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TRANSIENT MOMENTS OF THE WINDOW SIZE IN TCP

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Abstract

The window size in TCP can be modeled as a piecewise deterministic Markov process that increases linearly in time and experiences downward jumps at Poisson times. We present a transient analysis of this window size process. Our main result is the Laplace transform of the transient moments. Explicit formulae for the integer and fractional moments are derived, as well as an explicit characterization of the speed of convergence to steady-state. Central to our approach is the infinitesimal generator and Dynkin's martingale.

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1 Introduction

Data transfer over the Internet is predominantly controlled by TCP (Transmission Control Protocol), which adapts the window size (transmission rate) of data transfers to the congestion of the network. A TCP connection between a source and a destination progressively increases the window size, until it receives a signal that its path in the network is too congested, upon which the window size is drastically reduced. The most common implementation of TCP uses an additive-increase multiplicative-decrease (AIMD) algorithm. This allows the window size to increase linearly in the absence of congestion signals, whereas when congestion is detected, the window size is reduced by a multiplicative factor.

The emergence of TCP has spurred an enormous amount of research. In the pioneering work of Ott et al. [15], the window size process is approximated as a fluid model that constitutes a piecewise deterministic Markov process (PDMP). Our framework incorporates the model in [15], as well as some of the extensions made in Altman et al. [1], Altman et al. [2] and Guillemin et al. [12]. All these works restrict to the stationary behavior of the PDMP, which is tantamount to the assumption that the TCP connection is long enough so that its throughput is governed by the stationary regime. We obtain results on the transient moments of the PDMP. Our

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results may help in judging the effects that parameters have on the dynamics of the system.

We model the window size as a Markov process $(X_t)_{t\geq 0}$ that increases linearly with rate 1. Congestion signals arrive according to a Poisson process with rate λ, and upon receipt of the ith signal, the window size is reduced by multiplication with a random variable Q_i. We assume that $(Q_i)_{i\in\mathbb{N}}$ is a sequence of i.i.d. random variables with range in $[0, 1)$. Let Q denote a generic random variable equal in distribution to Q_i. TCP corresponds to the special case $Q = q$ for some constant q in $[0, 1)$, with $q = 1/2$ as most common choice.

We obtain all transient moments of the window size for any possible starting point $X_0 = x$. The main mathematical technique we use stems from the field of PDMPs and involves the analysis of the infinitesimal generator using Dynkin's formula.

A quantity of particular interest is the *relaxation time*, loosely defined as the time it takes for the PDMP to reach stationarity. Being in possession of the explicit formulae for the transient moments, we can measure relaxation time in terms of the difference between the transient and the stationary moments. Let $E_x(X_t^n)$ denote the nth moment of the Markov process at time t with $X_0 = x$. We find that (see Theorem 8)

$$E_x(X_t^n) = E(X_\infty^n) + \sum_{k=1}^{n} c_{k,n,x} e^{-\theta_k t}, \tag{1}$$

with $\theta_k = \lambda(1 - E(Q^k))$, and where both $E(X_\infty^n)$ and $c_{k,n,x}$ are fully expressed in terms of $\theta_1, \ldots, \theta_n$. From (1) we see that there is an exponential speed of convergence to the stationary moments. The relaxation time can be defined as the time until the difference between $E_x(X_t^n)$ and its stationary counterpart $E(X_\infty^n)$ is smaller than some predetermined value.

It seems evident that knowledge on transient behavior is useful for design and dimensioning purposes. By considering a more general model, we aim to account for a wide range of control mechanisms other than TCP and for future enhancements to congestion control.

There are some connections to other fields. First, our PDMP is part of a larger class of models known as growth-collapse processes, which are real-valued processes that grow between random collapse times, at which they jump down according to some distribution depending on their current level. This evolutionary pattern is encountered in a large variety of physical phenomena, see Eliazar & Klafter [9], like build-up of friction, earthquakes, avalanches, neuron firing, shot noise, and so on. Insurance mathematics [16], inventory theory [17] and queueing theory [4] are other fields where growth rate and occasional disasters are witnessed and analyzed. There is a second connection to the field of *stochastic recursive equations* of the type $X \overset{d}{=} Q \cdot X + Z$, where X, Q, Z are random variables, and X is independent of Q and Z. Indeed, the limiting random variable X_∞ of our Markov process satisfies such an equation. Vervaat [18] provides a detailed study of these equations and several examples of explicit solutions for particular choices of (Q, Z) (see also Gjessing & Paulsen [10]). A third connection is shown in Section 6 As it turns out, our Markov process is in distribution equal to the *exponential functional* associated to a Lévy process (compound Poisson process).

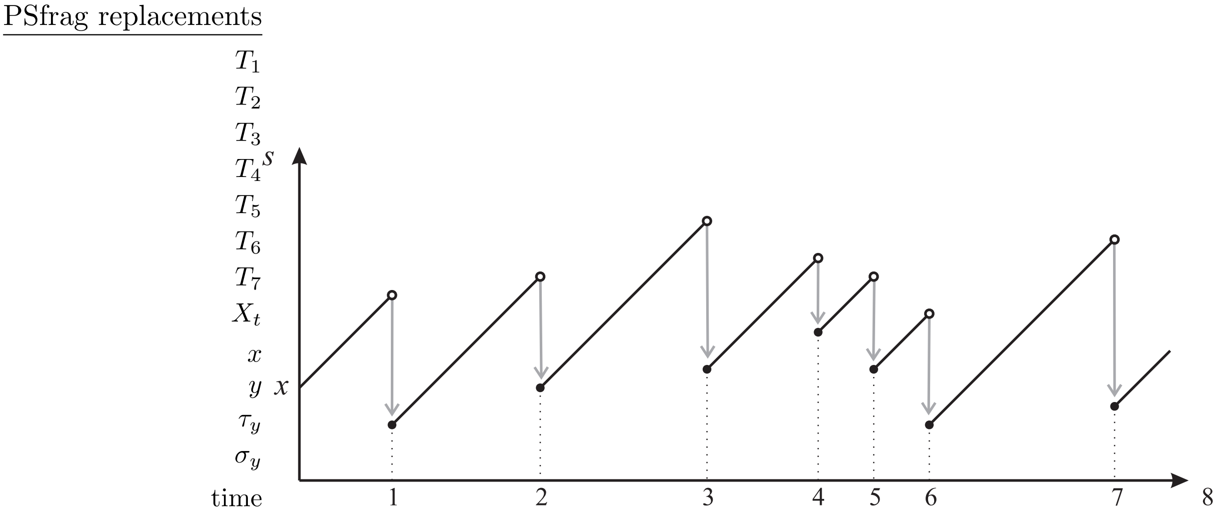

Figure 1: The process $(X_t)_{t\geq 0}$. Linear increase with slope one and random jumps at Poisson times $(T_i)_{i\in\mathbb{N}}$. The ith jump goes from X_{T_i-} to $X_{T_i} = Q_i \cdot X_{T_i-}$.

The paper is structured as follows. In Section 2 we give a detailed model description. The infinitesimal generator of the PDMP plays a fundamental role in our paper. We prove that a certain class of non-locally bounded functions belongs to the domain of the generator. In Section 3 we investigate the stationary distribution of the Markov process, making use only of the generator. Section 4 presents the transient analysis of the Markov process and comprises the core of this paper. We start with a derivation of the Laplace transform of the transient moments. The proof uses the generator, Dynkin's martingale, the solution of an inhomogeneous linear difference equation and the Bohr-Mollerup theorem. The Laplace transform is then shown to lead to fractional and integer moments of the stationary and transient distribution. A brief discussion follows in Section 5. We conclude the paper with the connection to Lévy processes in Section 6.

2 Model description

Consider a Markov process $(X_t)_{t\geq 0}$ that increases with slope one and has random jumps at Poisson times $(T_i)_{i\in\mathbb{N}}$. The ith jump goes from X_{T_i-} to $X_{T_i} = Q_i \cdot X_{T_i-}$, where $(Q_i)_{i\in\mathbb{N}}$ is a sequence of independent and identically distributed random variables with range $[0,1)$ and probability distribution function H. Let λ be the intensity of the Poisson process and let N_t count the number of jumps in $[0,t]$. Moreover, let Q denote a generic random variable with distribution function H and let

$$\theta_a = \lambda\left(1 - E(Q^a)\right),$$

for $a > a_{min} = \inf_{c\in\mathbb{R}}\{E(Q^c) < \infty\}$. It turns out that these quantities are crucial for the description of the transient behavior of $(X_t)_{t\geq 0}$. A connection with the Laplace exponent of an associated Lévy process is given in Section 6.

The process $(X_t)_{t\geq 0}$ is an example of a piecewise deterministic Markov process introduced by Davis [8]. The state space $\mathbb{S}$ consists of all non-negative real numbers, thus $\mathbb{S} = [0,\infty)$. If no jumps to zero occur, then we agree to exclude zero from $\mathbb{S}$ so that $\mathbb{S} = (0,\infty)$ in this case.

The full infinitesimal generator of the Markov process $(X_t)_{t\geq 0}$ is given by

$$\mathcal{A}f(x) = f'(x) - \lambda f(x) + \lambda \int_0^1 f(yx)\, dH(y) \ , \quad x \in \mathbb{S}. \tag{2}$$

The domain of the generator consists of all measurable functions $f : \mathbb{S} \to \mathbb{R}$ for which the process $f(X_t) - \int_0^t \mathcal{A}f(X_s)\, ds$ is a martingale. According to Davis [8], a function $f : \mathbb{S} \to \mathbb{R}$ belongs to the domain of $\mathcal{A}$ if it is absolutely continuous on $\mathbb{S}$ and the expectation of $\sum_{k=1}^{N_t} |f(X_{T_k -}) - f(X_{T_k})|$ is finite for every choice of $t \geq 0$ and $x > 0$. This is the case, for example, if f is absolutely continuous and locally bounded on $\mathbb{S}$ and this subclass will be sufficiently rich in most cases. However, since we also deal with certain non-locally bounded functions we show the following result, which identifies a subclass of the domain of $\mathcal{A}$ embracing functions like the negative powers x^a, $a < 0$.

Lemma 1. *Let $f : \mathbb{S} \to [0, \infty)$ be a non-increasing function with $f(x \cdot y) \leq f(x) \cdot f(y)$ for all $x, y \in \mathbb{S}$. Then f belongs to the domain of the generator if $E_x f(Q) < \infty$ for all $x \in \mathbb{S}$.*

Proof. Let $M_t = \min_{s \in [T_1 \wedge t, \leq t]} X_s$. Then, since for $k \leq N_t$ and thus $T_k \leq t$,

$$|f(X_{T_k -}) - f(X_{T_k})| \leq f(X_{T_k -}) + f(X_{T_k}) \leq 2f(X_{T_k}) \leq 2f(M_t).$$

We thus obtain

$$E_x \Big(\sum_{k=1}^{N_t} |f(X_{T_k -}) - f(X_{T_k})| \Big) \leq 2E_x \left(f(M_t) \cdot N_t \right).$$

Clearly $M_t = X_{T_k}$ for some $0 \leq k \leq N_t$, if we let $T_0 = 0$. Then

$$M_t = (\cdots((X_0 + Z_1) \cdot Q_1 + Z_2) \cdot Q_2 + \cdots) \cdot Q_k,$$

where $(Z_k)_{k\geq 1}$ are independent and exponentially distributed random variables. Thus $M_t \geq X_{(t\wedge T_1)-} \cdot W_t$, where $W_t = Q_1 \cdot Q_2 \cdots Q_{N_t}$. Consequently, using the monotonicity of f,

$$E_x(f(M_t) \cdot N_t) \leq E_x(f(X_{(t\wedge T_1)-} \cdot W_t) \cdot N_t).$$

Conditioning on the first jump time T_1 yields

$$\begin{aligned} E_x(f(X_{t\wedge T_1 -} \cdot W_t) \cdot N_t) &= \int_0^\infty \lambda e^{-\lambda y} E_x(f(X_{(t\wedge y)-} \cdot W_t) \cdot N_t \mid T_1 = y)\, dy \\ &= \int_0^t \lambda e^{-\lambda y} E_x(f((x+y) \cdot W_{t-y}) \cdot N_{t-y})\, dy. \end{aligned}$$

Next we condition on N_{t-y} to obtain

$$\begin{aligned} &\int_0^t \lambda e^{-\lambda y} E_x(f((x+y) \cdot W_{t-y}) \cdot N_{t-y})\, dy \\ &\qquad = \int_0^t \sum_{n=0}^\infty n \cdot P(N_{t-y} = n) \lambda e^{-\lambda y} E_x(f((x+y) \cdot Q_1 \cdot \ldots \cdot Q_n))\, dy \\ &\qquad \leq \int_0^t \sum_{n=0}^\infty n \cdot P(N_{t-y} = n) E_x f(Q)^n \lambda e^{-\lambda y} f(x+y)\, dy, \end{aligned}$$

which is clearly finite for all $x > 0$ and all $t \geq 0$ as long as $E_x f(Q) < \infty$. $\square$

3 Stationary behavior

Since for large values of X_t the downward jumps always dominate the deterministic linear increase and since the jump intensity λ is constant, it is plausible that the process will always be stable, in the sense that a limiting distribution for X_t exists as t tends to infinity. That this is actually the case is established in the following theorem.

Theorem 2. *The process $(X_t)_{t\geq 0}$ always has a stationary distribution.*

Proof. Let $z = \frac{1+\delta}{\theta_1}$ and $\tau_z = \inf\{t > 0 | X_t = z\}$. We first show that the mean of τ_z is finite if we start the process in $x \leq z$. Let $\tilde{X}_t$ be a process with the same deterministic behavior and the same jump times T_i as X_t but with $X_0 = 0$ and jumps that always go to zero. If we show that the expectation of $\tilde{\tau}_z = \inf\{t > 0 | \tilde{X}_t = z\}$ is finite, the same follows for τ_z, since $\tilde{X}_t$ always stays below X_t. Let N be the number of jumps before $\tilde{\tau}_z$. Then N has a geometric distribution with parameter $e^{-\lambda z}$ and $\tilde{\tau}_z$ is a geometric sum of random variables, which is bounded by z. It follows that $E_0 \tilde{\tau}^* < \infty$ and then that indeed $E_x \tau_z < \infty$.

Next we start the process X_t in $x \geq z$ and show that again $E_x \tau_z < \infty$. Let $p_1(x) = x$, then p_1 is in the domain of $\mathcal{A}$ and $\mathcal{A} p_1(x) = 1 - \theta_1 x$.

Choose a $\delta > 0$ and let $\tau = \inf\{t > 0 | X_t \leq z\}$. Up to time τ the process $X_t + \theta_1 \int_0^t X_s \, ds - t$ is a supermartingale, bounded below by $z + \delta t > 0$. It follows that

$$f(x) \geq E_x \Big(X_\tau + \theta_1 \int_0^\tau X_s \, ds - \tau \Big) \geq \frac{1+\delta}{\theta_1} + \delta E_x(\tau),$$

implying that $E_x \tau < \infty$. Hence, the expected time the process needs to go from z back to z, which is $E_z \tau_z + E(E_{X_{\tau_z}} \tau)$, is finite. From the theory of regenerative processes it follows that $(X_t)_{t\geq 0}$ has a stationary distribution (cf. Asmussen [4]). □

Theorem 3. *The density ν' of the stationary distribution ν satisfies the equation*

$$\nu'(z) = \lambda \left(\int_0^1 \nu(z/y) \, dH(y) - \nu(z) \right). \tag{3}$$

If $\psi(s) = \int_0^\infty e^{-st} \, d\nu(t)$ denotes the Laplace transform of ν then

$$\psi(s) = \frac{\lambda}{\lambda + s} \int_0^1 \psi(sy) \, dH(y). \tag{4}$$

Moreover, at least for $0 \leq s \leq \lambda$,

$$\psi(s) = \sum_{n=0}^\infty \frac{(-s)^n}{\prod_{k=1}^n \theta_k}. \tag{5}$$

Proof. We first consider the Laplace transform $\psi(s)$. The function $f(x) = e^{-sx}$ is a bounded member of the domain of $\mathcal{A}$ and

$$\mathcal{A} f(x) = -(\lambda + s) e^{-sx} + \lambda \int_0^1 e^{-syx} \, dH(y),$$

so that (4) follows by integrating with respect to ν, since for all bounded f in the domain of $\mathcal{A}$ we have $\int_0^\infty \mathcal{A}f(x)\, d\nu(x) = 0$. Equation (3) follows immediately by inversion.

Since $\liminf \theta_k \leq \lambda$ as $k \to \infty$, the radius of convergence of the series (5) is clearly larger than λ. By inserting (5) into (4) we find

$$\begin{aligned}
\frac{\lambda}{\lambda+s}\int_0^1 \sum_{n=0}^{\infty} \frac{(-sy)^n}{\prod_{k=1}^n \theta_k}\, dH(y) &= \frac{\lambda}{\lambda+s}\sum_{n=0}^{\infty} \frac{(-s)^n \int_0^1 y^n\, dH(y)}{\prod_{k=1}^n \theta_k} \\
&= \frac{\lambda}{\lambda+s}\sum_{n=0}^{\infty} \frac{(-s)^n \left(1-\frac{\theta_n}{\lambda}\right)}{\prod_{k=1}^n \theta_k} \\
&= \frac{\lambda}{\lambda+s}\left(\sum_{n=0}^{\infty} \frac{(-s)^n}{\prod_{k=1}^n \theta_k} + \frac{s}{\lambda}\sum_{n=0}^{\infty} \frac{(-s)^{n-1}}{\prod_{k=1}^{n-1} \theta_k}\right) \\
&= \sum_{n=0}^{\infty} \frac{(-s)^n}{\prod_{k=1}^n \theta_k},
\end{aligned}$$

and hence the series (5) actually represents ψ. □

The results of Theorem 3 are not new. In fact, the Laplace transform of the stationary distribution has been derived in Guillemin et al. [12], although both the setting and the method of proof is different from ours. They consider the same process, but represent our random variable Q as q^R with $q \in [0, 1)$ and R some non-negative random variable. Guillemin et al. [12] consider the stochastic recursive equation

$$X_\infty \stackrel{d}{=} Q \cdot X_\infty + Z, \tag{6}$$

where X_∞, Q, Z are independent and Z is exponential with mean $1/\lambda$. It follows then that

$$\psi(s) = Ee^{-sZ}Ee^{-sQX_\infty} = Ee^{-sZ}\int_0^1 Ee^{-syX_\infty}\, dH(y),$$

yielding (4). Also, for $n \in \mathbb{N}$,

$$E(X_\infty^n) = E((QX_\infty + Z)^n) = \sum_{k=0}^{n} \binom{n}{k} E(Q^k)E(X_\infty^k)E(Z^{n-k}),$$

(see also Gnedin et al. [11], p. 481) which gives

$$E(X_\infty^n) = \frac{n!}{\lambda^n \theta_n}\sum_{k=0}^{n-1}(\lambda-\theta_k)\frac{\lambda^k}{k!}E(X_\infty^k) = \frac{n!}{\prod_{i=1}^n \theta_i}. \tag{7}$$

These integer moments, with $\psi(s) = \sum_{n=0}^\infty E(X_\infty^n)\frac{(-s)^n}{n!}$ and after checking Carleman's criterion, then lead to (5); see Proposition 8 in [12].

Having this said, we point out that our proof of Theorem 3 strongly builds on the properties of the infinitesimal generator, an approach that shall prove its value in the upcoming section on transient analysis.

4 Transient moments

In what follows we derive our main result, namely a formula for the Laplace transform of the transient moments of the process. More precisely Theorem 5 provides a formula for

$$\mu_x^a(u) = \int_0^\infty e^{-ut} E_x(X_t^a)\, dt. \tag{8}$$

We start by observing that this function satisfies a certain difference equation in a.

Lemma 4. *If $a > a_{min}$, $-a \notin \mathbb{N} \cup \{0\}$, then $E_x(X_t^{a-1}) < \infty$ and*

$$\mu_x^a(u) \quad = \quad \frac{x^a + a\mu_x^{a-1}(u)}{\theta_a + u}. \tag{9}$$

Proof. The function $p_a(x) = x^a$ is absolutely continuous and if $E(Q^a) < \infty$ then p_a satisfies the conditions of Lemma 1. Indeed, p_a is non-increasing and $p_a(xy) = p_a(x)p_a(y)$. Consequently, p_a is in the domain of the generator. We have from (2) with $f(x) = p_a(x)$:

$$\mathcal{A}p_a(x) = ax^{a-1} - x^a \int_0^1 \lambda(1 - y^a)\, dH(y) = ax^{a-1} - \theta_a x^a.$$

Then, by Dynkin's formula (cf. Davis [8], Proposition 14.13),

$$p_a(X_t) - \int_0^t \mathcal{A}p_a(X_s)\, ds = X_t^a - \int_0^t aX_s^{a-1}\, ds - \theta_a \int_0^t X_s^a\, ds$$

is a martingale. In particular, its mean is constantly equal to $E_x X_0^a = x^a$, thus

$$E_x\Big(X_t^a - \int_0^t \left(aX_s^{a-1} - \theta_a X_s^a\right)\, ds\Big) = x^a. \tag{10}$$

Since $E_x X_t^a < \infty$ for $a \geq 0$ it follows from this formula that $E_x X_t^{a-1} < \infty$ for $a > -1$. Once $E_x X_t^a < \infty$ is established for $a \in (-1, 0)$, induction leads to $E_x X_t^a < \infty$ for $a > a_{min}$ and $-a \notin \mathbb{N} \cup \{0\}$. If $a \leq a_{min}$ then (10) is not assured since p_a may be not a member of the domain of $\mathcal{A}$.

Letting $f_a(t) = E_x(X_t^a)$, differentiation yields $f_a'(t) + \theta_a f_a(t) = af_{a-1}(t)$. Applying Laplace transforms we obtain

$$u\mu_x^a(u) - f_a(0) + \theta_a \mu_x^a(u) = a\mu_x^{a-1}(u).$$

and (9) follows from the initial condition $f_a(0) = x^a$. □

Once Lemma 4 is established, determining the Laplace transform (8) reduces to the solution of an inhomogeneous linear difference equation.

Theorem 5. *If $a > a_{min}$, $-a \notin \mathbb{N} \cup \{0\}$, then*

$$\begin{aligned} \mu_x^a(u) \quad &= \quad \frac{\Gamma(a+1)}{(\lambda+u)^a} \prod_{k=1}^\infty \frac{\theta_{a+k} + u}{\theta_k + u} \left(\frac{1}{u} + \sum_{m=1}^\infty \frac{x^m}{m!} \prod_{j=1}^{m-1} (\theta_j + u) \right) \\ &\quad - \sum_{k=1}^\infty \frac{\Gamma(a+1)x^{a+k}}{\Gamma(a+1+k)} \prod_{j=1}^{k-1} (\theta_{a+j} + u). \end{aligned} \tag{11}$$

Proof. We have to solve

$$A_a(u) \quad = \quad \frac{\theta_{a+1}+u}{a+1} A_{a+1}(u) - \frac{x^{a+1}}{a+1}. \tag{12}$$

The general solution $A_a(u)$ is given by $\omega(a,u)\tilde{A}_a(u) + A^*_a(u)$, where $\tilde{A}_a(u)$ is a solution of the homogeneous equation

$$\tilde{A}_a(u) \quad = \quad \frac{\theta_{a+1}+u}{a+1} \tilde{A}_{a+1}(u), \tag{13}$$

$A^*_a(u)$ is a particular solution of (12) and ω is an arbitrary periodic function with $\omega(a,u) = \omega(a+1,u)$ (see Milne-Thomson [14]). A naive solution is obtained from repeated application of (13) which leads to the expression $\prod_{k=1}^{\infty} \frac{\theta_{a+k}+u}{a+k}$. Unfortunately, the product is zero and yields only the trivial solution. We need a solution for which $\frac{\tilde{A}_a(u)}{\tilde{A}_{a+1}(u)} \approx \frac{\theta_{a+1}+u}{a+1}$ as $a \to \infty$ to prevent convergence of the infinite product to infinity or zero. We choose

$$\tilde{A}_a(u) \quad = \quad \frac{\Gamma(a+1)}{(\lambda+u)^a} \lim_{N\to\infty} \prod_{k=1}^{N} \frac{\theta_{a+k}+u}{\theta_k+u}.$$

It follows that

$$\begin{aligned}
\tilde{A}_a(u) \quad &= \quad \frac{\Gamma(a+2)}{(\lambda+u)^{a+1}} \frac{\lambda+u}{(a+1)} \lim_{N\to\infty} \prod_{k=1}^{N} \frac{\theta_{a+k}+u}{\theta_k+u} \\
&= \quad \frac{\Gamma(a+2)}{(\lambda+u)^{a+1}} \frac{\lambda+u}{(a+1)} \lim_{N\to\infty} \frac{\theta_{a+1}+u}{\theta_{a+N+1}+u} \prod_{k=1}^{N} \frac{\theta_{a+k+1}+u}{\theta_k+u} \\
&= \quad \frac{\theta_{a+1}+u}{(a+1)} \frac{\Gamma(a+2)}{(\lambda+u)^{a+1}} \lim_{N\to\infty} \prod_{k=1}^{N} \frac{\theta_{a+k+1}+u}{\theta_k+u} = \frac{\theta_{a+1}+u}{a+1} \tilde{A}_{a+1}(u),
\end{aligned}$$

as required by (13). Next we search for a particular solution of the inhomogeneous equation (12). This time the repeated application of (12) yields a valid particular solution, as can be checked by calculation:

$$A^*_a(u) = -\sum_{k=1}^{\infty} \frac{\Gamma(a+1)}{\Gamma(a+1+k)} \prod_{j=1}^{k-1} (\theta_{a+j}+u) x^{a+k}.$$

Thus

$$A_a(u) \quad = \quad \omega(a,u) \frac{\Gamma(a+1)}{(\lambda+u)^a} \prod_{k=1}^{\infty} \frac{\theta_{a+k}+u}{\theta_k+u} - \sum_{k=1}^{\infty} \frac{\Gamma(a+1)}{\Gamma(a+1+k)} \prod_{j=1}^{k-1} (\theta_{a+j}+u) x^{a+k}.$$

So far we do not know which choice of $\omega(a,u)$ yields the Laplace transform $\mu^a_x(u) = \int_0^\infty e^{-ut} E_x(X^a_t)\, dt$. We claim that $A_a(u) = \mu^a_x(u)$ if $\omega(a,u) = C/u$ for some constant C. It follows then from the initial conditions that this constant takes the form

$$C = 1 + u \sum_{k=1}^{\infty} \frac{x^k}{k!} \prod_{j=1}^{k-1} (\theta_j + u).$$

To prove our claim we mimic the inventive proof of Proposition 7 in Guillemin et al. [12] (see also Maulik & Zwart [13], Proposition 2.2, where the special case $x = 0$ and $u = 0$ is proved). Let $\omega(a, u) = C/u$ and define the function

$$G(a) = \frac{\mu_x^{a-1}(u)}{A_{a-1}(u)} \cdot \Gamma(a).$$

Our goal is to show that $G(a) = \Gamma(a)$. According to Bohr-Mollerup's theorem (see Andrews et al. [3]) it is enough to show that $G(a+1) = aG(a)$, $G(1) = 1$ and that G is convex.

Due to Equation (12), which is valid for both μ and A_a, we find that

$$x^a \left(1 - \frac{G(a+1)}{\Gamma(a+1)}\right) = \left(\frac{G(a+1)}{G(a)} - a\right) \mu_x^{a-1}(u)$$

and in particular

$$\left(\frac{G(a+1)}{G(a)} - a\right) \mu_0^{a-1}(u) = 0,$$

so that the required functional equation $G(a+1) = aG(a)$ holds. That $G(1) = 1$ follows from $A_0(u) = \mu_x^0(u)$. Moreover

$$\begin{aligned} \log G(a) &= \log \mu_0^{a-1}(u) + \log u - \log C \\ &\quad + (a-1) + \log(\lambda + u) + \sum_{k=1}^{\infty} \log\left(\frac{\theta_k + u}{\theta_{a-1+k} + u}\right). \end{aligned} \tag{14}$$

The function $\theta_a \mapsto \lambda(1 - EQ^a)$ is clearly concave since $a \mapsto EQ^a$ is convex, so that the series on the right in (14) is convex. The convexity of the middle terms in (14) is clear. To show that $\log \mu_0^{a-1}(u)$ is convex, note that $\mu_0^a(u) = \int_0^\infty e^{-ut} E_0 X_t^a \, dt$ and $E_0 X_t^a$ are convex (cf. the proof in [13]). □

If we multiply $\mu_x^a(u)$ in equation (11) by u and let $u \to 0$ we get the following corollary, identifying the fractional stationary moments.

Corollary 6. *If $a > a_{min}$ and $-a \notin \mathbb{N} \cup \{0\}$ then the fractional moments of the limiting distribution are given by*

$$E(X^a) = \frac{\Gamma(a+1)}{\lambda^a} \prod_{k=1}^{\infty} \frac{\theta_{a+k}}{\theta_k}.$$

A simple calculation leads to the following result, which can be found for the $x = 0$ case in Bertoin & Yor [5].

Corollary 7. *For integer values $a = n \in \mathbb{N}$ formula* (11) *reduces to*

$$\mu_x^n(u) = \frac{n!}{\prod_{k=1}^n (\theta_k + u)} \left(\frac{1}{u} + \sum_{k=1}^{n} \frac{x^k}{k!} \prod_{j=1}^{k-1} (\theta_j + u)\right). \tag{15}$$

An inversion of (15) is possible and results in the following theorem. It provides the precise rate of convergence of $E_x(X_t^n)$ to the stationary limit $E(X_\infty^n)$ in terms of exponential functions $e^{-\theta_m t}$.

Theorem 8. *If $n \in \mathbb{N}$ the n-th transient moment of X_t is*

$$E_x(X_t^n) = \frac{n!}{\prod_{i=1}^n \theta_i} + n! \sum_{m=1}^n \Big(\sum_{k=0}^m \frac{x^k}{k!} \prod_{j=k, j\neq m}^n \frac{1}{(\theta_j - \theta_m)} \Big) \cdot e^{-\theta_m t}. \tag{16}$$

Proof. By partial fraction expansion we obtain

$$\frac{1}{\prod_{j=k}^n (\theta_j + u)} = \sum_{m=k}^n \frac{1}{(\theta_m + u) \prod_{j=k, j\neq m}^n (\theta_j - \theta_m)},$$

so that we can write (15) as

$$\frac{\mu_x^n(u)}{n!} = \sum_{m=1}^n \frac{1}{\theta_m + u} \left(\frac{1}{u} \frac{1}{\prod_{j=1, j\neq m}^n (\theta_j - \theta_m)} + \sum_{k=1}^m \frac{x^k}{k!} \frac{1}{\prod_{j=k, j\neq m}^n (\theta_j - \theta_m)} \right).$$

Since $1/(\theta_m + u)$ is the Laplace transform of $e^{-\theta_m t}$ and $1/u \cdot 1/(\theta_m + u)$ is the transform of $(-e^{-\theta_m t})/\theta_m$, we have

$$\begin{aligned} \frac{E_x X_t^n}{n!} &= \sum_{m=1}^n \left(\frac{1 - e^{-\theta_m t}}{\theta_m \prod_{j=1, j\neq m}^n (\theta_j - \theta_m)} + \sum_{k=1}^m \frac{x^k}{k!} \frac{e^{-\theta_m t}}{\prod_{j=k, j\neq m}^n (\theta_j - \theta_m)} \right) \\ &= \sum_{m=1}^n \left(\frac{1}{\theta_m \prod_{j=1, j\neq m}^n (\theta_j - \theta_m)} + \left(\sum_{k=0}^m \frac{x^k}{k! \prod_{j=k, j\neq m}^n (\theta_j - \theta_m)} \right) \cdot e^{-\theta_m t} \right). \end{aligned}$$

A further application of partial fraction expansion yields (16). □

We remark that (16) may also be written as a polynomial in x:

$$E_x(X_t^n) = n! \sum_{k=0}^n \frac{x^k}{k!} \sum_{m=k}^n \prod_{j=k, j\neq m}^n \frac{e^{-\theta_m t}}{(\theta_j - \theta_m)}.$$

5 Discussion

Formula (16) leads to the following expression for the mean:

$$E_x X_t = \frac{1 - (1 - \theta_1 x) e^{-\theta_1 t}}{\theta_1}, \tag{17}$$

which shows exponential convergence to the stationary mean $EX_\infty = 1/\theta_1$. If the process is started in $x = 1/\theta_1$, then the transient mean stays constant, if started above (below) $1/\theta_1$, the mean stays above (below) $1/\theta_1$.

If the relaxation time would be defined as

$$r_x(\varepsilon) = \inf\{t \geq 0 \ : \ |1 - E_x X_t / EX_\infty| < \varepsilon\}, \tag{18}$$

the following result follows immediately from (17).

Corollary 9.

$$r_x(\varepsilon) = \frac{1}{\theta_1} \log \frac{|1-\theta_1 x|}{\varepsilon}, \tag{19}$$

for $\varepsilon < |1/\theta_1 - x|$.

A further measure for the speed of convergence to the steady state may follow from the formula for the variance, which is given by

$$\begin{aligned} \mathrm{Var}_x(X_t) &= \frac{2\theta_1-\theta_2}{\theta_1^2\theta_2} + 2\left(\frac{2\theta_1-\theta_2}{\theta_1^2(\theta_1-\theta_2)} - \frac{2\theta_1-\theta_2}{\theta_1(\theta_1-\theta_2)}x\right)e^{-\theta_1 t} \\ &\quad - \left(\frac{2}{\theta_2(\theta_1-\theta_2)} - \frac{2}{\theta_1-\theta_2}x - x^2\right)e^{-\theta_2 t} \\ &\quad - \left(\frac{1}{\theta_1^2} - \frac{2}{\theta_1}x + x^2\right)e^{-2\theta_1 t}. \end{aligned} \tag{20}$$

Note that the exponential terms appear according to their asymptotic order since

$$\theta_1 \le \theta_2 = \lambda(1-E(Q^2)) \le \lambda(1-(EQ)^2) = \lambda(1-EQ)(1+EQ) \le 2\theta_1.$$

If we start the process in $EX_\infty = 1/\theta_1$ then (20) reduces to the handy expression

$$\mathrm{Var}_x(X_t) \quad = \quad \mathrm{Var}X_\infty \cdot \left(1-e^{-\theta_2 t}\right).$$

For the case $Q \overset{d}{=} U$, with U a uniformly distributed random variable on [0,1), and $x=0$, Theorems 4 and 5 in Boxma et al. [6] coincide with (17) and (20), respectively.

6 Connection to Lévy processes

The transformation $X_t \mapsto \log X_t$ converts the multiplicative jumps of the process X_t into jumps with i.i.d. jump sizes. Let $L_t = -\log W_t = -\sum_{k=1}^{N_t} \log Q_k$ be the associated compound Poisson process. Then the process $Y_t = X_t \cdot e^{L_t}$ has absolutely continuous paths. Moreover e^{L_t} is piecewise constant and X_t has slope one between the jumps, so that the density of Y_t is given by $Y_t' = e^{L_t}X_t' + 0 = e^{L_t}$. Hence

$$X_t = e^{-L_t}X_0 + \int_0^t e^{L_s - L_t}\, ds \overset{d}{=} e^{-L_t}X_0 + \int_0^t e^{L_{s-t}}\, ds,$$

since L_t has stationary increments and $X_0 = Y_0$. It follows that

$$X_t \overset{d}{=} e^{-L_t}X_0 + \int_0^t e^{-L_s}\, ds. \tag{21}$$

The function $a \mapsto \theta_a$ is the Laplace exponent of the Lévy process L_t, since

$$\theta_a = \lambda\left(1 - \int_0^1 q^a\, dH(q)\right) \quad = \quad \lambda\left(1-\int_0^\infty e^{-ua}\, dH(e^{-u})\right) = \lambda\left(1-\beta(a)\right),$$

where $\beta(a)$ is the Laplace transform of $-\log Q$.

For $t \to \infty$ it is readily seen from (21) that

$$X_\infty \stackrel{d}{=} \int_0^\infty e^{-L_s}\, ds,$$

which relates the stationary distribution of the Markov process $(X_t)_{t\geq 0}$ to the terminating value of the exponential functional. This relation was already observed in Guillemin et al. [12], Section 3 (see also Carmona et al. [7]).

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