Duality and exact correlations for a model of heat conduction

Cristian Giardinà [*]
Jorge Kurchan [†]
Frank Redig [‡]

May 22, 2007

Abstract

We study a model of heat conduction with stochastic diffusion of energy. We obtain a dual particle process which describes the evolution of all the correlation functions. An exact expression for the covariance of the energy exhibits long-range correlations in the presence of a current. We discuss the formal connection of this model with the simple symmetric exclusion process.

1   Introduction

Simple systems of particles on a lattice have received considerable attention in the last years, as they are a testing ground for exploring the properties of far from equilibrium states, for which at present no general theory is available.

Amongst the most studied models are the so called exclusion processes in which particles diffuse amongst empty sites on a lattice. The problem has been attacked with techniques of statistical mechanics and probability, and many analytic results are available [15, 20, 17], in particular for the simple symmetric exclusion process (SEP)

One is also interested in the transport of continuous quantities, especially energy. Kipnis, Marchioro and Presutti (KMP) [11] have introduced a model of energy transport in which the energy of neighboring sites is randomly redistributed. This model

---

[*]Department of Mathematics and Computer Science, Eindhoven University, P.O. Box 513 - 5600 MB Eindhoven, The Netherlands, *c.giardina@tue.nl*

[†]CNRS-ESPCI, rue Vauquelin 10, 75231 Paris, France, *jorge@pmmh.espci.fr*

[‡]Mathematisch Instituut Universiteit Leiden, Niels Bohrweg 1, 2333 CA Leiden, The Netherlands, *redig@math.leidenuniv.nl*

has also been thoroughly studied, although analytic solutions - in particular for the steady-state correlation functions - are harder to obtain than in the SEP. Both the KMP and the SEP have an interesting (and rather exceptional) feature: the evolution of the K-point correlation functions can be exactly mapped onto a diffusion equation for K particles. This so-called 'duality property' [15] is a powerful tool, which in the study of the SEP yields the complete ergodic theory (see [15]).

Despite the resemblance between energy and particle transport, studying both models is not a redundant exercise, as the SEP and the KMP models also show intriguing differences [4], mainly in their large-deviation functions. Perhaps the most striking is the fact that in the implementation of an 'additivity principle' of the large deviation functions of two subsystems, the density at the interface has to be maximized in one case, and minimized on the other, in order to obtain the correct result [6].

In this paper we shall study a family of models of energy transport which, we shall argue, are the natural counterpart of the SEP models. In each site i of the lattice there is a free particle with momentum x_i. Between any two neighboring sites $(i, i+1)$ and for any small time interval there is a random exchange of momentum that leaves $\{x_i^2 + x_{i+1}^2\}$ invariant. The same transport terms were already considered in models of wave propagation in random media [21],[1]. They later appeared as the high-energy limit of a chain with deterministic dynamics [9], and were also used as a stochastic perturbation (mimicking nonlinearities) of an oscillator chain [3],[8].

In the present paper we construct for our energy diffusion model a dual process that expresses the evolution of the K-point correlation functions of the kinetic energies in terms of a process of K interacting random walkers. We also give a closed expression for the stationary covariance $\langle x_i^2; x_j^2 \rangle$ which confirms the presence of long-range correlations in the non-equilibrium stationary state (as already found before in the SEP [19]).

The energy diffusion model is clearly very close from the physical point of view to the KMP model. But at the same time, as we have mentioned, it can be viewed as the continuation of the SEP family, in the following sense: on the one hand, the SEP can naturally be generalized [18] to processes in which up to n particles are allowed per site (the usual SEP has $n = 1$), and hopping rates are proportional to the number of particles at the departure and the number of 'holes' at the arrival site. On the other hand, the energy diffusion model can be generalized to having in each site i of the lattice m free particles with momenta $x_{i,\alpha}$, $\alpha = 1, ..., m$, and the random exchange between $x_{i,\alpha}$ and $x_{i+1,\beta}$ leaves $\sum_\gamma \{x_{i,\gamma}^2 + x_{i+1,\gamma}^2\}$ invariant. We shall see below that the energy transport model with m particles per site formally corresponds to the continuation of the SEP family to *negative* occupation number: $m = -n/2$. In other words, the energy diffusion model as defined above is, formally, *'the SEP with* $-1/2$ *particles per site'.*

The rest of our paper is organized as follows. In section 2 we define our energy diffusion model. In section 3 we construct the dual process of interacting random walkers and derive the basic corollaries of duality, namely, existence and uniqueness of the stationary measure and expression of the stationary correlation functions in

terms of absorption probabilities. In sections 4 and 5 we derive the exact stationary temperature profile and energy-energy correlation function. In section 6 we show that our model satisfies local equilibrium. In section 7 we show a formal connection between our model and the SEP family, showing that the energy diffusion model can be viewed as a bosonic version of the SEP. In the last section we discuss possible asymmetric extensions of our model.

2 The model

The model is defined as a stochastic process on N-dimensional vectors $(x_1, \ldots, x_N) \in \mathbb{R}^N$ which have to be interpreted as momenta associated to lattice sites $\{1, \ldots, N\}$. Additionally, lattice sites 1 and N are in contact with a heat reservoir at temperature T_L resp. T_R.

The process is defined by its generator L (acting on the core of $\mathcal{C}^\infty$ functions f with compact support)

$$L = L_1 + L_N + \sum_{i=1}^{N-1} L_{i,i+1} \tag{2.1}$$

with

$$L_1 f = T_L \frac{\partial^2 f}{\partial x_1^2} - x_1 \frac{\partial f}{\partial x_1} \tag{2.2}$$

$$L_N f = T_R \frac{\partial^2 f}{\partial x_N^2} - x_N \frac{\partial f}{\partial x_N} \tag{2.3}$$

$$L_{i,i+1} f = \left(x_i \frac{\partial}{\partial x_{i+1}} - x_{i+1} \frac{\partial}{\partial x_i} \right)^2 (f) \tag{2.4}$$

This corresponds, in the language of Fokker-Planck equation (or master equation) to the following evolution equation for the time-dependent probability density $p(x,t)$:

$$\frac{\partial p(x,t)}{\partial t} = L^* p(x,t) \tag{2.5}$$

where L^* is the adjoint (in $L^2(dx)$) of L, i.e., more explicitly,

$$L^* = L_1^* + L_N^* + \sum_{i=1}^{N-1} L_{i,i+1}$$

$$L_1^* f = T_L \frac{\partial^2 f}{\partial x_1^2} + \frac{\partial}{\partial x_1}(x_1 f)$$

$$L_N^* f = T_R \frac{\partial^2 f}{\partial x_N^2} + \frac{\partial}{\partial x_N}(x_N f)$$

3

Let us first motivate the choice of this generator (2.1). The L_1 and L_N part is the generator of the usual Ornstein-Uhlenbeck process which represents thermalising noise corresponding to heat baths at temperatures T_L, resp. T_R.

To explain the other part, consider the operator

$$\mathcal{A} = \left(x\frac{\partial}{\partial y} - y\frac{\partial}{\partial y} \right)^2 \tag{2.6}$$

In polar coordinates $x = r\cos\theta$, $y = r\sin\theta$, this operator simply reads

$$\mathcal{A} = \frac{\partial^2}{\partial\theta^2}$$

which means that in the process $(x(t), y(t))$ corresponding to (2.6), $r(t) = r(0)$ and $\theta(t)$ performs a Brownian motion on the interval $[0, 2\pi]$. More precisely, from the Itô formula, it follows that the generator (2.6) corresponds to the stochastic differential equations (in Itô sense)

$$
\begin{aligned}
dx(t) &= -x(t)dt + \sqrt{2}\,y(t)dB(t) \\
dy(t) &= -y(t)dt - \sqrt{2}\,x(t)dB(t)
\end{aligned}
\tag{2.7}
$$

where $B(t)$ is standard Brownian motion. In other words $r^2(t) = x^2(t) + y^2(t) = r^2(0)$ with probability one and the angular variable $\theta(t) = \arctan(y(t)/x(t))$ is a martingale.

The bulk part $\sum_i L_{i,i+1}$ of the process $(x_1(t), \ldots, x_N(t))$ corresponds then to the stochastic differential equations

$$
\begin{aligned}
dx_1(t) &= -x_1(t)dt + \sqrt{2}x_2(t)dB_{1,2}(t) \\
dx_i(t) &= -2x_i(t)dt + \sqrt{2}\,x_{i+1}(t)dB_{i,i+1}(t) - \sqrt{2}\,x_{i-1}(t)dB_{i-1,i}(t) && i \in [2, N-1] \\
dx_N(t) &= -x_N(t)dt - \sqrt{2}x_{N-1}(t)dB_{N-1,N}(t)
\end{aligned}
\tag{2.8}
$$

where $B_{i,i+1}(t)$ $(i = 1, \ldots, N-1)$ are independent Brownian motions and $B_{0,1}(t) = B_{N,N+1}(t) = 0$. In this process, the total kinetic energy $\sum_i x_i^2(t)$ is conserved, i.e., $\sum_i x_i^2(t) = \sum_i x_i^2(0)$ with probability one as can be seen easily from Itô's formula.

The full process, i.e., the process with generator (2.1) (including the boundary terms L_1, L_N) corresponds to the system of stochastic differential equations

$$
\begin{aligned}
dx_1(t) &= -2x_1(t)dt + \sqrt{2}x_2(t)dB_{1,2}(t) + \sqrt{2T_L}dW(t) \\
dx_i(t) &= -2x_i(t)dt + \sqrt{2}\,x_{i+1}(t)dB_{i,i+1}(t) - \sqrt{2}\,x_{i-1}(t)dB_{i-1,i}(t) && i \in [2, N-1] \\
dx_N(t) &= -2x_N(t)dt - \sqrt{2}x_{N-1}(t)dB_{N-1,N}(t) + \sqrt{2T_R}dW'(t)
\end{aligned}
\tag{2.9}
$$

where $W(t), W'(t)$ are two independent Brownian motions, independent of all the other $B_{i,i+1}(t)$.

3 Duality

The main tool which considerably simplifies the analysis of this model is duality. First note that the equations for the evolution of correlation functions of degree n for the x process are *closed*, i.e., the time derivative of the expectation of a polynomial of degree n in the variables $x_1, \ldots, x_n$ does not involve expectations of polynomials of higher order [9].

We now show that the evolution in time of well-chosen polynomials reduces to a Markovian evolution of their *indices*. If we interpret the indices of the polynomials as discrete (indistinguishable) particle configurations (i.e., specifying for each site $i \in \{0, \ldots, N+1\}$ the number of particles), then the evolution of the indices turns out to become a jump process that conserves the total number of particles. This jump process is called the dual process. A similar situation arises in the case of the SEP (which is self-dual) [15], in the case of an infinite system of independent random walkers (where the Poisson polynomials have the self-dual property) [10], and in the KMP model [11].

We index our polynomials by a vector $\xi = (\xi_0, \ldots, \xi_{N+1})$, $\xi_i \in \mathbb{N}$ and introduce the notation $\vec{x} = (x_0, x_1, \ldots, x_N, x_{N+1})$, where the first and last components are *fixed* by $x_0 = \sqrt{T_L}$ and $x_{N+1} = \sqrt{T_R}$. The polynomial $D(\xi, \vec{x})$ is then defined by

$$D(\xi, \vec{x}) = T_L^{\xi_0} T_R^{\xi_{N+1}} \prod_{i=1}^{N} \frac{x^{2\xi_i}}{(2\xi_i - 1)!!} \tag{3.1}$$

Only even powers of x_i need to be considered, since other stationary correlations vanish due to the invariance of the generator under the transformation $x_i \to -x_i$. We interpret $\xi \in \Omega = \mathbb{N}^{N+2}$ as prescribing the number of particles in each lattice site $i \in \{0, \ldots, N+1\}$.

In order to introduce the generator of the dual process, we define, for $\xi \in \Omega$, $i, j \in \{0, \ldots, N+1\}$, the configuration $\xi^{i,j}$ to be the configuration obtained from ξ by removing one particle at i and adding one particle at j. On the dual variables $\xi \in \Omega$, we then define the following generator.

$$
\begin{aligned}
\mathcal{L}\varphi(\xi) \; := \; & 2\xi_1[\varphi(\xi^{1,0}) - \varphi(\xi)] + 2\xi_1(2\xi_2 + 1)[\varphi(\xi^{1,2}) - \varphi(\xi)] \\
& + \; \sum_{i=2}^{N-1} \Big(2\xi_i(2\xi_{i-1} + 1)[\varphi(\xi^{i,i-1}) - \varphi(\xi)] + \; 2\xi_i(2\xi_{i+1} + 1)[\varphi(\xi^{i,i+1}) - \varphi(\xi)] \Big) \\
& + \; 2\xi_N(2\xi_{N-1} + 1)[\varphi(\xi^{N,N-1}) - \varphi(\xi)] + 2\xi_N[\varphi(\xi^{N,N+1}) - \varphi(\xi)] \qquad (3.2)
\end{aligned}
$$

where $\varphi : \Omega \to \mathbb{R}$ is an arbitrary function of the finite particle configurations.

In words, we can describe the process generated by $\mathcal{L}$ as follows: a configuration $\xi = (\xi_0, \ldots, \xi_{N+1})$ represents K particles (or walkers) on $\{0, 1, \ldots, N+1\}$ with $K = \sum_{i=0}^{N+1} \xi_i$. The walkers can only jump to neighboring sites and are stuck when arriving to sites 0 or $N+1$. The rate at which there is a jump of a walker depends on how many walkers there are at neighboring sites. If we have ξ_i walkers at site i, ξ_{i-1} walkers at site $i-1$ and ξ_{i+1} walkers at site $i+1$ (for $i = 2, \ldots N-1$) then each of

the walkers at site i jumps to site $i-1$ at rate $2(2\xi_{i-1}+1)$ and to site $i+1$ at rate $2(2\xi_{i+1}+1)$. At the boundaries: each of the ξ_1 walkers at site 1 is absorbed at site 0 at rate 2 and it jumps to site 2 at rate $2(2\xi_2+1)$; each of the ξ_N walkers at site N is absorbed at site $N+1$ at rate 2 and it jumps to site $N-1$ at rate $2(2\xi_{N-1}+1)$. Note that this process conserves the number of particles, i.e.,

$$|\xi(t)| = \sum_{i=0}^{N+1} \xi_i(t) = \sum_{i=0}^{N+1} \xi_i(0) = |\xi(0)|$$

For a single particle, i.e., $\xi = \delta_i$, the dual process is then $\xi(t) = \delta_{X(t)}$ where $X(t)$ is a continuous time simple symmetric nearest neighbor random walk jumping at rate 2 and absorbed upon hitting 0 or $N+1$. For two particles, i.e., $\xi = \delta_i + \delta_j$ the particles perform independent symmetric nearest neigbor random walks at rate 2, *except* when they are sitting at *neighboring sites*. In that case, i.e., if the two walkers are at neigboring places, one of them jumps to the place of the other one at rate 6 (and other jumps are still at rate 2).

Remark 1. *This attractive interaction between the dual walkers has to be compared with the repulsive (hard-core) interaction between the walkers in the SEP.*

In order to formulate our duality result and its consequences, we denote by $\hat{\mathbb{E}}_\xi$ expectation in the dual process (i.e., the process with generator (3.2)) starting from ξ. We now can formulate the duality result.

Theorem 1. *Let $\xi(t)$ denote the process with generator (3.2) and $\vec{x}(t)$ the process $(x_0(t), x_1(t), x_2(t), \ldots, x_N(t), x_{N+1}(t))$ where $(x_1(t), \ldots, x_N(t))$ is the process with generator (2.1), and where $x_0(t) = \sqrt{T_L}$, $x_{N+1}(t) = \sqrt{T_R}$. Then we have*

$$\mathbb{E}_{\vec{x}}[D(\xi, \vec{x}(t))] = \hat{\mathbb{E}}_\xi[D(\xi(t), \vec{x})] \tag{3.5}$$

Proof. Start from (2.1). For $i = 1, \ldots, N-1$ we have

$$L_{i,i+1}D(\xi, x) = T_L^{\xi_0} T_R^{\xi_{N+1}} \left(\prod_{k \notin \{i, i+1\}} \frac{x_k^{2\xi_k}}{(2\xi_k - 1)!!} \right) \times$$

$$\left(2\xi_{i+1}(2\xi_{i+1}-1)\frac{x_i^{2\xi_i+2}}{(2\xi_i-1)!!}\frac{x_{i+1}^{2\xi_{i+1}-2}}{(2\xi_{i+1}-1)!!} - 2\xi_i(2\xi_{i+1}+1)\frac{x_i^{2\xi_i}}{(2\xi_i-1)!!}\frac{x_{i+1}^{2\xi_{i+1}}}{(2\xi_{i+1}-1)!!} \right.$$

$$\left. -2\xi_{i+1}(2\xi_i+1)\frac{x_i^{2\xi_i}}{(2\xi_i-1)!!}\frac{x_{i+1}^{2\xi_{i+1}}}{(2\xi_{i+1}-1)!!} + 2\xi_i(2\xi_i-1)\frac{x_i^{2\xi_i-2}}{(2\xi_i-1)!!}\frac{x_{i+1}^{2\xi_{i+1}+2}}{(2\xi_{i+1}-1)!!} \right)$$

which implies

$$\begin{aligned} L_{i,i+1}D(\xi, x) &= \Big(2\xi_{i+1}(2\xi_i+1)[D(\xi^{i+1,i}, x) - D(\xi, x)] \\ &\quad + 2\xi_i(2\xi_{i+1}+1)[D(\xi^{i,i+1}, x) - D(\xi, x)]\Big) \end{aligned}$$

6

Furthermore

$$
\begin{aligned}
L_1 D(\xi, x) &= T_R^{\xi_{N+1}} \left(\prod_{k \notin \{1\}} \frac{x_k^{2\xi_k}}{(2\xi_k - 1)!!} \right) \times \\
&\qquad \left(T_L^{\xi_0+1} \, 2\xi_1(2\xi_1 - 1) \frac{x_1^{2\xi_1-2}}{(2\xi_1 - 1)!!} - T_L^{\xi_0} \, 2\xi_1 \frac{x_1^{2\xi_1}}{(2\xi_1 - 1)!!} \right) \\
&= 2\xi_1 [D(\xi^{1,0}, x) - D(\xi, x)]
\end{aligned}
$$

and

$$
\begin{aligned}
L_N D(\xi, x) &= T_L^{\xi_0} \left(\prod_{k \notin \{N\}} \frac{x_k^{2\xi_k}}{(2\xi_k - 1)!!} \right) \times \\
&\qquad \left(T_R^{\xi_{N+1}+1} \, 2\xi_N(2\xi_N - 1) \frac{x_N^{2\xi_N-2}}{(2\xi_N - 1)!!} - T_R^{\xi_{N+1}} \, 2\xi_N \frac{x_N^{2\xi_N}}{(2\xi_N - 1)!!} \right) \\
&= 2\xi_N [D(\xi^{N,N+1}, x) - D(\xi, x)]
\end{aligned}
$$

Therefore, we obtain

$$
LD(\xi, \vec{x}) = \mathcal{L}D(\xi, \vec{x}) \tag{3.6}
$$

where in the lhs of (3.6) the operator L is working on the $\vec{x}$ variable, and in the rhs the operator $\mathcal{L}$ is working on the ξ variable.

This relation then lifts to the semigroups and the processes via a standard argument, see e.g. the proof of theorem 1.1 p. 363 in [15]. (Notice that informally, by (3.6) $\mathbb{E}_x D(\xi, \vec{x}(t)) = e^{tL} D(\xi, \cdot)(\vec{x}) = e^{t\mathcal{L}} D(\cdot, \vec{x})(\xi) = \hat{\mathbb{E}}_\xi D(\xi(t), \vec{x})$). $\qquad\square$

Notice that in the dual process all particles are eventually absorbed at one of the boundaries $0, N + 1$. Hence, the limiting configuration of the dual process starting from $\xi \in \Omega$ will always be of the form $k\delta_0 + l\delta_{N+1}$ where $k + l = \sum_i \xi_i$. We say that a function $f : \Omega \to \mathbb{R}$ is harmonic for the process generated by $\mathcal{L}$ if $f(\xi) = \hat{\mathbb{E}}_\xi[f(\xi(t))]$. Since the configurations of the type $k\delta_0 + l\delta_{N+1}$ are absorbing, the corresponding absorption probabilities

$$
c_{kl}(\xi) = \hat{\mathbb{P}}_\xi(\xi(\infty) = k\delta_0 + l\delta_{N+1}) \tag{3.7}
$$

are harmonic.

The following propositions gives some consequences of duality.

Proposition 1. *If $T_L = T_R = T$, then the unique stationary measure of the process $x(t) = (x_1(t), \ldots, x_N(t))$ is the Gaussian product measure with probability density function*

$$
\rho_T(x) = \frac{1}{(2\pi T)^{N/2}} \exp\left(-\sum_{i=1}^{N} \frac{x_i^2}{2T} \right) \tag{3.9}
$$

From every initial condition $(x_1, \ldots, x_N)$ the process converges in distribution to ρ_T.

Proof. We first show stationarity of ρ_T. If $T_L = T_R = T$, for the Gaussian measure (3.9) we have

$$\int D(\xi, \vec{x}) \rho_T(x) dx = T^{|\xi|}$$

where $|\xi| = \sum_{i=1}^N \xi_i$. Therefore, using (3.5)

$$
\begin{aligned}
\int \mathbb{E}_{\vec{x}}[D(\xi, \vec{x}(t))] \rho_T(x) dx &= \hat{\mathbb{E}}_\xi \int D(\xi(t), \vec{x}) \rho_T(x) dx \\
&= \hat{\mathbb{E}}_\xi T^{|\xi(t)|} \\
&= T^{|\xi|} = \int D(\xi, \vec{x}) \rho_T(x) dx \qquad (3.10)
\end{aligned}
$$

where in the third equality we used that in the dual process the number of particles is conserved. Next, to prove the convergence, remember that in the dual process eventually all particles are absorbed at 0 or at $N+1$. Therefore, if $T_L = T_R = T$,

$$
\begin{aligned}
\lim_{t \to \infty} \mathbb{E}_{\vec{x}}(D(\xi, \vec{x}(t))) &= \sum_{k,l : k+l=|\xi|} \hat{\mathbb{P}}_\xi(\xi(\infty) = k\delta_0 + l\delta_{N+1}) T^k T^l = T^{|\xi|} \\
&= \int D(\xi, \vec{x}) \rho_T(x) dx \qquad (3.11)
\end{aligned}
$$

$\square$

Proposition 2. *For all $T_L \neq T_R$, there is a unique stationary measure μ_{T_L, T_R} which has finite moments of every order, and for this measure one has*

$$\mu_{T_L, T_R}(D(\xi, \vec{x})) = \sum_{k,l \,:\, k+l=|\xi|} T_L^k T_R^l c_{kl}(\xi) \qquad (3.13)$$

where $c_{kl}(\xi)$ are the absorption probabilities defined in (3.7).

Proof. If μ_{T_L, T_R} is invariant and has finite moment of all order, then, by (3.5) we have

$$
\begin{aligned}
\int D(\xi, \vec{x}) \mu_{T_L, T_R}(dx) &= \lim_{t \to \infty} \int \mathbb{E}_x(D(\xi, \vec{x}(t))) \mu_{T_L, T_R}(dx) \\
&= \int \left(\lim_{t \to \infty} \hat{\mathbb{E}}_\xi(D(\xi(t), \vec{x})) \right) \mu_{T_L, T_R}(dx) \\
&= \sum_{k,l \,:\, k+l=|\xi|} T_L^k T_R^l \hat{\mathbb{P}}_\xi(\xi(\infty) = k\delta_0 + l\delta_{N+1}) \qquad (3.14)
\end{aligned}
$$

where in the last equation we used that eventually all particle are absorbed at one of the absorbing states 0 or $N+1$. Conversely, if (3.13) holds, then, since the absorption probabilities are harmonic for the dual process, by (3.5) we have

$$\int D(\xi, \vec{x}) \mu_{T_L, T_R}(dx) = \int \hat{\mathbb{E}}_\xi[D(\xi(t), \vec{x})] \mu_{T_L, T_R}(dx) = \int \mathbb{E}_{\vec{x}}[D(\xi, \vec{x}(t))] \mu_{T_L, T_R}(dx)$$

which shows invariance of μ_{T_L, T_R}. $\square$

8

4 Temperature profile.

The following proposition shows the convergence to and linearity of the limiting temperature profile (linearity of the stationary temperature profile has been shown before in [9]).

Proposition 3.

a) *Starting from $x \in \mathbb{R}^N$, the local temperature at site i satisfies*

$$\mathbb{E}_x(x_i^2(t)) = \hat{\mathbb{E}}_i(x_{X(t)}^2) = \sum_j p_t(i,j)x_j^2 \tag{4.2}$$

where $X(t)$ is continuous-time simple symmetric random walk jumping at rate 2, and absorbed at 0, $N+1$, and where $p_t(i,j)$ denotes the transition probability of this random walk to go from i to j in time t.

b) *The stationary temperature profile is given by*

$$
\begin{aligned}
T_i := \mu_{T_L,T_R}(x_i^2) &= T_L \,\hat{\mathbb{P}}_i(X(\infty) = 0) + T_R \,\hat{\mathbb{P}}_i(X(\infty) = N + 1) \\
&= T_L \left(1 - \frac{i}{N+1}\right) + T_R \left(\frac{i}{N+1}\right) \tag{4.3}
\end{aligned}
$$

c) *For the stationary local temperature autocorrelation function we have*

$$\mathbb{E}_{\mu_{T_L,T_R}}(x_i^2(0)x_j^2(t)) - T_iT_j = \sum_{k=1}^N \hat{\mathbb{P}}_j(X_t = k, \tau > t)\mu_{T_L,T_R}(x_i^2x_k^2) - T_iT_j\hat{\mathbb{P}}_j(\tau > t) \tag{4.4}$$

where τ is the absorption time of the random walk $\{X_t : t \geq 0\}$. As a consequence,

$$\lim_{t\to\infty} \frac{\mathbb{E}_{\mu_{T_L,T_R}}(x_i^2(0)x_j^2(t)) - T_iT_j}{\hat{\mathbb{P}}_j(\tau > t)} = -T_iT_j + \frac{1}{N}\sum_{k=1}^N \mu_{T_L,T_R}(x_i^2x_k^2) \tag{4.5}$$

Proof.

a) This follows from (3.5) and the fact that the dual process starting from a single particle at i is given by the random walk $X(t)$.

b) This follows by taking the limit $t \to \infty$ in (4.2) and the well-known formula for the absorption probabilities of the random walk $X(t)$.

c) The first statement (4.4) follows from (4.2) and (4.3). The second statement (4.5) follows from the fact that the quasistationary distribution of the random walk $X(t)$ is the uniform measure on $\{1, \ldots, N\}$, i.e., for any function $f : \{1, \ldots, N\} \to \mathbb{R}$:

$$\lim_{t\to\infty} \hat{\mathbb{E}}_i(f(X_t)|\tau > t) = \frac{1}{N}\sum_{j=1}^N f(j)$$

$\square$

9

5 Stationary two-point correlation function

We now study the two point function $\mu_{T_L,T_R}(x_i^2 x_j^2)$. In the dual process we then have two particles starting at sites i and j of which we denote the positions at time t by (X_t, Y_t). From (3.2) we see that these two particles behave as two independent random walkers (with rate 2) as long as they are not at neighboring sites. When they are at neighbor sites then on the next step they prefer to be on same site compared to being further separated.

From (3.13) we infer

$$
\begin{aligned}
\mu_{T_L,T_R}(x_i^2 x_j^2) &= (1 + 2\delta_{i,j})\Big(T_L^2\,\hat{\mathbb{P}}_{ij}(\{X_\infty, Y_\infty\} = \{0\}) + T_R^2\,\hat{\mathbb{P}}_{ij}(\{X_\infty, Y_\infty\} = \{N+1\}) \\
&+ T_L T_R\,\hat{\mathbb{P}}_{ij}(\{X_\infty, Y_\infty\} = \{0, N+1\})\Big)
\end{aligned}
\tag{5.1}
$$

Unfortunately, now the absorption probabilities cannot be obtained by elementary probabilistic considerations since the two random walkers of the dual process are interacting (attracting each other when they are at nearest neighbor sites). We therefore proceed by directly solving the linear equations for $y_{i,j} = \mu_{T_L,T_R}(x_i^2 x_j^2)$. We restrict ourself to the case $1 \leq i \leq j \leq N$ since $\mu_{T_L,T_R}(x_i^2 x_j^2)$ is obviously symmetric. We further denote $y_i = T_i = \mu_{T_L,T_R}(x_i^2)$.

The $y_{i,j}$ then satisfy the following system of linear equations:

$$
\begin{aligned}
3\,y_{i-1,i} - 2\,y_{i,i} + 3\,y_{i,i+1} = 0 && j = i \\
y_{i-1,i+1} + y_{i+1,i+1} + y_{i,i} + y_{i,i+2} - 8\,y_{i,i+1} = 0 && j = i+1 \\
y_{i-1,j} + y_{i+1,j} + y_{i,j-1} + y_{i,j+1} - 4\,y_{i,j} = 0 && j \geq i+2
\end{aligned}
\tag{5.2}
$$

with boundary conditions given by

$$
\begin{aligned}
y_{i,N+1} = y_i y_{N+1} = y_i T_R && i = 0, \ldots, N+1 \\
y_{0,j} = y_0 y_j = T_L y_j && j = 0, \ldots, N+1
\end{aligned}
\tag{5.3}
$$

Plugging in the ansatz

$$
y_{i,j} = A + Bi + Cj + Dij
$$

we find that these equations have the following solution: if $j > i$ then

$$
y_{i,j} = T_L^2 + i\,\frac{T_R - T_L}{N+1}\left(T_L + 2\frac{T_R - T_L}{N+3}\right) + j\,\frac{T_L\,(T_R - T_L)}{N+1} + ij\,\frac{(T_R - T_L)^2}{(N+1)(N+3)}
\tag{5.4}
$$

and for the diagonal $i = j$ we have

$$
\begin{aligned}
y_{i,i} &= 3\left(T_L^2 - \frac{(T_R - T_L)^2}{(N+1)(N+3)}\right) \\
&+ 6i\,\frac{T_R - T_L}{N+1}\left(T_L + \frac{T_R - T_L}{N+3}\right) \\
&+ 3i^2\,\frac{(T_R - T_L)^2}{(N+1)(N+3)}
\end{aligned}
\tag{5.5}
$$

The solution can also be written as a quadratic form in T_L and T_R and then the coefficients of the quadratic form add up to 1 outside of diagonal and to 3 on the diagonal (as can be seen directly from (5.1)). Explicitly

$$y_{i,j} = \left(1 - \frac{i}{N+3}\right)\left(1 - \frac{j}{N+1}\right)T_L^2 + \frac{i(2+j)}{(N+1)(N+3)}T_R^2 +$$
$$\left[1 - \left(1 - \frac{i}{N+3}\right)\left(1 - \frac{j}{N+1}\right) - \frac{i(2+j)}{(N+1)(N+3)}\right]T_L T_R \quad (5.6)$$

and

$$y_{i,i} = 3\left[\left(1 - \frac{i}{N+3}\right)\left(1 - \frac{i}{N+1}\right) - \frac{1}{(N+1)(N+3)}\right]T_L^2$$
$$+ \quad 3\left[\frac{i(2+i)}{(N+1)(N+3)} - \frac{1}{(N+1)(N+3)}\right]T_R^2$$
$$+ \quad 3\left[1 - \left(1 - \frac{i}{N+3}\right)\left(1 - \frac{i}{N+1}\right) - \frac{i(2+i)-2}{(N+1)(N+3)}\right]T_L T_R \quad (5.7)$$

Comparing this with (5.1) we can *read of* the expressions for the absorption probabilities. E.g., for $i < j$

$$\hat{\mathbb{P}}_{ij}(X_\infty = 0, Y_\infty = 0) = \left(1 - \frac{i}{N+3}\right)\left(1 - \frac{j}{N+1}\right) \quad (5.8)$$

Remark that this probability is larger than in the case of independent random walkers where the expression would be $(1 - \frac{i}{N+1})(1 - \frac{j}{N+1})$. This means that due to the attractive interaction of the walkers, they have a (small) preference to be absorbed at the same site. This effect is however negligible (of order $1/N$) as $N \to \infty$.

Finally, we compute the covariance

$$c_{i,j} = \mu_{T_L,T_R}(x_i^2 x_j^2) - \mu_{T_L,T_R}(x_i^2)\mu_{T_L,T_R}(x_j^2)$$

and we obtain

$$c_{i,j} = \frac{2i(N+1-j)}{(N+3)(N+1)^2}(T_R - T_L)^2 \quad (5.9)$$

$$c_{i,i} = \frac{(2N^2 + 8N + 3)T_L^2 + 6T_L T_R - 3T_R^2}{(N+1)(N+3)} + i\frac{2(T_R - T_L)((3+2N)T_L + 3T_R)}{(N+1)(N+3)}$$
$$+ \quad i^2\frac{2N(T_R - T_L)^2}{(N+1)^2(N+3)} \quad (5.10)$$

Remark 2. *Notice that the covariance $c_{i,j}$ manifests the presence of long range correlations which are believed to be typical for non-equilibrium steady states [19]. The fact that the covariance is positive is due to the attractive character of the interaction between the dual walkers. This has to be contrasted with the case of the SEP model where the covariance is negative and the interaction between walkers is repulsive.*

We notice that in $c_{i,j}$ appears the Green's function of the simple (continuous time) random walk X_t,

$$G(i, j) = \frac{2i(N + 1 - j)}{(N + 1)}$$

i.e., the expected total time spent at i starting at j before being absorbed at 0 or $N + 1$. Moreover, for $0 < \alpha_1 < \alpha_2 < 1$,

$$\lim_{N \to \infty} N c_{N\alpha_1, N\alpha_2} = 2(T_R - T_L)^2 \alpha_1 (1 - \alpha_2) \tag{5.12}$$

From (5.12), in the spirit of [14], one expects that in the limit $N \to \infty$ the random distributions

$$Y_\alpha^N := \frac{1}{\sqrt{N}} \sum_{i=1}^{N} (x_i^2 - T_\alpha) \delta(\alpha - i/N)$$

converge jointly to a Gaussian random field with covariance

$$C(\alpha_1, \alpha_2) = 2(T_L - T_R)^2 \alpha_1 (1 - \alpha_2) + 2T^2(\alpha_1) \delta(\alpha_1 - \alpha_2)$$

where

$$T(\alpha) = \lim_{N \to \infty} T_{\lfloor \alpha N \rfloor} = (1 - \alpha)T_L + \alpha T_R \tag{5.13}$$

with $\lfloor \alpha N \rfloor$ denoting the integer part of αN.

6 Local equilibrium

From the computation of the stationary two point correlation, we infer in particular that for $\alpha \in (0, 1)$

$$\lim_{N \to \infty} \mu_{T_L, T_R}(x_{\lfloor \alpha N \rfloor}^4) = 3T^2(\alpha) = \frac{1}{\sqrt{2\pi T(\alpha)}} \int x^4 e^{-x^2/2T(\alpha)} dx$$

while for each $i \in \mathbb{N}, i > 0$

$$\lim_{N \to \infty} \mu_{T_L, T_R}(x_{\lfloor \alpha N \rfloor}^2 x_{\lfloor \alpha N \rfloor + i}^2) = T^2(\alpha) = \left(\frac{1}{\sqrt{2\pi T(\alpha)}} \int x^2 e^{-x^2/2T(\alpha)} dx \right)^2$$

where T_α is defined in (5.13).

This suggests that "around each macroscopic point α", which we associate to the micro-point $\lfloor \alpha N \rfloor$, there is a Gaussian "local equilibrium" distribution with variance $T(\alpha)$. More precisely we give the following definition cf. [11].

Definition 1. *Let T_L and T_R be fixed and let $\alpha \in (0, 1)$. We say that local equilibrium holds if for all $n \in \mathbb{N}$, for all $k_1, \ldots, k_n \in \mathbb{N}$ and for all $i_1, \ldots, i_n \in \{1, \ldots, N\}$ fixed, we have:*

$$\lim_{N \to \infty} \int D \left(\sum_{l=1}^{n} k_l \delta_{\lfloor \alpha N \rfloor + i_l}, \vec{x} \right) \mu_{T_L, T_R}(dx) = \prod_{l=1}^{n} \int \frac{x^{2k_l}}{(2k_l - 1)!!} \rho_{T(\alpha)}(dx) = [T(\alpha)]^{\sum_{l=1}^{n} k_l}$$

where $\rho_{T(\alpha)}$ denotes the measure of a centered guassian variable with variance $T(\alpha)$.

12

This is equivalent with the requirement that

$$\lim_{N\to\infty} \tau_{\lfloor \alpha N \rfloor}(\mu_{T_L, T_R}) = \mathcal{G}(T(\alpha)) \tag{6.2}$$

where $\tau_{\lfloor \alpha N \rfloor}$ denotes spatial shift, $\mathcal{G}(\sigma)$ denotes the product measure on $\mathbb{R}^{\mathbb{N}}$ with marginals that are normally distributed with mean zero and variance σ^2, and where the limit is in the sense that expectations of polynomials of type $D(\xi, \vec{x})$ converge to the corresponding expectations in the Gaussian measure.

The following lemma shows that factorization of the absorption probabilities is sufficient for local equilibrium.

Lemma 1. *Let T_L and T_R be* fixed *and let $\alpha \in (0,1)$. Suppose that for all $n \in \mathbb{N}$, for all $k_1, \ldots, k_n \in \mathbb{N}$, for all $i_1, \ldots, i_n \in \{1, \ldots, N\}$ fixed *and for all $K, L \in \mathbb{N}$ with $K + L = \sum_{l=1}^{n} k_l$*

$$\lim_{N\to\infty} \hat{\mathbb{P}}_{\sum_{l=1}^{n} k_l \delta_{\lfloor \alpha N \rfloor + i_l}}(\xi(\infty) = K\delta_0 + L\delta_{N+1})$$
$$= \lim_{N\to\infty} \hat{\mathbb{P}}_{\lfloor \alpha N \rfloor}(X(\infty) = 0)^K \hat{\mathbb{P}}_{\lfloor \alpha N \rfloor}(X(\infty) = N+1)^L \tag{6.4}$$

then local equilibrium holds.

Proof. Combination of (3.13) with (6.4) gives

$$\lim_{N\to\infty} \int D\left(\sum_{l=1}^{n} k_l \delta_{\lfloor \alpha N \rfloor + i_l}, \vec{x}\right) \mu_{T_L, T_R}(dx)$$
$$= \sum_{K,M:K+M=k_1+\ldots+k_n} \lim_{N\to\infty} \hat{\mathbb{P}}_{\lfloor \alpha N \rfloor}(X(\infty) = 0)^M \hat{\mathbb{P}}_{\lfloor \alpha N \rfloor}(X(\infty) = N+1)^K T_L^M T_R^K$$
$$= \lim_{N\to\infty} \left(\hat{\mathbb{P}}_{\lfloor \alpha N \rfloor}(X(\infty) = 0)T_L + \hat{\mathbb{P}}_{\lfloor \alpha N \rfloor}(X(\infty) = N+1)T_R\right)^{k_1+\ldots+k_n}$$
$$= \lim_{N\to\infty} T_{\lfloor \alpha N \rfloor}^{k_1+\ldots+k_n} = T(\alpha)^{k_1+\ldots+k_n} \tag{6.5}$$

$\square$

In order to see heuristically why the factorization property for the absorption probabilities holds, it suffices to see that if we start n dual random walkers $X_1(t), \ldots, X_n(t)$ from initial positions $\lfloor \alpha N \rfloor + i_l$, $l = 1, \ldots, n$ then we can couple them with n independent random walkers $X_1'(t), \ldots, X_n'(t)$ started at the same initial positions such that for all $\epsilon > 0$ and for all $l \in \{1, \ldots, n\}$

$$X_l(t) - X_l'(t) \le \epsilon \sqrt{t} \tag{6.6}$$

with probability close to one for t large enough. Indeed, if dual walker l is absorbed at 0, then his happens at a time of the order N^2, and then, for large N, (6.6) tells that the corresponding independent walker is at distance less than ϵN from 0 at that time. Therefore, the probability that this independent walker is absorbed at $N+1$ is at most $\epsilon N/(N+1) \le \epsilon$. So the probability (in the coupling) that dual

walker X_l and corresponding independent walker X'_l are absorbed at different points is less than ϵ. Therefore, if we have the coupling with property (6.6), we have for all $\theta_1, \ldots, \theta_n \in \{0, N+1\}$

$$\lim_{N \to \infty} \left(\hat{\mathbb{P}}(X_1(\infty) = \theta_1, \ldots, X_n(\infty) = \theta_l) - \mathbb{P}'(X'_1(\infty) = \theta_1, \ldots, X'_n(\infty) = \theta_l) \right) = 0$$

where $\hat{\mathbb{P}}$ refers to the probability measure on path space for dual walkers starting at positions $\lfloor \alpha N \rfloor + i_1, \ldots, \lfloor \alpha N \rfloor + i_n$, and $\mathbb{P}'$ refers to the probability for independent walkers starting at positions $\lfloor \alpha N \rfloor + i_1, \ldots, \lfloor \alpha N \rfloor + i_n$.

To see that such a coupling exists, we observe that in the dual process there is only interaction of the walkers when they are at neigboring positions. In that case they jump with higher rate (than independent walkers) to the same position. The coupling then consists in letting the independent walkers and the dual walkers perform the same jumps, and having extra jumps for the dual walkers (which are not performed by the independent walkers) when they are at neigboring positions. The total time that dual walkers are at neigboring positions in the time interval $[0, t]$ is less than $t^{\frac{1}{2}+\delta}$ with probability close to one. So the difference between the position of the independent walker and the dual walker is a sum of the order of $t^{\frac{1}{2}+\delta}$ independent increments taking values ± 1, which is bounded by $(t^{\frac{1}{2}+\delta})^{\frac{1}{2}+\delta} \leq t^{\frac{1}{2}-\delta'}$ with high probability.

A similar idea of coupling has been implemented in the context of the simple symmetric exclusion process, see [5]. For the full proof of the factorization property along these lines we however refer to future work.

7 The energy-diffusion model and the SEP family: bosons versus fermions.

In this section we shall see the formal relation between the SEP family and the energy-diffusion model. This can help as a guide to see the similarities in methods for treating both cases.

We shall first consider a generalization of the energy-diffusion model to the case in which there are m momenta per site, and kinetic energy is exchanged between any two momenta in neighboring sites. The generator is again (2.1), with now

$$L_1 = \sum_{\alpha=1}^{m} \left[T_L \frac{\partial^2}{\partial x_{1,\alpha}^2} - x_{1,\alpha} \frac{\partial}{\partial x_{1,\alpha}} \right] \tag{7.1}$$

$$L_N = \sum_{\alpha=1}^{m} \left[T_R \frac{\partial^2}{\partial x_{N,\alpha}^2} - x_{N,\alpha} \frac{\partial}{\partial x_{N,\alpha}} \right] \tag{7.2}$$

$$L_{i,i+1} = \frac{1}{m} \sum_{\alpha,\beta=1}^{m} \left(x_{i,\alpha} \frac{\partial}{\partial x_{i+1,\beta}} - x_{i+1,\beta} \frac{\partial}{\partial x_{i,\alpha}} \right)^2 \tag{7.3}$$

The factor $1/m$ multiplying $L_{i,i+1}$ is rather arbitrary, it sets the time-scale. We wish to show the connection of this family of models (labelled by m) with the (partial) exclusion process, with maximal occupancy n, and in which the jumping rate is proportional both to the number of particles on the departure configuration, and to the number of 'holes' (n minus the number of particles) in the arrival configuration. The evolution operator of this process can be written as the Hamiltonian H of the spin $j = n/2$ ferromagnet [18]

$$H = -L^*_{SEP} \tag{7.4}$$

with

$$
\begin{aligned}
L^*_{SEP} &= \frac{1}{j} \sum_i \left(J_i^+ J_{i+1}^- + J_i^- J_{i+1}^+ + 2J_i^o J_{i+1}^o - 2j^2 \right) \\
&+ \alpha(J_1^- - J_1^o - j) + \gamma(J_1^+ + J_1^o - j) + \delta(J_L^- - J_L^o - j) + \beta(J_L^+ + J_L^o - j)
\end{aligned} \tag{7.5}
$$

The factor $1/j$ is analogous to the factor $1/m$ in (7.3). The operators J_i^+, J_i^-, J_i^o act on the Hilbert space corresponding to $0 \le r \le n$ particles per site $\otimes_i |r\rangle_i$ as follows:

$$
\begin{aligned}
J_i^+ |r\rangle_i &= (2j - r)|r\rangle_i \\
J_i^- |r\rangle_i &= r|r\rangle_i \\
J_i^o |r\rangle_i &= (r - j)|r\rangle_i
\end{aligned} \tag{7.6}
$$

They satisfy the commutation relations of the $SU(2)$ algebra:

$$[J_i^o, J_i^\pm] = \pm J_i^\pm \quad ; \quad [J_i^-, J_i^+] = -2J_i^o \tag{7.7}$$

and they can be transformed to the conventional $SU(2)$ matrices by a similarity transformation.

Representations are labeled by the squared angular momentum operator:

$$J^2 |jM> = j(j+1)|jM> \tag{7.8}$$

with $j = n/2$, so that the ordinary SEP (with $(0,1)$ occupation) corresponds to a representation of spin $1/2$.

Going back to the generalized model defined above, we can rewrite the generator of the energy diffusion process as the Hamiltonian $H = -L^*$ with

$$
\begin{aligned}
L^* &= \frac{4}{m} \sum_i \left(K_i^+ K_{i+1}^- + K_i^- K_{i+1}^+ - 2K_i^o K_{i+1}^o + \frac{m^2}{8} \right) \\
&+ 2\left(T_1 K_1^+ - K_1^o - \frac{m}{4} \right) + 2\left(T_L K_L^+ - K_L^o - \frac{m}{4} \right)
\end{aligned} \tag{7.9}
$$

where we have defined, in each site:

$$
\begin{aligned}
K_i^+ &= \frac{1}{2} \sum_\alpha x_{i,\alpha}^2 \\
K_i^- &= \frac{1}{2} \sum_\alpha \frac{\partial^2}{\partial x_{i,\alpha}^2} \\
K_i^o &= \frac{1}{4} \sum_\alpha \left\{ \frac{\partial}{\partial x_{i,\alpha}} x_{i,\alpha} + x_{i,\alpha} \frac{\partial}{\partial x_{i,\alpha}} \right\}
\end{aligned} \tag{7.10}
$$

15

These satisfy the SU(1,1) relations:

$$[K_i^o, K_i^\pm] = \pm K_i^\pm \quad ; \quad [K_i^-, K_i^+] = 2K_i^o \tag{7.11}$$

(note the sign difference with respect to $SU(2)$). Representations are labeled in a manner analogous to $SU(2)$:

$$
\begin{aligned}
K_i^2 |kM> &= [K_i^o]^2 - \frac{1}{2}[K_i^+ K_i^- + K_i^- K_i^+]|kM> = k(k-1)|kM> \\
K_i^o |kM> &= m|kM>
\end{aligned}
\tag{7.12}
$$

To identify which is the representation (i.e., the value of k), we check the value of K_i^2 as applied to the constant (which is the zero eigenvalue eigenvector of L^*):

$$K_i^- |1> = 0 \ \ ; \ \ K_i^o|1> = k|1> = \frac{m}{4}|1> \ \ ; \ \ K_i^2|1> = k(k-1)|1> = \frac{m}{4}(\frac{m}{4} - 1)|1> \tag{7.13}$$

Hence $k = m/4$, and in particular $k = 1/4$ for the process with one velocity per site.

Consider the coherent-state representation of vectors and operators (cf.[16, 12, 13]):

$$
\begin{aligned}
\psi(z_i) &= <z_i|\psi> \quad \text{with} \quad |z_i> = e^{z_i^* K_i^+}|0> \\
\psi(z_i) &= <z_i|\psi> \quad \text{with} \quad |z_i> = e^{z_i^* J_i^+}|0>
\end{aligned}
\tag{7.14}
$$

(where $|0>$ is the state anihilated by J_i^- or K_i^-). The group operators act on such states as:

$$
\begin{aligned}
<z_i|K_i^+|\psi> &= \left[z_i^2 \frac{\partial}{\partial z_i} + 2kz_i \right] <z_i|\psi> \\
<z_i|K_i^-|\psi> &= \frac{\partial}{\partial z_i} <z_i|\psi> \\
<z_i|K_i^o|\psi> &= \left[z_i \frac{\partial}{\partial z_i} + k \right] <z_i|\psi>
\end{aligned}
\tag{7.15}
$$

and

$$
\begin{aligned}
<z_i|J_i^+|\psi> &= \left[-z_i^2 \frac{\partial}{\partial z_i} + 2jz_i \right] <z_i|\psi> \\
<z_i|J_i^-|\psi> &= \frac{\partial}{\partial z_i} <z_i|\psi> \\
<z_i|J_i^o|\psi> &= \left[z_i \frac{\partial}{\partial z_i} - j \right] <z_i|\psi>
\end{aligned}
\tag{7.16}
$$

Now, writing the generator of the energy diffusion model and the SEP models in these representations, it is easy to check that models with $k = m/4$ of one class formally map into models with $-j = -n/2$ of the other class, albeit with somewhat different boundary terms.

16

Another way to see the relation between these two families of models is to note that we can rewrite the generator L of the energy transport model *with even m* in terms of $m/2$ bosons $a_{i\alpha}$ and $m/2$ bosons $b_{i\alpha}$ in each site

$$a_{i\alpha}a_{j\beta}^{\dagger} - a_{j\beta}^{\dagger}a_{i\alpha} = \delta_{ij}\delta_{\alpha\beta} \quad ; \quad b_{i\alpha}b_{j\beta}^{\dagger} - b_{j\beta}^{\dagger}b_{i\alpha} = \delta_{ij}\delta_{\alpha\beta} \tag{7.17}$$

(all other commutators vanish) and writing

$$K_i^+ = \sum_\alpha a_{i\alpha}^{\dagger}b_{i\alpha}^{\dagger} \quad ; \quad K_i^- = \sum_\alpha b_{i\alpha}a_{i\alpha}$$
$$K_i^o = \frac{1}{2}\sum_\alpha \left(a_{i\alpha}^{\dagger}a_{i\alpha} + b_{i\alpha}^{\dagger}b_{i\alpha}\right) + r \tag{7.18}$$

We can similarly write the generalized SEP with allowed occupancy n with n *fermions* $a_{i\alpha}$ and n fermions $b_{i\alpha}$ in each site

$$a_{i\alpha}a_{j\beta}^{\dagger} + a_{j\beta}^{\dagger}a_{i\alpha} = \delta_{ij}\delta_{\alpha\beta} \quad ; \quad b_{i\alpha}b_{j\beta}^{\dagger} + b_{j\beta}^{\dagger}b_{i\alpha} = \delta_{ij}\delta_{\alpha\beta} \tag{7.19}$$

(all other anti-commutators vanish) and writing

$$J_i^+ = \sum_\alpha a_{i\alpha}^{\dagger}b_{i\alpha}^{\dagger} \quad ; \quad J_i^- = \sum_\alpha b_{i\alpha}a_{i\alpha}$$
$$J_i^o = \frac{1}{2}\sum_\alpha \left(a_{i\alpha}^{\dagger}a_{i\alpha} + b_{i\alpha}^{\dagger}b_{i\alpha}\right) - n \tag{7.20}$$

Hence, SEP and energy diffusion models are essentially fermionic and bosonic counterparts of the same models. This is perhaps not surprising, in view of the fact that Poisson processes (as the SEP) are related to Fermions, just as Gaussian processes are related to bosons [7, 2].

8 Asymmetric generalizations

In this last section we briefly indicate how to introduce asymmetry in our model without breaking the bulk energy conservation law. This is in the spirit of introducing a bias to move in a preferred direction in the simple symmetric exclusion process.

Consider as the first example two neighboring sites with momenta x, y. Going back to Eq. (2.6), we may add to the diffusion term a bias:

$$\mathcal{A}^{drift} = \left(x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x}\right)^2 + E(x,y)\left(x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x}\right) \tag{8.1}$$

and its obvious generalization to a chain then becomes the asymmetric analog of the bulk part of (2.1)

$$\sum_i L_{i,i+1}^{drift} = \sum_i \left(x_i\frac{\partial}{\partial x_{i+1}} - x_{i+1}\frac{\partial}{\partial x_i}\right)^2 + E(x_i, x_{i+1})\left(x_i\frac{\partial}{\partial x_{i+1}} - x_{i+1}\frac{\partial}{\partial x_i}\right) \tag{8.2}$$

Possible choices for the drift function $E(x, y)$ are:

1. $E(x, y) = E$ is constant. In that case, the product of Gaussian measures (3.9) is still invariant if we add thermalising noise at both ends of the chain at the same temperature T. Moreover, in the case of different temperatures, the equations for the stationary correlation functions of energies at different sites are still closed, i.e., the equations for the expectations of polynomials of degree K only involve polynomials of degree K. However, duality is lost just as in the case of the asymmetric exclusion process.

2. $E(x, y) = E(x^2 + y^2)$. This corresponds to the most general situation where the product of Gaussian measures is invariant (under the circumstances of the previous item). In the case of unequal temperatures, equations for stationary correlations are no longer closed.

3. $E(x, y) = Exy$. This form of the drift has the advantage of being a product of two K-operators in (7.10), which is also the case if one goes from the symmetric to the asymmetric exclusion process (in the quantum spin chain language).

Acknowledgements

References

[1] L. Balents, M. P. A. Fisher, and M. R. Zirnbauer, Chiral metal as a ferromagnetic super spin chain, *Nucl. Phys. B* **483**, 601 (1996).

[2] M. Beccaria, C. Presilla, G.F. De Angelis, G. Jona-Lasinio, An exact represenation of the fermion dynamics in terms of Poisson processes and its connection with Monte Carlo algorithms, *Europhys. Lett.* **48** 243 (1999).

[3] C. Bernardin, S. Olla, Fourier's law for a microscopic model of heat conduction, *J. Stat. Phys.* **121** 271 (2005).

[4] L. Bertini, D. Gabrielli, J.L. Lebowitz, Large deviations for a stochastic model of heat flow, *J. Stat. Phys.* **121** 843-885 (2005).

[5] A. De Masi and E. Presutti, Probability estimates for symmetric simple exclusion random walks. *Ann. Inst. H. Poincaré* **19** 71–85 (1983).

[6] B. Derrida, J.L. Lebowitz, E.R. Speer, Large deviations of the density profile in the stady state of the open symmetric simple exclusion process, *J. Stat. Phys.* **107** 599- (2002).

[7] S.E. Esipov, T,J, Newman, New formulation of restricted growth process, *J. Stat. Phys.* **70** 691- (1993).

[8] J. Fritz, K. Nagy, S. Olla, Equilibrium Fluctuations for a System of Harmonic Oscillators with Conservative Noise, *J. Stat. Phys.* **122** 399- (2006).

[9] C. Giardinà, J. Kurchan, The Fourier law in a momentum-conserving chain, *J. Stat. Mech.* P05009 (2005).

[10] C. Kipnis and C. Landim, *Scaling limits of interacting particle systems*, Springer, Berlin, (1999).

[11] C. Kipnis, C. Marchioro, E. Presutti, Heat flow in an exactly solvable model, *J. Stat. Phys.* bf 27 65-74 (1982).

[12] J. R. Klauder and B. Skagerstam, *Coherent States Applications in Physics and Mathematical Physics* World Scientific, Singapore, (1985).

[13] J. Kurchan, P. Leboeuf, and M. Saraceno, Semiclassical approximations in the coherent-state representation, *Phys. Rev. A* **40** 6800-6813 (1989).

[14] C. Landim, A. Milanés, S. Olla, Stationary and Nonequilibrium Fluctuations in Boundary Driven Exclusion Processes, math.PR/0608165 (2006).

[15] T.M. Liggett, *Interacting Particle Systems*, Springer, Berlin, (1985).

[16] A. Perelomov, *Generalized Coherent States and their Applications*, Springer, New York, (1986).

[17] G. M. Schutz, Exactly solvable models for many-body systems far from equilibrium, in *Phase Transitions and Critical Phenomena.* Vol. **19**, Eds. C. Domb and J. Lebowitz, Academic Press, London, (2000).

[18] G. Schutz and S. Sandow, Non-Abelian symmetries of stochastic processes: Derivation of correlation functions for random-vertex models and disordered-interacting-particle systems, Phys. Rev. **E 49** 2726- (1994).

[19] H. Spohn, Long range correlations for stochastic lattice gases in a non-equilibrium steady state, *J. Phys. A* **16** 4275-4291 (1983).

[20] R. Stinchcombe, Stochastic non-equilibrium systems, *Advances in Physics*, **50** 431-496 (2001).

[21] Yi-Kuo Yu, H. Mathur, Are Directed Waves Multifractal?, *Phys. Rev. Lett.* **81** 3924- (1998).