Step 1: Rewriting (11)

Using the fact that $\phi_i(\omega) = r_i\omega - \lambda_i + \lambda_i\beta_i(\omega)$, $i = 1, 2$, we may rewrite the fraction of Lévy exponents as follows:

$$\begin{aligned} \frac{\phi_1(\omega) - \phi_2(\omega)}{\phi_1(\omega)} &= \frac{(r_1 - r_2)\omega + \frac{\lambda_2}{\lambda_1}(-\lambda_1\beta_2(\omega) + \lambda_1\beta_1(\omega))}{\phi_1(\omega)} + \left(1 - \frac{\lambda_2}{\lambda_1}\right) \frac{-\lambda_1 + \lambda_1\beta_1(\omega)}{\phi_1(\omega)} \\ &= -r_2 \frac{\omega}{\phi_1(\omega)} + \frac{\lambda_2}{\lambda_1} \frac{r_1\omega - \lambda_1\beta_2(\omega) + \lambda_1\beta_1(\omega)}{r_1\omega - \lambda_1 + \lambda_1\beta_1(\omega)} + 1 - \frac{\lambda_2}{\lambda_1}. \end{aligned}$$

Although this form is rather involved, we have decomposed it into three familiar terms. Specifically, the first one is related to the standard M/G/1 queue, the second one corresponds to an M/G/1 queue with exceptional first service (up to a constant), see e.g. [28, 29] or Example A.1, and the third is just a constant. Substituting the above in (11) provides

$$\begin{aligned} \zeta(\omega) &= \frac{\mathbb{E}L_1^{(\tau_1)}}{\mathbb{E}\tau_2} \frac{\omega}{\phi_1(\omega)} + \left(-r_2 \frac{\omega}{\phi_1(\omega)} \right. \\ &\quad \left. + \frac{\lambda_2}{\lambda_1} \frac{r_1\omega - \lambda_1\beta_2(\omega) + \lambda_1\beta_1(\omega)}{r_1\omega - \lambda_1 + \lambda_1\beta_1(\omega)} + 1 - \frac{\lambda_2}{\lambda_1} \right) \int_K^\infty e^{-\omega x} d\mathbb{P}(Z < x). \end{aligned} \tag{16}$$

Step 2: Workload distribution on $(0, K]$

The first term on the rhs of (16) can be readily inverted using Theorem 2.1 and Definition 2.1 yielding the scale function $W_1(\cdot)$. For details on the exact form of $W_1(\cdot)$ in the M/G/1 case, we refer to Subsection 5.2.

For the remaining terms, we apply the decomposition of Step 1 and consider each term separately. Crucial is the observation that $\int_K^\infty e^{-\omega x} d\mathbb{P}(Z < x)$ is the transform of a function with mass on $[K, \infty)$. For the first decomposed term

$$\frac{\omega}{\phi_1(\omega)} \int_K^\infty e^{-\omega x} d\mathbb{P}(Z < x),$$

we note that this corresponds to the convolution of two functions. Because the inverse of $\omega/\phi_1(\omega)$ has mass on $[0, \infty)$ and $\int_K^\infty e^{-\omega x} d\mathbb{P}(Z < x)$ is the transform of a function with mass on $[K, \infty)$, this convolution has mass on $[K, \infty)$.

The same argument holds for the second decomposed term

$$\frac{r_1\omega - \lambda_1\beta_2(\omega) + \lambda_1\beta_1(\omega)}{r_1\omega - \lambda_1 + \lambda_1\beta_1(\omega)} \int_K^\infty e^{-\omega x} d\mathbb{P}(Z < x), \quad (17)$$

interpreting the ratio in terms of an M/G/1 queue with exceptional first service requirement. Specifically, the first transform in (17) corresponds to the Laplace transform of the workload in such a queue (see for instance [28, 29] or Example A.1) and thus has mass on $[0, \infty)$. The convolution with $\mathbb{P}(Z < x)$ on $[K, \infty)$ clearly has no mass on $[0, K)$. For the final term it is readily seen that it has only mass on $[K, \infty)$.

Summarizing the above we obtain, for $x \in (0, K]$,

$$\mathbb{P}(Z < x) = \frac{\mathbb{E}L_1^{(\tau_1)}}{\mathbb{E}\tau_2} W_1(x), \quad (18)$$

which is proportional to regular M/G/1 behavior with $\phi_1(\omega) = r_1\omega - \lambda_1 + \lambda_1\beta_1(\omega)$.

Step 3: Workload distribution on (K, ∞)

We apply (10) and a similar decomposition as in Step 1 to obtain the workload distribution on (K, ∞) . In particular, using the same calculations as in Step 1, (10) may be equivalently written as

$$\begin{aligned} \zeta(\omega) = & \frac{\mathbb{E}L_1^{(\tau_1)}}{\mathbb{E}\tau_2} \frac{\omega}{\phi_2(\omega)} + \left(-r_1 \frac{\omega}{\phi_2(\omega)} \right. \\ & \left. + \frac{\lambda_1}{\lambda_2} \frac{r_2\omega - \lambda_2\beta_1(\omega) + \lambda_2\beta_2(\omega)}{r_2\omega - \lambda_2 + \lambda_2\beta_2(\omega)} + 1 - \frac{\lambda_1}{\lambda_2} \right) \int_0^K e^{-\omega x} d\mathbb{P}(Z < x). \end{aligned} \quad (19)$$

Since we have determined $\mathbb{P}(Z < x)$ for $x \in (0, K]$, we next apply Laplace inversion to each of the above terms separately. Note that the inverse of the first term is directly given by the scale function $W_2(\cdot)$ (again, see Subsection 5.2 for the precise form of $W_2(x)$). The second term involves the convolution of $W_2(\cdot)$ and the distribution of Z on $(0, K]$, giving $\int_0^K W_2(x-y) dW_1(y)$ times a constant.

For the third transform, we introduce $W_2^{\text{exc}}(\cdot)$ as the distribution of the workload in an M/G/1 queue with service rate r_2 , arrival rate λ_2 , and generic service requirement B_2 , but with exceptional first service B_1 in a busy period. Similar to the second term we obtain the convolution $\int_0^K W_2^{\text{exc}}(x-y) dW_1(y)$ up to a constant. (For the constant, multiply and divide by $(1 - \rho_2)/(1 + \lambda_2\beta_1/r_2 - \rho_2)$.)

The final term corresponds to the transform of a function with mass only on $(0, K]$. Hence, for $x > K$, the inverse only appears as the constant $\mathbb{P}(Z < K)$.

Summarizing the above we have, for $x > K$,

$$\begin{aligned} \mathbb{P}(Z < x) = & \frac{\mathbb{E}L_1^{(\tau_1)}}{\mathbb{E}\tau_2} \left(W_2(x) - r_1 \int_0^K W_2(x-y) dW_1(y) \right. \\ & \left. + \frac{\lambda_1}{\lambda_2} \frac{1 + \frac{\lambda_2\beta_1}{r_2} - \rho_2}{1 - \rho_2} \int_0^K W_2^{\text{exc}}(x-y) dW_1(y) + \left(1 - \frac{\lambda_1}{\lambda_2} \right) W_1(K) \right) \end{aligned} \quad (20)$$

Step 4: Determination of the constant

Again, the remaining constant can be determined by normalization and (18). In particular, letting $\omega \downarrow 0$ in (10) and using (18) with $x = K$, we obtain

$$\frac{\mathbb{E}L_1^{(\tau_1)}}{\mathbb{E}\tau_2} = \frac{r_2(1 - \rho_2)}{1 + (r_2(1 - \rho_2) - r_1(1 - \rho_1))W_1(K)},$$

where the specific form of $W_1(\cdot)$ is given in Subsection 5.2. This completes the analysis of the general M/G/1 queues with continuously adaptable Lévy exponents.

5 Lévy processes with Poisson observer

In this section, we consider the reflected Lévy process where the Lévy exponent is adapted at Poisson instants, see Section 2. To determine the steady-state workload distribution, we use Steps 0–4 as outlined in Section 3. In particular, Steps 0 and 1 are obtained using a direct approach. Depending on the form of $(\phi_i(\omega) - \phi_{3-i}(\omega))/\phi_i(\omega)$, $i = 1, 2$, it is possible to give a direct and intuitive derivation of Steps 2–4, leading to intuitively appealing expressions for the steady-state workload distribution in various special cases. In Subsection 5.1, we consider the special case of a change of drift. The M/G/1 case (i.e. compound Poisson with a negative drift) is addressed in Subsection 5.2 and the case of Brownian motion is the subject of Subsection 5.3.

In general, it is not possible to rewrite $(\phi_i(\omega) - \phi_{3-i}(\omega))/\phi_i(\omega)$, $i = 1, 2$, into tractable terms. The steady-state workload distribution for this general case can be found in Section 6.

In the following lemma we present the two equations for $\zeta(\cdot)$ as outlined in the procedure of Section 3. First we introduce the constant Q :

$$Q := \xi \int_0^K \frac{e^{-\eta_1(\xi)x}}{\eta_1(\xi)} d\mathbb{P}(Z < x) + \xi \int_K^\infty \frac{e^{-\eta_2(\xi)x}}{\eta_2(\xi)} d\mathbb{P}(Z < x). \quad (21)$$

Lemma 5.1. $\zeta(\omega)$ satisfies the following two equations, for $\text{Re } \omega \geq 0$,

$$\begin{aligned} \zeta(\omega) &= Q \frac{\omega}{\phi_1(\omega)} \\ &+ \frac{\phi_1(\omega) - \phi_2(\omega)}{\phi_1(\omega)} \frac{\xi}{\xi - \phi_2(\omega)} \left[\int_K^\infty e^{-\omega x} d\mathbb{P}(Z < x) - \omega \int_K^\infty \frac{e^{-\eta_2(\xi)x}}{\eta_2(\xi)} d\mathbb{P}(Z < x) \right], \end{aligned} \quad (22)$$

and

$$\begin{aligned} \zeta(\omega) &= Q \frac{\omega}{\phi_2(\omega)} \\ &+ \frac{\phi_2(\omega) - \phi_1(\omega)}{\phi_2(\omega)} \frac{\xi}{\xi - \phi_1(\omega)} \left[\int_0^K e^{-\omega x} d\mathbb{P}(Z < x) - \omega \int_0^K \frac{e^{-\eta_1(\xi)x}}{\eta_1(\xi)} d\mathbb{P}(Z < x) \right]. \end{aligned} \quad (23)$$

Here, the constant Q is defined in (21) above and $\eta_i(\xi)$, $i = 1, 2$, are defined in Theorem 2.2.

Proof. Starting point is the recursion relation (1) between V_n and V_{n+1} , where V_n denotes the workload just before the n th arrival instant of the observer. To determine the LST of V_{n+1} , we condition on V_n , use (1), and apply Theorem 2.2, to obtain

$$\begin{aligned}\mathbb{E} [e^{-\omega V_{n+1}}] &= \int_0^\infty \mathbb{E} [e^{-\omega V_{n+1}} | V_n = x] d\mathbb{P}(V_n < x) \\ &= \frac{\xi}{\xi - \phi_1(\omega)} \left(\int_0^K e^{-\omega x} d\mathbb{P}(V_n < x) - \omega \int_0^K \frac{e^{-\eta_1(\xi)x}}{\eta_1(\xi)} d\mathbb{P}(V_n < x) \right) \\ &\quad + \frac{\xi}{\xi - \phi_2(\omega)} \left(\int_K^\infty e^{-\omega x} d\mathbb{P}(V_n < x) - \omega \int_K^\infty \frac{e^{-\eta_2(\xi)x}}{\eta_2(\xi)} d\mathbb{P}(V_n < x) \right).\end{aligned}$$

To analyze the steady-state behavior of V_n , we let $n \rightarrow \infty$. Using PASTA, it follows that the steady-state distribution of V_n equals the distribution of the workload at an arbitrary instant $\mathbb{P}(Z < x)$. Using the above, we may obtain two alternative equations for $\zeta(\omega)$ as described in Step 0 of the general procedure. That is, one equation involving the incomplete LST $\int_0^\infty e^{-\omega x} d\mathbb{P}(Z < x)$ and one equation involving the incomplete LST $\int_0^K e^{-\omega x} d\mathbb{P}(Z < x)$.

Note that, for $i = 1, 2$,

$$1 - \frac{\xi}{\xi - \phi_i(\omega)} = \frac{-\phi_i(\omega)}{\xi - \phi_i(\omega)}. \quad (24)$$

Then, for the first equation, we deduce after some basic manipulations, using (6) and (24) for $i = 1$, and dividing by the term on the rhs of Equation (24) with $i = 1$, that

$$\begin{aligned}\zeta(\omega) &= \frac{\omega}{\phi_1(\omega)} \xi \int_0^K \frac{e^{-\eta_1(\xi)x}}{\eta_1(\xi)} d\mathbb{P}(Z < x) + \frac{\omega}{\phi_1(\omega)} \frac{\xi - \phi_1(\omega)}{\xi - \phi_2(\omega)} \xi \int_K^\infty \frac{e^{-\eta_2(\xi)x}}{\eta_2(\xi)} d\mathbb{P}(Z < x) \\ &\quad + \frac{1}{\phi_1(\omega)} \left(1 - \frac{\xi - \phi_1(\omega)}{\xi - \phi_2(\omega)} \right) \xi \int_K^\infty e^{-\omega x} d\mathbb{P}(Z < x) \\ &= \frac{\omega}{\phi_1(\omega)} \left[\xi \int_0^K \frac{e^{-\eta_1(\xi)x}}{\eta_1(\xi)} d\mathbb{P}(Z < x) + \xi \int_K^\infty \frac{e^{-\eta_2(\xi)x}}{\eta_2(\xi)} d\mathbb{P}(Z < x) \right] \\ &\quad + \frac{\phi_1(\omega) - \phi_2(\omega)}{\phi_1(\omega)} \frac{\xi}{\xi - \phi_2(\omega)} \left[\int_K^\infty e^{-\omega x} d\mathbb{P}(Z < x) - \omega \int_K^\infty \frac{e^{-\eta_2(\xi)x}}{\eta_2(\xi)} d\mathbb{P}(Z < x) \right].\end{aligned} \quad (25)$$

The second equation can be derived in a similar fashion. In particular, using (6) and (24) for $i = 2$, and dividing by the term on the rhs of Equation (24) with $i = 2$, we obtain after some calculations that

$$\begin{aligned}\zeta(\omega) &= \frac{\omega}{\phi_2(\omega)} \frac{\xi - \phi_2(\omega)}{\xi - \phi_1(\omega)} \xi \int_0^K \frac{e^{-\eta_1(\xi)x}}{\eta_1(\xi)} d\mathbb{P}(Z < x) + \frac{\omega}{\phi_2(\omega)} \xi \int_K^\infty \frac{e^{-\eta_2(\xi)x}}{\eta_2(\xi)} d\mathbb{P}(Z < x) \\ &\quad + \frac{1}{\phi_2(\omega)} \left(1 - \frac{\xi - \phi_2(\omega)}{\xi - \phi_1(\omega)} \right) \xi \int_0^K e^{-\omega x} d\mathbb{P}(Z < x) \\ &= \frac{\omega}{\phi_2(\omega)} \left[\xi \int_0^K \frac{e^{-\eta_1(\xi)x}}{\eta_1(\xi)} d\mathbb{P}(Z < x) + \xi \int_K^\infty \frac{e^{-\eta_2(\xi)x}}{\eta_2(\xi)} d\mathbb{P}(Z < x) \right] \\ &\quad + \frac{\phi_2(\omega) - \phi_1(\omega)}{\phi_2(\omega)} \frac{\xi}{\xi - \phi_1(\omega)} \left[\int_0^K e^{-\omega x} d\mathbb{P}(Z < x) - \omega \int_0^K \frac{e^{-\eta_1(\xi)x}}{\eta_1(\xi)} d\mathbb{P}(Z < x) \right].\end{aligned} \quad (26)$$

Combining (21) with (25) and (26) yields (22) and (23) in Lemma 5.1, respectively. This completes the proof. \square

Step 1: Rewriting (22)

In this step we rewrite the second term in the right-hand side of Equation (22) into the sum of two LST; (i) an LST with mass only on $[K, \infty)$, and (ii) an LST that only depends on the distribution of Z through a constant. This step is based on the following intuition. Consider the workload process continuous in time and note that a Lévy exponent of $\phi_2(\cdot)$ implies that the workload at the previous observer-arrival instant was larger than K . When the observer arrives finding a workload smaller than K , then the exponent is set to $\phi_1(\cdot)$. Thus, periods with exponent $\phi_2(\cdot)$ and workloads smaller than K are always initiated with an arriving observer finding a workload larger than K followed by a downcrossing of level K before the next arrival instant of the observer. Due to the lack-of-memory property of the Poisson arrival process, the remaining interarrival time of the observer is still exponential at a downcrossing of K . Hence, the precise distribution of Z on (K, ∞) does only affect the distribution of Z on $[0, K]$ through a constant.

Applying the above intuition, we condition on $V_n > K$ and use Theorem 2.2 to obtain

$$\begin{aligned} & \int_K^\infty \mathbb{E}[e^{-\omega V_{n+1}} | V_n = x] d\mathbb{P}(V_n < x) \\ &= \frac{\xi}{\xi - \phi_2(\omega)} \left[\int_K^\infty e^{-\omega x} d\mathbb{P}(V_n < x) - \omega \int_K^\infty \frac{e^{-\eta_2(\xi)x}}{\eta_2(\xi)} d\mathbb{P}(V_n < x) \right]. \end{aligned}$$

Observe that the rhs corresponds to the final part of the second term on the rhs of (22) in case $n \rightarrow \infty$. Let $\tau_K := \inf\{t \geq 0 : Z^{(t)} = K\}$ be the first hitting time of K . Recall that T is exponentially distributed with mean $1/\xi$. Using (1) in the first step and the lack-of-memory property of Lévy processes and the exponential interarrival time distribution of the observer in the second step, we deduce that

$$\begin{aligned} & \int_K^\infty \mathbb{E}[e^{-\omega V_{n+1}} | V_n = x] d\mathbb{P}(V_n < x) \\ &= \int_K^\infty \mathbb{E}[e^{-\omega Z_2^{(T)}} I(T < \tau_K) | Z_2^{(0)} = x] d\mathbb{P}(V_n < x) \\ & \quad + \int_K^\infty \mathbb{E}[e^{-\omega Z_2^{(T)}} I(T \geq \tau_K) | Z_2^{(0)} = x] d\mathbb{P}(V_n < x) \\ &= \int_K^\infty \mathbb{E}[e^{-\omega Z_2^{(T)}} I(T < \tau_K) | Z_2^{(0)} = x] d\mathbb{P}(Z < x) + P_{\downarrow K} \mathbb{E}[e^{-\omega Z_2^{(T)}} | Z_2^{(0)} = K], \end{aligned} \tag{27}$$

where in the final step we let $n \rightarrow \infty$ and use PASTA. Here, $P_{\downarrow K}$ is the probability of downcrossing of level K before an exponential time starting from $x > K$ according to the distribution $\mathbb{P}(Z < x)$. Using results on LST of first-exit times, see e.g. [20], it may be checked that

$$P_{\downarrow K} = \xi e^{\eta_2(\xi)K} \int_K^\infty e^{-\eta_2(\xi)x} d\mathbb{P}(Z < x). \tag{28}$$

Observe that the first term on the rhs of (27) corresponds to an LST with mass on $[K, \infty)$, and the second term of (27) only depends on the distribution of Z on (K, ∞) through a

constant. Combining the above, Equation (22) in Lemma 5.1 can be written as

$$\begin{aligned}\zeta(\omega) &= Q \frac{\omega}{\phi_1(\omega)} \\ &\quad + \frac{\phi_1(\omega) - \phi_2(\omega)}{\phi_1(\omega)} \int_K^\infty \mathbb{E}[e^{-\omega Z_2^{(T)}} I(T < \tau_K) \mid Z_2^{(0)} = x] d\mathbb{P}(Z < x) \\ &\quad + \frac{\phi_1(\omega) - \phi_2(\omega)}{\phi_1(\omega)} P_{\downarrow K} \mathbb{E}[e^{-\omega Z_2^{(T)}} \mid Z_2^{(0)} = K],\end{aligned}\tag{29}$$

where $\mathbb{E}[e^{-\omega Z_2^{(T)}} \mid Z_2^{(0)} = K]$ may be directly obtained from Theorem 2.2. This completes Step 1.

We note that $(\phi_i(\omega) - \phi_{3-i}(\omega))/\phi_i(\omega)$, $i = 1, 2$, can not be reduced in general. This is the reason for applying alternating Lévy processes as defined in Appendix A for the general case, see Section 6. However, in several important special cases, as in Subsections 5.1–5.3 below, the derivation is more insightful leading to intuitively appealing expressions.

5.1 Poisson observer: Change of drift

In this subsection, we consider the important special case of a change of drift, i.e., we assume that $\phi_i(\omega) = r_i\omega + \tilde{\phi}(\omega)$, with $i = 1, 2$, for some Lévy exponent $\tilde{\phi}(\omega)$. We then have

$$\frac{\phi_1(\omega) - \phi_2(\omega)}{\phi_1(\omega)} = (r_1 - r_2) \frac{\omega}{\phi_1(\omega)},\tag{30}$$

where the rhs can be readily inverted using the scale function $W_1(\cdot)$.

Step 2: Workload distribution on $(0, K]$

In this step, we apply Laplace inversion to each of the three terms on the rhs of (29) separately. The inverse of the first term can be directly obtained from Definition 2.1, see also Theorem 2.1 and (2), yielding $QW_1(\cdot)$.

Using (30), the second transform reduces to

$$(r_1 - r_2) \frac{\omega}{\phi_1(\omega)} \int_K^\infty \mathbb{E}[e^{-\omega Z_2^{(T)}} I(T < \tau_K) \mid Z_2^{(0)} = x] d\mathbb{P}(Z < x).$$

We note that this involves the product of two LST's, corresponding to the convolution of two functions. Since the first function has mass on $[0, \infty)$ and the second only on $[K, \infty)$, its convolution has no mass on $[0, K)$.

The third transform on the rhs of (29) also corresponds to a convolution of two functions. It may be readily verified that the Laplace inverse of $(\omega/\phi_1(\omega))\mathbb{E}[e^{-\omega Z_2^{(T)}} \mid Z_2^{(0)} = K]$ reads

$$W_1(x) * W_2^{(T)}(x; K),$$

where $*$ denotes a convolution and $W_2^{(T)}(x; K)$ is given by (5).

Combining the above, we have for $x \in (0, K]$,

$$\mathbb{P}(Z < x) = QW_1(x) + (r_1 - r_2)P_{\downarrow K} \int_0^x W_2^{(T)}(x - y; K) dW_1(y).\tag{31}$$

Step 3: Workload distribution on (K, ∞)

Substituting the results from Step 1 into (23) provides $\zeta(\omega)$. In this step we use Equation (23) directly to derive the workload distribution on (K, ∞) .

For the first term on the rhs of (23) the Laplace inverse is directly given by the scale function $W_2(\cdot)$. For the special case of a change of drift, the second term reduces to

$$(r_2 - r_1) \frac{\omega}{\phi_2(\omega)} \frac{\xi}{\xi - \phi_1(\omega)} \left[\int_0^K e^{-\omega x} d\mathbb{P}(Z < x) - \omega \int_0^K \frac{e^{-\eta_1(\xi)x}}{\eta_1(\xi)} d\mathbb{P}(Z < x) \right].$$

Inversion of the above term corresponds to a convolution of the scale function $W_2(\cdot)$ (given by $\omega/\phi_2(\omega)$) and a second function. For the latter one, we note that, by applying Theorem 2.2, the inverse is given by

$$\tilde{W}^{(T)}(x) := \int_0^K W_1^{(T)}(x; y) d\mathbb{P}(Z < y),$$

that is, the amount of work of a reflected Lévy process with exponent $\phi_1(\omega)$ after an exponential time, starting according to $\mathbb{P}(Z < x)$. Since we have determined the distribution of Z on $[0, K]$, we also have found $\tilde{W}^{(T)}(\cdot)$ (although its precise form may be rather involved).

Combining the above, we obtain, for $x > K$,

$$\mathbb{P}(Z < x) = QW_2(x) + (r_2 - r_1) \int_0^x W_2(x - y) d\tilde{W}^{(T)}(y).$$

Step 4: Determination of the constants

The remaining constants can be determined in a similar fashion as in [4]. First, the constants $\int_0^K e^{-\eta_1(\xi)x} d\mathbb{P}(Z < x)$ and $\int_K^\infty e^{-\eta_2(\xi)x} d\mathbb{P}(Z < x)$ can be expressed in terms of Q and $\mathbb{P}(Z < K)$ using Equations (21) and (31). (For the latter equation, multiply by $\exp(-\eta_1(\xi)x)$ and integrate over the interval $[0, K]$.)

Letting $\omega \downarrow 0$ in (23) and applying l'Hôpital's rule gives

$$Q = \phi_2'(0) - (\phi_2'(0) - \phi_1'(0)) \mathbb{P}(Z < K).$$

Moreover, substituting $x = K$ in (31) provides

$$\mathbb{P}(Z < K) = QW_1(K) + (r_1 - r_2)P_{\downarrow K} \int_0^K W_2^{(T)}(K - y; K) dW_1(y),$$

with $P_{\downarrow K}$ given in (28). These equations determine the remaining constants Q and $\mathbb{P}(Z < K)$.

5.2 M/G/1 queues

In this subsection we assume that $\phi_i(\omega) = r_i\omega - \lambda_i + \lambda_i\beta_i(\omega)$, for $i = 1, 2$. Recall that the stability condition reads $\rho_2 < 1$, with $\rho_i = \lambda_i\beta_i/r_i$, $i = 1, 2$. In this case, the scale function $W_i(\cdot)$ can be explicitly determined (we note that for $\rho_1 \geq 1$ the explicit form is rather involved).

To describe the scale function $W_i(\cdot)$, we define

$$H_i(x) := \beta_i^{-1} \int_0^x (1 - B_i(y)) dy$$

as the stationary residual service requirement distribution of a generic service requirement B_i , $i = 1, 2$. In case $\rho_i < 1$, it is well-known that

$$W_i(x) = \frac{1}{r_i} \sum_{n=0}^{\infty} \rho_i^n H_i^{n*}(x),$$

which is directly related to the steady-state workload distribution in an M/G/1 queue. In fact, it may be checked that the LST of $(1 - \rho_i)r_i W_i(\cdot)$ equals

$$\mathbb{E}e^{-\omega Z} = \frac{(1 - \rho_i)r_i\omega}{r_i\omega - \lambda_i + \lambda_i\beta_i(\omega)},$$

which is given by Theorem 2.1. Since we assumed that $\rho_2 < 1$ for stability, this directly provides $W_2(\cdot)$. However, we allow that $\rho_1 \geq 1$. To obtain $W_1(\cdot)$, we apply arguments of [12, 13]. Let $\delta_1 = 0$ for $\rho_1 \leq 1$ and for $\rho_1 > 1$ let δ_1 be the unique positive zero of the function

$$\int_0^{\infty} e^{-xy} \rho_1 dH_1(y) - 1.$$

Then, for $x > 0$, define

$$L(x) := \int_0^x e^{-\delta_1 y} \rho_1 dH_1(y),$$

and

$$W_1(x) := \frac{1}{r_1} \int_{0^-}^x e^{\delta_1 y} d \left\{ \sum_{n=0}^{\infty} L^{n*}(y) \right\},$$

where $L^{n*}(\cdot)$ denotes the n -fold convolution of $L(\cdot)$ with itself. It may be checked that, as in [12, 13], the Laplace transform of $W_1(\cdot)$ equals $1/\phi_1(\omega)$.

As mentioned, for $\rho_i < 1$, $(1 - \rho_i)r_i W_i(\cdot)$ corresponds to the steady-state waiting-time distribution in an M/G/1 queue with service speed r_i . In case $\rho_1 \geq 1$, $W_1(\cdot)$ may be interpreted in terms of a dam with release rate r_1 and capacity K . Specifically, the stationary waiting-time distribution for such a dam equals $W_1(\cdot)/W_1(K)$, see for instance [12], [15], p. 536, or Equation (3).

The special case in which only the service speed is adapted can be directly obtained from Subsection 5.1 and the explicit form of the scale function. Moreover, the results become especially tractable in case the service requirements have an exponential distribution function. A probabilistic approach for this M/M/1 model can be found in [5]. In the general M/G/1 setting, the steady-state workload distribution can be obtained from Equations (29) and (23) and rewriting the fraction of Lévy exponents as in Subsection 4.2. Because of similarities with Subsections 4.2 and 5.1, we only give an outline.

Step 2: Workload distribution on $(0, K]$

Again, we apply Laplace inversion to each of the three terms on the rhs of (29). The inverse of the first term is given by $QW_1(\cdot)$, see Definition 2.1. The second term corresponds to the LST of a function with mass only on $[K, \infty)$. To see this, we note that $\int_K^{\infty} \mathbb{E}[e^{-\omega Z_2^{(T)}} I(T <$

$\tau_K) | Z_2^{(0)} = x]d\mathbb{P}(Z < x)$ is the transform of a function with mass on $[K, \infty)$, while the term $(\phi_1(\omega) - \phi_2(\omega))/\phi_1(\omega)$ can be treated as in Subsection 4.2.

For the third term on the rhs of (29), we note that $\mathbb{E}[e^{-\omega Z_2^{(T)}} | Z_2^{(0)} = K]$ is the LST of $W^{(T)}(\cdot; K)$ corresponding to the amount of work after an exponential time starting from K . Rewriting the term $(\phi_1(\omega) - \phi_2(\omega))/\phi_1(\omega)$ again, and applying similar arguments as in Subsection 4.2, we may invert the third term. Specifically, let $W_1^{\text{exc}}(\cdot)$ be the steady-state workload distribution in an M/G/1 queue with service rate r_1 , arrival rate λ_1 , and generic service requirement B_1 , but with exceptional first service B_2 in a busy period (see e.g. [28, 29] or Example A.1). For simplicity, we assume here that $\rho_1 < 1$. Applying Laplace inversion then yields, for $x \in (0, K]$,

$$\begin{aligned} \mathbb{P}(Z < x) = & QW_1(x) + P_{\downarrow K} \left(\frac{\lambda_2}{\lambda_1} \frac{1 + \frac{\lambda_1 \beta_2}{r_1} - \rho_1}{1 - \rho_1} \int_0^x W^{(T)}(x - y; K) dW_1^{\text{exc}}(y) \right. \\ & \left. - r_2 \int_0^x W^{(T)}(x - y; K) dW_1(y) + \left(1 - \frac{\lambda_2}{\lambda_1} \right) W^{(T)}(x; K) \right). \end{aligned}$$

Step 3: Workload distribution on (K, ∞)

Using the result from Step 2, we next apply Laplace inversion to each of the terms on the rhs of (23). The Laplace inverse of the first term is directly given by $QW_2(\cdot)$. For the second term on the rhs of (23), it follows from Theorem 2.2 that

$$\frac{\xi}{\xi - \phi_1(\omega)} \left[\int_0^K e^{-\omega x} d\mathbb{P}(Z < x) - \omega \int_0^K \frac{e^{-\eta_1(\xi)x}}{\eta_1(\xi)} d\mathbb{P}(Z < x) \right].$$

is the LST of $\tilde{W}^{(T)}(\cdot)$, i.e., the amount of work of a reflected Lévy process with exponent $\phi_1(\omega)$ after an exponential time, starting according to $\mathbb{P}(Z < x)$. Since we have determined the distribution of Z on $[0, K]$, this also formally gives $\tilde{W}^{(T)}(\cdot)$ (although the precise form is again rather involved).

Now, rewriting the term $(\phi_2(\omega) - \phi_1(\omega))/\phi_2(\omega)$ similarly as in Subsection 4.2, we can invert each term separately. Recall that $W_2^{\text{exc}}(\cdot)$ denotes the workload distribution in an M/G/1 queue with service rate r_2 , arrival rate λ_2 , and generic service requirement B_2 , but with exceptional first service B_1 in a busy period. Applying similar arguments as in Subsections 4.2 and 5.1, we find by Laplace inversion that, for $x > K$,

$$\begin{aligned} \mathbb{P}(Z < x) = & QW_2(x) + \frac{\lambda_1}{\lambda_2} \frac{1 + \frac{\lambda_2 \beta_1}{r_2} - \rho_2}{1 - \rho_2} \int_0^x W_2^{\text{exc}}(x - y) d\tilde{W}^{(T)}(y) \\ & - r_1 \int_0^x W_2(x - y) d\tilde{W}^{(T)}(y) + \left(1 - \frac{\lambda_1}{\lambda_2} \right) \tilde{W}^{(T)}(x). \end{aligned}$$

Step 4: Determination of the constants

The constants can be determined in a similar fashion as in Subsection 5.1, see also [4].

5.3 Brownian motion

In this subsection we consider the special case of Brownian motion, i.e., we assume $\phi_i(\omega) = \omega^2 \sigma_i^2 / 2 - \mu_i \omega$, $i = 1, 2$. For stability we require that $\mu_2 < 0$. In the case of Brownian motion, the steady-state workload distribution has a rather tractable form.

First, consider the q -scale function $W_i^{(q)}(\cdot)$. Define $\delta_i(q) := \sqrt{\mu_i^2 + 2q\sigma_i^2}$ and let $\eta_i^\pm(q)$ be the positive and negative solution, respectively, of $\phi_i(\omega) - q$, that is

$$\eta_i^\pm(q) := \frac{\mu_i \pm \delta_i(q)}{\sigma_i^2} = \frac{\mu_i \pm \sqrt{\mu_i^2 + 2q\sigma_i^2}}{\sigma_i^2}.$$

We note that $\eta_i^+(q)$ equals $\eta_i(q)$ as defined in Theorem 2.2. The q -scale function now reads

$$W_i^{(q)}(x) = \frac{1}{\delta_i(q)} \left(e^{\eta_i^+(q)x} - e^{\eta_i^-(q)x} \right),$$

see also [21] for the spectrally negative case with $\sigma_i = 1$. Given Theorem 2.1, of special interest is the 0-scale function. In case $\mu_i < 0$, we have $W_i(x) = (1 - e^{2\mu_i x/\sigma_i^2})/|\mu_i|$ and Equation (2) indeed reduces to the familiar steady-state distribution of a reflected Brownian motion.

Finally, the density of the amount of work after an exponential time starting from v follows from the derivative with respect to x of Equation (5), or can be obtained from Laplace inversion in Theorem 2.2. Specifically, for $x \in (0, K)$, we have

$$\frac{d}{dx} W_i^{(T)}(x; v) = \frac{\xi}{\delta_i(\xi)} e^{-\eta_i^+(\xi)v} \left(e^{\eta_i^+(\xi)x} - \frac{\eta_i^-(\xi)}{\eta_i^+(\xi)} e^{\eta_i^-(\xi)x} \right), \quad (32)$$

and for $x \in [K, \infty)$, we have

$$\frac{d}{dx} W_i^{(T)}(x; v) = \frac{\xi}{\delta_i(\xi)} \left(e^{-\eta_i^-(\xi)v} - \frac{\eta_i^-(\xi)}{\eta_i^+(\xi)} e^{-\eta_i^+(\xi)v} \right) e^{\eta_i^-(\xi)x}. \quad (33)$$

Step 1: Rewriting (29)

Using the specific form of $\phi_i(\cdot)$, we may rewrite the fraction of Lévy exponents on the rhs of Equation (29) as follows:

$$\frac{\phi_1(\omega) - \phi_2(\omega)}{\phi_1(\omega)} = \frac{\mu_2 - \mu_1 + \frac{1}{2}(\sigma_1^2 - \sigma_2^2)\omega}{\frac{\sigma_2^2}{2}\omega - \mu_1} = 1 - \frac{\sigma_2^2}{\sigma_1^2} + \left(\frac{\mu_2}{\mu_1} - \frac{\sigma_2^2}{\sigma_1^2} \right) \frac{\frac{2\mu_1}{\sigma_1^2}}{\omega - \frac{2\mu_1}{\sigma_1^2}}. \quad (34)$$

Substitution in (29) then gives

$$\begin{aligned} \zeta(\omega) &= Q \frac{\omega}{\phi_1(\omega)} \\ &+ \left(1 - \frac{\sigma_2^2}{\sigma_1^2} + \left(\frac{\mu_2}{\mu_1} - \frac{\sigma_2^2}{\sigma_1^2} \right) \frac{\frac{2\mu_1}{\sigma_1^2}}{\omega - \frac{2\mu_1}{\sigma_1^2}} \right) \int_K^\infty \mathbb{E}[e^{-\omega Z_2^{(T)}} I(T < \tau_K) \mid Z_2^{(0)} = x] d\mathbb{P}(Z < x) \\ &+ \left(1 - \frac{\sigma_2^2}{\sigma_1^2} + \left(\frac{\mu_2}{\mu_1} - \frac{\sigma_2^2}{\sigma_1^2} \right) \frac{\frac{2\mu_1}{\sigma_1^2}}{\omega - \frac{2\mu_1}{\sigma_1^2}} \right) P_{\downarrow K} \mathbb{E}[e^{-\omega Z_2^{(T)}} \mid Z_2^{(0)} = K]. \end{aligned} \quad (35)$$

Step 2: Workload density on $(0, K]$

To obtain the workload density on $(0, K]$, we apply Laplace inversion to each of the terms

on the rhs of (35) separately. The Laplace inverse of the first term is simply given by a constant times the scale function $W_1(\cdot)$. The second term

$$\left(1 - \frac{\sigma_2^2}{\sigma_1^2} + \left(\frac{\mu_2}{\mu_1} - \frac{\sigma_2^2}{\sigma_1^2}\right) \frac{\frac{2\mu_1}{\sigma_1^2}}{\omega - \frac{2\mu_1}{\sigma_1^2}}\right) \int_K^\infty \mathbb{E}[e^{-\omega Z_2^{(T)}} I(T < \tau_K) \mid Z_2^{(0)} = x] d\mathbb{P}(Z < x)$$

is the LST of a function that has no mass on $[0, K]$. To see this, we first note that the latter part (involving the integral) corresponds to a function with mass on (K, ∞) . The first part consists of a constant and the LST of the scale function $W_1(\cdot)$ (having mass on $[0, \infty)$) times a constant. Hence, the convolution between the two parts has only mass on (K, ∞) .

Using (32) and the fact that a product of two transforms corresponds to the convolution of two functions, it is a matter of tedious calculations to determine the Laplace inverse of the third term. Denote by $f_Z(\cdot)$ the density of Z . For $0 < x \leq K$, we then have

$$f_Z(x) = Q_1 e^{2\mu_1 x / \sigma_1^2} + Q_2 e^{\eta_2^+(\xi)x} + Q_3 e^{\eta_2^-(\xi)x}, \quad (36)$$

for some constants Q_1 , Q_2 and Q_3 .

Step 3 *Workload density on (K, ∞)*

Combining Step 2 with (23) gives the LST $\zeta(\omega)$. The density $f_Z(\cdot)$ on (K, ∞) can be determined by applying Laplace inversion again. Using Theorem 2.2 and a similar calculation as in (34), we may rewrite (23) as

$$\begin{aligned} \zeta(\omega) &= Q \frac{\omega}{\phi_2(\omega)} \\ &+ \left(1 - \frac{\sigma_1^2}{\sigma_2^2} + \left(\frac{\mu_1}{\mu_2} - \frac{\sigma_1^2}{\sigma_2^2}\right) \frac{\frac{2\mu_2}{\sigma_2^2}}{\omega - \frac{2\mu_2}{\sigma_2^2}}\right) \int_0^K \mathbb{E}[e^{-\omega Z_1^{(T)}} \mid Z_1^{(0)} = y] f_Z(y) dy. \end{aligned} \quad (37)$$

Using (36) of Step 2 combined with (32) and (33), we may determine the distribution function $\tilde{W}^{(T)}(\cdot)$ (and its density). However, it turns out to be sufficient to determine the density $\tilde{W}^{(T)}(\in dx)$ only for $x \geq K$, see also Step 4 below. In that case, it follows from (36) and (33) and some straightforward algebra that, for $x > K$,

$$\frac{d}{dx} \tilde{W}^{(T)}(x) = \tilde{Q} e^{\eta_1^-(\xi)x}.$$

Now, applying Laplace inversion to each of the terms in (37) we obtain after some calculations that, for $x \geq K$,

$$f_Z(x) = Q_4 e^{\eta_1^-(\xi)x} + Q_5 e^{2\mu_2 x / \sigma_2^2},$$

for some constants Q_4 and Q_5 .

Step 4: *Determination of the constants*

It remains to determine the constants Q_i , $i = 1, \dots, 5$. We note that for the case of Brownian motion, there is no atom in 0 (to see this, observe that $W_i^{(q)}(0) = 0$). First, we have the normalizing condition $\int_0^\infty f_Z(x) dx = 1$. The lengthy calculations to determine Q_i , $i = 1, \dots, 4$, in Steps 2 and 3 also provide us four equations. Together we then have five independent equations to determine the five unknowns Q_i , $i = 1, \dots, 5$. Since we also find the constant Q_5 , there is no need to specify Q_5 any further in Step 3.

6 Poisson observer: General solution

In this section, we consider the general solution of the reflected Lévy process where the Lévy exponent is adapted at Poisson instants. Again, we apply the solution procedure as outlined in Section 3. However, since it is in general not possible to reduce $(\phi_i(\omega) - \phi_{3-i}(\omega))/\phi_i(\omega)$, $i = 1, 2$, we use a slightly different approach in this section than in Section 5. In particular, in Step 0, we use the Palm inversion formula and results on alternating Lévy processes as defined in Appendix A to obtain two sets of equations for $\zeta(\cdot)$. The interpretation of the specific alternating Lévy processes allows us to carry out Steps 1–4 in the general case. We believe that the more direct and insightful derivation in Section 5 is of independent interest, leading to more tractable expressions for $\mathbb{P}(Z < x)$. In Step 0, we derive a first set of equations using a specific alternating Lévy process. Since we apply a different alternating Lévy process for the second set of equations, we have chosen to derive this second set of equations in Step 3, where the specific alternating process is introduced.

Step 0: Determining equations

Assume that the process is in stationarity and consider the workload embedded at epochs when the Poisson observer arrives finding a workload larger than K . To describe the workload behavior in periods between these embedded epochs, we introduce the following artificial regenerative alternating Lévy process $\{\hat{Z}^{(t)}, t \geq 0\}$: Let the process $\hat{Z}^{(t)}$ start from some level $x \geq K$ according to $\mathbb{P}(Z < x \mid Z \geq K)$, independent of the past evolution, and define $\theta_1 = T$, the first observer arrival epoch after 0. During this first period the Lévy exponent is taken to be $\phi_2(\cdot)$. At time θ_1 the Lévy exponent is changed into $\phi_1(\cdot)$. We define $\theta_2 > 0$ as the first arrival instant after time 0 of the observer finding a workload larger than or equal to K . It should be noted that it is possible that θ_2 coincides with θ_1 . Then the period during which the Lévy exponent equals $\phi_1(\cdot)$ has length zero. Times 0 and θ_2 are two consecutive arrival epochs of the observer finding a workload larger than K . Due to stationarity, we have $\mathbb{E}e^{-\omega\hat{Z}^{(0)}} = \mathbb{E}e^{-\omega\hat{Z}^{(\theta_2)}}$. Applying Lemma A.1, we obtain the following equation for this process

$$\mathbb{E}[e^{-\omega\hat{Z}}] = \frac{\mathbb{E}\hat{L}_2^{(\theta_2-\theta_1)} + \mathbb{E}\hat{L}_1^{(\theta_1)}}{\mathbb{E}\theta_2} \frac{\omega}{\phi_1(\omega)} + \frac{\phi_1(\omega) - \phi_2(\omega)}{\phi_1(\omega)} \frac{1}{\mathbb{E}\theta_2} \mathbb{E}\left[\int_{s=0}^{\theta_1} e^{-\omega\hat{Z}^{(s)}} ds\right]. \quad (38)$$

In fact, the LST of the actual process Z and the regenerative process \hat{Z} are identical. To see this, we consider the arrival instants of the observer finding a workload larger than or equal to K of the actual process Z (i.e. the embedded epochs) as event times and apply the Palm inversion formula [3, 27]:

$$\zeta(\omega) = \frac{1}{\mathbb{E}\theta_2} \mathbb{E}\left[\int_0^{\theta_2} e^{-\omega Z^{(s)}} ds\right] = \frac{1}{\mathbb{E}\theta_2} \mathbb{E}\left[\int_0^{\theta_2} e^{-\omega\hat{Z}^{(s)}} ds\right] = \mathbb{E}[e^{-\omega\hat{Z}}].$$

Here, the second step is by construction of the process \hat{Z} and the third follows from regeneration theory. Thus, Equation (38) gives a first set of equations for the LST of Z .

Remark 6.1. We note that Equations (38) and (22) are identical. Write, for $x \geq K$, $\mathbb{P}(Z < x) = \mathbb{P}(Z < K) + \mathbb{P}(Z < x \mid Z \geq K)\mathbb{P}(Z \geq K)$ and let x in $\mathbb{E}_x[\cdot]$ denote the initial

position of the process. We then have

$$\begin{aligned}
\mathbb{E}\left[\int_0^{\theta_1} e^{-\omega \hat{Z}^{(s)}} ds\right] &= \int_K^\infty \mathbb{E}_x\left[\int_0^{\theta_1} e^{-\omega \hat{Z}^{(s)}} ds\right] d\mathbb{P}(Z < x \mid Z \geq K) \\
&= \frac{1}{\mathbb{P}(Z \geq K)} \int_K^\infty \mathbb{E}_x\left[\int_0^{\theta_1} e^{-\omega \hat{Z}^{(s)}} ds\right] d\mathbb{P}(Z < x) \\
&= \frac{\mathbb{E}\theta_1}{\mathbb{P}(Z \geq K)} \int_K^\infty \mathbb{E}_x[e^{-\omega Z_2^{(T)}}] d\mathbb{P}(Z < x),
\end{aligned}$$

where the final step follows from (4). The fraction of time that the Lévy exponent is $\phi_2(\cdot)$ equals the fraction of observers finding a workload larger than K , and hence, using PASTA, also equals the fraction of time that the workload is larger than K . An application of the Palm inversion formula then provides $\mathbb{P}(Z \geq K) = \mathbb{E}\theta_1/\mathbb{E}\theta_2$. Combining the above and using Theorem 2.2, it follows that the second terms on the rhs of Equations (38) and (22) are identical, and (38) can be rewritten as (7).

For the first terms, we have

$$\frac{\mathbb{E}\hat{L}_2^{(\theta_2-\theta_1)} + \mathbb{E}\hat{L}_1^{(\theta_1)}}{\mathbb{E}\theta_2} = \lim_{t \rightarrow \infty} \frac{\mathbb{E}L^{(t)}}{t} = Q,$$

where the first step follows from an application of the Palm inversion formula and the second step is due to starting in stationarity. \diamond

Step 1: Rewriting (38)

In this step, we rewrite Equation (38) using a similar intuition as in Step 1 of Section 5. In particular, we use the fact that periods with Lévy exponent $\phi_2(\cdot)$ and workloads smaller than K are initiated by an observer finding a workload larger than K (to set the exponent to $\phi_2(\cdot)$) followed by a downcrossing of level K before the next arrival instant of the observer. Due to the lack-of-memory property of the Poisson arrival process, the remaining interarrival time of the observer is still exponential at a downcrossing of K . Using the stationary and independent increments property of Lévy processes, it is intuitively clear that the precise distribution of Z on (K, ∞) does only affect the distribution of Z on $[0, K]$ through a constant.

The above intuition can be applied as follows. Denote $\mathbb{E}_{Z \geq K}[e^{-\omega \hat{Z}}]$ for $\int_K^\infty \mathbb{E}_x[e^{-\omega \hat{Z}}] d\mathbb{P}(Z < x \mid Z \geq K)$. Define, with some abuse of notation (see already the hitting time τ_K in Section 5), $\tau_K := \inf\{t \geq 0 : \hat{Z}^{(t)} = K\}$ as the first hitting time of K and let $I(\cdot)$ again denote the indicator function. Now, using (4) in the second and fourth equality below and the lack-of-memory property of the Poisson arrival process and the Lévy process in the

third equality, we have

$$\begin{aligned}
& \mathbb{E}_{Z \geq K}[e^{-\omega \hat{Z}}] \\
&= \frac{1}{\mathbb{E}\theta_2} \mathbb{E}_{Z \geq K} \left[\int_0^{\theta_2} e^{-\omega \hat{Z}(s)} I(T < \tau_K) ds \right] + \frac{1}{\mathbb{E}\theta_2} \mathbb{E}_{Z \geq K} \left[\int_0^{\theta_2} e^{-\omega \hat{Z}(s)} I(T \geq \tau_K) ds \right] \\
&= \frac{1}{\mathbb{E}\theta_2} \mathbb{E}_{Z \geq K} \left[\int_0^T e^{-\omega \hat{Z}(s)} I(T < \tau_K) ds \right] \\
&\quad + \frac{1}{\xi \mathbb{E}\theta_2} \mathbb{E}_{Z \geq K} [e^{-\omega \hat{Z}(T)} I(T \geq \tau_K) ds] + \frac{1}{\mathbb{E}\theta_2} \mathbb{E}_{Z \geq K} \left[\int_{\theta_1}^{\theta_2} e^{-\omega \hat{Z}(s)} I(T \geq \tau_K) ds \right] \\
&= \frac{1}{\mathbb{E}\theta_2} \mathbb{E}_{Z \geq K} \left[\int_0^T e^{-\omega \hat{Z}(s)} I(T < \tau_K) ds \right] \\
&\quad + \frac{1}{\xi \mathbb{E}\theta_2} \hat{P}_{\downarrow K} \mathbb{E}_K [e^{-\omega \hat{Z}(T)} ds] + \frac{1}{\mathbb{E}\theta_2} \hat{P}_{\downarrow K} \mathbb{E}_K \left[\int_{\theta_1}^{\theta_2} e^{-\omega \hat{Z}(s)} ds \right] \\
&= \frac{1}{\mathbb{E}\theta_2} \mathbb{E}_{Z \geq K} \left[\int_0^T e^{-\omega \hat{Z}(s)} I(T < \tau_K) ds \right] + \hat{P}_{\downarrow K} \frac{\mathbb{E}_K \theta_2}{\mathbb{E}\theta_2} \mathbb{E}_K [e^{-\omega \hat{Z}}], \tag{39}
\end{aligned}$$

where $\hat{P}_{\downarrow K}$ is the probability of downcrossing K before an exponential time with parameter ξ , starting from x according to $\mathbb{P}(Z < x \mid Z \geq K)$. Using results on LST of first-exit times, see e.g. [20], it may be checked that

$$\begin{aligned}
\hat{P}_{\downarrow K} &= \xi e^{\eta_2(\xi)K} \int_K^\infty e^{-\eta_2(\xi)x} d\mathbb{P}(Z < x \mid Z \geq K) \\
&= \frac{\xi}{\mathbb{P}(Z \geq K)} e^{\eta_2(\xi)K} \int_K^\infty e^{-\eta_2(\xi)x} d\mathbb{P}(Z < x). \tag{40}
\end{aligned}$$

Note that for the first transform on the last line of (39), it holds that $\hat{Z}^{(s)} > K$ for all $s \in [0, T]$. Hence, it has no mass on $[0, K]$. Thus the inverse of \hat{Z} , and thus of Z , on the interval $[0, K]$ is completely determined by the second transform $\mathbb{E}_K[e^{-\omega \hat{Z}}]$, where the initial condition only depends on the distribution of Z on $[K, \infty)$ through the constant $(\hat{P}_{\downarrow K} \mathbb{E}_K \theta_2) / \mathbb{E}\theta_2$.

Step 2: Workload distribution on $(0, K]$

To obtain the workload distribution on $(0, K]$, we apply Laplace inversion to $\mathbb{E}_K[e^{-\omega \hat{Z}}]$. Using Lemma A.2, this transform satisfies

$$\mathbb{E}_K[e^{-\omega \hat{Z}}] = \frac{\mathbb{E} \hat{L}_2^{(\theta_2 - \theta_1)}}{\mathbb{E}_K \theta_2} \frac{\omega}{\phi_1(\omega)} + \frac{1}{\mathbb{E}_K \theta_2} \frac{\mathbb{E} e^{-\omega \hat{Z}(\theta_2)} - \mathbb{E} e^{-\omega \hat{Z}(\theta_1)}}{\phi_1(\omega)} + \frac{1}{\mathbb{E}_K \theta_2} \mathbb{E} \left[\int_{s=0}^{\theta_1} e^{-\omega \hat{Z}(s)} ds \right]. \tag{41}$$

In this step, the initial position of the process is K . For notational convenience, we suppress this initial position, except for $\mathbb{E}_K \theta_2$ (to distinguish from $\mathbb{E}\theta_2$).

Using Theorem 2.1, (4) and Theorem 2.2, we directly obtain the inverse of the first and third term on the rhs of the above equation as constants times $W_1(\cdot)$ and $W_2^{(T)}(\cdot; K)$, respectively. It remains to find the Laplace inverse on $(0, K]$ of the second term.

Observe that $\theta_1 = \theta_2$ if and only if $\hat{Z}^{(\theta_1)} \geq K$. Thus, we may write

$$\begin{aligned}
& \mathbb{E}e^{-\omega\hat{Z}^{(\theta_2)}} - \mathbb{E}e^{-\omega\hat{Z}^{(\theta_1)}} \\
&= \mathbb{E}[e^{-\omega\hat{Z}^{(\theta_2)}} I(\hat{Z}^{(\theta_1)} < K)] - \mathbb{E}[e^{-\omega\hat{Z}^{(\theta_1)}} I(\hat{Z}^{(\theta_1)} < K)] \\
&= \mathbb{P}(\hat{Z}^{(\theta_1)} < K) \left(\mathbb{E}[e^{-\omega\hat{Z}^{(\theta_2)}} \mid \theta_2 > \theta_1] - \mathbb{E}[e^{-\omega\hat{Z}^{(\theta_1)}} \mid \hat{Z}^{(\theta_1)} < K] \right) \\
&= \mathbb{P}(\hat{Z}^{(\theta_1)} < K)e^{-\omega K} \left(\mathbb{E}[e^{-\omega(\hat{Z}^{(\theta_2)} - K)} \mid \theta_2 > \theta_1] - 1 + 1 - \mathbb{E}[e^{\omega U} \mid U > 0] \right),
\end{aligned}$$

where $U := K - \hat{Z}^{(\theta_1)}$ is the undershoot under K at the end of the first exponential time θ_1 starting from level K . For the second term on the rhs of (41), we then have

$$\begin{aligned}
\frac{\mathbb{E}e^{-\omega\hat{Z}^{(\theta_2)}} - \mathbb{E}e^{-\omega\hat{Z}^{(\theta_1)}}}{\phi_1(\omega)} &= -\mathbb{P}(\hat{Z}^{(\theta_1)} < K) \frac{\omega}{\phi_1(\omega)} e^{-\omega K} \\
&\quad \times \left(\frac{1 - \mathbb{E}[e^{-\omega(\hat{Z}^{(\theta_2)} - K)} \mid \theta_2 > \theta_1]}{\omega} + \frac{1 - \mathbb{E}[e^{\omega U} \mid U > 0]}{-\omega} \right).
\end{aligned}$$

For the first term on the rhs of the above equation,

$$\frac{\omega}{\phi_1(\omega)} e^{-\omega K} \frac{1 - \mathbb{E}[e^{-\omega(\hat{Z}^{(\theta_2)} - K)} \mid \theta_2 > \theta_1]}{\omega},$$

we note that $\hat{Z}^{(\theta_2)} - K \geq 0$ corresponds to the overshoot over K . Hence, $e^{-\omega K}(1 - \mathbb{E}[e^{-\omega(\hat{Z}^{(\theta_2)} - K)} \mid \theta_2 > \theta_1])/\omega$ corresponds to the sum of the constant K and the residual overshoot at time θ_2 , given that $\theta_2 > \theta_1$. Clearly, the corresponding convolution has no mass on $[0, K]$. Since $\omega/\phi_1(\omega)$ has mass on $[0, \infty)$, this first term corresponds to the LST of a function with mass on (K, ∞) .

The second term

$$\frac{\omega}{\phi_1(\omega)} e^{-\omega K} \frac{1 - \mathbb{E}[e^{\omega U} \mid U > 0]}{-\omega \mathbb{E}[U \mid U > 0]} \mathbb{E}[U \mid U > 0]$$

can be interpreted as the transform of a sum as well. First note that the conditional undershoot $U \mid U > 0$ has mass on $[0, K]$. The transform $e^{-\omega K}(1 - \mathbb{E}[e^{\omega U} \mid U > 0]) / (-\omega \mathbb{E}[U \mid U > 0])$ then corresponds to $K - U^{\text{res}}$, where U^{res} represents a generic residual undershoot given that $U > 0$. Using Theorem 2.1 and Definition 2.1, the Laplace inverse of the second term reads

$$W_1(x) * \mathbb{P}(K - U^{\text{res}} \leq x \mid U > 0) \mathbb{E}[U \mid U > 0].$$

Summarizing, we have, for $x \in (0, K]$,

$$\begin{aligned}
\mathbb{P}(Z < x) &= \frac{\hat{P}_{\downarrow K}}{\mathbb{E}\theta_2} \left(\mathbb{E}\hat{L}_2^{(\theta_2 - \theta_1)} W_1(x) + \frac{1}{\xi} W_2^{(T)}(x; K) \right. \\
&\quad \left. - W_2^{(T)}(K; K) \mathbb{E}[U \mid U > 0] \int_0^x \mathbb{P}(K - U^{\text{res}} < x - y \mid U > 0) dW_1(y) \right). \tag{42}
\end{aligned}$$

Although the distribution of $K - U^{\text{res}}$ may be rather involved, we note that $\mathbb{P}(Z < x)$ for $x \in (0, K]$ does only depend on the distribution of Z on (K, ∞) through a constant.

Remark 6.2. We may also give another representation of the Laplace inverse of the second term on the rhs of (41). Using Theorem 2.1 and Definition 2.1, we easily find that $1/\phi_1(\omega)$ is the LST of $\int_0^x W_1(y)dy$. Clearly, this function has mass on $[0, \infty)$. Rewriting the second term on the rhs of (41), we obtain

$$\frac{\mathbb{E}e^{-\omega\hat{Z}^{(\theta_2)}} - \mathbb{E}e^{-\omega\hat{Z}^{(\theta_1)}}}{\phi_1(\omega)} = \frac{1}{\phi_1(\omega)}\mathbb{E}e^{-\omega\hat{Z}^{(\theta_2)}} - \frac{1}{\phi_1(\omega)}\mathbb{E}e^{-\omega\hat{Z}^{(\theta_1)}}.$$

Both terms are products of LST, corresponding to the convolution of two functions. For the first term $\mathbb{E}e^{-\omega\hat{Z}^{(\theta_2)}}/\phi_1(\omega)$, observe that $\mathbb{E}e^{-\omega\hat{Z}^{(\theta_2)}}$ is the LST of a function with mass on (K, ∞) . Hence, its convolution with $1/\phi_1(\omega)$ has no mass on $(0, K]$.

As mentioned, the second term $\mathbb{E}e^{-\omega\hat{Z}^{(\theta_1)}}/\phi_1(\omega)$ is also a convolution of two functions. Applying Laplace inversion, it is easy to see that this term is the LST of $\int_0^x W_1(y)dy * W_2^{(T)}(x; K)$. Summarizing, we obtain, for $x \in (0, K]$,

$$\mathbb{P}(Z < x) = \frac{\hat{P}_{\downarrow K}}{\mathbb{E}\theta_2} \left(\mathbb{E}\hat{L}_2^{(\theta_2 - \theta_1)} W_1(x) + \frac{1}{\xi} W_2^{(T)}(x; K) - \int_0^x W_2^{(T)}(x - y; K) W_1(y) dy \right). \quad (43)$$

Note that the distribution of Z on $(0, K]$ again only depends on $\mathbb{P}(Z < x)$ for $x \in (K, \infty)$ through a constant. \diamond

Step 3: Workload distribution on (K, ∞)

Using (23), we have completely determined the LST of Z . In this step we apply Laplace inversion similar to Step 2 to obtain its distribution on (K, ∞) .

In particular, assume that the process is in steady state and consider the workload embedded at epochs when the Poisson observer arrives finding a workload smaller than K . Again, to describe the workload process between these embedded epochs we introduce the following regenerative alternating Lévy process $\{\tilde{Z}^{(t)}, t \geq 0\}$: Let $\tilde{Z}^{(0)}$ be determined according to the conditional distribution $\mathbb{P}(Z < x \mid Z < K)$ for $x \in (0, K)$, independent of the past evolution, and define $\theta_1 = T$. During this first period the Lévy exponent is taken to be $\phi_1(\cdot)$. At time θ_1 the Lévy exponent is changed into $\phi_2(\cdot)$. We define $\tilde{\theta}_2 > 0$ as the first arrival instant of the observer finding a workload smaller than K . As in Step 0, $\tilde{\theta}_2$ may coincide with θ_1 . Because of the stationarity of the embedded process we have $\mathbb{E}e^{-\omega\tilde{Z}^{(0)}} = \mathbb{E}e^{-\omega\tilde{Z}^{(\tilde{\theta}_2)}}$. Lemma A.1 then reads

$$\mathbb{E}[e^{-\omega\tilde{Z}}] = \frac{\mathbb{E}\tilde{L}_1^{(\theta_1)} + \mathbb{E}\tilde{L}_2^{(\tilde{\theta}_2 - \theta_1)}}{\mathbb{E}\tilde{\theta}_2} \frac{\omega}{\phi_2(\omega)} + \frac{\phi_2(\omega) - \phi_1(\omega)}{\phi_2(\omega)} \frac{1}{\mathbb{E}\tilde{\theta}_2} \mathbb{E}\left[\int_{s=0}^{\theta_1} e^{-\omega\tilde{Z}^{(s)}} ds\right]. \quad (44)$$

It follows from similar arguments as in Step 2 that the steady-state distribution of the actual process Z and of the regenerative process \tilde{Z} are identical. Using the Palm inversion formula [3, 27], it readily follows that the time stationary LST equals $\zeta(\omega)$:

$$\zeta(\omega) = \frac{1}{\mathbb{E}\tilde{\theta}_2} \mathbb{E}\left[\int_0^{\tilde{\theta}_2} e^{-\omega Z^{(s)}} ds\right] = \frac{1}{\mathbb{E}\tilde{\theta}_2} \mathbb{E}\left[\int_0^{\tilde{\theta}_2} e^{-\omega\tilde{Z}^{(s)}} ds\right] = \mathbb{E}[e^{-\omega\tilde{Z}}],$$

where the second step is by construction of \tilde{Z} and the third step follows from regeneration theory.

Remark 6.3. Using similar arguments as in Remark 6.1, it follows that Equations (44) and (23) are identical. Specifically, for $x \in (0, K]$, writing $\mathbb{P}(Z < x) = \mathbb{P}(Z < x \mid Z < K)\mathbb{P}(Z < K)$, we have

$$\begin{aligned} \mathbb{E}\left[\int_0^{\theta_1} e^{-\omega \tilde{Z}^{(s)}} ds\right] &= \int_0^K \mathbb{E}_x\left[\int_0^{\theta_1} e^{-\omega \tilde{Z}^{(s)}} ds\right] d\mathbb{P}(Z < x \mid Z < K) \\ &= \frac{\mathbb{E}\theta_1}{\mathbb{P}(Z < K)} \int_0^K \mathbb{E}_x[e^{-\omega Z_1^{(T)}}] d\mathbb{P}(Z < x), \end{aligned}$$

where the second step follows from (4). The fraction of time that the Lévy exponent is $\phi_1(\cdot)$ equals the fraction of observers finding a workload smaller than K , and hence, using PASTA again, also equals the fraction of time that the workload is smaller than K . An application of the Palm inversion formula then provides $\mathbb{P}(Z < K) = \mathbb{E}\theta_1/\mathbb{E}\tilde{\theta}_2$. Using Theorem 2.2 and combining the above, we deduce that the second terms on the rhs of Equations (44) and (23) are identical, and (44) may thus be presented in the form of (8). For the first term on the rhs of (44), we have

$$\frac{\mathbb{E}\tilde{L}_2^{(\tilde{\theta}_2 - \theta_1)} + \mathbb{E}\tilde{L}_1^{(\theta_1)}}{\mathbb{E}\tilde{\theta}_2} = \lim_{t \rightarrow \infty} \frac{\mathbb{E}L^{(t)}}{t} = Q,$$

where the first step is an application of the Palm inversion formula and the second step follows from starting in stationarity. \diamond

It thus remains to find the time stationary distribution of \tilde{Z} . Using (4) and Theorem 2.2, it is easy to obtain the distribution on the first interval of the alternating process. Using Lemma A.2, we have

$$\mathbb{E}[e^{-\omega \tilde{Z}}] = \frac{\mathbb{E}\tilde{L}_2^{(\tilde{\theta}_2 - \theta_1)}}{\mathbb{E}\tilde{\theta}_2} \frac{\omega}{\phi_2(\omega)} + \frac{1}{\mathbb{E}\tilde{\theta}_2} \frac{\mathbb{E}e^{-\omega \tilde{Z}^{(\tilde{\theta}_2)}} - \mathbb{E}e^{-\omega \tilde{Z}^{(\theta_1)}}}{\phi_2(\omega)} + \frac{1}{\mathbb{E}\tilde{\theta}_2} \mathbb{E}\left[\int_{s=0}^{\theta_1} e^{-\omega \tilde{Z}^{(s)}} ds\right]. \quad (45)$$

From Theorem 2.1, the Laplace inverse of the first term is readily obtained, giving $W_2(\cdot)$ times a constant. Applying (4) and Theorem 2.2, we also directly obtain the inverse of the third term as $\mathbb{E}\theta_1/\mathbb{E}\tilde{\theta}_2$ times

$$\begin{aligned} \tilde{W}_{\text{cond}}^{(T)}(x) &:= \int_0^K W_1^{(T)}(x; y) d\mathbb{P}(Z < y \mid Z < K) \\ &= \frac{1}{\mathbb{P}(Z < K)} \int_0^K W_1^{(T)}(x; y) d\mathbb{P}(Z < y). \end{aligned}$$

Since we have determined the distribution of Z on $[0, K]$, we have also found $\tilde{W}_{\text{cond}}^{(T)}(\cdot)$ (as mentioned in Section 5, we note that its precise form may be rather involved). It remains to find the Laplace inverse on (K, ∞) of the second term.

For this second term, we may write

$$\frac{\mathbb{E}e^{-\omega \tilde{Z}^{(\tilde{\theta}_2)}} - \mathbb{E}e^{-\omega \tilde{Z}^{(\theta_1)}}}{\phi_2(\omega)} = \frac{\omega}{\phi_2(\omega)} \left(\frac{1 - \mathbb{E}e^{-\omega \tilde{Z}^{(\theta_1)}}}{\omega} - \frac{1 - \mathbb{E}e^{-\omega \tilde{Z}^{(\tilde{\theta}_2)}}}{\omega} \right).$$

Using Theorem 2.1 and Definition 2.1, it follows directly that $\omega/\phi_2(\omega)$ is the LST of $W_2(\cdot)$. Observing that we again have the difference of two convolutions, we may apply

Laplace inversion to both terms separately. Note that $(1 - \mathbb{E}e^{-\omega\tilde{Z}^{(\theta_1)}})/\omega$ is the LST of $\int_0^x (1 - \tilde{W}_{\text{cond}}^{(T)}(y))dy$. Equivalently, $(1 - \mathbb{E}e^{-\omega\tilde{Z}^{(\tilde{\theta}_2)}})/\omega$ is the LST of $\int_0^x (1 - \mathbb{P}(Z < y \mid Z < K))dy$, with $x \in (0, K]$. In fact, both terms correspond to the integrated tail distribution. Using the fact that a product of LSTs corresponds to a convolution of two functions, it is now easy to apply Laplace inversion to this second term.

Summarizing, we have, for $x \in (K, \infty)$,

$$\begin{aligned} \mathbb{P}(Z < x) &= \frac{1}{\mathbb{E}\tilde{\theta}_2} \left(\mathbb{E}\tilde{L}_2^{(\tilde{\theta}_2 - \theta_1)} W_2(x) + \int_0^x W_2(x-y)(1 - \tilde{W}_{\text{cond}}^{(T)}(y))dy \right. \\ &\quad \left. - \int_0^K W_2(x-y)(1 - \mathbb{P}(Z < y \mid Z < K))dy + \frac{1}{\xi} \tilde{W}_{\text{cond}}^{(T)}(x) \right). \end{aligned}$$

Remark 6.4. Similar to Remark 6.2, we may give a different representation for the second term on the rhs of (45). In particular, for this second term, we may write

$$\frac{\mathbb{E}e^{-\omega\tilde{Z}^{(\tilde{\theta}_2)}} - \mathbb{E}e^{-\omega\tilde{Z}^{(\theta_1)}}}{\phi_2(\omega)} = \frac{1}{\phi_2(\omega)} \mathbb{E}e^{-\omega\tilde{Z}^{(\tilde{\theta}_2)}} - \frac{1}{\phi_2(\omega)} \mathbb{E}e^{-\omega\tilde{Z}^{(\theta_1)}}.$$

Using Theorem 2.1 and Definition 2.1, we easily find that $1/\phi_2(\omega)$ is the LST of $\int_0^x W_2(y)dy$. Observing that we again have the sum of two convolution terms, we may easily apply Laplace inversion to both terms separately. In particular, for the first one $\mathbb{E}e^{-\omega\tilde{Z}^{(\tilde{\theta}_2)}}/\phi_2(\omega)$, we get an incomplete convolution between $\int_0^x W_2(y)dy$ and $\mathbb{P}(Z < x \mid Z < K)$ (see also below). For the second term $\mathbb{E}e^{-\omega\tilde{Z}^{(\theta_1)}}/\phi_2(\omega)$, we have a convolution of $\int_0^x W_2(y)dy$ with $\tilde{W}_{\text{cond}}^{(T)}(\cdot)$. Summarizing, we get, for $x \in (K, \infty)$,

$$\begin{aligned} \mathbb{P}(Z < x) &= \frac{1}{\mathbb{E}\tilde{\theta}_2} \left(\mathbb{E}\tilde{L}_2^{(\tilde{\theta}_2 - \theta_1)} W_2(x) + \int_0^K \int_0^{x-y} W_2(z)dz d\mathbb{P}(Z < y \mid Z < K) \right. \\ &\quad \left. - \int_0^x \tilde{W}_{\text{cond}}^{(T)}(x-y)W_2(y)dy + \frac{1}{\xi} \tilde{W}_{\text{cond}}^{(T)}(x) \right). \end{aligned}$$

Alternatively, conditioning on $\int_0^x W_2(y)dy$ for the second convolution, we may represent the distribution of Z on (K, ∞) as

$$\begin{aligned} \mathbb{P}(Z < x) &= \frac{1}{\mathbb{E}\tilde{\theta}_2} \left(\mathbb{E}\tilde{L}_2^{(\tilde{\theta}_2 - \theta_1)} W_2(x) + \int_{x-K}^x \mathbb{P}(Z < x-y \mid Z < K)W_2(y)dy \right. \\ &\quad \left. - \int_0^x \tilde{W}_{\text{cond}}^{(T)}(x-y)W_2(y)dy + \frac{1}{\xi} \tilde{W}_{\text{cond}}^{(T)}(x) \right). \end{aligned}$$

◇

Step 4: Determination of the constants

It follows directly from Remarks 6.1 and 6.3 that

$$\frac{\mathbb{E}\hat{L}_2^{(\theta_2 - \theta_1)} + \mathbb{E}\hat{L}_1^{(\theta_1)}}{\mathbb{E}\theta_2} = \frac{\mathbb{E}\tilde{L}_2^{(\tilde{\theta}_2 - \theta_1)} + \mathbb{E}\tilde{L}_1^{(\theta_1)}}{\mathbb{E}\tilde{\theta}_2} = Q. \quad (46)$$

Using (4) and the final observation in Remark 2.1 it can be easily verified (see also [10, Equation (3.6)]) that

$$\frac{\mathbb{E}\tilde{L}_1^{(\theta_1)}}{\mathbb{E}\tilde{\theta}_2} = \xi \int_0^K \frac{e^{-\eta_1(\xi)x}}{\eta_1(\xi)} d\mathbb{P}(Z < x),$$

which can be determined in terms of Q and $\mathbb{P}(Z < K)$ by multiplying either (42) or (43) by $\exp(-\eta_1(\xi)x)$ and integrating over the interval $[0, K]$. Similarly, we have

$$\frac{\mathbb{E}\hat{L}_1^{(\theta_1)}}{\mathbb{E}\theta_2} = \xi \int_K^\infty \frac{e^{-\eta_2(\xi)x}}{\eta_2(\xi)} d\mathbb{P}(Z < x),$$

which can also be expressed in terms of Q and $\mathbb{P}(Z < K)$ by combining the above with the definition of Q , i.e. (21). From the observations in Remarks 6.1 and 6.3, we also have $1/\mathbb{E}\theta_2 = \xi\mathbb{P}(Z \geq K)$ and $1/\mathbb{E}\hat{\theta}_2 = \xi\mathbb{P}(Z < K)$. Now, letting $\omega \downarrow 0$ in (38), we obtain

$$Q = \phi_2'(0) - (\phi_2'(0) - \phi_1'(0))\mathbb{P}(Z < K).$$

Note that letting $\omega \downarrow 0$ in (44) gives the same equation (this also provides the first equality in (46)). In addition, $\mathbb{P}(Z < K)$ can be directly obtained by substituting $x = K$ in either (42) or (43), with $\hat{P}_{\downarrow K}$ given by (40). These latter two equations determine the constants Q and $\mathbb{P}(Z < K)$ and thus the remaining constants.

A Alternating Lévy processes

Here, we analyze alternating Lévy processes without negative jumps. Consider a regeneration cycle and let some (possibly random) level $Z^{(0)} \geq 0$ be the starting level of the cycle. At these regeneration points a first period starts, consisting of a reflected Lévy process without negative jumps and Lévy exponent $\phi_I(\cdot)$. At some stopping time τ_I the first interval ends, and the Lévy exponent is changed into $\phi_{II}(\cdot)$ until the end of the regeneration cycle, denoted by time τ_{II} . For convenience, the first period is referred to as interval I and the second period as interval II . Also, the reflected process is denoted by $\{Z^{(t)}, t \geq 0\}$, and the steady-state version, assuming that it exists, by Z .

We note that the model introduced above is not very natural in its full generality. However, various natural models might be considered as a special case. The most prominent one is the M/G/1 dam, see Section 3; define $Z^{(0)} = K$, τ_I as the first upcrossing of K and τ_{II} as the subsequent downcrossing of K . The equations in Lemma's A.1 and A.2 appear in various parts of the paper.

Lemma A.1. *For an alternating Lévy process as described above, the LST of the steady-state workload satisfies the following equations*

$$\begin{aligned} \mathbb{E}[e^{-\omega Z}] &= \frac{\mathbb{E}L_{II}^{(\tau_{II}-\tau_I)} + \mathbb{E}L_I^{(\tau_I)}}{\mathbb{E}\tau_{II}} \frac{\omega}{\phi_{II}(\omega)} + \frac{\mathbb{E}e^{-\omega Z^{(\tau_{II})}} - \mathbb{E}e^{-\omega Z^{(0)}}}{\phi_{II}(\omega)\mathbb{E}\tau_{II}} \\ &\quad + \frac{\phi_{II}(\omega) - \phi_I(\omega)}{\phi_{II}(\omega)} \frac{1}{\mathbb{E}\tau_{II}} \mathbb{E}\left[\int_{s=0}^{\tau_I} e^{-\omega Z^{(s)}} ds\right], \end{aligned} \quad (47)$$

where $L_i^{(\cdot)}$, $i = I, II$, represents the local time in 0 during interval i . Also

$$\begin{aligned} \mathbb{E}[e^{-\omega Z}] &= \frac{\mathbb{E}L_{II}^{(\tau_{II}-\tau_I)} + \mathbb{E}L_I^{(\tau_I)}}{\mathbb{E}\tau_{II}} \frac{\omega}{\phi_I(\omega)} + \frac{\mathbb{E}e^{-\omega Z^{(\tau_{II})}} - \mathbb{E}e^{-\omega Z^{(0)}}}{\phi_I(\omega)\mathbb{E}\tau_{II}} \\ &\quad + \frac{\phi_I(\omega) - \phi_{II}(\omega)}{\phi_I(\omega)} \frac{1}{\mathbb{E}\tau_{II}} \mathbb{E}\left[\int_{s=\tau_I}^{\tau_{II}} e^{-\omega Z^{(s)}} ds\right]. \end{aligned} \quad (48)$$

Proof. To derive the steady-state distribution of this process, we use the following martingale [1, 18], for $i = I, II$:

$$M_i^{(t)} = \phi_i(\omega) \int_{s=0}^t e^{-\omega Z^{(s)}} ds - e^{-\omega Z^{(t)}} + e^{-\omega Z^{(0)}} - \omega L_i^{(t)}. \quad (49)$$

Application of the optional sampling theorem, with stopping time τ_I , to this martingale (with $i = I$) yields (cf. [1, 18]):

$$\phi_I(\omega) \mathbb{E} \left[\int_{s=0}^{\tau_I} e^{-\omega Z^{(s)}} ds \right] = \mathbb{E} e^{-\omega Z^{(\tau_I)}} - \mathbb{E} e^{-\omega Z^{(0)}} + \omega \mathbb{E} L_I^{(\tau_I)}. \quad (50)$$

Note that the end point of this first period, i.e. $Z^{(\tau_I)}$, is also the starting point of the second interval. Rewriting the above, we have

$$\mathbb{E} e^{-\omega Z^{(\tau_I)}} = \phi_I(\omega) \mathbb{E} \left[\int_{s=0}^{\tau_I} e^{-\omega Z^{(s)}} ds \right] + \mathbb{E} e^{-\omega Z^{(0)}} - \omega \mathbb{E} L_I^{(\tau_I)}. \quad (51)$$

For the second interval, we apply the optional sampling theorem, with stopping time τ_{II} , to this martingale (starting at τ_I instead of at 0), yielding

$$\begin{aligned} & \phi_{II}(\omega) \mathbb{E} \left[\int_{s=\tau_I}^{\tau_{II}} e^{-\omega Z^{(s)}} ds \right] \\ &= \mathbb{E} e^{-\omega Z^{(\tau_{II})}} - \mathbb{E} e^{-\omega Z^{(\tau_I)}} + \omega \mathbb{E} L_{II}^{(\tau_{II}-\tau_I)} \end{aligned} \quad (52)$$

$$= -\phi_I(\omega) \mathbb{E} \left[\int_{s=0}^{\tau_I} e^{-\omega Z^{(s)}} ds \right] + \mathbb{E} e^{-\omega Z^{(\tau_{II})}} - \mathbb{E} e^{-\omega Z^{(0)}} + \omega \mathbb{E} L_{II}^{(\tau_{II}-\tau_I)} + \omega \mathbb{E} L_I^{(\tau_I)}, \quad (53)$$

where the second step follows from (51). From regeneration theory it follows that the LST of the steady-state workload Z in a queue with alternating exponents is given by

$$\mathbb{E}[e^{-\omega Z}] = \frac{1}{\mathbb{E}\tau_{II}} \left(\mathbb{E} \left[\int_{s=0}^{\tau_I} e^{-\omega Z^{(s)}} ds \right] + \mathbb{E} \left[\int_{s=\tau_I}^{\tau_{II}} e^{-\omega Z^{(s)}} ds \right] \right). \quad (54)$$

Hence, dividing (53) by $\phi_{II}(\omega)$, adding $\mathbb{E}[\int_{s=0}^{\tau_I} e^{-\omega Z^{(s)}} ds]$ to both sides and then dividing by $\mathbb{E}\tau_{II}$, we obtain

$$\begin{aligned} \mathbb{E}[e^{-\omega Z}] &= \frac{\mathbb{E} L_{II}^{(\tau_{II}-\tau_I)} + \mathbb{E} L_I^{(\tau_I)}}{\mathbb{E}\tau_{II}} \frac{\omega}{\phi_{II}(\omega)} + \frac{\mathbb{E} e^{-\omega Z^{(\tau_{II})}} - \mathbb{E} e^{-\omega Z^{(0)}}}{\phi_{II}(\omega) \mathbb{E}\tau_{II}} \\ &\quad + \frac{1}{\mathbb{E}\tau_{II}} \frac{\phi_{II}(\omega) - \phi_I(\omega)}{\phi_{II}(\omega)} \mathbb{E} \left[\int_{s=0}^{\tau_I} e^{-\omega Z^{(s)}} ds \right]. \end{aligned} \quad (55)$$

Hence, we have derived Equation (47). Now, using (54) to rewrite the final term in (55) and some rewriting provides Equation (48). \square

In addition to the M/G/1 dam, the equations of Lemma A.1 can be useful in several special cases, especially when $Z^{(0)} = Z^{(\tau_{II})}$ and ϕ_I and ϕ_{II} are related. Typical examples are Lévy storage models where the output is shut off every time the system reaches zero (as in, for instance, vacation models or service according to D -policies), see also [19]. As an easy application, we next consider an M/G/1 queue with service rate r and an exceptional first service during a busy period. Moreover, the result is a slight extension of familiar results, where service at unit speed is assumed, see e.g. [29], or [28], p. 128.

Example A.1. In this example we consider an M/G/1 queue with arrival rate λ , service speed r , and generic service requirement B_{II} . Each first customer in a busy period receives an exceptional service that is generically denoted by B_I . Let $\beta_i(\cdot)$, $i = I, II$, be the LST of B_i and denote by β_i its mean.

This model can easily be analyzed using Lemma A.1 and some trivial observations. In the general model with alternating exponents, let $Z^{(0)} = 0$, define τ_I as the first customer arrival epoch and let $\tau_{II} := \inf\{t > \tau_I : Z^{(t)} = 0\}$ be the end of the busy cycle. We may then take $\phi_i(\omega) = r\omega - \lambda + \lambda\beta_i(\omega)$, $i = I, II$. Note that there is no reflection in the second interval, hence $\mathbb{E}L_{II}^{(\tau_{II}-\tau_I)} = 0$. By definition of τ_I we have for the first interval that $Z^{(s)} = 0$, with $s \in [0, \tau_I)$, implying that $\mathbb{E}[\int_0^{\tau_I} e^{-\omega Z^{(s)}} ds] = \mathbb{E}\tau_I$. Moreover, since it holds for the free process that $X^{(\tau_I^-)} = -r\tau_I$, we have $\mathbb{E}L_I^{(\tau_I)} = r\mathbb{E}\tau_I$. Substituting the above in (47) and some straightforward rewriting yields

$$\mathbb{E}[e^{-\omega Z}] = \frac{\mathbb{E}\tau_I}{\mathbb{E}\tau_{II}} \frac{r\omega - \lambda\beta_I(\omega) + \lambda\beta_{II}(\omega)}{r\omega - \lambda + \lambda\beta_{II}(\omega)}.$$

The constant $\mathbb{E}\tau_I/\mathbb{E}\tau_{II}$ can be obtained by letting $\omega \downarrow 0$ and applying l'Hôpital's rule, giving the final result

$$\mathbb{E}[e^{-\omega Z}] = \frac{1 - \rho}{1 + \frac{\lambda\beta_I}{r} - \rho} \frac{r\omega - \lambda\beta_I(\omega) + \lambda\beta_{II}(\omega)}{r\omega - \lambda + \lambda\beta_{II}(\omega)},$$

where $\rho = \lambda\beta_{II}/r$. We refer to, e.g., [29], or [28], p. 128, in case $r \equiv 1$.

Another equation that is useful when $\mathbb{E}e^{-\omega Z^{(\tau_I)}}$ and the steady-state workload distribution of the first interval can be determined separately, is presented in the following lemma.

Lemma A.2. *For an alternating Lévy process as described above, we have the following relations*

$$\mathbb{E}\left[\int_{s=0}^{\tau_{II}} e^{-\omega Z^{(s)}} ds\right] = \frac{\mathbb{E}L_{II}^{(\tau_{II}-\tau_I)}\omega}{\phi_{II}(\omega)} + \frac{\mathbb{E}e^{-\omega Z^{(\tau_{II})}} - \mathbb{E}e^{-\omega Z^{(\tau_I)}}}{\phi_{II}(\omega)} + \mathbb{E}\left[\int_{s=0}^{\tau_I} e^{-\omega Z^{(s)}} ds\right]. \quad (56)$$

Proof. Dividing both sides of (52) by $\phi_{II}(\omega)$ and adding $\mathbb{E}[\int_{s=0}^{\tau_I} e^{-\omega Z^{(s)}} ds]$ to both sides directly gives the result. \square

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