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Abstract

In this paper, a random graph process $\{G(t)\}_{t\geq 1}$ is studied and its degree sequence is analyzed. Let $\{W_t\}_{t\geq 1}$ be an i.i.d. sequence. The graph process is defined so that, at each integer time t, a new vertex, with W_t edges attached to it, is added to the graph. The new edges added at time t are then preferentially connected to older vertices, i.e., conditionally on $G(t-1)$, the probability that a given edge is connected to vertex i is proportional to $d_i(t-1)+\delta$, where $d_i(t-1)$ is the degree of vertex i at time $t-1$, independently of the other edges. The main result is that the asymptotical degree sequence for this process is a power law with exponent $\tau = \min\{\tau_{\rm W}, \tau_{\rm P}\}$, where $\tau_{\rm W}$ is the power-law exponent of the initial degrees $\{W_t\}_{t\geq 1}$ and $\tau_{\rm P}$ the exponent predicted by pure preferential attachment. This result extends previous work by Cooper and Frieze, which is surveyed.

1 Introduction

Empirical studies on real life networks, such as the Internet, the World-Wide Web, social networks, and various types of technological and biological networks, show fascinating similarities. Many of the networks are *small worlds*, meaning that typical distances in the network are small, and many of them have *power-law degree sequences*, meaning that the number of vertices with degree k falls off as $k^{-\tau}$ for some exponent $\tau > 1$. See [18] for an example of these phenomena in the Internet, and [24, 25] for an example on the World-Wide Web. Also, Table 3.1 in [26] gives an overview of a large number of networks and their properties.

Incited by these empirical findings, random graphs have been proposed to model and/or explain these phenomena – see [14] for an introduction to random graph models for complex networks. Two particular classes of models that have been studied from a mathematical viewpoint are (i) graphs where the edge probabilities depend on certain weights associated with the vertices, see e.g. [7, 10, 11, 12, 28], and (ii) so-called preferential attachment models, see e.g. [2, 6, 8, 9, 13]. The first class can be viewed as generalizations of the classical Erdős-Rényi graph allowing for power-law degrees. In [10], for instance, a model is considered in which each vertex i is assigned a random weight W_i and an edge is drawn between two vertices i and j with a probability depending on W_iW_j. This model, which is referred to as the *generalized random graph*, leads to a graph where vertex i has an asymptotic degree distribution equal to a Poisson random variable with (random) parameter W_i as the number of vertices tends to infinity, that is, the asymptotic degree of a vertex is determined by its weight. We refer to [5, 22] for introductions to classical random graphs.

Preferential attachment models are different in spirit in that they are dynamic, more precisely, a new vertex is added to the graph at each integer time. Each new vertex comes with a number of

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edges attached to it and these edges are connected to the old vertices in such a way that vertices with high degree are more likely to be attached to. It can be shown that this leads to graphs with power-law degree sequences. Note that, in preferential attachment models, the degree of a vertex increases over time, implying that the oldest vertices tend to have the largest degrees.

Consider the degree of a vertex as an indication of its *success*, so that vertices with large degree correspond to successful vertices. In preferential attachment models, vertices with large degrees are the most likely vertices to obtain even larger degrees, that is, successful vertices are likely to become even more successful. In the literature this is sometimes called the *rich-get-richer* effect. In the generalized random graph, on the other hand, a vertex is *born* with a certain weight and this weight determines the degree of the vertex, as described above. This will be referred to as the *rich-by-birth* effect in what follows.

Naturally, in reality, both the rich-get-richer and the rich-by-birth effect may play a role. To see this, consider for instance a social network, where we identify vertices with individuals and edges with social links, that is, an edge is added between two individuals if they have some kind of social relation with each other. Then, indeed, we would expect to see both effects: Firstly, the rich-get-richer effect should be apparent, since individuals with a high number of social links will in time acquire more new social links than individuals with few social links. Evidently, more social contacts imply that the individual is socially more active, so that he/she meets more people, and, in turn, each meeting offers a possibility to create a lasting social link. Thus, these individuals are more likely to get acquainted with even more individuals. Secondly, the rich-by-birth effect comes in due to the fact that some individuals are better in turning a meeting into a lasting social link than others. The social activity and skill varies from individual to individual and could be measured for instance by weights associated with the individuals. Hence, in reality, both the previous success of a vertex and an initial weight may play a role in the final success of the vertex. Naturally, there are several ways to model how the weight influences the success of a vertex. In the model considered in this paper, individuals arrive in the network with a different initial number of contacts (given to them at birth) and these initial numbers form the basis for their future success. Later on, we shall also discuss other ways of how this effect can be modeled.

The aim of the current paper is to formulate and analyze a model that combines the rich-get-richer and the rich-by-birth effect. The model is a preferential attachment model where the number of edges added upon the addition of a new vertex is a random variable associated to the vertex. This indeed gives the desired combination of preferential attachment and vertex-dependency of degrees upon vertex-birth. For bounded initial degrees, the model is included in the very general class of preferential attachment models treated in [13], but the novelty of the model lies in that the initial degrees can have an arbitrary distribution. In particular, we can take the weight distribution to be a power law, which gives a model with two "competing" power laws: the power law caused by the preferential attachment mechanism and the power law of the initial degrees. In such a situation it is indeed not clear which of the power laws will dominate in the resulting degrees of the graph. Our main result implies that the most heavy-tailed power law wins, that is, the degrees in the resulting graph will follow a power law with the same exponent as the initial degrees in case this is smaller than the exponent induced by the preferential attachment, and with an exponent determined by the preferential attachment in case this is smaller.

The proof of our main result requires finite moment of order $1+\varepsilon$ for the initial degrees. However, we believe that the conclusion is true also in the infinite mean case. More specifically, we conjecture that, when the distribution of the initial degrees is a power law with infinite mean, the degree sequence in the graph will obey a power law with the same exponent as the the one of the initial degrees. Indeed, the power law of the initial degrees will always be the "strongest" in this case, since preferential attachment mechanisms only seem to be able to produce power laws with finite mean. In reality, power laws with infinite mean are not uncommon, see e.g. Table 3.1 in [26] for some examples, and hence it is desirable to find a model that can capture this. Unfortunately, we have not been able to give a full proof for the infinite mean case, but we present partial results in Section 1.2.

We now proceed with a formal definition of the model and the formulation of the main results.

1.1 Definition of the model

The model that we consider is described by a graph process $\{G(t)\}_{t\geq 1}$. To define it, let $\{W_i\}_{i\geq 1}$ be an i.i.d. sequence of positive integer-valued random variables and let $G(1)$ be a graph consisting of two vertices v_0 and v_1 with W_1 edges joining them. For $t \geq 2$, the graph $G(t)$ is constructed from $G(t-1)$ in such a way that a vertex v_t, with associated weight W_t, is added to the graph $G(t-1)$, and the edge set is updated by adding W_t edges between the vertex v_t and the vertices $\{v_0, v_1, \ldots, v_{t-1}\}$. Thus, W_t is the *initial degree* of vertex v_t. Write $d_0(s), \ldots, d_{t-1}(s)$ for the degrees of the vertices $\{v_0, v_1, \ldots, v_{t-1}\}$ at time $s \geq t-1$. The endpoints of the W_t edges emanating from vertex v_t are chosen independently (with replacement) from $\{v_0, \ldots, v_{t-1}\}$, and the probability that v_i is chosen as the endpoint of a fixed edge is equal to

$$\frac{d_i(t-1)+\delta}{\sum_{j=0}^{t-1}(d_j(t-1)+\delta)} = \frac{d_i(t-1)+\delta}{2L_{t-1}+t\delta}, \qquad 0 \leq i \leq t-1, \tag{1.1}$$

where $L_t = \sum_{i=1}^{t} W_i$, and δ is a fixed parameter of the model. Write S_{W} for the support of the distribution of the initial degrees. To ensure that the above expression defines a probability, we require that

$$\delta + \min\{x : x \in S_{\mathrm{W}}\} > 0. \tag{1.2}$$

This model will be referred to as the PARID-model (Preferential Attachment with Random Initial Degrees). Note that, when $W_i \equiv 1$ and $\delta = 0$, we retrieve the original preferential attachment model from Barabási-Albert [2].

Remark 1.1 In the PARID-model, we assume that the different edges of a vertex are attached in an *independent* way, and we also take a simple choice for the initial graph $G(1)$. However, the proofs given below are rather insensitive to the precise model definitions, and can be applied to slightly different settings as well. For example, the proof can also be applied to the model used in [9], where $W_i \equiv m$ for some integer $m \geq 1$, and the degrees are updated during the attachment of the successive edges (i.e., the m edges are *not* independent).

Remark 1.2 We shall give special attention to the case where $\mathbb{P}(W_i = m) = 1$ for some integer $m \geq 1$. This model is closest in spirit to the Barabási-Albert model, and it turns out that sharper bounds are possible for the error terms in this case. These results will be used in [15], where we study the diameter in preferential attachment models.

1.2 Heuristics and main result

Our main result concerns the degree sequence in the resulting graph $G(t)$. To formulate it, let $N_k(t)$ be the number of vertices with degree k in $G(t)$ and define $p_k(t) = N_k(t)/(t+1)$ as the fraction of vertices with degree k. We are interested in the limiting distribution of $p_k(t)$ as $t \to \infty$. This distribution arises as the solution of a certain recurrence relation, of which we will now give a short heuristic derivation. First note that, obviously,

$$\mathbb{E}[N_k(t)|G(t-1)] = N_k(t-1) + \mathbb{E}[N_k(t) - N_k(t-1)|G(t-1)]. \tag{1.3}$$

Asymptotically, for t large, it is very unlikely that a vertex will be hit by more than one of the W_t edges added upon the addition of vertex v_t. Let us hence ignore this possibility for the moment. The difference $N_k(t) - N_k(t-1)$ between the number of vertices with degree k at time t and time $t-1$ respectively, is then obtained as follows:

(a) Vertices with degree k in $G(t-1)$ that are hit by one of the W_t edges emanating from v_t are subtracted from $N_k(t-1)$. The conditional probability that a fixed edge is attached to a vertex with degree k is $(k+\delta)N_k(t-1)/(2L_{t-1}+t\delta)$, so that the expected number of edges connected to vertices with degree k is $W_t(k+\delta)N_k(t-1)/(2L_{t-1}+t\delta)$. This coincides with the mean number of vertices with degree k in $G(t-1)$ hit by edges from v_t, apart from the case when two edges are attached to the same vertex.

(b) Vertices with degree $k-1$ in $G(t-1)$ that are hit by one of the W_t edges emanating from vertex v_t are added to $N_k(t-1)$. By reasoning as above, it follows that the mean number of such vertices is $W_t(k-1+\delta)N_{k-1}(t-1)/(2L_{t-1}+t\delta)$.

(c) The new vertex v_t should be added to $N_k(t-1)$ if it has degree k, that is, if $W_t = k$.

Combining this gives

$$\begin{aligned}&\mathbb{E}\left[N_k(t)-N_k(t-1)|G(t-1)\right]\\&\qquad\approx \frac{(k-1+\delta)W_t}{2L_{t-1}+t\delta}N_{k-1}(t-1)-\frac{(k+\delta)W_t}{2L_{t-1}+t\delta}N_k(t-1)+\mathbf{1}_{\{W_t=k\}},\end{aligned}\qquad(1.4)$$

where the approximation sign refers to the fact that we have ignored the possibility of two or more edges of an arriving vertex being connected to the same end vertex. Now assume that $p_k(t)$ converges to some limit p_k as $t\to\infty$, so that hence $N_k(t)\sim(t+1)p_k$. Also assume that the distribution of the initial degrees has finite mean μ, so that $L_{t-1}/t\to\mu$. Finally, let $\{r_k\}_{k\geq 1}$ be the probabilities associated with the weight distribution, that is,

$$r_k=\mathbb{P}(W_1=k),\qquad k\geq 1.\qquad(1.5)$$

Substituting (1.4) into (1.3) and replacing L_{t-1} by μt, we arrive, after taking double expectations, at

$$\mathbb{E}\left[N_k(t)\right]\approx\mathbb{E}[N_k(t-1)]+\frac{(k-1+\delta)}{t\theta}\mathbb{E}[N_{k-1}(t-1)]-\frac{(k+\delta)}{t\theta}\mathbb{E}[N_k(t-1)]+r_k,\qquad(1.6)$$

where $\theta=2+\delta/\mu$. Sending $t\to\infty$, and observing that $\mathbb{E}[N_k(t)]-\mathbb{E}[N_k(t-1)]\to p_k$ implies that $\frac{1}{t}\mathbb{E}[N_k(t)]\to p_k$, for all k, then yields the recursion

$$p_k=\frac{k-1+\delta}{\theta}p_{k-1}-\frac{k+\delta}{\theta}p_k+r_k.\qquad(1.7)$$

By iteration, it can be seen that this recursion is solved by

$$p_k=\frac{\theta}{k+\delta+\theta}\sum_{i=0}^{k-1}r_{k-i}\prod_{j=1}^{i}\frac{(k-j+\delta)}{(k-j+\delta+\theta)},\qquad k\geq 1,\qquad(1.8)$$

where the empty product, arising when $i=0$, is defined to be equal to one. Furthermore, since $\{p_k\}_{k\geq 1}$ satisfies (1.7), we have that $\sum_{k=1}^{\infty}p_k=\sum_{k=1}^{\infty}r_k=1$. Hence, $\{p_k\}_{k\geq 1}$ defines a probability distribution, and the above reasoning indicates that the limiting degree distribution in the PARID-model should be given by $\{p_k\}_{k\geq 1}$. Our main result confirms this heuristics:

Theorem 1.3 *If the initial degrees $\{W_i\}_{i\geq 1}$ have finite moment of order $1+\varepsilon$ for some $\varepsilon>0$, then there exists a constant $\gamma\in(0,\frac{1}{2})$ such that*

$$\lim_{t\to\infty}\mathbb{P}\left(\max_{k\geq 1}|p_k(t)-p_k|\geq t^{-\gamma}\right)=0,$$

where $\{p_k\}_{k\geq 1}$ is defined in (1.8). When $r_m=1$ for some integer $m\geq 1$, then $t^{-\gamma}$ can be replaced by $C\sqrt{\frac{\log t}{t}}$ for some sufficiently large constant C.

To analyze the distribution $\{p_k\}_{k\geq 1}$, first consider the case when the initial degrees are almost surely constant, that is, when $r_m = 1$ for some positive integer m. Then $r_j = 0$ for all $j \neq m$, and (1.8) reduces to

$$p_k = \begin{cases} \frac{\theta\Gamma(k+\delta)\Gamma(m+\delta+\theta)}{\Gamma(m+\delta)\Gamma(k+1+\delta+\theta)} & \text{for } k \geq m; \\ 0 & \text{for } k < m, \end{cases}$$

where $\Gamma(\cdot)$ denotes the gamma-function. By Stirling's formula, we have that $\Gamma(s+a)/\Gamma(s) \sim s^a$ as $s \to \infty$, and from this it follows that $p_k \sim ck^{-(1+\theta)}$ for some constant $c > 0$. Hence, the degree sequence obeys a power law with exponent $1 + \theta = 3 + \delta/m$. Note that, by choosing $\delta > -m$ appropriately, any value of the exponent larger than 2 can be obtained. For other choices of $\{r_k\}_{k\geq 1}$, the behavior of $\{p_k\}_{k\geq 1}$ is less transparent. The following proposition asserts that, if $\{r_k\}_{k\geq 1}$ is a power law, then $\{p_k\}_{k\geq 1}$ is a power law as well. It also gives the aforementioned characterization of the exponent as the minimum of the exponent of the r_k's and an exponent induced by the preferential attachment mechanism.

Proposition 1.4 *Assume that $r_k = \mathbb{P}(W_1 = k) = k^{-\tau_{\mathrm{W}}}L(k)$ for some $\tau_{\mathrm{W}} > 2$ and some function $k \mapsto L(k)$ which is slowly varying. Then $p_k = k^{-\tau}\hat{L}(k)$ for some slowly varying function $k \mapsto \hat{L}(k)$ and with power-law exponent τ given by*

$$\tau = \min\{\tau_{\mathrm{W}}, \tau_{\mathrm{P}}\}, \tag{1.9}$$

where τ_{P} is the power-law exponent of the pure preferential attachment model given by $\tau_{\mathrm{P}} = 3+\delta/\mu$. When r_k decays faster than a power law, then (1.9) remains true with the convention that $\tau_{\mathrm{W}} = \infty$.

In deriving the recursion (1.7) we assumed that the initial degrees $\{W_i\}_{i\geq 1}$ have finite mean μ. Assume now that the mean of the initial degrees is infinite. More specifically, suppose that $\{r_k\}_{k\geq 1}$ is a power law with exponent $\tau_{\mathrm{W}} \in [1, 2]$. Then, we conjecture that the main result above remains true.

Conjecture 1.5 *When $\{r_k\}_{k\geq 1}$ is a power law distribution with exponent $\tau_{\mathrm{W}} \in (1, 2)$, then the degree sequence in PARID-model obeys a power law with the same exponent τ_{W}.*

Unfortunately, we cannot quite prove Conjecture 1.5. However, we shall prove a slightly weaker version of it. To this end, write $N_{\geq k}(t)$ for the number of vertices with degree larger than or equal to k at time t, that is, $N_{\geq k}(t) = \sum_{i=0}^{t} \mathbf{1}_{\{d_i(t)\geq k\}}$, and let $p_{\geq k}(t) = N_{\geq k}(t)/(t+1)$. Since $d_i(t) \geq W_i$, obviously

$$\mathbb{E}[p_{\geq k}(t)] = \frac{\mathbb{E}[N_{\geq k}(t)]}{t+1} \geq \frac{\mathbb{E}[\sum_{i=1}^{t} \mathbf{1}_{\{W_i\geq k\}}]}{t+1} = \mathbb{P}(W_1 \geq k)\frac{t}{t+1} = \mathbb{P}(W_1 \geq k)(1 + o(1)), \tag{1.10}$$

that is, the expected degree sequence in the PARID-model is always bounded from below by the weight distribution. In order to prove a related upper bound, we start by investigating the expectation of the degrees.

Theorem 1.6 *Suppose that $\sum_{k>x} r_k = \mathbb{P}(W_1 > x) = x^{1-\tau_{\mathrm{W}}}L(x)$, where $\tau_{\mathrm{W}} \in (1, 2)$ and $x \mapsto L(x)$ is a slowly varying function at infinity. Then, for every $s < \tau_{\mathrm{W}} - 1$, there exists a constant $C > 0$ and a slowly varying function $x \mapsto l(x)$ such that, for $i \in \{0, \ldots, t\}$, we have that*

$$\mathbb{E}[d_i(t)^s] \leq C\Big(\frac{t}{i \vee 1}\Big)^{s/(\tau_{\mathrm{W}}-1)}\Big(\frac{l(t)}{l(i)}\Big)^s,$$

where $x \vee y = \max\{x, y\}$.

Theorem 1.6 gives an upper bound for the expected degree sequence:

Corollary 1.7 *If $\sum_{k>x} r_k = \mathbb{P}(W_1 > x) = x^{1-\tau_W} L(x)$, where $\tau_W \in (1,2)$ and $x \mapsto L(x)$ is a slowly varying function at infinity, then, for every $s < \tau_W - 1$, there exists an M (independent of t) such that*

$$\mathbb{E}[p_{\geq k}(t)] \leq Mk^{-s}.$$

Proof. For $s < \tau_W - 1$, it follows from Theorem 1.6 and Markov's inequality that

$$\begin{aligned} \mathbb{E}[p_{\geq k}(t)] &= \frac{1}{t+1}\sum_{i=0}^{t} \mathbb{P}(d_i(t) \geq k) = \frac{1}{t+1}\sum_{i=0}^{t} \mathbb{P}(d_i(t)^s \geq k^s) \\ &\leq \frac{1}{t+1}\sum_{i=0}^{t} k^{-s}\mathbb{E}[d_i(t)^s] \leq k^{-s}\frac{C}{t+1}\sum_{i=0}^{t}\Big(\frac{t}{i\vee 1}\Big)^{s/(\tau_W-1)}\Big(\frac{l(t)}{l(i)}\Big)^s \leq Mk^{-s}, \end{aligned} \tag{1.11}$$

since, for $s < \tau - 1$ and using [17, Theorem 2, p. 283], there exists a constant $c > 0$ such that

$$\sum_{i=0}^{t} (i \vee 1)^{-s/(\tau_W-1)} l(i)^{-s} = ct^{1-\frac{s}{\tau_W-1}} l(t)^{-s}(1 + o(1)).$$

□

Combining Corollary 1.7 with (1.10) yields that, when the weight distribution is a power law with exponent $\tau_W \in (1,2)$, the only possible power law for the degrees has exponent equal to τ_W. This statement is obviously not as strong as Theorem 1.3, but it does offer convincing evidence for Conjecture 1.5. Theorem 1.6 is proved in Section 3.

1.3 Related work

Before proceeding with the proofs, we describe some related work. In Section 2.5, the proof of Theorem 1.3 is compared to related proofs that have appeared in the literature, and we refer there for the extensive literature on power laws for preferential attachment models. In this section, we describe work on related models.

As mentioned in the introduction, the paper by Cooper and Frieze [13] deals with a very general class of preferential attachment models, including the PARID-model with bounded initial degrees. Another way of introducing the rich-by-birth effect in a preferential attachment model, is the *fitness model*, formulated by Barabási and Bianconi [3, 4], and later generalized by Ergün and Rodgers [16]. We will shortly describe the model of Ergün and Rodgers and some (non-rigorous) results for the degree sequence.

The idea with the model is that vertices have different ability – referred to as *fitness* – to compete for edges. More precisely, each vertex has two types of fitness, a multiplicative and an additive fitness associated to it. These are given by independent copies of random variables η and ζ, respectively. The dynamics of the model is then very similar to the dynamics of the PARID-model. However, instead of adding a random number of edges together with each new vertex, new vertices come with a *fixed* number m of edges. Also, instead of connecting an edge to a given vertex with a probability proportional to the degree plus δ, the probability of connecting to a given vertex is proportional to the degree times the multiplicative fitness plus the additive fitness. Thus, the expression (1.1) for the probability of choosing v_i $(0 \leq i \leq t-1)$ as the endpoint of a fixed edge is replaced by

$$\frac{\eta_i d_i(t-1) + \zeta_i}{\sum_{j=0}^{t-1} \eta_j d_j(t-1) + \sum_{j=0}^{t-1} \zeta_j}, \quad 0 \leq i \leq t-1. \tag{1.12}$$

The original fitness model of Barabási and Bianconi is obtained when $\zeta \equiv 0$. If, in addition, the multiplicative fitness is the same for all vertices, the model reduces further to the Albert-Barabási model.

The rich-by-birth effect is present since relatively young vertices, with a small degree, can acquire edges at a high rate if the multiplicative fitness or the additive fitness is large. Therefore, this model is sometimes called the *fit-get-rich* model. Excluding trivial choices for the distribution of η and ζ, it is not clear before hand if the graph process is driven by the rich-get-richer effect, by the rich-by-birth effect or by a combination of them. When the additive fitness is zero, Barabási and Bianconi [4] show that the distribution of the average degree sequence of $G(t)$ depends on the distribution of η. For η uniformly distributed on $[0,1]$, they show (non-rigorously) that the degree sequence $\{p_k\}_{k\geq 1}$ is given by

$$p_k \sim c\frac{k^{-(1+C^*)}}{\log k},$$

where C^* is the solution of the equation $\exp(-2/C) = 1 - 1/C$ and $c > 0$ a constant, that is, the average degree sequence follows a generalized power law. When η is exponentially distributed, numerical simulations indicate that the degree sequence also behaves like an exponential distribution. For the general model with non-zero additive fitness there are no explicit expressions for the p_k's. See however [16] for some special cases. We mention also that [3] provides a coupling of the fitness model to a so-called Bose gas. This coupling gives a way of predicting (non-rigorously) whether the rich-get-richer or the rich-by-birth effect will be dominant.

2 Proof of Theorem 1.3 and Proposition 1.4

In this section, we prove Theorem 1.3 and Proposition 1.4. We start by proving Proposition 1.4, since the proof of Theorem 1.3 makes use of it.

2.1 Proof of Proposition 1.4

Recall the definition (1.8) of p_k. Assume that $\{r_k\}_{k\geq 1}$ is a power law distribution with exponent $\tau_{\mathrm{W}} > 2$, that is, assume that $r_k = L(k)k^{-\tau_{\mathrm{W}}}$, for some slowly varying function $k \mapsto L(k)$. We want to show that then p_k is a power law distribution as well, more precisely, we want to show that $p_k = \hat{L}(k)k^{-\tau}$, where $\tau = \min\{\tau_{\mathrm{W}}, 1+\theta\}$ and $k \mapsto \hat{L}(k)$ is again a slowly varying function. To this end, first note that the expression for p_k can be rewritten in terms of gamma-functions as

$$p_k = \frac{\theta \cdot \Gamma(k+\delta)}{\Gamma(k+\delta+1+\theta)} \sum_{m=1}^{k} \frac{\Gamma(m+\delta+\theta)}{\Gamma(m+\delta)} r_m. \tag{2.1}$$

By Stirling's formula, we have that

$$\frac{\Gamma(k+\delta)}{\Gamma(k+\delta+1+\theta)} = k^{-(1+\theta)}\left(1 + O\big(k^{-1}\big)\right), \quad k \to \infty, \tag{2.2}$$

and

$$\frac{\Gamma(m+\delta+\theta)}{\Gamma(m+\delta)} = m^{\theta}\left(1 + O\big(m^{-1}\big)\right), \quad m \to \infty. \tag{2.3}$$

Furthermore, by the assumption, $r_m = L(m)m^{-\tau_{\mathrm{W}}}$. It follows that

$$\sum_{m=1}^{k} \frac{\Gamma(m+\delta+\theta)}{\Gamma(m+\delta)} r_m \tag{2.4}$$

is convergent as $k \to \infty$ if $\theta - \tau_{\mathrm{W}} < -1$, that is, if $\tau_{\mathrm{W}} > 1+\theta$. For such values of τ_{W}, the distribution p_k is hence a power law with exponent $\tau_{\mathrm{P}} = 1+\theta$. When $\theta - \tau_{\mathrm{W}} \geq -1$, that is, when $\tau_{\mathrm{W}} \leq \tau_{\mathrm{P}}$, the series in (2.4) diverges and, by [17, Lemma, p. 280], it can be seen that

$$k \mapsto \sum_{m=1}^{k} \frac{\Gamma(m+\delta+\theta)}{\Gamma(m+\delta)} r_m$$

varies regularly with exponent $\theta - \tau_W + 1$. Combining this with (2.2) yields that p_k (compare (2.1)) varies regularly with exponent τ_W, as desired. □

2.2 Proof of Theorem 1.3

The proof of Theorem 1.3 consists of two parts: in the first part, we prove that the degree sequence is concentrated around its mean, and in the second part, the mean degree sequence is identified. We formulate these results in two separate propositions – Proposition 2.1 and 2.2 – which are proved in Section 2.3 and 2.4 respectively.

The result on the concentration of the degree sequence is as follows:

Proposition 2.1 *If the initial degrees $\{W_i\}_{i\geq 1}$ in the PARID-model have finite moment of order $1+\varepsilon$, for some $\varepsilon > 0$, then there exists a constant $\alpha \in (\frac{1}{2}, 1)$ such that*

$$\lim_{t\to\infty} \mathbb{P}\left(\max_{k\geq 1} \left|N_k(t) - \mathbb{E}[N_k(t)]\right| \geq t^{\alpha}\right) = 0.$$

When $r_m = 1$ for some $m \geq 1$, then t^{α} can be replaced by $C\sqrt{t\log t}$ for some sufficiently large C. Identical concentration estimates hold for $N_{\geq k}(t)$.

As for the identification of the mean degree sequence, the following proposition says that the expected number of vertices with degree k is close to $(t+1)p_k$ for large t. More precisely, it asserts that the difference between $\mathbb{E}[N_k(t)]$ and $(t+1)p_k$ is bounded, uniformly in k, by a constant times t^{β}, for some $\beta \in [0, 1)$.

Proposition 2.2 *Assume that the initial degrees $\{W_i\}_{i\geq 1}$ in the PARID-model have finite moment of order $1+\varepsilon$ for some $\varepsilon > 0$, and let $\{p_k\}_{k\geq 1}$ be defined as in (1.8). Then there exist constants $c > 0$ and $\beta \in [0, 1)$ such that*

$$\max_{k\geq 1} |\mathbb{E}[N_k(t)] - (t+1)p_k| \leq ct^{\beta}. \tag{2.5}$$

When $r_m = 1$ for some $m \geq 1$, then the above estimate holds with $\beta = 0$.

With Propositions 2.1 and 2.2 at hand it is not hard to prove Theorem 1.3:

Proof of Theorem 1.3: Combining (2.5) with the triangle inequality, it follows that

$$\mathbb{P}\left(\max_{k\geq 1} \left|N_k(t) - (t+1)p_k\right| \geq ct^{\beta} + t^{\alpha}\right) \leq \mathbb{P}\left(\max_{k\geq 1} \left|N_k(t) - \mathbb{E}[N_k(t)]\right| \geq t^{\alpha}\right).$$

By Proposition 2.1, the right side tends to 0 as $t \to \infty$ and hence, since $p_k(t) = N_k(t)/(t+1)$, we have that

$$\lim_{t\to\infty} \mathbb{P}\left(\max_{k\geq 1} |p_k(t) - p_k| \geq \frac{ct^{\beta} + t^{\alpha}}{t+1}\right) = 0.$$

The theorem follows from this by picking $0 < \gamma < 1 - \max\{\alpha, \beta\}$. Note that, since $0 \leq \beta < 1$ and $\frac{1}{2} < \alpha < 1$, we have $0 < \gamma < \frac{1}{2}$. The proof for $r_m = 1$ is analogous. □

2.3 Proof of Proposition 2.1

This proof is an adaption of a martingale argument, which first appeared in [9], and has been used for all proofs of power-law degree sequences since. The idea is to express the difference $N_k(t) - \mathbb{E}[N_k(t)]$ in terms of a Doob martingale. After bounding the martingale differences, which are bounded in terms of the random number of edges $\{W_i\}_{i\geq 1}$, the Azuma-Hoeffding inequality can be applied to conclude that the probability of observing large deviations is suitably small, at least when the initial number of edges has bounded support. When the initial degrees $\{W_i\}_{i\geq 1}$ are unbounded, and extra coupling step is required. The argument for $N_{\geq k}(t)$ is identical, so we focus on $N_k(t)$.

We start by giving an argument when $W_i \leq t^a$ for all $i \leq t$ and some $a \in (0, \frac{1}{2})$. First note that

$$N_k(t) \leq \frac{1}{k}\sum_{l=k}^{\infty} lN_l(t) \leq \frac{1}{k}\sum_{l=1}^{\infty} lN_l(t) = \frac{L_t}{k}. \tag{2.6}$$

Thus, $\mathbb{E}[N_k(t)] \leq \mu t/k$. For $\alpha \in (\frac{1}{2}, 1)$, let $\eta > 0$ be such that $\eta + \alpha > 1$ (the choice of α will be specified in more detail below). Then, for any $k > t^\eta$, the event $|N_k(t) - \mathbb{E}[N_k(t)]| \geq t^\alpha$ implies that $N_k(t) \geq t^\alpha$, and hence that $L_t \geq kN_k(t) > t^{\eta+\alpha}$. It follows from Boole's inequality that

$$\mathbb{P}\left(\max_{k\geq 1} |N_k(t) - \mathbb{E}[N_k(t)]| \geq t^\alpha\right) \leq \sum_{k=1}^{t^\eta} \mathbb{P}\Big(|N_k(t) - \mathbb{E}[N_k(t)]| \geq t^\alpha\Big) + \mathbb{P}(L_t > t^{\eta+\alpha}).$$

Since $\eta + \alpha > 1$ and $L_t/t \to \mu$, the event $L_t > t^{\eta+\alpha}$ has small probability. To estimate the probability $\mathbb{P}(|N_k(t) - \mathbb{E}[N_k(t)]| \geq t^\alpha)$, introduce

$$M_n = \mathbb{E}[N_k(t)|G(n)], \qquad n = 0, \ldots, t,$$

where $G(0)$ is defined as the empty graph. Since $\mathbb{E}[M_n] < \infty$, the process is a Doob martingale with respect to $\{G(n)\}_{n=0}^t$. Furthermore, we have that $M_t = N_k(t)$ and $M_0 = \mathbb{E}[N_k(t)]$, so that

$$N_k(t) - \mathbb{E}[N_k(t)] = M_t - M_0.$$

Also, conditionally on the initial degrees $\{W_i\}_{i=1}^t$, the increments satisfy $|M_n - M_{n-1}| \leq 2W_n$. To see this, note that the additional information contained in $G(n)$ compared to $G(n-1)$ consists in how the W_n edges emanating from v_n are attached. This can affect the degrees of at most $2W_n$ vertices. By the assumption that $W_i \leq t^a$ for all $i = 1, \ldots, t$, we obtain that $|M_n - M_{n-1}| \leq 2t^a$. Combining all this, it follows from the Azuma-Hoeffding inequality – see e.g. [19, Section 12.2] – that

$$\mathbb{P}\Big(|N_k(t) - \mathbb{E}[N_k(t)]| \geq t^\alpha\Big) \leq 2\exp\Big\{-\frac{t^{2\alpha}}{8\sum_{i=1}^t t^{2a}}\Big\} = 2\exp\left\{-t^{2\alpha-1-2a}/8\right\},$$

so that we end up with the estimate

$$\mathbb{P}\Big(\max_{k\geq 1} |N_k(t) - \mathbb{E}[N_k(t)]| \geq t^\alpha\Big) \leq 2t^\eta \exp\left\{-t^{2\alpha-1-2a}/8\right\} + \mathbb{P}(L_t > t^{\eta+\alpha}). \tag{2.7}$$

Since $a < 1/2$, the above exponential tends to 0 for $\alpha < 1$ satisfying that $\alpha > a + 1/2$. When the initial degrees are *bounded*, the above argument can be adapted to yield that the probability that $\max_{k\geq 1} |N_k(t) - \mathbb{E}[N_k(t)]|$ exceeds $C\sqrt{t\log t}$ is $o(1)$ for some $C > 0$ sufficiently large. We omit the details of this argument.

We conclude that we have proved the statement for graphs satisfying that $W_i \leq t^a$ for some $a \in (0, 1/2)$ and all $i = 1, \ldots, t$. Naturally, this assumption may not be true. When the initial degrees are bounded, the assumption is true, even with t^a replaced by m, but we are interested in graphs having initial degrees with finite $(1+\varepsilon)$-moments. We next extend the proof to this setting by a coupling argument.

Write

$$W_i' = W_i \wedge t^a, \qquad L_s' = \sum_{i=1}^{s} W_i', \tag{2.8}$$

where $x \wedge y = \min\{x, y\}$. Then, the above argument shows that the PARID-model with initial degrees $\{W_i'\}_{i=1}^t$ satisfies the claim in Proposition 2.1. Denote the graph process with initial degrees

$\{W_i'\}_{i=1}^t$ by $\{G'(i)\}_{i=1}^t$ and, for $i \leq s$, the degree of vertex i in $G'(s)$ by $d_i'(s)$. We now present a coupling between $\{G(i)\}_{i=1}^t$ and $\{G'(i)\}_{i=1}^t$ which is such that, with high probability, the number of edges that differ is bounded by t^b for some $b \in (0,1)$.

Define the attachment probabilities in $\{G(i)\}_{i=1}^t$ and $\{G'(i)\}_{i=1}^t$ by

$$p_i(s) = \frac{d_i(s-1)+\delta}{2L_{s-1}+\delta s}, \quad p_i'(s) = \frac{d_i'(s-1)+\delta}{2L_{s-1}'+\delta s}. \tag{2.9}$$

Now, we couple the edge attachments such that the l^{th} edge of vertex s in *both* graphs is attached to i with probability $p_i(s) \wedge p_i'(s)$. Otherwise, the edge is *miscoupled*. We shall give a bound on the expected number of miscouplings. The number of miscouplings in $G(s)$ and $G'(s)$ is denoted by U_s, and is defined in more detail as follows. We define $U_0 = 0$ and explain recursively how to construct U_s from U_{s-1}.

The number of miscouplings is adjusted after each edge which is connected. We consider the edges to be *directed*, and call a directed edge pointing towards i an *in-edge for* i, and a directed edge pointing away from i an *out-edge for* i. For convenience later on, we regard an edge from s to i as *both* an in-edge for i as well as an out-edge for s. By the above definitions, the number of in-edges of i at time s is the in-degree of i at time s, and the number of out-edges of i at time s is the out-degree of i at time s. If we denote the in- and out-degrees of vertex i in $G(s)$ by $d_{i,\text{in}}(s)$ and $d_{i,\text{out}}(s)$, then clearly $d_i(s) = d_{i,\text{in}}(s) + d_{i,\text{out}}(s)$. The same holds for the in- and out-degrees $d_{i,\text{in}}'(s)$ and $d_{i,\text{out}}'(s)$ of vertex i in $G'(s)$.

The edges which are attached from vertex s are numbered $1, \ldots, W_s$. When $W_s > t^a$, then an edge with a number between t^a and W_s adds 2 to U_{s-1}, and we call both the in-edge of i and the out-edge of s as belonging to the miscoupled set. The size of the miscoupled set at any time $s = 0, \ldots, t$ will be equal to U_s.

When the edge number is in between 1 and W_s', then we add 1 to U_{s-1} precisely when the edge is connected *differently* for $G(s)$ and $G'(s)$. In this case, we say that the in-edge of i belongs to the miscoupled set, but the out-edge of s does not. The miscoupled set remains unchanged when an edge is attached in the same way in $G(s)$ and in $G'(s)$.

We next define the *weight* of every in- and out-edge of vertex i at time s to be equal to 1. The *total weight* of a vertex i in $G(s)$ at time s is the sum of weights of the in- and out-edges of i in $G(s)$ plus δ. The total weight of a vertex i in $G'(s)$ is defined in a similar manner.

The probabilities in (2.9) are precisely proportional to the *total* weight of the vertex i at time $s-1$. As a result, for an out-edge of vertex s with number in between 1 and W_s', a miscoupling occurs with probability equal to U_{s-1}/TW_{s-1}, where TW_s is the total weight of all vertices at time s (i.e., all weights in $G(s)$ and $G'(s)$ combined) plus δs. To see this, we can choose an edge with probability equal to the total weight of the end vertex of the edge divided by TW_s. If this edge is not in the miscoupled set, then we are done, and the two (directed) edges in $G(s)$ and $G'(s)$ are equal to an in-edge in the vertex which is connected to the chosen (directed) edge, and an out-edge from the vertex s. If the chosen (directed) edge is in the miscoupled set, then it is an edge either for $G(s-1)$ or for $G'(s-1)$, but not for both, and it is chosen with the correct (conditional) probability. The above rule then constructs the in- and out-edges corresponding to the edge we wish to attach. Say the chosen edge is an edge for $G(s-1)$, then we choose the edge for $G'(s-1)$ from the edges of $G'(s-1)$ with probability equal to $p_i'(s-1)$. As a result, we do not create any miscoupling when the initial edge drawn was not in the miscoupled set. The probability of a miscoupling at time s is therefore equal to U_{s-1}/TW_{s-1}. Note that $\text{TW}_s \geq 2L_s + \delta(s+1)$.

The following lemma bounds the expected value of U_t:

Lemma 2.3 *There exists a constants $K > 0$ and $b \in (0,1)$ such that*

$$\mathbb{E}[U_t] \leq Kt^b. \tag{2.10}$$

Proof of Lemma 2.3: We prove Lemma 2.3 by induction. The induction hypothesis is that, for all $0 \leq s \leq t$,

$$\mathbb{E}[U_s] \leq K\left(\frac{s}{t}\right) \cdot t^b. \tag{2.11}$$

The bound in (2.10) follows from the one in (2.11) by substituting $s = t$.

We now prove (2.11). For $s = 0$, we have $U_0 = 0$, which initializes the induction hypothesis. To advance the induction hypothesis, we note that U_s is equal to $U_{s-1} + 2(W_s - W_s') + R_s$, where R_s is the number of out-edges for s with number in between 1 and W_s' that are miscoupled. As a result, we have

$$\mathbb{E}[U_s] = \mathbb{E}[U_{s-1}] + 2\mathbb{E}[W_s - W_s'] + \mathbb{E}[R_s]. \tag{2.12}$$

By the fact that for each out-edge for s with number in between 1 and W_s', a miscoupling occurs with probability equal to $U_{s-1}/\mathrm{TW}_{s-1}$, we have that

$$\mathbb{E}[R_s] = \mathbb{E}\Big[\mathbb{E}[R_s|W_s]\Big] = \mathbb{E}\Big[W_s'\frac{U_{s-1}}{\mathrm{TW}_{s-1}}\Big] = \mathbb{E}[W_s']\mathbb{E}\Big[\frac{U_{s-1}}{\mathrm{TW}_{s-1}}\Big], \tag{2.13}$$

where the last equality follows from the independence of W_s and $(U_{s-1}, \mathrm{TW}_{s-1})$. Now, we use that $\mathrm{TW}_{s-1} \geq 2L_{s-1} + \delta(s-1)$, together with the fact that L_s is concentrated around its mean, to conclude that $L_{s-1} \geq (\mu - \varepsilon)(s-1)$ with high probability. Thus,

$$\mathbb{E}[R_s] = \mathbb{E}\Big[\mathbb{E}[R_s|W_s]\Big] \leq \frac{\mu}{(s-1)(2\mu+\delta-2\varepsilon)}\mathbb{E}[U_{s-1}] + \mu\mathbb{P}\big(L_{s-1} \leq (\mu-\varepsilon)(s-1)\big). \tag{2.14}$$

Using the induction hypothesis, we arrive at

$$\begin{aligned}\mathbb{E}[R_s] &\leq \frac{\mu}{(s-1)(2\mu+\delta-2\varepsilon)}K\left(\frac{s-1}{t}\right)t^b + \mu\mathbb{P}(L_{s-1} \leq (\mu-\varepsilon)(s-1)) \\ &\leq Kt^{b-1}\frac{\mu}{(2\mu+\delta-2\varepsilon)} + \mu\mathbb{P}\big(L_{s-1} \leq (\mu-\varepsilon)(s-1)\big).\end{aligned} \tag{2.15}$$

We further bound

$$\mathbb{E}[W_s - W_s'] = \mathbb{E}[(W_s - t^a)\mathbf{1}_{\{W_s > t^a\}}] \leq t^{-a\varepsilon}\mathbb{E}[W_s^{1+\varepsilon}] \leq Ct^{-a\varepsilon}. \tag{2.16}$$

Therefore, by taking $b - 1 = -a\varepsilon$, we get that

$$\begin{aligned}\mathbb{E}[U_s] &\leq \mathbb{E}[U_{s-1}] + 2\mathbb{E}[W_s - W_s'] + \mathbb{E}[R_s] \\ &\leq K(s-1)t^{b-1} + 2Ct^{b-1} + Kt^{b-1}\frac{\mu}{(2\mu+\delta-2\varepsilon)} + \mu\mathbb{P}(L_{s-1} \leq (\mu-\varepsilon)(s-1)) \\ &= Kt^{b-1}\left\{(s-1) + 2C/K + \frac{\mu}{(2\mu+\delta-2\varepsilon)}\right\} + \mu\mathbb{P}(L_{s-1} \leq (\mu-\varepsilon)(s-1)) \\ &\leq Kst^{b-1},\end{aligned} \tag{2.17}$$

by noting that $\mathbb{P}\big(L_s \leq (\mu-\varepsilon)s\big)$ is exponentially small in s, for $s \to \infty$, and using that, since $2\mu + \delta > \mu$, we can take $\varepsilon > 0$ so small that $\mu/(2\mu+\delta-2\varepsilon) < 1$, and, after this, we can take K so large that

$$\frac{\mu}{(2\mu+\delta-2\varepsilon)} + \frac{2C}{K} < 1.$$

With these choices, we have advanced the induction hypothesis. □

We now complete the proof of Proposition 2.1. The Azuma-Hoefding argument proves that $N_k'(t)$, the number of vertices with degree k in $G'(t)$, satisfies the bound in Proposition 2.1, i.e., that (recall (2.7))

$$\mathbb{P}\left(\max_{k\geq 1}|N_k'(t) - \mathbb{E}\big[N_k'(t)\big]| \geq t^{\alpha}\right) \leq 2t^{\eta}\exp\left\{-t^{2\alpha-1-2a}/8\right\} + \mathbb{P}(L_t' > t^{\eta+\alpha}), \tag{2.18}$$

for $\alpha \in (\frac{1}{2}, 1)$ and $\eta > 0$ such that $\alpha + \eta > 1$ and $a \in (0, \frac{1}{2})$. Moreover, we have for every $k \geq 1$, that

$$|N_k(t) - N'_k(t)| \leq U_t, \tag{2.19}$$

since every miscoupling can change the degree of at most one vertex. By (2.19) and (2.10), there is a $b \in (0,1)$ such that

$$\Big|\mathbb{E}[N_k(t)] - \mathbb{E}[N'_k(t)]\Big| \leq \mathbb{E}[U_t] \leq Kt^b. \tag{2.20}$$

Also, by the Markov inequality, (2.19) and (2.10), for every $\alpha \in (b, 1)$, we have that

$$\mathbb{P}\Big(\max_{k\geq 1} |N_k(t) - N'_k(t)| > t^{\alpha}\Big) \leq \mathbb{P}\big(U_t > t^{\alpha}\big) \leq t^{-\alpha}\mathbb{E}[U_t] = o(1). \tag{2.21}$$

Now fix $\alpha \in (b \vee (a + \frac{1}{2}), 1)$, where $x \vee y = \max\{x, y\}$, and decompose

$$\max_{k\geq 1} \big|N_k(t) - \mathbb{E}[N_k(t)]\big| \leq \max_{k\geq 1} \big|N'_k(t) - \mathbb{E}[N'_k(t)]\big| + \max_{k\geq 1} \big|\mathbb{E}[N_k(t)] - \mathbb{E}[N'_k(t)]\big| + \max_{k\geq 1} \big|N_k(t) - N'_k(t)\big|. \tag{2.22}$$

The first term on the right hand side is bounded by t^{α} with high probability by (2.18), the second term is, for t sufficiently large and with probability one, bounded by t^{α} by (2.20) while the third term is bounded by t^{α} with high probability by (2.21). This completes the proof. □

2.4 Proof of Proposition 2.2

For $k \geq 1$, let

$$\bar{N}_k(t) = \mathbb{E}[N_k(t) | \{W_i\}_{i=1}^t] \tag{2.23}$$

denote the expected number of vertices with degree k at time t given the initial degrees $W_1, \ldots, W_t$, and define

$$\varepsilon_k(t) = \bar{N}_k(t) - (t+1)p_k, \qquad k \geq 1. \tag{2.24}$$

Also, for $Q = \{Q_k\}_{k\geq 1}$ a sequence of real numbers, define the supremum norm of Q as $\|Q\| = \sup_{k\geq 1} |Q_k|$. Using this notation, since $\mathbb{E}[\bar{N}_k(t)] = \mathbb{E}[N_k(t)]$, we have to show that there are constants $c > 0$ and $\beta \in [0, 1)$ such that

$$\|\mathbb{E}[\varepsilon(t)]\,\| = \sup_{k\geq 1} |\mathbb{E}[\bar{N}_k(t)] - (t+1)p_k| \leq ct^{\beta}, \qquad \text{for} \quad t = 0, 1, \ldots, \tag{2.25}$$

where $\varepsilon(t) = \{\varepsilon_k(t)\}_{k=1}^{\infty}$. The plan to do this is to formulate a recursion for $\varepsilon(t)$, and then use induction in t to establish (2.25). The recursion for $\varepsilon(t)$ is obtained by combining a recursion for $\bar{N}(t) = \{\bar{N}_k(t)\}_{k\geq 1}$, that will be derived below, and the recursion for p_k in (1.7). The hard work then is to bound the error terms in this recursion; see Lemma 2.4 below.

Let us start by deriving a recursion for $\bar{N}(t)$. To this end, for a real-valued sequence $Q = \{Q_k\}_{k\geq 0}$, with $Q_0 = 0$, introduce the operator T_t, defined as (compare to (1.6))

$$(T_t Q)_k = \left(1 - \frac{k+\delta}{2L_{t-1} + t\delta}\right) Q_k + \frac{k-1+\delta}{2L_{t-1} + t\delta} Q_{k-1}, \qquad k \geq 1. \tag{2.26}$$

When applied to $\bar{N}(t-1)$, the operator T_t describes the effect of the addition of a single edge emanating from the vertex v_t, the vertex v_t itself being excluded from the degree sequence. Indeed, there are on the average $\bar{N}_{k-1}(t-1)$ vertices with degree $k-1$ at time $t-1$ and a new edge is connected to such a vertex with probability $(k-1+\delta)/(2L_{t-1}+t\delta)$. After this connection is made, the vertex will have degree k. Similarly, there are on the average $\bar{N}_k(t-1)$ vertices with degree k at time $t-1$. Such a vertex is hit by a new edge with probability $(k+\delta)/(2L_{t-1}+t\delta)$, and will then have degree $k+1$. The expected number of vertices with degree k after the addition of one edge is hence given by the operator in (2.26) applied to $\bar{N}(t)$.

Write T_t^n for the n-fold application of T_t, and define $T_t' = T_t^{W_t}$. Then T_t' describes the change in the expected degree sequence $\bar{N}(t)$ when all the W_t edges emanating from vertex v_t have been connected, ignoring vertex v_t itself. Hence, $\bar{N}(t)$ satisfies

$$\bar{N}_k(t) = (T_t'\bar{N}(t-1))_k + \mathbf{1}_{\{W_t=k\}}, \qquad k \geq 1. \tag{2.27}$$

Introduce a second operator S on sequences of real numbers $Q = \{Q_k\}_{k\geq 0}$, with $Q_0 = 0$, by (compare to (1.7))

$$(SQ)_k = \frac{k-1+\delta}{\theta} Q_{k-1} - \frac{k+\delta}{\theta} Q_k, \qquad k \geq 1, \tag{2.28}$$

where $\theta = 2 + \delta/\mu$ and μ is the expectation of W_1.

The recursion (1.7) is given by $p_k = (Sp)_k + r_k$, with initial condition $p_0 = 0$. It is solved by $p = \{p_k\}_{k\geq 1}$, as defined in (1.8). Observe that

$$(t+1)p_k \quad = \quad tp_k + (Sp)_k + r_k = t(T_t'p)_k + r_k - \kappa_k(t), \qquad k \geq 1, \tag{2.29}$$

where

$$\kappa_k(t) = t(T_t'p)_k - (Sp)_k - tp_k. \tag{2.30}$$

Combining (2.24), (2.27) and (2.29), and using the linearity of T_t', it follows that $\varepsilon(t) = \{\varepsilon_k(t)\}_{k\geq 1}$ satisfies the recursion

$$\varepsilon_k(t) = (T_t'\varepsilon(t-1))_k + \mathbf{1}_{\{W_t=k\}} - r_k + \kappa_k(t), \tag{2.31}$$

indeed,

$$\begin{aligned}
\varepsilon_k(t) &= \bar{N}_k(t) - (t+1)p_k \\
&= (T_t'\bar{N}(t-1))_k + \mathbf{1}_{\{W_t=k\}} - t(T_t'p)_k - r_k + \kappa_k(t) \\
&= (T_t'\varepsilon(t-1))_k + \mathbf{1}_{\{W_t=k\}} - r_k + \kappa_k(t).
\end{aligned}$$

Now we define $k_t = \eta t$, where $\eta \in (\mu, 2\mu+\delta)$. Since, by (1.2), $\delta > -\min\{x : x \in S_{\mathrm{W}}\} \geq -\mu$, the interval $(\mu, 2\mu+\delta) \neq \varnothing$. Also, by the law of large numbers, $L_t \leq k_t$, as $t \to \infty$, with high probability. Further, we define $\tilde{\varepsilon}_k(t) = \varepsilon_k(t)\mathbf{1}_{\{k\leq k_t\}}$ and note that, for $k \leq k_t$, the sequence $\{\tilde{\varepsilon}_k(t)\}_{k\geq 1}$ satisfies

$$\tilde{\varepsilon}_k(t) = \mathbf{1}_{\{k\leq k_t\}}(T_t'\varepsilon(t-1))_k + \mathbf{1}_{\{W_t=k\}} - r_k + \tilde{\kappa}_k(t), \tag{2.32}$$

where $\tilde{\kappa}_k(t) = \kappa_k(t)\mathbf{1}_{\{k\leq k_t\}}$. It follows from $\mathbb{E}\big[\mathbf{1}_{\{W_t=k\}}\big] = r_k$ and the triangle inequality that

$$\begin{aligned}
\|\mathbb{E}[\varepsilon(t)]\| &\leq \|\mathbb{E}[\varepsilon(t) - \tilde{\varepsilon}(t)]\| + \|\mathbb{E}[\tilde{\varepsilon}(t)]\| \\
&\leq \|\mathbb{E}[\varepsilon(t) - \tilde{\varepsilon}(t)]\| + \|\mathbb{E}\big[\mathbf{1}_{(-\infty,k_t]}(\cdot)T_t'\varepsilon(t-1)\big]\| + \|\mathbb{E}[\tilde{\kappa}(t)]\|,
\end{aligned} \tag{2.33}$$

where $\mathbf{1}_{(-\infty,k_t]}(k) = \mathbf{1}_{\{k\leq k_t\}}$. Inequality (2.33) is the key ingredient in the proof of Proposition 2.2. We will derive the following bounds for the terms in (2.33).

Lemma 2.4 *There are constants $C_{\tilde{\varepsilon}}$, $C_\varepsilon^{(1)}$, $C_\varepsilon^{(2)}$ and $C_{\tilde{\kappa}}$, independent of t, such that for t sufficiently large and some $\beta \in [0,1)$,*

(a) $\|\mathbb{E}[\varepsilon(t) - \tilde{\varepsilon}(t)]\| \leq \frac{C_{\tilde{\varepsilon}}}{t^{1-\beta}}$,

(b) $\|\mathbb{E}\big[\mathbf{1}_{(-\infty,k_t]}(\cdot)T_t'\varepsilon(t-1)\big]\| \leq \Big(1 - \frac{C_\varepsilon^{(1)}}{t}\Big)\|\mathbb{E}[\varepsilon(t-1)]\| + \frac{C_\varepsilon^{(2)}}{t^{1-\beta}}$,

(c) $\|\mathbb{E}[\tilde{\kappa}(t)]\| \leq \frac{C_{\tilde{\kappa}}}{t^{1-\beta}}$.

When $r_m = 1$ for some integer $m \geq 1$, then the above bounds hold with $\beta = 0$.

Given these bounds, Proposition 2.2 is easily established.

Proof of Proposition 2.2: Recall that we want to establish (2.25). We shall prove this by induction on t. Fix $t_0 \in \mathbb{N}$. We start by verifying the induction hypothesis for $t \le t_0$, thus initializing the induction hypothesis. For any $t \le t_0$, we have

$$\|\mathbb{E}[\varepsilon(t)]\| \le \sup_{k\ge1} \mathbb{E}\big[\bar{N}_k(t)\big] + (t_0+1)\sup_{k\ge1} p_k \le 2(t_0+1), \tag{2.34}$$

since there are precisely t_0+1 vertices at time t_0 and $p_k \le 1$. This initializes the induction hypothesis, when c is so large that $2(t_0+1) \le ct_0^{\beta}$. Next, we advance the induction hypothesis. Assume that (2.25) holds at time $t-1$ and apply Lemma 2.4 to (2.33) to get that

$$\begin{aligned}\|\mathbb{E}[\varepsilon(t)]\| &\le \|\mathbb{E}[\varepsilon(t)-\tilde{\varepsilon}(t)]\| + \|\mathbb{E}\big[\mathbf{1}_{(-\infty,k_t]}(\cdot)T_t'\varepsilon(t-1)\big]\| + \|\mathbb{E}[\tilde{\kappa}(t)]\| \\ &\le \frac{C_{\tilde{\varepsilon}}}{t^{1-\beta}} + \left(1-\frac{C_\varepsilon^{(1)}}{t}\right)c(t-1)^{\beta} + \frac{C_\varepsilon^{(2)}}{t^{1-\beta}} + \frac{C_{\tilde{\kappa}}}{t^{1-\beta}} \\ &\le ct^{\beta} - \frac{c\cdot C_\varepsilon^{(1)} - (C_\varepsilon^{(2)} + C_{\tilde{\varepsilon}} + C_{\tilde{\kappa}})}{t^{1-\beta}},\end{aligned}$$

as long as $1-\frac{C_\varepsilon^{(1)}}{t} \ge 0$, which is equivalent to $t \ge C_\varepsilon^{(1)}$. If we then choose c large so that $c\cdot C_\varepsilon^{(1)} \ge C_\varepsilon^{(2)} + C_\varepsilon + C_{\tilde{\kappa}}$ and $c \ge 2(t_0+1)t_0^{-\beta}$ (recall (2.34)) and $t_0 \ge C_\varepsilon^{(1)}$, then we have that $\|\mathbb{E}[\varepsilon(t)]\| \le ct^{\beta}$, and (2.25) follows by induction in t. □

It remains to prove Lemma 2.4. We shall prove Lemma 2.4 (a)-(c) one by one, starting with (a).

Proof of Lemma 2.4(a): We have $\|\mathbb{E}[\varepsilon(t)-\tilde{\varepsilon}(t)]\| \le \mathbb{E}[\|\varepsilon(t)-\tilde{\varepsilon}(t)\|]$, and, using the definition of $\tilde{\varepsilon}(t)$, we get that

$$\|\varepsilon(t)-\tilde{\varepsilon}(t)\| \;=\; \sup_{k>k_t}\big|\bar{N}_k(t)-(t+1)p_k\big| \le \sup_{k>k_t}\bar{N}_k(t) + (t+1)\sup_{k>k_t}p_k.$$

The maximal possible degree of a vertex at time t is L_t, implying that $\sup_{k>k_t}\bar{N}_k(t)=0$, when $L_t \le k_t$. The latter is true almost surely when $r_m = 1$ for some integer m, when t is sufficiently large, since for t large $L_t = mt \le \eta t = k_t$, where $\eta \in (m, 2m+\delta)$, by the fact that $\mu = m$ and $\delta > -m$. On the other hand, by (2.6), with $N_k(t)$ replaced by $\bar{N}_k(t)$ we find $\bar{N}_k(t) \le \frac{L_t}{k_t}$ for $k \ge k_t$, and we obtain that

$$\mathbb{E}[\sup_{k>k_t}\bar{N}_k(t)] \le (k_t)^{-1}\mathbb{E}[L_t\mathbf{1}_{\{L_t>k_t\}}]. \tag{2.35}$$

With $k_t = \eta t$ for some $\eta \in (\mu, 2\mu+\delta)$, we have that

$$\mathbb{E}[L_t\mathbf{1}_{\{L_t>k_t\}}] \le k_t^{-\varepsilon}\mathbb{E}[L_t^{1+\varepsilon}\mathbf{1}_{\{L_t>k_t\}}] \le k_t^{-\varepsilon}\mathbb{E}[|L_t-\mu t|^{1+\varepsilon}] + (\mu t)^{1+\varepsilon}k_t^{-\varepsilon}\mathbb{P}(L_t>k_t), \tag{2.36}$$

and, by the Markov inequality

$$\mathbb{P}(L_t>k_t) \le \mathbb{P}\Big(|L_t-\mu t|^{1+\varepsilon} > (k_t-\mu t)^{1+\varepsilon}\Big) \le (k_t-\mu t)^{-(1+\varepsilon)}\mathbb{E}[|L_t-\mu t|^{1+\varepsilon}].$$

Combining the two latter results, we obtain

$$\mathbb{E}[L_t\mathbf{1}_{\{L_t>k_t\}}] \le k_t^{-\varepsilon}\Big(1+\Big(\frac{\mu}{\eta-\mu}\Big)^{1+\varepsilon}\Big)\mathbb{E}[|L_t-\mu t|^{1+\varepsilon}]. \tag{2.37}$$

To bound the last expectation, we will use a consequence of the Marcinkiewicz-Zygmund inequality, see e.g [20, Corollary 8.2 in §3], which runs as follows. Let $q \in [1,2]$, and suppose that

$\{X_i\}_{i\geq 1}$ is an i.i.d. sequence with $\mathbb{E}[X_1]=0$ and $\mathbb{E}[|X_1|^q]<\infty$. Then there exists a constant c_q depending only on q, such that

$$\mathbb{E}\Big[\Big|\sum_{i=1}^{t} X_i\Big|^q\Big] \leq c_q t\mathbb{E}[|X_1{}^q|]\,. \tag{2.38}$$

Applying (2.38) with $q=1+\varepsilon$, we obtain

$$\mathbb{E}[\sup_{k>k_t}\bar{N}_k(t)] \leq k_t^{-(1+\varepsilon)}\Big(1+\Big(\frac{\mu}{\eta-\mu}\Big)^{1+\varepsilon}\Big)\mathbb{E}[|L_t-\mu t|^{1+\varepsilon}] \leq c_{1+\varepsilon}t^{-\varepsilon}. \tag{2.39}$$

Furthermore, since by Proposition 1.4, we have $p_k\leq ck^{-\gamma}$, for some $\gamma>2$ (see also (1.9)), we have that $\sup_{k>k_t}p_k\leq ct^{-\gamma}$ for some constant c. It follows that

$$(t+1)\sup_{k>k_t}p_k \quad\leq\quad \frac{C_p}{t^{\gamma-1}},$$

and, since $\gamma>2$, part (a) is established with $C_{\tilde{\varepsilon}}=c_{1+\varepsilon}+C_p$, and $1-\beta=(\varepsilon\wedge\gamma)-1$. □

Proof of Lemma 2.4(b): Moving on to (b), we will start by showing that for t sufficiently large,

$$\|\mathbb{E}\big[\mathbf{1}_{(-\infty,k_t]}(\cdot)T_t\varepsilon(t-1)\big]\| \leq \bigg(1-\frac{C_\varepsilon^{(1)}}{t}\bigg)\|\mathbb{E}\big[\mathbf{1}_{(-\infty,k_t]}(\cdot)\varepsilon(t-1))\big]\| + \frac{C_\varepsilon^{(3)}}{t^{1-\beta}}, \tag{2.40}$$

which is (b) when we condition on $W_t=1$. We shall extend the proof to the case where $W_t\geq 1$ at a later stage. To prove (2.40), we shall prove a related bound, which also proves useful in the extension to $W_t\geq 1$. Indeed, we shall prove, for any real-valued sequence $Q=\{Q_k\}_{k\geq 0}$, satisfying (i) $Q_0=0$ and (ii)

$$\sup_{k\geq 1}|k+\delta||Q_k|\leq C_Q L_{t-1}, \tag{2.41}$$

that there exists a $\beta\in(0,1)$ (independent of Q) and a constant $c>0$ such that for t sufficiently large,

$$\|\mathbb{E}\big[\mathbf{1}_{(-\infty,k_t]}(\cdot)T_tQ\big]\| \leq \bigg(1-\frac{C_\varepsilon^{(1)}}{t}\bigg)\|\mathbb{E}\big[\mathbf{1}_{(-\infty,k_t]}(\cdot)Q\big]\| + \frac{cC_Q}{t^{1-\beta}}. \tag{2.42}$$

Here we stress that Q can be *random*, for example, we shall apply (2.42) to $\varepsilon(t-1)$ in order to derive (2.40).

In order to prove (2.42), we recall that

$$\mathbb{E}[(T_tQ)_k] = \mathbb{E}\left[\left(1-\frac{k+\delta}{2L_{t-1}+t\delta}\right)Q_k + \frac{k-1+\delta}{2L_{t-1}+t\delta}Q_{k-1}\right], \qquad k\geq 1. \tag{2.43}$$

In bounding this expectation we will encounter a problem in that Q_k, which is allowed to be random, and L_{t-1} are not independent (for example when $Q=\varepsilon(t-1)$). To get around this, we add and subtract the expression on the right hand side but with the random quantities replaced by their expectations, that is, for $k\geq 1$, we write

$$\begin{aligned}\mathbb{E}[(T_tQ)_k] \quad&=\quad \left(1-\frac{k+\delta}{2\mu(t-1)+t\delta}\right)\mathbb{E}[Q_k] + \frac{k-1+\delta}{2\mu(t-1)+t\delta}\mathbb{E}[Q_{k-1}] &(2.44)\\ &\quad+\ (k+\delta)\mathbb{E}\bigg[Q_k\frac{2L_{t-1}-2\mu(t-1)}{(2L_{t-1}+t\delta)(2\mu(t-1)+t\delta)}\bigg] &(2.45)\\ &\quad+\ (k+\delta-1)\mathbb{E}\bigg[Q_{k-1}\frac{2\mu(t-1)-2L_{t-1}}{(2L_{t-1}+t\delta)(2\mu(t-1)+t\delta)}\bigg]. &(2.46)\end{aligned}$$

Note that, when $r_m = 1$ for some integer $m \geq 1$, then $L_t = \mu t = mt$. Hence the terms in (2.45, 2.46) are both equal to zero, and only (2.44) contributes. We first deal with (2.44). Observe that $k \leq k_t = \eta t$, with $\eta \in (\mu, 2\mu + \delta)$, implies that $k \leq (2\mu + \delta)(t-1)$ for t sufficiently large, and hence

$$1 - \frac{k+\delta}{2\mu(t-1)+t\delta} \geq 0. \tag{2.47}$$

It follows that, for t sufficiently large,

$$\begin{aligned} &\sup_{k\leq k_t}\left|\left(1-\frac{k+\delta}{2\mu(t-1)+t\delta}\right)\mathbb{E}[Q_k] + \frac{k-1+\delta}{2\mu(t-1)+t\delta}\mathbb{E}[Q_{k-1}]\right| \qquad (2.48)\\ &\quad\leq \left(1-\frac{1}{2\mu(t-1)+t\delta}\right)\|\mathbb{E}\big[\mathbf{1}_{(-\infty,k_t]}(\cdot)Q\big]\| \leq \big(1-\frac{C_\varepsilon^{(1)}}{t}\big)\|\mathbb{E}\big[\mathbf{1}_{(-\infty,k_t]}(\cdot)Q\big]\|, \end{aligned}$$

for some constant $C_\varepsilon^{(1)}$. This proves (2.42) – with $C_Q = 0$ – when the number of edges is a.s. constant since (2.45, 2.46) are zero. It remains to bound the terms (2.45, 2.46) in the case where the number of edges is not a.s. constant. We will prove that the supremum over k of the absolute values of both these terms are bounded by constants divided by $t^{1-\beta}$ for some $\beta \in [0,1)$. Starting with (2.45), by using the assumption (ii) in (2.41), as well as $2L_{t-1} + \delta t \geq L_{t-1}$ for t sufficiently large, it follows that

$$\sup_{k\geq 1}\left|(k+\delta)\mathbb{E}\left[Q_k\frac{2L_{t-1}-2\mu(t-1)}{(2L_{t-1}+t\delta)(2\mu(t-1)+t\delta)}\right]\right| \leq \frac{cC_Q}{t}\mathbb{E}[|L_{t-1}-\mu(t-1)|]\,.$$

To bound the latter expectation, we combine (2.38) for $q = 1+\varepsilon$, with Hölders inequality, to obtain

$$\mathbb{E}[|L_t-\mu t|] \;\leq\; \left(\mathbb{E}\Big[|L_t-\mu t|^{1+\varepsilon}\Big]\right)^{1/(1+\varepsilon)} \leq \left(c_{1+\varepsilon}t\mathbb{E}\Big[|W_1-\mu|^{1+\varepsilon}\Big]\right)^{1/(1+\varepsilon)} \leq ct^{1/(1+\varepsilon)}, \tag{2.49}$$

since W_i have finite moment of order $1+\varepsilon$ by assumption, where, without loss of generality, we can assume that $\varepsilon \leq 1$. Hence, we have shown that the supremum over k of the absolute value of (2.45) is bounded from above by a constant divided by $t^{1-\beta}$, where $\beta = 1/(1+\varepsilon)$. That the same is true for the term (2.46) can be seen analogously. This completes the proof of (2.42).

To prove (2.40), we note that, by convention, $\varepsilon_0(t-1) = 0$, so that we only need to prove that $\sup_{k\geq 1}|k+\delta||\varepsilon_k(t-1)| \leq cL_{t-1}$. For this, note from (2.6), the bound $p_k \leq ck^{-\gamma}$, $\gamma > 2$, and from the lower bound $L_t \geq t$ that

$$\begin{aligned} \sup_{k\geq 1}|k+\delta||\varepsilon_k(t-1)| &\leq \sum_{k\geq 1}(k+|\delta|)|\varepsilon_k(t-1)| \\ &\leq \sum_{k\geq 1}(k+|\delta|)\bar{N}_k(t-1) + t\sum_{k\geq 1}(k+|\delta|)p_k \\ &\leq L_{t-1} + |\delta|(t-1) + t\sum_{k\geq 1}(k+|\delta|)p_k \leq cL_{t-1}, \end{aligned} \tag{2.50}$$

for some constant c. This completes the proof of (2.40).

To complete the proof of Lemma 2.4(b), we first show that (2.42) implies, for every $1 \leq n \leq t$, and all $k \geq 1$,

$$\mathbb{E}\big[\mathbf{1}_{\{k\leq k_t\}}\big(T_t^n\varepsilon(t-1)\big)_k\big] \;\leq\; \left(1-\frac{C_\varepsilon^{(1)}}{t}\right)\|\mathbb{E}\big[\mathbf{1}_{(-\infty,k_t]}(\cdot)\varepsilon(t-1)\big]\| + \frac{nC_\varepsilon^{(3)}}{t^{1-\beta}}. \tag{2.51}$$

To see (2.51), we use induction on n. We note that (2.51) for $n = 1$ is precisely equal to (2.40), and this initializes the induction hypothesis. To advance the induction hypothesis, we note that

$$\mathbf{1}_{\{k\leq k_t\}}\big(T_t^n\varepsilon(t-1)\big)_k = \mathbf{1}_{\{k\leq k_t\}}T_t\big(Q(n-1)\big)_k, \tag{2.52}$$

where $Q_k(n-1) = \mathbf{1}_{\{k\le k_t\}}\Big(T_t^{n-1}\varepsilon(t-1)\Big)_k$. We wish to use (2.42), and we first check the assumptions (i-ii). By definition, $Q_0(n-1)=0$, which establishes (i). For assumption (ii), we need to do some more work. According to (2.26), and using that $2L_{t-1}+t\delta > L_{t-1}\ge t-1$, for t sufficiently large,

$$\sum_{k=1}^{\infty}(k+|\delta|)(T_tQ)_k \le \Big(1+\frac{1}{t}\Big)\sum_{k=1}^{\infty}(k+|\delta|)Q_k,$$

and hence, by induction,

$$\sum_{k=1}^{\infty}(k+|\delta|)(T_t^{n-1}Q)_k \le \Big(1+\frac{1}{t}\Big)^{n-1}\sum_{k=1}^{\infty}(k+|\delta|)Q_k.$$

Substituting $Q_k = \varepsilon_k(t-1)$ and using $|\varepsilon_k(t-1)|\le N_k(t-1)+tp_k$, yields

$$\begin{aligned}
&\sum_{k\le k_t}(k+|\delta|)(T_t^{n-1}N(t-1))_k + t\sum_{k\le k_t}(k+|\delta|)(T_t^{n-1}p)_k\\
&\quad\le \Big(1+\frac{1}{t}\Big)^{n-1}\sum_{k=1}^{\infty}(k+|\delta|)N_k(t-1)+\Big(1+\frac{1}{t}\Big)^{n-1}t\sum_{k=1}^{\infty}(k+|\delta|)p_k\\
&\quad\le \Big(1+\frac{1}{t}\Big)^{n-1}\cdot cL_{t-1},
\end{aligned}\tag{2.53}$$

according to (2.50). Using the inequality $1+x\le e^x$, $x\ge 0$, together with $n\le t$, this in turn yields,

$$\sup_{k\ge 1}|k+\delta||Q_k(n-1)|\le \exp(1)cL_{t-1},\tag{2.54}$$

which implies assumption (ii).

By the induction hypothesis, we have that, for $k\le k_t$,

$$\mathbb{E}[Q_k(n-1)] \le \Big(1-\frac{C_\varepsilon^{(1)}}{t}\Big)\,\|\mathbb{E}\big[\mathbf{1}_{(-\infty,k_t]}(\cdot)\varepsilon(t-1)\big]\,\| + \frac{(n-1)C_\varepsilon^{(3)}}{t^{1-\beta}},\tag{2.55}$$

so that we obtain, from (2.42), with $Q=\mathbf{1}_{(-\infty,k_t]}(\cdot)T_t\varepsilon(t-1)$,

$$\mathbb{E}\big[\mathbf{1}_{\{k\le k_t\}}\big(T_t^n\varepsilon(t-1)\big)_k\big] \le \Big(1-\frac{C_\varepsilon^{(1)}}{t}\Big)\,\|\mathbb{E}\big[\mathbf{1}_{(-\infty,k_t]}(\cdot)\varepsilon(t-1)\big]\,\| + \frac{(n-1)C_\varepsilon^{(3)}+cC_Q}{t^{1-\beta}},\tag{2.56}$$

which advances the induction hypothesis when $C_\varepsilon^{(3)} > cC_Q$.

By (2.56), we obtain that, for $W_t\le t$,

$$\begin{aligned}
\mathbb{E}\big[\mathbf{1}_{\{k\le k_t\}}\big(T_t'\varepsilon(t-1)\big)_k\big|W_t\big] &\le \Big(1-\frac{C_\varepsilon^{(1)}}{t}\Big)\,\|\mathbb{E}[\varepsilon(t-1)|W_t]\,\| + \frac{W_tC_\varepsilon^{(3)}}{t^{1-\beta}}\\
&= \Big(1-\frac{C_\varepsilon^{(1)}}{t}\Big)\,\|\mathbb{E}[\varepsilon(t-1)]\,\| + \frac{W_tC_\varepsilon^{(3)}}{t^{1-\beta}},
\end{aligned}$$

where we use that $\varepsilon(t-1)$ is independent of W_t. In the case that $W_t>t$, we bound, similarly as in (2.50),

$$\sup_{k\le k_t}|\big(T_t'\varepsilon(t-1)\big)_k|\le cL_t,\tag{2.57}$$

so that

$$\mathbb{E}\big[\mathbf{1}_{\{k\le k_t\}}\big(T_t'\varepsilon(t-1)\big)_k\big|W_t\big] \le \Big(1-\frac{C_\varepsilon^{(1)}}{t}\Big)\,\|\mathbb{E}[\varepsilon(t-1)]\,\| + \frac{W_tC_\varepsilon^{(3)}}{t^{1-\beta}} + c\mathbb{E}[L_t\mathbf{1}_{\{W_t>t\}}\big|W_t].\tag{2.58}$$

The bound in (b) follows from this by taking expectations on both sides, using

$$\mathbb{E}[L_t\mathbf{1}_{\{W_t>t\}}] = \mu(t-1)\mathbb{P}(W_t>t)+\mathbb{E}[W_t\mathbf{1}_{\{W_t>t\}}]\le\Big(\frac{\mu}{t^{\varepsilon}}+\frac{1}{t^{\varepsilon}}\Big)\mathbb{E}[W_t^{1+\varepsilon}],\tag{2.59}$$

after which we use that $\beta = 1/(1+\varepsilon)\ge 1-\varepsilon$ and choose the constants appropriately. This completes the proof of Lemma 2.4(b). □

Proof of Lemma 2.4(c): For part (c) of the lemma, recall that

$$\tilde{\kappa}_k(t) = \kappa_k(t)\mathbf{1}_{\{k\leq k_t\}} \qquad \text{with} \qquad \kappa_k(t) = t((T_t'-I)p)_k - (Sp)_k, \tag{2.60}$$

where T_t is defined in (2.26), $T_t' = T_t^{W_t}$, S is defined in (2.28), and where I denotes the identity operator. In what follows, we will assume that $k \leq k_t$, so that $\tilde{\kappa}_k(t) = \kappa_k(t)$. We start by proving a trivial bound on $\kappa_k(t)$. By (2.31), we have that

$$\kappa_k(t) = \varepsilon_k(t) - (T_t'\varepsilon(t-1))_k - \mathbf{1}_{\{W_t=k\}} + r_k, \tag{2.61}$$

where $\sup_{k\geq 1}|\varepsilon_k(t)| \leq cL_t$ by (2.50) and $\sup_{1\leq k\leq k_t}|(T_t'\varepsilon(t-1))_k| \leq cL_t$ by (2.57), so that hence

$$\sup_{k\leq k_t}|\kappa_k(t)| \leq C_\eta L_t. \tag{2.62}$$

(recall that $k_t = \eta t$ where $\eta \in (\mu, 2\mu+\delta)$). For $x\in[0,1]$ and $w \in \mathbb{N}$, we denote

$$f_k(x;w) = \big((I + x(T_t - I))^w p\big)_k.$$

Then $\kappa_k(t) = \kappa_k(t; W_t)$, where

$$\kappa_k(t;w) = t[f_k(1;w) - f_k(0;w)] - (Sp)_k, \tag{2.63}$$

and $x \mapsto f_k(x;w)$ is a polynomial in x of degree w. By a Taylor expansion around $x=0$,

$$f_k(1;w) = p_k + w\big((T_t - I)p\big)_k + \frac{1}{2}f_k''(x_k;w), \tag{2.64}$$

for some $x_k \in (0,1)$, and, since $I + x(T_t - I)$ and $T_t - I$ commute,

$$f_k''(x;w) = w(w-1)\Big((I + x(T_t-I))^{w-2}(T_t-I)^2 p\Big)_k.$$

We next claim that, on the event $\{k_t \leq 2L_{t-1} + (t-1)\delta\}$,

$$\sup_{k\leq k_t}\Big|\big((I + x(T_t - I))Q\big)_k\Big| \leq \sup_{k\leq k_t}|Q_k|.$$

Indeed, $I + x(T_t - I) = (1-x)I + xT_t$, so the claim follows when $\sup_{k\leq k_t}|(T_tQ)_k| \leq \sup_{k\leq k_t}|Q_k|$. The latter is the case, since, on the event that $k+\delta \leq 2L_{t-1} + t\delta$, and arguing as in (2.48), we have

$$\begin{aligned}\sup_{k\leq k_t}|(T_tQ)_k| &\leq \sup_{k\leq k_t}\Big[\Big(1 - \frac{k+\delta}{2L_{t-1}+t\delta}\Big)|Q_k| + \frac{k-1+\delta}{2L_{t-1}+t\delta}|Q_{k-1}|\Big] \\ &\leq \Big(1 - \frac{1}{2L_{t-1}+t\delta}\Big)\sup_{k\leq k_t}|Q_k|.\end{aligned}$$

Since $k \leq k_t$, the inequality $k + \delta \leq 2L_{t-1} + t\delta$ follows when $k_t \leq 2L_{t-1} + (t-1)\delta$.

As a result, on the event $\{k_t \leq 2L_{t-1} + (t-1)\delta\}$, we have that

$$\max_{x\in[0,1]}\sup_{k\leq k_t}|f_k''(x;w)| \leq w(w-1)\sup_{k\leq k_t}\big|\big((T_t-I)^2p\big)_k\big|. \tag{2.65}$$

Now recall the definition (2.28) of the operator S, and note that, for any sequence $Q = \{Q_k\}_{k=1}^\infty$, we can write

$$((T_t - I)Q)_k = \frac{\theta}{(2L_{t-1}+t\delta)}(SQ)_k = \frac{1}{t\mu}(SQ)_k + (R_tQ)_k, \tag{2.66}$$

where the remainder operator R_t is defined as

$$(R_tQ)_k = \left(\frac{k+\delta}{2t\mu + t\delta} - \frac{k+\delta}{2L_{t-1}+t\delta}\right)Q_k + \left(\frac{k-1+\delta}{2L_{t-1}+t\delta} - \frac{k-1+\delta}{2t\mu+t\delta}\right)Q_{k-1}. \tag{2.67}$$

Combining (2.63), (2.64), (2.65) and (2.66), on the event $\{k_t \le 2L_{t-1}+(t-1)\delta\}$ and uniformly for $k \le k_t$, we obtain that

$$\kappa_k(t;w) \le \left(\frac{w}{\mu}-1\right)(Sp)_k + wt\sup_{k\le k_t}|(R_tp)_k| + \frac{1}{2}w(w-1)t\sup_{k\le k_t}\left|\left((T_t-I)^2p\right)_k\right|, \tag{2.68}$$

together with a similar lower bound with minus signs in front of the last two terms. Indeed,

$$\begin{aligned}\kappa_k(t;w) &= t[f_k(1;w)-f_k(0;w)]-(Sp)_k\\ &= tw\left((T_t-I)p\right)_k+\frac{1}{2}tf_k''(x_k;w)-(Sp)_k\\ &= \frac{wt}{\mu t}(Sp)_k+wt(Rp)_k-(Sp)_k+\frac{1}{2}tf_k''(x_k;w),\end{aligned}$$

and (2.68) follows from this identity and (2.65).

With (2.68) at hand, we are now ready to complete the proof of (c). We start by treating the case where $r_m=1$ for some integer $m\ge 1$. In this case, with $w=W_t=m=\mu$, we have that $(\frac{w}{\mu}-1)(Sp)_k\equiv 0$. Furthermore, the inequality $k_t\le 2L_{t-1}+(t-1)\delta$ is true almost surely when t is sufficiently large. Hence, we are done if we can bound the last two terms in (2.68) with $w=W_t$. To do this, note that, by the definition (2.26) of T_t and the fact that $2L_{t-1}+t\delta\ge k_t=\eta t$, with $\eta>\mu$,

$$\sup_{k\ge1}\left|\left((T_t-I)Q\right)_k\right| \le \frac{2}{\eta t}\sup_{k\ge1}(k+|\delta|)|Q_k|. \tag{2.69}$$

Applying (2.69) twice yields that

$$\left|\left((T_t-I)^2p\right)_k\right| \le \frac{4}{\eta^2t^2}\sup_{k\ge1}(k+|\delta|)^2p_k,$$

and hence, since by Proposition 1.4, $p_k\le ck^{-\gamma}$ for some $\gamma>2$, there is a constant $\tilde{C}_p$ such that

$$\sup_{k\ge1}(k+|\delta|)^2p_k\le\tilde{C}_p. \tag{2.70}$$

Finally, since $L_t=mt$, we have that

$$\left|(R_tp)_k\right| \le \frac{2}{m(t-1)t}\sup_{k\ge1}(k+|\delta|)p_k \le \frac{2\tilde{C}_p}{m(t-1)t}.$$

Summarizing, we arrive at the statement that there exists $c_{m,\delta}$ such that

$$\sup_{k\le k_t}|\kappa_k(t;m)|\le\frac{c_{m,\delta}}{t},$$

which proves the claim in (c) when $r_m=1$, and with $\beta=0$.

We now move to random initial degrees. For any $a\in(0,1)$, we can split

$$\kappa_k(t)=\kappa_k(t)\mathbf{1}_{\{W_t\le t^a\}}+\kappa_k(t)\mathbf{1}_{\{W_t>t^a\}}. \tag{2.71}$$

On the event $\{k_t\le 2L_{t-1}+(t-1)\delta\}$, the first term of (2.71) can be bounded by the right side of (2.68), i.e.,

$$\begin{aligned}&\kappa_k(t)\mathbf{1}_{\{W_t\le t^a\}}\\ &\quad\le\Big((W_t/\mu-1)(Sp)_k+tW_t\sup_{k\le k_t}|(R_tp)_k|+\frac{W_t(W_t-1)}{2}t\sup_{k\le k_t}\left|\left((T_t-I)^2p\right)_k\right|\Big)\mathbf{1}_{\{W_t\le t^a\}},\end{aligned}$$

with a similar lower bound where the last two terms have a minus sign. From (2.62), we obtain the upper bound

$$\kappa_k(t)\mathbf{1}_{\{W_t>t^a\}} \le C_\eta L_t \mathbf{1}_{\{W_t>t^a\}}.$$

Combining these two upper bounds with (2.71), and adding the term $(W_t/\mu-1)(Sp)_k\mathbf{1}_{\{W_t>t^a\}}$ to the right side, yields that on the event that $\{k_t \le 2L_{t-1}+(t-1)\delta\}$,

$$\begin{aligned}\kappa_k(t) \le \left(\frac{W_t}{\mu}-1\right)(Sp)_k + tW_t\mathbf{1}_{\{W_t\le t^a\}}\sup_{k\le k_t}|(R_tp)_k| \\ + tW_t^2\mathbf{1}_{\{W_t\le t^a\}}\sup_{k\le k_t}\left|\left((T_t-I)^2p\right)_k\right| + \mathbf{1}_{\{W_t>t^a\}}C_\eta L_t,\end{aligned} \tag{2.72}$$

and similarly we get as a lower bound,

$$\begin{aligned}\kappa_k(t) \ge \left(\frac{W_t}{\mu}-1\right)(Sp)_k - tW_t\mathbf{1}_{\{W_t\le t^a\}}\sup_{k\le k_t}|(R_tp)_k| \\ - tW_t^2\mathbf{1}_{\{W_t\le t^a\}}\sup_{k\le k_t}\left|\left((T_t-I)^2p\right)_k\right| - \mathbf{1}_{\{W_t>t^a\}}\left(C_s\Big|\frac{W_t}{\mu}-1\Big| + C_\eta L_t\right),\end{aligned} \tag{2.73}$$

where we used that $\sup_{k\ge1}|(Sp)_k|\le C_s$. We use (2.72) and (2.73) on $\{k_t\le 2L_{t-1}+(t-1)\delta\}$, and (2.62) on the event $\{k_t > 2L_{t-1}+(t-1)\delta\}$ to arrive at

$$\begin{aligned}\kappa_k(t) \le \left(\frac{W_t}{\mu}-1\right)(Sp)_k + tW_t\mathbf{1}_{\{W_t\le t^a\}}\sup_{k\le k_t}|(R_tp)_k| \\ + tW_t^2\mathbf{1}_{\{W_t\le t^a\}}\sup_{k\le k_t}\left|\left((T_t-I)^2p\right)_k\right| + \left(\mathbf{1}_{\{W_t>t^a\}} + \mathbf{1}_{\{k_t>2L_{t-1}+(t-1)\delta\}}\right)\left(C_s\Big|\frac{W_t}{\mu}-1\Big| + C_\eta L_t\right),\end{aligned} \tag{2.74}$$

with a similar lower bound where the last three terms have a minus sign. We now take expectations on both sides of (2.74) and take advantage of the equality $\mathbb{E}[W_t/\mu]=1$ and the property that $(Sp)_k$ is deterministic, so that the first term on the right side drops out. Moreover, using that W_t and L_{t-1} are independent, as well as that $k_t > 2L_{t-1}+(t-1)\delta$ implies that $L_{t-1}\le k_t$, we arrive at

$$|\mathbb{E}[\kappa_k(t)]| \le \mathbb{E}\Big[\mathbf{1}_{\{W_t>t^a\}}\Big(C_s\Big|\frac{W_t}{\mu}-1\Big| + C_\eta t\Big)\Big] \tag{2.75}$$

$$+\Big(C_\eta\big(k_t+\mathbb{E}[W_t]\big) + C_s\mathbb{E}\big[\big|\frac{W_t}{\mu}-1\big|\big]\Big)\mathbb{P}(k_t > 2L_{t-1}+(t-1)\delta) \tag{2.76}$$

$$+t\mathbb{E}\Big[\sup_{k\le k_t}|(R_tp)_k|\Big]\mathbb{E}\Big[W_t\mathbf{1}_{\{W_t>t^a\}}\Big] \tag{2.77}$$

$$+t\mathbb{E}[W_t^2\mathbf{1}_{\{W_t\le t^a\}}]\mathbb{E}\Big[\sup_{k\le k_t}\big|\big((T_t-I)^2p\big)_k\big|\Big]. \tag{2.78}$$

We now bound each of these four terms one by one. To bound (2.75), we use that W_t has finite $(1+\varepsilon)$-moment, to obtain that

$$\mathbb{E}\big[\mathbf{1}_{\{W_t>t^a\}}W_t\big] = \mathbb{E}\big[\mathbf{1}_{\{W_t>t^a\}}W_t^{-\varepsilon}W_t^{1+\varepsilon}\big] \le t^{-a\varepsilon}\mathbb{E}\big[W_t^{1+\varepsilon}\big] = O(t^{-a\varepsilon}),$$

and,

$$t\mathbb{E}\big[\mathbf{1}_{\{W_t>t^a\}}\big] = t\mathbb{P}\big(W_t^{1+\varepsilon} > t^{a(1+\varepsilon)}\big) \le t^{1-a(1+\varepsilon)}\mathbb{E}\big[W_t^{1+\varepsilon}\big] = O(t^{1-a(1+\varepsilon)}),$$

which bounds (2.75) as

$$\mathbb{E}\Big[\mathbf{1}_{\{W_t>t^a\}}\Big(C_s\Big|\frac{W_t}{\mu}-1\Big| + C_\eta t\Big)\Big] = O(t^b), \tag{2.79}$$

with $b=\max\{-a\varepsilon, 1-a(1+\varepsilon)\}$.

To bound (2.76), we use that when $k_t > 2L_{t-1} + (t-1)\delta$, then $L_{t-1} < \frac{1}{2}(\eta t - \delta(t-1)) = \frac{1}{2}(\eta - \delta)(t-1) + \frac{1}{2}\eta$. Now, since $\eta \in (\mu, 2\mu + \delta)$, we have that $\frac{1}{2}(\eta - \delta) < \mu$. Standard Large Deviation theory and the fact that the initial degrees W_i are non-negative give that the probability that $L_{t-1} < \sigma(t-1)$, with $\sigma < \mu$, is exponentially small in t. As a result, we obtain that

$$\Big(C_\eta\big(k_t + \mathbb{E}[W_t]\big) + C_s\mathbb{E}\big[\big|\frac{W_t}{\mu} - 1\big|\big]\Big)\mathbb{P}\Big(k_t > 2L_{t-1} + (t-1)\delta\Big) = O(t^{-1}). \tag{2.80}$$

To bound (2.77), we use that $2L_{t-1} + t\delta \geq L_{t-1} \geq t - 1 \geq t/2$, and also use (2.70), to obtain that

$$\mathbb{E}\Big[\sup_{k\leq k_t} |(R_t p)_k|\Big] \leq \frac{c}{t^2}\mathbb{E}|L_{t-1} - t\mu| \sup_{k\geq 1}(k + |\delta|)p_k \leq \frac{c}{t^2}\mathbb{E}|L_{t-1} - t\mu|.$$

Thus,

$$t\mathbb{E}\Big[\sup_{k\leq k_t} |(R_t p)_k|\Big]\mathbb{E}\Big[W_t\mathbf{1}_{\{W_t > t^a\}}\Big] \leq \frac{c}{t}\mathbb{E}|L_{t-1} - t\mu| \cdot t^{-a\varepsilon} \leq O\left(t^{-a\varepsilon - \varepsilon/(1+\varepsilon)}\right), \tag{2.81}$$

where the final bound follows from (2.49).

Finally, to bound (2.78), note that

$$\mathbb{E}[W_t^2\mathbf{1}_{\{W_t \leq t^a\}}] = \mathbb{E}[W_t^{1-\varepsilon}W_t^{1+\varepsilon}\mathbf{1}_{\{W_t \leq t^a\}}] \leq t^{a(1-\varepsilon)}\mathbb{E}[W_t^{1+\varepsilon}] = O\left(t^{a(1-\varepsilon)}\right),$$

and, by (2.26) and the fact that $2L_{t-1} + t\delta \geq \eta t$ for some $\eta > 0$, we have

$$\mathbb{E}\left[\sup_{k\leq k_t} \left|\big((T_t - I)^2 p\big)_k\right|\right] \leq \frac{c}{t^2}\sup_{k\geq 1}(k + |\delta|)^2 p_k. \tag{2.82}$$

This leads to the bound that

$$t\mathbb{E}[W_t^2\mathbf{1}_{\{W_t \leq t^a\}}]\mathbb{E}\left[\sup_{k\leq k_t} \left|\big((T_t - I)^2 p\big)_k\right|\right] \leq O\left(t^{a(1-\varepsilon)-1}\right). \tag{2.83}$$

Combining the bounds in (2.79), (2.80), (2.81) and (2.83) completes the proof of part (c) of Lemma 2.4, for any a such that $1/(\varepsilon + 1) < a < 1$. □

2.5 Discussion and related results

In this section, we discuss the similarities and the differences of our proof of the asymptotic degree sequence in Theorem 1.3 as compared to other proofs that have appeared in the literature.

Virtually all proofs of asymptotic power laws in preferential attachment models use the two steps presented here in Propositions 2.1 and 2.2. For bounded support of W_i, the concentration result in Proposition 2.1 and its proof are identical in all proofs. For unbounded W_i, an additional coupling argument is required. The differences arise in the statement and proof of Proposition 2.2. Our Proposition 2.2 proves a stronger result than that for $\delta = 0$ appearing in [9] for the fixed number of edges case, and in [13] in the random number of edges case, in that the result is valid for a wider range of k values and the error term is smaller. Indeed, in [13], the equivalent of Proposition 2.2 is proved for $k \leq t^{1/21}$, and in [9] for $k \leq t^{1/15}$.

In [13], also a random number of edges $\{W_i\}_{i\geq 1}$ is allowed. However, it is assumed that the support of W_i is *bounded*, in which case the Azuma-Hoeffding argument presented in Section 2.3 simplifies considerably. The nice feature of allowing an unbounded number of edges is the competition of the exponents in (1.9). The model in [13] is much more general than the model discussed here, and at every time allows for the creation of a new vertex with a random number of edges or the addition of a random number of edges to an old vertex. Under reasonable assumptions on the parameters of the model, a power law is proved for the degree sequence of the graph, indicating that the occurrence of power laws is rather robust in the model definition (in contrast to the value of the power-law exponent). Due to the complexity of the model in [13], the asymptotic degree sequence satisfies a more involved recurrence relation (see [13, Eq. (2)]) than the one in (1.7).

We close this discussion by reviewing some related results. Similar results for various random graph processes where a *fixed* number of edges is added can be found in [21], where also similar error bounds are proved for models where a fixed number of edges is added. In [6], a *directed* preferential attachment model is investigated, and it is proved that the degrees obey a power law similar to the one in [9]. Finally, in [1], the error bound in Proposition 2.1 is proved for $m = 1$ for several models. The result for fixed $\{W_t\}_{t\geq 1}$ and $m > 1$ is, however, not contained there. We intend to make use of this result in order to study distances in preferential attachment models in [15]. For related references, see [21] and [29]. We finally mention the results in [23]. There, a scale-free graph process is studied where, conditionally on $G(t)$, edges are added *independently* with a probability which is proportional to the degree of the vertex. In this case, as in [9], the power-law exponent can only take the value $\tau = 3$, but it can be expected that by incorporating an additive δ-term as in (1.1), the model can be generalized to $\tau \geq 3$. Since $\delta < 0$ is not allowed in this model (by the independence of the edges, a degree is zero with positive probability), we expect that only $\tau \geq 3$ is possible.

3 Proof of Theorem 1.6

In this section, we write $F(x) = \mathbb{P}(W_1 \leq x)$, and assume that $1 - F(x) = x^{1-\tau}L(x)$ for some slowly varying function $x \mapsto L(x)$. Throughout this section, we write $\tau = \tau_{\mathrm{W}}$.

From (1.1) it is immediate that

$$d_i(t) = d_i(t-1) + X_{i,t}, \qquad \text{for } i = 0, 1, 2, \ldots, t-1, \tag{3.1}$$

where, conditionally on $d_i(t-1)$ and $\{W_j\}_{j=1}^t$, the distribution of $X_{i,t}$ is binomial with parameters W_t and success probability

$$q_i(t) = \frac{d_i(t-1) + \delta}{2L_{t-1} + t\delta}. \tag{3.2}$$

Hence, for $t > i$,

$$\begin{aligned}\mathbb{E}[(d_i(t)+\delta)^s] &= \mathbb{E}[\mathbb{E}\big[\,(d_i(t-1)+\delta+X_{i,t})^s\,|d_i(t-1), \{W_j\}_{j=1}^t\big]] \\ &\leq \mathbb{E}\big[\big(d_i(t-1)+\delta+\mathbb{E}\big[X_{i,t}|d_i(t-1), \{W_j\}_{j=1}^t\big]\big)^s\big]\,,\end{aligned} \tag{3.3}$$

where we have used the Jensen inequality $\mathbb{E}[(a+X)^s] \leq (a+\mathbb{E}[X])^s$, which follows from concavity of $t \mapsto (a+t)^s$ for $0 < s < 1$. Next, we substitute $\mathbb{E}\Big[X_{i,t}|d_i(t-1), \{W_j\}_{j=1}^t\Big] = W_t q_i(t)$ and use the inequality $2L_{t-1} + t\delta \geq L_{t-1} + \delta$, to obtain that

$$\begin{aligned}\mathbb{E}[(d_i(t)+\delta)^s] &\leq \mathbb{E}\Big[(d_i(t-1)+\delta)^s \left(1 + \frac{W_t}{2L_{t-1}+t\delta}\right)^s\Big] \\ &\leq \mathbb{E}\Big[(d_i(t-1)+\delta)^s \left(1 + \frac{W_t}{L_{t-1}+\delta}\right)^s\Big] = \mathbb{E}\Big[(d_i(t-1)+\delta)^s \left(\frac{L_t+\delta}{L_{t-1}+\delta}\right)^s\Big].\end{aligned}$$

The above two steps can be repeated, for $t > i+1$, to yield

$$\begin{aligned}\mathbb{E}[(d_i(t)+\delta)^s] \quad &\leq \quad \mathbb{E}\Big[\mathbb{E}\Big[(d_i(t-1)+\delta)^s\Big|d_i(t-2), \{W_j\}_{j=1}^t\Big]\left(\frac{L_t+\delta}{L_{t-1}+\delta}\right)^s\Big] \\ &\leq \quad \mathbb{E}\Big[(d_i(t-2)+\delta)^s\left(\frac{L_{t-1}+\delta}{L_{t-2}+\delta}\right)^s\left(\frac{L_t+\delta}{L_{t-1}+\delta}\right)^s\Big].\end{aligned}$$

Thus, by induction, and because $d_i(i) = W_i$, we get that, for all $t > i \geq 1$,

$$\mathbb{E}[(d_i(t)+\delta)^s] \leq \mathbb{E}\left[(W_i+\delta)^s \prod_{n=i+1}^{t} \left(\frac{L_n+\delta}{L_{n-1}+\delta}\right)^s\right] = \mathbb{E}\left[(W_i+\delta)^s \left(\frac{L_t+\delta}{L_i+\delta}\right)^s\right]. \tag{3.4}$$

The case $i = 0$ can be treated by $\mathbb{E}[(d_0(t)+\delta)^s] = \mathbb{E}[(d_1(t)+\delta)^s]$, which is immediate from the definition of $G(1)$.

Define $f(W_i) = (W_i+\delta)^s$ and

$$g(W_i) = \left(\frac{L_t+\delta}{L_i+\delta}\right)^s = \left(1 + \frac{W_{i+1}+W_{i+2}+\cdots+W_t}{W_1+W_2+\ldots+W_i+\delta}\right)^s,$$

and notice that when we condition on all W_j, $1 \leq j \leq t$, except W_i, then the map $W_i \mapsto f(W_i)$ is increasing in its argument, whereas $W_i \mapsto g(W_i)$ is decreasing. This implies that,

$$\mathbb{E}[f(W_i)g(W_i)] \leq \mathbb{E}[f(W_i)]\mathbb{E}[g(W_i)]. \tag{3.5}$$

Hence,

$$\mathbb{E}[(d_i(t)+\delta)^s] \leq \mathbb{E}[(W_i+\delta)^s]\mathbb{E}\left[\left(\frac{L_t+\delta}{L_i+\delta}\right)^s\right] \leq \mathbb{E}[(W_i+\delta)^s]\mathbb{E}\left[(L_t+\delta)^s\right]\mathbb{E}\left[(L_i+\delta)^{-s}\right], \tag{3.6}$$

where in the final step we have applied the inequality (3.5) once more.

For $i \to \infty$,

$$\mathbb{E}\left[(L_i+\delta)^{-s}\right] = (1+o(1))\mathbb{E}\left[L_i^{-s}\right], \qquad \mathbb{E}\left[(L_t+\delta)^s\right] = (1+o(1))\mathbb{E}\left[L_t^s\right]. \tag{3.7}$$

The moment of order s of $W_i + \delta$ can be bounded by

$$\mathbb{E}[(W_i+\delta)^s] \leq \mathbb{E}\left[W_i^s\left(1+\frac{|\delta|}{W_i}\right)^s\right] \leq (1+|\delta|)^s\mathbb{E}[W_i^s] = (1+|\delta|)^s\mathbb{E}[W_1^s], \tag{3.8}$$

since $W_i \geq 1$. Combining (3.6), (3.7) and (3.8) gives for i sufficiently large and $t > i$,

$$\mathbb{E}[(d_i(t)+\delta)^s] \leq (1+|\delta|)^s\mathbb{E}[W_1^s]\,\mathbb{E}\left[L_i^{-s}\right]\mathbb{E}\left[L_t^s\right](1+o(1)). \tag{3.9}$$

We will bound each of the terms $\mathbb{E}[W_1^s]$, $\mathbb{E}\left[L_t^s\right]$ and $\mathbb{E}\left[L_i^{-s}\right]$ separately.

It is well known that a positive random variable, where the tail of the survival function is regularly varying with exponent $1-\tau$ possesses all moments of order $s < \tau - 1$, so that $\mathbb{E}[W_1^s] < \infty$

Take a norming sequence $\{a_n\}_{n\geq 1}$ such that

$$a_n = \sup\left\{x : 1 - F(x) \geq \frac{1}{n}\right\}, \tag{3.10}$$

then it is immediate that $a_n = n^{1/(\tau-1)}l(n)$, where $n \mapsto l(n)$ is slowly varying. Also, conveniently, we have that $1 - F(a_n) \geq 1/n$. In the Appendix (cf. Lemma 4.1, part (a)), we show that for some constant C_s

$$\mathbb{E}[L_t^s] = C_s a_t^s(1+o(1)) \leq C_s t^{s/(\tau-1)} l(t)^s. \tag{3.11}$$

As a second result, we prove in the Appendix (cf. Lemma 4.1, part (b)), that, for i sufficiently large,

$$\mathbb{E}\left[L_i^{-s}\right] \leq C_s a_i^{-s} = C_s i^{-s/(\tau-1)} l(i)^{-s}. \tag{3.12}$$

Combining the equations (3.9), (3.11) and (3.12), we obtain

$$\mathbb{E}[(d_i(t)+\delta)^s] \leq C\Big(\frac{t}{i \vee 1}\Big)^{s/(\tau-1)}\Big(\frac{l(t)}{l(i)}\Big)^s. \tag{3.13}$$

Finally, we note that, since $d_i(t) \geq \min\{x : x \in S_{\mathrm{W}}\} \equiv \delta + \nu$ where $\nu > 0$, and using (1.2), we can bound $\mathbb{E}[d_i(t)^s] \leq (1 \vee \nu^{-1})^s\mathbb{E}[(d_i(t)+\delta)^s]$, which together with (3.13) establishes the proof of Theorem 1.6 subject to the proof of Lemma 4.1, parts (a) and (b). □

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4 Appendix

Lemma 4.1 *Let $F(x) = \mathbb{P}(W_1 \leq x)$, and assume that $1 - F(x) = x^{1-\tau}L(x)$, and let $L_t = \sum_{i=1}^{t} W_i$ where $\{W_i\}_{i=1}^{t}$ are i.i.d. copies of W_1. Then, there exists a constant C_s such that*

$$\text{(a)} \qquad \mathbb{E}[L_t^s] = C_s a_t^s (1 + o(1)) \leq C_s t^{s/(\tau-1)} l(t)^s.$$

and

$$\text{(b)} \qquad \mathbb{E}\big[L_t^{-s}\big] \leq C_s a_t^{-s} = C_s t^{-s/(\tau-1)} l(t)^{-s}.$$

Proof: We start with part (a). We bound

$$L_t^s = (U_t + V_t)^s \leq 2^s (U_t^s + V_t^s),$$

where

$$U_t = \sum_{j=1}^{t} W_j \mathbf{1}_{\{W_j > a_t\}}, \qquad V_t = \sum_{j=1}^{t} W_j \mathbf{1}_{\{W_j \leq a_t\}} \tag{4.1}$$

By concavity of $x \mapsto x^s$,

$$\mathbb{E}[V_t^s] \le (\mathbb{E}[V_t])^s \le \Big(t \int_0^{a_t} [1 - F(x)]\, dx\Big)^s \sim (2-\tau)^{-s} a_t^s,$$

according to [17, Theorem 2(i), p. 448]. For the other term, we use that

$$U_t \le X W_{(t)},$$

where $W_{(t)} = \max_{1 \le j \le t} W_j$, the maximum summand and X the number of W_j, $1 \le j \le t$, which are larger than a_t. Then from the Hölder inequality with $p, q > 1$ such that $p^{-1} + q^{-1} = 1$,

$$\mathbb{E}[U_t^s] \le (\mathbb{E}[X^{sp}])^{1/p} (\mathbb{E}\big[W_{(t)}^{sq}\big])^{1/q}. \tag{4.2}$$

From $t[1 - F(a_t)] \to 1$, we see that all moments of X are bounded, so taking $q > 1$, but arbitrarily close to 1, we can assume that $sq < \tau_{\mathrm{W}} - 1$, and it follows from Theorem 2.1 of [27] that

$$a_t^{-sq} \mathbb{E}\big[W_{(t)}^{sq}\big] \to \mathbb{E}[\zeta^{sq}]\,,$$

where ζ is the limit in distribution of $W_{(t)}$, since $\mathbb{E}[\zeta^{sq}]$ is finite when $sq < \tau - 1$. Hence,

$$\mathbb{E}[U_t^s] \le (\mathbb{E}[X^{sp}])^{1/p} (\mathbb{E}\big[W_{(t)}^{sq}\big])^{1/q} = O\big((a_t)^{sq}\big)^{1/q} = O\big((a_t)^s\big). \tag{4.3}$$

Combining these bounds yields the bound (a) of the lemma.

We now turn to part (b) of the proof. We use that $L_t \ge W_{(t)}$, so that

$$\mathbb{E}\left[L_t^{-s}\right] \le \mathbb{E}\left[W_{(t)}^{-s}\right] = -\mathbb{E}\left[(-Y_{(t)})^s\right], \tag{4.4}$$

where $Y_i = -W_i^{-1}$ and $Y_{(t)} = \max_{1 \le i \le t} Y_i$. Clearly, $Y_i \in [-1, 0]$, so that $\mathbb{E}[(-Y_1)^s] < \infty$. Also, $a_t Y_{(t)} = -a_t / W_{(t)}$ converges in distribution to $-E^{-1/(\tau_{\mathrm{W}}-1)}$, where E is exponential with mean 1, so again it follows from [27, Theorem 2.1] that

$$\mathbb{E}\left[(a_t/L_t)^s\right] \le -\mathbb{E}\left[(-a_t Y_{(t)})^s\right] \to \mathbb{E}[E^{-1/(\tau-1)}] < \infty. \tag{4.5}$$

□