Pattern theorems, ratio limit theorems and Gumbel maximal clusters for random fields

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Abstract

We study occurrences of patterns on clusters of size n in random fields on $\mathbb{Z}^d$. We prove that for a given pattern, there is a constant $a > 0$ such that the probability that this pattern occurs at most an times on a cluster of size n is exponentially small. Moreover, for random fields obeying a certain Markov property, we show that the ratio between the numbers of occurrences of two distinct patterns on a cluster is concentrated around a constant value. This leads to an elegant and simple proof of the ratio limit theorem for these random fields, which states that the ratio of the probabilities that the cluster of the origin has sizes $n+1$ and n converges as $n \to \infty$. Implications for the maximal cluster in a finite box are discussed.

1 Introduction and main results

We consider random fields on the lattice $\mathbb{Z}^d$ in dimensions $d \geq 2$, with a finite state space $S = \{0, 1, \ldots, q-1\}$ per site for some $q \geq 2$. Thus, the configuration space for the models studied here is $\Omega = S^{\mathbb{Z}^d}$. This space is endowed with a probability measure $\mathbb{P}$, which we assume to be translation-invariant. The results in this paper hold under different further conditions on the measure $\mathbb{P}$, which we will define and discuss first. To do so, for any configuration $\omega \in \Omega$ and any $V \subset \mathbb{Z}^d$, we will write ω_V for the configuration restricted to the set V, that is, ω_V is considered to be an element of S^V.

Definition 1.1 (Finite-energy property) We say that the measure $\mathbb{P}$ has the *finite-energy property* if there exists an $h \in (0,1)$ such that for all

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states $s \in S$,

$$h \leq \inf_{\sigma \in \Omega} \mathbb{P}\big(\omega(x) = s \bigm| \omega_{\mathbb{Z}^d \setminus \{x\}} = \sigma_{\mathbb{Z}^d \setminus \{x\}}\big) \\ \leq \sup_{\sigma \in \Omega} \mathbb{P}\big(\omega(x) = s \bigm| \omega_{\mathbb{Z}^d \setminus \{x\}} = \sigma_{\mathbb{Z}^d \setminus \{x\}}\big) \leq 1 - h. \tag{1.1}$$

Definition 1.2 (Markov property) We say that the measure $\mathbb{P}$ has the *Markov property* if the state of a site $x \in \mathbb{Z}^d$ depends only on the states of its nearest neighbours in $\mathbb{Z}^d$ and not on the rest of the field, that is, if for all $x \in \mathbb{Z}^d$ and $s \in S$,

$$\mathbb{P}\big(\omega(x) = s \bigm| \omega_{\mathbb{Z}^d \setminus \{x\}}\big) = \mathbb{P}\big(\omega(x) = s \bigm| \omega_{N_x}\big), \tag{1.2}$$

where $N_x = \{y \in \mathbb{Z}^d : |y - x| = 1\}$ is the set of neighbours of x.

Observe that if the random field is Markovian, then by translation invariance and (1.2) it has the finite-energy property (1.1) if and only if for a given vertex x, each state s has strictly positive probability regardless of the states of the neighbours of x. The Markov property implies the following "boundary-s Markov property" for every state $s \in S$, which will actually be sufficient for our purposes:

Definition 1.3 (Boundary-s Markov property) For a finite $V \subset \mathbb{Z}^d$, write $N_V = \bigcup_{x \in V} (N_x \setminus V)$ for the set of neighbours of V. We say that $\mathbb{P}$ has the *boundary-s Markov property* if for every finite $V \subset \mathbb{Z}^d$, given that all sites of N_V are in the state $s \in S$, the configuration on V is conditionally independent of the configuration on $\mathbb{Z}^d \setminus (V \cup N_V)$.

A special case of the boundary-s Markov property is the so-called *empty-boundary Markov property* as defined in [8], which is the boundary-0 Markov property in our terminology.

Note that in the discussion so far, we have considered Markovian properties under nearest-neighbour dependencies only. Indeed, in the conditional probabilities (1.2) we only consider nearest neighbours of x, and in our formulation of the boundary-s Markov property, it suffices that the nearest neighbours of V are in the state s to have independence between V and the outside world. It is not difficult, however, to extend the methods in this paper to random fields with more general dependencies between sites, as long as these dependencies have finite range.

As a typical example of the kind of random fields we want to study, we consider site percolation. In this case, $q = 2$ and for given $0 < p < 1$, we take the measure $\mathbb{P}$ to be the Bernoulli product measure $\mathbb{P}_p$ such that $\mathbb{P}_p(\omega(x) = 1) = p$ and $\mathbb{P}_p(\omega(x) = 0) = 1 - p$ for each $x \in \mathbb{Z}^d$. Since the states of different sites are independent in this case, it is clear that (1.1) holds with $h = \min(p, 1-p)$, and the field is obviously Markovian as well. For a given percolation configuration $\omega \in \Omega$, we say that a site $x \in \mathbb{Z}^d$ is *occupied* if $\omega(x) = 1$ and *vacant* if $\omega(x) = 0$. For convenience, the same terminology will be used for the states 0 and 1 of any random field. This does not attach any special meaning to the states 0 and 1, since one can always study the random fields obtained by permuting the states of S.

Using the terminology introduced above, we write $C(x) = C(x, \omega)$ for the occupied cluster of the site $x \in \mathbb{Z}^d$. That is, $C(x)$ is the set of occupied sites (i.e., sites in state 1) that can be connected to x by a nearest-neighbour path passing only through occupied sites. By convention, we set $C(x) = \varnothing$ if $\omega(x) \neq 1$ and we write $C = C(0)$ for the cluster of the origin. In general, if $X \subset \mathbb{Z}^d$, then the number of sites in X is denoted by $|X|$. In particular, the number of sites in the occupied cluster of the origin will be denoted by $|C|$. The results in this paper hold under the assumption that the distribution of $|C|$ has an exponential or stretched exponential tail.

Definition 1.4 (Exponential tail) We say that the cluster-size distribution has an *exponential tail* if the limit

$$\mu = \lim_{n\to\infty} \big[\mathbb{P}(|C| = n)\big]^{1/n} = \lim_{n\to\infty} \big[\mathbb{P}(n \le |C| < \infty)\big]^{1/n} \tag{1.3}$$

exists for some $0 < \mu \le 1$.

Definition 1.5 (Stretched exponential tail) We say that the cluster-size distribution has a *stretched exponential tail with exponent* $\beta \in (0,1)$ if the limit

$$\nu = \lim_{n\to\infty} \big[\mathbb{P}(|C| = n)\big]^{1/n^\beta} = \lim_{n\to\infty} \big[\mathbb{P}(n \le |C| < \infty)\big]^{1/n^\beta} \tag{1.4}$$

exists for some $0 < \nu < 1$.

We note that, for instance, for subcritical percolation the cluster of the origin has an exponential tail with μ strictly between 0 and 1 [7, Theorems 6.78, 8.61, 8.65]. The result (1.3) also holds for critical and supercritical percolation, but with $\mu = 1$, which indicates that the distribution decays slower than exponentially. For supercritical percolation, it is in fact known that $\mathbb{P}(|C| = n)$ decays like a stretched exponential with $\beta = (d-1)/d$. This result appears in [1, 4] for $d = 2$, in [5] for $d = 3$, and in [6] for $d \ge 4$.

It is believed that the behaviour of the cluster-size distribution described above is rather typical. That is, (1.3) is expected to be true quite generally for random fields in the non-percolating regime, i.e., when all clusters are finite. This result follows for instance if one can show that there exists a constant $A > 0$ such that

$$\frac{\mathbb{P}(|C| = n+m)}{n+m} \ge A\,\frac{\mathbb{P}(|C| = n)}{n}\,\frac{\mathbb{P}(|C| = m)}{m} \quad \text{for all } n, m \ge 1. \tag{1.5}$$

See [11, p. 91] for an example of this supermultiplicativity result in the case of percolation, which is reproduced in [7, Lemma 6.102]. Furthermore, it is believed that (1.4) holds quite generally in the percolating regime with an exponent $\beta = (d-1)/d$.

An example of a binary random field which does not satisfy the Markov property but does satisfy the boundary-s Markov property both for $s = 0$ and for $s = 1$, is the random-cluster model [8]. Although this model is defined in terms of edge occupation statuses rather than site occupation statuses (as considered here), it is not difficult to adapt our methods for the random-cluster model. Note that for the random-cluster model, the exponential decay in (1.3) can be shown by adapting the proof of Lemma 6.12 in [7], see the corrected version of Theorem 5.47 in [8]. Theorem 5.47 in [8] is stated in the more general setting of finite-energy FKG measures satisfying the empty-boundary (i.e., boundary-0) Markov property. An important open problem for the random-cluster model is whether $\mu < 1$ in the subcritical regime (see e.g. [8, Conjecture 5.54]).

1.1 Pattern theorems

Throughout this paper, we will use Q to denote the cube of diameter r at the origin,

$$Q = \{x \in \mathbb{Z}^d : 0 \leq x_i < r \text{ for all } i = 1, 2, \ldots, d\}, \tag{1.6}$$

where the diameter $r > 0$ is considered to be fixed once and for all. The *extended cube* $\overline{Q}$ is obtained by extending Q by 1 unit in all directions, that is,

$$\overline{Q} = \{x \in \mathbb{Z}^d : -1 \leq x_i < r+1 \text{ for all } i = 1, 2, \ldots, d\}. \tag{1.7}$$

The *boundary* of the cube Q is defined as $\partial Q = \overline{Q} \setminus Q$. Likewise, the boundary of $\overline{Q}$ is defined as $\partial\overline{Q} = \overline{\overline{Q}} \setminus \overline{Q}$, where, as before, $\overline{\overline{Q}}$ is obtained by extending $\overline{Q}$ by 1 unit in all directions. While Q always denotes the cube at the origin, we will write Q_x to denote the cube at the site x, that is, $Q_x = \{x + y : y \in Q\}$. The boundary of this cube, the extended cube at x, and its boundary are defined analogously as $\partial Q_x = \{x + y : y \in \partial Q\}$, $\overline{Q}_x = \{x + y : y \in \overline{Q}\}$ and $\partial\overline{Q}_x = \{x + y : y \in \partial\overline{Q}\}$.

Definition 1.6 (Pattern) A *pattern* P is a prescribed configuration of the states of the sites in the cube Q, that is, $P = (P(x) : x \in Q)$ is an element of S^Q.

Suppose that P is a pattern. Then for a given configuration $\omega \in \Omega$, we say that P *occurs at the site* x if $\omega(x + y) = P(y)$ for all $y \in Q$. We say that P occurs at x *on* C if P occurs at x and $\partial Q_x \subset C$. Thus, in our

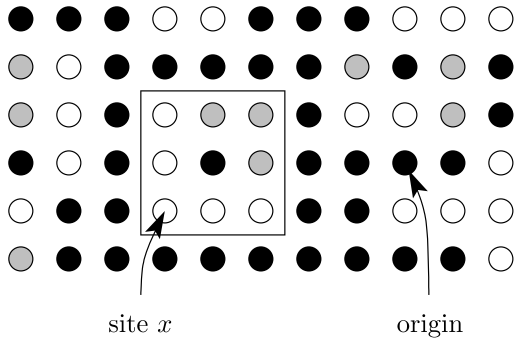

Figure 1: A piece of a three-state ($q = 3$) random-field configuration on $\mathbb{Z}^2$, with sites in states 0, 1 and 2 depicted in white, black and gray, respectively. The square highlights a pattern of diameter $r = 3$ occurring on the occupied cluster of the origin at the site x.

terminology, a pattern can only occur on the cluster of the origin if it is completely surrounded by this cluster. See Figure 1 for an illustration. The reason for defining occurrences of patterns on C in this way, is to guarantee that a pattern will always contribute the same number of sites to C whenever it occurs on C. This fact is crucial for the two-pattern theorems stated below. However, the standard pattern theorem (Theorem 1.7 below) still holds if we weaken our definition of an occurrence on C, for instance if we say that a pattern P occurs at x on C if P occurs at x and $\partial Q_x \cap C \neq \varnothing$. This is obvious, because there must be more occurrences of P on C under the weaker definition.

The standard pattern theorem states that for a given pattern P, if the cluster of the origin has size n, then, for some $a > 0$ sufficiently small, it is very unlikely that P occurs less than an times on this cluster. This statement is true whether or not one allows occurrences of patterns to overlap. However, in this paper we will also study the ratio between the numbers of occurrences of two distinct patterns, and for these results it is important that patterns cannot overlap. We will avoid this by imposing a condition on which occurrences of a pattern are counted. Let us now explain this in more detail.

First of all, we note that there are patterns P and P' for which overlaps are ruled out by definition. If that is the case, we do not have to impose restrictions on which occurrences of the patterns we count for the two-pattern theorems (Theorems 1.8 and 1.9) to hold. For generality, however, we will choose to consider only occurrences of patterns at sites of the grid $V := (r+2)\mathbb{Z}^d + (1, 1, \dots, 1)$. That is, we denote by $N_P = N_P(\omega)$ the total number of occurrences of the pattern P on C at distinct sites of V, and we will study N_P rather than the total number of distinct occurrences of P on C. To explain our choice of V, we note that later on, we are going to replace occurrences of P at sites of V by occurrences of P', and we want to guarantee that this does not change the state of the origin from occupied to non-occupied. This is why V has been chosen such that the origin does not belong to Q_x for any $x \in V$.

Henceforth, we will write $\mathbb{P}_n$ for the measure $\mathbb{P}$ conditioned on the event $\{|C| = n\}$, that is, $\mathbb{P}_n(\,\cdot\,) := \mathbb{P}\big(\,\cdot \;\big|\; |C| = n\big)$. In Section 2 we will prove the following result, which is slightly different from the pattern theorem for lattice clusters appearing in [13] because of the way we have defined N_P:

Theorem 1.7 (Pattern theorem) Suppose that $\mathbb{P}$ has the finite-energy property (1.1) and that the cluster-size distribution satisfies (1.3). Let P be a pattern. Then there exists an $a > 0$ such that

$$\limsup_{n\to\infty} \big[\mathbb{P}_n(N_P \le an)\big]^{1/n} < 1.$$

The main new results of this paper, however, concern ratios between the numbers of occurrences of two distinct patterns P and P' on C. In particular, we are interested in patterns P and P' such that one of these patterns contributes one more site to the cluster of the origin than the other pattern if it occurs on C. We shall see that for such patterns, bounds on the ratio between the number of occurrences of P and the number of occurrences of P' lead directly to bounds on $\mathbb{P}(|C| = n+1)/\mathbb{P}(|C| = n)$ if the cluster-size distribution is known to have an exponential tail.

To state our results, for a given pattern P, we shall write c_P for the number of occupied sites in the pattern P which will be part of the cluster of the origin whenever P occurs on C at some site. We also introduce the notation

$$\mathbb{P}\big(\Box P\big) := \mathbb{P}\big(\omega(x) = P(x)\ \forall x \in Q, \omega(x) = 1\ \forall x \in \partial Q\big) \tag{1.8}$$

for the "probability of an occurrence of P at the origin surrounded by an occupied cluster".

Now suppose that the cluster-size distribution satisfies (1.3). Then we define, for a given pattern P,

$$\gamma_P := \mu^{-c_P}\, \mathbb{P}\big(\Box P\big), \tag{1.9}$$

and, for distinct patterns P and P',

$$\gamma_{PP'} := \frac{\gamma_P}{\gamma_{P'}} = \frac{\mu^{c_{P'}}\, \mathbb{P}\big(\Box P\big)}{\mu^{c_P}\, \mathbb{P}\big(\Box P'\big)}. \tag{1.10}$$

We believe that a law of large numbers holds for patterns, stating that there exists a $\rho > 0$ such that for any pattern P, the number N_P is concentrated around $\rho\gamma_P$ for "almost all" configurations. Although we cannot prove this, the following theorem (which we shall prove in Section 3) implies that if one can prove a law of large numbers for one particular pattern, then it must hold for all patterns. It states that for "almost all" configurations, the ratio $N_P/N_{P'}$ is concentrated around $\gamma_{PP'} = \gamma_P/\gamma_{P'}$ for two distinct patterns P and P':

Theorem 1.8 (Two-pattern theorem) Let P and P' be two distinct patterns. Suppose that $\mathbb{P}$ satisfies the boundary-1 Markov property, and that the cluster-size distribution satisfies (1.3). Then for all $\epsilon > 0$ and $\gamma_{PP'}$ as defined in (1.10),

$$\limsup_{n\to\infty} \left[\mathbb{P}_n(|N_P - \gamma_{PP'}\,N_{P'}| \geq \epsilon n)\right]^{1/n} < 1.$$

A natural question is whether stronger bounds on the ratio $N_P/N_{P'}$ hold, showing for instance that the difference between this ratio and $\gamma_{PP'}$ is at most of order $|C|^\alpha$ for some $0 < \alpha < 1$. Indeed, there are two cases in which we have obtained such stronger bounds. The first case is the case $c_P = c_{P'}$, that is, the case where the cluster size does not change if one replaces an occurrence of P on C by an occurrence of P' on C. The second case is the case of stretched exponential decay of the cluster-size distribution (i.e., the supercritical case for percolation). In that case, our stronger bounds on $N_P - \gamma_{PP'}\,N_{P'}$ also lead to a stronger version of the ratio limit theorem, see Corollary 1.11 below. The stronger version of the two-pattern theorem reads as follows:

Theorem 1.9 (Strengthened two-pattern theorem) Consider a random field which has the boundary-1 Markov property, the finite-energy property (1.1) and a cluster-size distribution satisfying (1.3). Suppose that P and P' are two distinct patterns, and let $\gamma_{PP'}$ be defined as in (1.10). Then the following statements hold:

(i) If $c_P = c_{P'}$, then for all $\alpha > \frac{1}{2}$ and for every $\epsilon > 0$,

$$\limsup_{n\to\infty} \left[\mathbb{P}_n(|N_P - \gamma N_{P'}| \geq \epsilon n^\alpha)\right]^{1/n^{2\alpha-1}} < 1.$$

(ii) If the cluster-size distribution has a stretched exponential tail with exponent β, then for every $\alpha \geq \frac{1}{2}(1+\beta)$ and every $\epsilon > 0$,

$$\limsup_{n\to\infty} \left[\mathbb{P}_n(N_P \geq \gamma N_{P'} + \epsilon n^\alpha)\right]^{1/n^{2\alpha-1}} < 1.$$

Remark. In Section 3 we show that Theorem 1.9(ii) can be strengthened if the supermultiplicativity result (1.5) holds. Under (1.5), if $c_{P'} < c_P$ and the cluster-size distribution has a stretched exponential tail with exponent β, one can show that for $\alpha = (2-\beta)^{-1}$ there exists an $a_0 > 0$ such that for every $a \geq a_0$,

$$\limsup_{n\to\infty} \left[\mathbb{P}_n(N_P \geq \gamma N_{P'} + an^{\alpha})\right]^{1/n^{2\alpha-1}} < 1. \tag{1.11}$$

Examples where Theorems 1.7, 1.8 and 1.9(i) apply are percolation, the Ising and Potts models, and, in general, the random-cluster measure. For Theorem 1.9(ii), stretched exponential decay is needed.

1.2 Ratio limit theorems

We can use our pattern theorems to prove that the ratio of the probabilities that the occupied cluster of the origin has sizes $n+1$ and n converges with n. To prove this, one can in principle follow Kesten's argument in [10], where a ratio limit theorem is derived from a pattern theorem for self-avoiding walks. The same argument also appears in [12, 13]. It requires that the probability of a configuration changes by a constant factor whenever one replaces a single occurrence of a pattern P on the cluster of the origin by an occurrence of another pattern P'. In our present context, this means that we need to assume the boundary-1 Markov property. However, under this condition we can use our two-pattern theorem (Theorem 1.8) to give a more elegant and direct proof of the ratio limit theorem, avoiding the rather technical argument by Kesten. Our proof is presented in Section 4.

Corollary 1.10 (Ratio limit theorem) Suppose that $\mathbb{P}$ has the finite-energy property (1.1) and the boundary-1 Markov property, and that (1.3) holds for the cluster-size distribution. Then the limit μ in (1.3) also satisfies

$$\lim_{n\to\infty} \frac{\mathbb{P}(|C| = n+1)}{\mathbb{P}(|C| = n)} = \lim_{n\to\infty} \frac{\mathbb{P}(n+1 \leq |C| < \infty)}{\mathbb{P}(n \leq |C| < \infty)} = \mu.$$

Examples where Corollary 1.10 applies are percolation, Ising and Potts models, as well as the random-cluster model for general p and q. Note that in the case of percolation, for subcritical p, the result in Corollary 1.10 is stronger than (1.3), but still much weaker than the widely believed tail-behaviour of the cluster-size distribution, namely that there exist $\theta = \theta(d) \in \mathbb{R}$ and $A = A(p,d)$ such that

$$\mathbb{P}_p(|C| \geq n) = An^{\theta}\mu^n[1 + o(1)]. \tag{1.12}$$

For supercritical p, we can obtain a stronger result than Corollary 1.10 by virtue of our stronger version of the two-pattern theorem, Theorem 1.9:

Corollary 1.11 (Strengthened ratio limit theorem) Suppose that $\mathbb{P}$ has the boundary-1 Markov property and the finite-energy property (1.1), and that the cluster-size distribution has a stretched exponential tail with exponent $\beta \in (0,1)$. Then, for every $\epsilon > 0$,

$$\left| \frac{\mathbb{P}(|C| = n+1)}{\mathbb{P}(|C| = n)} - 1 \right| \leq \frac{\epsilon}{n^{(1-\beta)/2}}$$

for sufficiently large n. Hence, for every $x > 0$ and $0 < \alpha \leq (1-\beta)/2$,

$$\lim_{n\to\infty} \frac{\mathbb{P}(|C| = n + \lfloor x n^{\alpha} \rfloor)}{\mathbb{P}(|C| = n)} = 1.$$

Remark. If the supermultiplicativity result (1.5) holds, then we can derive a stronger lower bound for the ratio $\mathbb{P}(|C| = n+1)/\mathbb{P}(|C| = n)$, by virtue of the Remark below Theorem 1.9. Namely, in that case we can show that there exists a constant $A > 0$ such that for n sufficiently large,

$$\frac{\mathbb{P}(|C| = n+1)}{\mathbb{P}(|C| = n)} \geq 1 - \frac{A}{n^{(1-\beta)/(2-\beta)}}. \tag{1.13}$$

Examples where Corollary 1.11 applies are supercritical percolation, the Ising model and random-cluster measures in those cases where the Wulff-shape results have been proved (see [1, 4, 5, 6] and the references therein).

1.3 Consequences for maximal clusters

In this subsection, we describe the consequences of the ratio limit theorem for maximal clusters as derived in [9]. In order to state our results, we need some further notation. Let $B_n = [-n, n]^d \cap \mathbb{Z}^d$ be the cube of width $2n+1$. We let

$$|C_{\max}| = |C_{\max}(\omega)| = \max_{x \in B_n} |C(x)| \tag{1.14}$$

denote the size of the maximal cluster having a non-empty intersection with B_n. Furthermore, we define the cluster $C_{\rm le}(x)$ by

$$C_{\rm le}(x) = \begin{cases} C(x) & \text{if } x \text{ is the left-endpoint of } C(x), \\ \varnothing & \text{otherwise,} \end{cases} \tag{1.15}$$

where by the left-endpoint of a finite set $A \subset \mathbb{Z}^d$, we mean the minimum of A in the lexicographic order. In [9], results were shown for $|C_{\max}|$ assuming the ratio limit theorem for the cluster $C_{\rm le}(0)$ instead of $C(0)$. Therefore, we shall need the following corollary, which we shall prove simultaneously with Corollary 1.10 in Section 4:

Corollary 1.12 (Ratio limit theorem for $C_{\mathbf{le}}(0)$) Suppose that $\mathbb{P}$ has the finite-energy property (1.1) and the boundary-1 Markov property, and that (1.3) holds. Then the limit μ in (1.3) also satisfies

$$\lim_{n\to\infty} \frac{\mathbb{P}(|C_{\mathrm{le}}(0)| = n+1)}{\mathbb{P}(|C_{\mathrm{le}}(0)| = n)} = \lim_{n\to\infty} \frac{\mathbb{P}(n+1 \leq |C_{\mathrm{le}}(0)| < \infty)}{\mathbb{P}(n \leq |C_{\mathrm{le}}(0)| < \infty)} = \mu.$$

The results for $|C_{\max}|$ in [9] hold under further conditions on the measure $\mathbb{P}$, that we will formulate now. We start by introducing a so-called 'high mixing' condition. For $A \subseteq \mathbb{Z}^d$, we write E_A for an event depending only on the site variables in A. Let $\mathcal{F}_A$ denote the σ-field generated by the site variables in A. For $m > 0$, we define

$$\phi(m) = \sup \frac{1}{|A_1|} \left| \mathbb{P}\left(E_{A_1} \mid E_{A_2}\right) - \mathbb{P}\left(E_{A_1}\right) \right|, \tag{1.16}$$

where the supremum is taken over all finite subsets A_1, A_2 of $\mathbb{Z}^d$, with $d(A_1, A_2) \geq m$ (d denoting Euclidian distance) and over all $E_{A_i} \in \mathcal{F}_{A_i}$ with $\mathbb{P}(E_{A_2}) > 0$.

Note that this $\phi(m)$ differs from the usual φ-mixing function since we divide by the size of the dependence set of the event E_{A_1}. This is natural in the context of Gibbsian random fields, where the classical φ-mixing mostly fails (except for the simplest i.i.d. case and ad-hoc examples of independent copies of one-dimensional Gibbs measures). We are now ready to formulate the non-uniformly exponentially ϕ-mixing (NUEM) condition:

Definition 1.13 (NUEM) We say that a random field is *non-uniformly exponentially ϕ-mixing* (NUEM) if there exist constants $C, c > 0$ such that

$$\phi(m) \leq C \exp(-cm) \quad \text{for all } m > 0. \tag{1.17}$$

Examples of random fields satisfying the NUEM condition are Gibbs measures with exponentially decaying potential in the Dobrushin uniqueness regime, or local transformations of such measures. Of course, for site percolation, where we have independence, we have $\phi \equiv 0$. The NUEM condition is weaker than the more often used weak mixing (see [2, 3] for a definition of weak mixing and the related stronger notion of ratio weak mixing). Weak mixing holds for (i) the Ising model on $\mathbb{Z}^d$ for supercritical temperatures ([2, Corollary 3.8]); (ii) the Ising model on $\mathbb{Z}^2$ for all temperatures and non-zero external field ([2, Corollary 3.8]); (iii) the Potts model on $\mathbb{Z}^2$ with $q \geq 26$ ([2, Corollary 3.9] and [3, Theorem 1.8]); for general Potts models on $\mathbb{Z}^d$ under the assumption of exponential decay of connectivities and random-cluster uniqueness; (iv) general random-cluster measures on $\mathbb{Z}^d$ under the assumption of exponential decay of connectivities and random-cluster uniqueness ([3, Theorem 1.6]). We refer to [2, 3] and the references therein for a discussion on mixing aspects.

In the subcritical case, we shall deal with models where the cluster-size distribution has exponential tails, i.e., we shall assume that $\mathbb{P}$, the law of the random field, satisfies (1.3) with $\mu < 1$. We shall also assume a *second moment condition*, which is used in [9] to prove the asymptotics of the largest connected component. The precise assumption is that for all $\alpha > 1$,

$$\limsup_{n\to\infty} \sum_{0<|x|<n^\alpha} \frac{\mathbb{P}\big(n \le |C_{\rm le}(x)| < \infty,\, n \le |C_{\rm le}(0)| < \infty\big)}{\mathbb{P}\big(n \le |C_{\rm le}(0)| < \infty\big)} < \infty, \qquad (1.18)$$

In [9, Proposition 3.7], it is shown that for Markov models with the FKG property, (1.18) follows from (1.3) with $\mu < 1$. An inspection of the proof of [9, Proposition 3.7] shows that it also applies to models with the FKG property satisfying the boundary-s Markov property with $s = 0$.

Let us now state our results for $|C_{\max}|$. As an implication of Corollary 1.12, we obtain the following result on the subcritical maximal cluster:

Theorem 1.14 (Subcritical Gumbel maximal cluster) Assume that $\mathbb{P}$ has the finite-energy property (1.1), is NUEM, and satisfies (1.3) with $\mu < 1$ and (1.18). Then there exists a sequence $u_n \in \mathbb{N}$, with $u_n \to \infty$, real numbers $a, \rho > 0$ and a bounded sequence $a_n \in [a, 1]$, such that for all $x \in \mathbb{N}$,

$$\mathbb{P}(|C_{\max}| \le u_n + x) = e^{-a_n\, \mu^x} + O(n^{-\rho}).$$

Theorem 1.14 shows that $|C_{\max}|$ is bounded above and below by Gumbel laws, and shows in particular that the sequence $|C_{\max}| - u_n$ is tight. As explained in more detail in [9], the statement above in terms of the sequence a_n is necessary, and is, for instance, also present when dealing with the maximum of n i.i.d. geometric random variables.

Examples of random fields for which Theorem 1.14 applies are given in the following corollary:

Corollary 1.15 (Examples subcritical Gumbel maximal clusters) The conclusions in Theorem 1.14 hold in the following special cases:

(i) Subcritical percolation;

(ii) Subcritical Ising model;

(iii) Subcritical random-cluster models satisfying the FKG-property, for which $\mu < 1$.

The assumed FKG-property in Corollary 1.15(iii) is necessary to ensure that (1.18) applies. See e.g. [8] for a discussion of the FKG-property for random-cluster measures, and [8, Theorems 5.55, 5.86] for examples of parameter values for which the random-cluster measure satisfies $\mu < 1$.

Theorem 1.14 follows from [9, Theorem 3.6], and, in the case of percolation, from [9, Theorem 1.1] combined with [9, Theorem 1.5]. We note that in [9], it was assumed that the measure $\mathbb{P}$ has so-called *subcritical clusters*[1] [9, Definition 3.3(i)], which is implied by the ratio limit theorem with $\mu < 1$, Corollary 1.12 above. In fact, Corollary 1.12 implies that ξ and ζ, defined in [9, Definition 3.3(i)], are equal so that the strongest version in [9, Theorem 3.6] applies.

We now turn to the maximal supercritical cluster, which was investigated in [9] only in the context of site percolation, to which we will therefore restrict ourselves here as well. For supercritical p, we define

$$|C_{\max}| = \max_{x \in B_n : |C(x)| < \infty} |C(x)| \tag{1.19}$$

to be the largest *finite* cluster intersecting the cube. The ratio limit theorem implies that, in the language of [9, Definition 3.3(ii)], percolation has supercritical clusters. Therefore, [9, Theorem 3.9] implies the following Gumbel statistics for the largest finite supercritical cluster:

Theorem 1.16 (Supercritical Gumbel maximal cluster for percolation) Let $p_c < p < 1$ and let $\mathbb{P}_p$ denote the percolation measure with parameter p. There exists a sequence $u_n(x)$ with $u_n(x) \in \mathbb{N}$ and $u_n(x) \to \infty$ for all $x \in \mathbb{R}$ as $n \to \infty$, such that for all $x \in \mathbb{R}$,

$$\mathbb{P}_p(|C_{\max}| \le u_n(x)) = e^{-e^{-x}} + o(1),$$

where the error term may depend on x.

1.4 Further notation used in the proofs

Throughout the rest of the paper, we will write c_n for the probability that the cluster of the origin has size n, and p_n for the probability that the cluster of the origin is finite and has size at least n. That is, $c_n := \mathbb{P}(|C| = n)$ and $p_n := \mathbb{P}(n \le |C| < \infty) = \sum_{m=n}^{\infty} c_m$.

2 Proof of the pattern theorem

This section is devoted to the proof of the pattern theorem, Theorem 1.7. Our proof is similar to the proof in [13] of a pattern theorem stated in a different context. We recall that the pattern theorem states that for a given pattern P, there exists an $a > 0$ such that it is very unlikely that P

[1] We take this opportunity to correct a mistake in Definition 3.3(i) in [9]: the condition that $\mathbb{P}(|C_{\mathrm{le}}(0)| < \infty) = 1$ should read $\mathbb{P}(|C(0)| < \infty) = 1$. Similarly in Definition 3.3(ii).

occurs at less than an distinct sites of the grid $V = (r+2)\mathbb{Z}^d + (1, 1, \dots, 1)$ on a cluster of size n.

The proof proceeds roughly as follows. If we assume that one is likely to see at most an occurrences of P on C at sites of the grid V, then there are many sites left on this grid where we can create new occurrences of P on C. By creating a single new occurrence, we may change the size of C, but never by more than $|\overline{Q}| = (r+2)^d$ sites. Consider all configurations we can generate by introducing δn new occurrences of P on C. For every configuration we start from, there is an exponential number of ways to introduce δn occurrences of P on C, and all generated configurations contribute to c_m for some $m \in [n - |Q|\delta n, n + |Q|\delta n]$, and hence to $p_{n-|Q|\delta n}$ in particular. However, introducing occurrences of P will change the probability, and it is furthermore clear that many of the generated configurations will be obtained multiple times from different starting configurations. Still, if the random field has the finite-energy property and δ is small enough, we will generate so many distinct configurations that this contribution wins, and this will prove Theorem 1.7. Now let us fill in the details of the proof.

Proof of Theorem 1.7. We want to study the probability that the cluster of the origin has size n, and P occurs on C at no more than an distinct sites of V. So, for all $n \geq 1$ define the set of relevant configurations by

$$S_n = \{\omega \in \Omega : |C| = n, N_P \leq an\}. \tag{2.1}$$

Consider a configuration $\omega \in S_n$. Then C has size n, which implies that there are at least $n/|\overline{Q}| =: bn$ extended cubes at sites of V that intersect C. No more than an of these cubes can contain an occurrence of P. Thus, assuming $a < b$, with every configuration $\omega \in S_n$ we can associate a collection $X(\omega) \subset V$ of exactly $bn - an$ sites such that the extended cubes at these sites intersect C, and P does not occur at any of these sites.

For any configuration $\omega \in S_n$ and site $x \in X(\omega)$, we can create an occurrence of P at x on C by vacating all occupied sites of $\partial\overline{Q}_x$ that do not belong to C, occupying all sites of ∂Q_x, and changing the configuration inside Q_x into an occurrence of P at x. Here we emphasise that we first vacate sites on the boundary of $\overline{Q}_x$ to avoid connecting the occupied cluster of the origin to another occupied cluster, whose size we cannot control and might be infinite. Also note that by the finite-energy property (1.1), the factor by which the probability changes upon a single introduction of P is bounded from below by a constant of the form $\exp(-A\,|\partial\overline{Q} \cup \overline{Q}|) = \exp\bigl(-A\,(r+4)^d\bigr)$ for some fixed $A > 0$.

Now, for every $\omega \in S_n$, consider all possible ways of creating occurrences of P at exactly δn sites chosen from the collection $X(\omega)$, where $\delta > 0$ is a small number that will be fixed later on. Write S'_n for the collection of all configurations generated in this way. Then in general, the same configuration $\omega' \in S'_n$ can be obtained from multiple configurations in S_n. However, the only differences between these configurations can be the local configurations inside the extended cubes and their boundaries at those sites of V where P occurs on the cluster $C(\omega')$. One can have $q^{|\overline{Q}\cup\partial\overline{Q}|}$ different configurations inside an extended cube plus its boundary, where we recall that q is the size of the state space S per site. Also, there are at most $(\delta+a)n$ occurrences of P on C if we create δn new occurrences. Therefore, on the one hand,

$$\mathbb{P}(S'_n) \geq \binom{bn-an}{\delta n} e^{-A\,|\overline{Q}\cup\partial\overline{Q}|\,\delta n}\, q^{-(\delta+a)n\,|\overline{Q}\cup\partial\overline{Q}|}\, \mathbb{P}(S_n). \qquad (2.2)$$

On the other hand it is clear that for any $\omega' \in S'_n$, the occupied cluster of the origin is finite and has size at least $n - |Q|\,\delta n$. Therefore,

$$\mathbb{P}(S'_n) \leq p_{n-|Q|\,\delta n}. \qquad (2.3)$$

We now divide (2.2) and (2.3) by c_n, combine the two resulting inequalities, take n-th roots on both sides and then the $\limsup_{n\to\infty}$, using (1.3) and Stirling's formula. This leads to

$$\mu^{-|Q|\,\delta} \geq \frac{(b-a)^{b-a}}{\delta^{\delta}\,(b-a-\delta)^{b-a-\delta}}\, e^{-A\,|\overline{Q}\cup\partial\overline{Q}|\,\delta}\, q^{-(\delta+a)\,|\overline{Q}\cup\partial\overline{Q}|} \times \limsup_{n\to\infty} \left[\mathbb{P}_n(N_P \leq an)\right]^{1/n}. \qquad (2.4)$$

At this point we may just as well take $a = \delta$. Setting $t = \delta/(b-a)$ and $\tilde{\mu} = \mu^{-|Q|}\, e^{A|\overline{Q}\cup\partial\overline{Q}|}\, q^{2|\overline{Q}\cup\partial\overline{Q}|}$, the previous inequality can be rewritten as

$$\limsup_{n\to\infty} \left[\mathbb{P}_n(N_P \leq \delta n)\right]^{1/n} \leq \left(t^t\,(1-t)^{1-t}\,\tilde{\mu}^t\right)^{b-\delta}. \qquad (2.5)$$

The right-hand side is smaller than 1 whenever $t = \delta/(b-a) < \tilde{\mu}^{-1}$. Therefore, the left-hand side is smaller than 1 for sufficiently small $\delta > 0$, which proves Theorem 1.7. □

3 Proofs of the two-pattern theorems

In this section we are interested in the ratio between the numbers of occurrences of two distinct pattern P and P' on C. We will prove, as stated in Theorems 1.8 and 1.9, that if the random field has the boundary-1 Markov property, then the ratio $N_P/N_{P'}$ must be close to a fixed number $\gamma = \gamma_{PP'}$ defined in (1.10). The basic strategy of the proofs is as follows. We will consider those configurations such that $N_P - \gamma N_{P'}$ differs from 0 by at least

ϵn^α occurrences for some $\epsilon > 0$ and $0 < \alpha \leq 1$. Then we can make this difference smaller by either turning occurrences of P into occurrences of P' or the other way around. By deriving a bound on the probability of the collection of configurations generated in this way, in a similar way as in the proof of Theorem 1.7, we can then show that the probability that $|N_P - \gamma N_{P'}|$ is at least ϵn^α must be small. We start by proving Theorem 1.8, in which $\alpha = 1$, after which we prove Theorem 1.9.

Proof of Theorem 1.8. We will show that under the conditions stated in the theorem, for every choice of P and P' (and every possible value of the corresponding $\gamma = \gamma_{PP'}$), and for all $\epsilon > 0$,

$$\limsup_{n\to\infty} \big[\mathbb{P}_n(N_P \leq \gamma N_{P'} - \epsilon n)\big]^{1/n} \leq \frac{\big(1 + \frac{\epsilon}{1+\gamma}\big)^{1+(1+\epsilon)\,\gamma^{-1}}}{(1+\epsilon)^{(1+\epsilon)\,\gamma^{-1}}} < 1. \tag{3.1}$$

Since this holds for any choice of the two patterns, we can interchange the roles of P and P' in (3.1), which also replaces $\gamma = \gamma_{PP'}$ by $\gamma^{-1} = \gamma_{P'P}$ and ϵ by $\epsilon\gamma^{-1}$, to obtain

$$\limsup_{n\to\infty} \big[\mathbb{P}_n(N_P \geq \gamma N_{P'} + \epsilon n)\big]^{1/n} \leq \frac{\big(1 + \frac{\epsilon}{1+\gamma}\big)^{1+\gamma+\epsilon}}{\big(1 + \frac{\epsilon}{\gamma}\big)^{\gamma+\epsilon}} < 1. \tag{3.2}$$

These two results together evidently imply Theorem 1.8. Hence, it suffices to derive (3.1).

To derive (3.1), let S_n be the collection of configurations such that $|C| = n$ and $N_P \leq \gamma N_{P'} - \epsilon n$. For every $\omega \in S_n$, consider all possible ways in which we can change $\delta\,\epsilon n$ occurrences of P' on C at sites of V into occurrences of P on C, where $0 < \delta < 1$ will be fixed later on. Then, for given N_P and $N_{P'}$, the ratio of the number of configurations generated in this way to the number of configurations from which they were obtained, is given by the factor

$$\begin{aligned} &\binom{N_P + N_{P'}}{N_P + \delta\,\epsilon n} \binom{N_P + N_{P'}}{N_P}^{-1} \\ &\qquad = \frac{N_P!}{(N_P + \delta\,\epsilon n)!} \, \frac{N_{P'}!}{(N_{P'} - \delta\,\epsilon n)!} \\ &\qquad \geq \frac{N_P!}{(N_P + \delta\,\epsilon n)!} \, \frac{(\gamma^{-1} N_P + \gamma^{-1}\,\epsilon n)!}{(\gamma^{-1} N_P + (\gamma^{-1} - \delta)\,\epsilon n)!}, \end{aligned} \tag{3.3}$$

where the last inequality follows from $N_P \leq \gamma N_{P'} - \epsilon n$.

Now we observe (by differentiating with respect to N) that for all $k = 1, 2, \ldots, \delta\,\epsilon n$, the fraction $(\gamma^{-1}N + (\gamma^{-1} - \delta)\,\epsilon n + k)/(N + k)$ is non-increasing in N. Therefore, using the fact that N_P is necessarily smaller than n, we can bound the factor (3.3) by

$$\prod_{k=1}^{\delta\,\epsilon n} \frac{\gamma^{-1}N_P + (\gamma^{-1}-\delta)\,\epsilon n + k}{N_P + k}$$

$$\geq \prod_{k=1}^{\delta\,\epsilon n} \frac{\gamma^{-1}n + (\gamma^{-1}-\delta)\,\epsilon n + k}{n+k} \tag{3.4}$$

$$= \frac{n!}{(n+\delta\,\epsilon n)!}\,\frac{(\gamma^{-1}n + \gamma^{-1}\,\epsilon n)!}{(\gamma^{-1}n + (\gamma^{-1}-\delta)\,\epsilon n)!}.$$

For every configuration obtained from $\omega \in S_n$ by changing $\delta\,\epsilon n$ occurrences of P' on C into occurrences of P on C, the cluster of the origin has size $|C| = n - \delta\,\epsilon n\,\delta_c$, where $\delta_c = c_{P'} - c_P$. Moreover, by virtue of the boundary-1 Markov property, every change of P' on C into P changes the probability of a configuration by the factor $\mathbb{P}(\square P)/\mathbb{P}(\square P')$, where we recall the definition (1.8) of $\mathbb{P}(\square P)$. Therefore, we can write

$$\frac{c_{n-\delta\,\epsilon n\,\delta_c}}{c_n} \geq \left(\frac{\mathbb{P}(\square P)}{\mathbb{P}(\square P')}\right)^{\delta\,\epsilon n} \frac{n!}{(n+\delta\,\epsilon n)!}\,\frac{(\gamma^{-1}n + \gamma^{-1}\,\epsilon n)!}{(\gamma^{-1}n + (\gamma^{-1}-\delta)\,\epsilon n)!} \times \mathbb{P}_n(N_P \leq \gamma N_{P'} - \epsilon n). \tag{3.5}$$

Taking the n-th root on both sides and then the $\limsup_{n\to\infty}$, using (1.3), (1.10) and Stirling's formula, leads to

$$1 \geq f_\delta(\epsilon)\,\limsup_{n\to\infty}\left[\mathbb{P}_n(N_P \leq \gamma N_{P'} - \epsilon n)\right]^{1/n}, \tag{3.6}$$

where

$$f_\delta(x) = \frac{(1+x)^{(1+x)\gamma^{-1}}}{(1+(1-\delta\gamma)x)^{\gamma^{-1}+(\gamma^{-1}-\delta)x}\,(1+\delta x)^{1+\delta x}}. \tag{3.7}$$

Observe that $f_\delta(0) = 1$, and taking the derivative of $f_\delta(x)$ with respect to x yields

$$\frac{f'_\delta(x)}{f_\delta(x)} = \frac{1}{\gamma}\,\log\left(\frac{1+x}{1+x-\delta\gamma x}\right) + \delta\,\log\left(\frac{1+x-\delta\gamma x}{1+\delta x}\right). \tag{3.8}$$

It follows that for all $x > 0$ and $\delta \leq (1+\gamma)^{-1}$, $f'_\delta(x) > 0$ and therefore $f_\delta(x) > 1$. It is not difficult to see that for fixed $x > 0$, $f_\delta(x)$ is actually maximal at $\delta = (1+\gamma)^{-1}$. We therefore set $\delta = (1+\gamma)^{-1}$, and then the desired result (3.1) follows from (3.6). This proves Theorem 1.8. $\square$

Remark. The bound appearing on the right-hand side of (3.1) is not best possible in general. Indeed, if we restrict ϵ to the range $(0,\gamma)$, then the combinatorial factor (3.3) can also be bounded by

$$\begin{aligned}\frac{N_P!}{(N_P+\delta\,\epsilon n)!}\,\frac{N_{P'}!}{(N_{P'}-\delta\,\epsilon n)!} &\geq \frac{(\gamma N_{P'}-\epsilon n)!}{(\gamma N_{P'}-(1-\delta)\,\epsilon n)!}\,\frac{N_{P'}!}{(N_{P'}-\delta\,\epsilon n)!} \\ &\geq \frac{(\gamma n-\epsilon n)!}{(\gamma n-(1-\delta)\,\epsilon n)!}\,\frac{n!}{(n-\delta\,\epsilon n)!}.\end{aligned} \tag{3.9}$$

Here, as before, the first inequality follows from $N_P \leq \gamma N_{P'} - \epsilon n$ and the second is a consequence of the fact that $N_{P'}$ is necessarily less than n. The same reasoning as in the previous proof then leads us again to an inequality of the form (3.6), where now the function $f_\delta(x)$ is given by

$$f_\delta(x) = \frac{(1-x\gamma^{-1})^{\gamma-x}}{(1-(1-\delta)x\gamma^{-1})^{\gamma-(1-\delta)x}\,(1-\delta x)^{1-\delta x}}. \tag{3.10}$$

As before, for fixed $0<x<\gamma$, this function is maximal and larger than 1 at $\delta=(1+\gamma)^{-1}$. This leads for $0<\epsilon<\gamma$ to the bound

$$\limsup_{n\to\infty}\left[\mathbb{P}_n(N_P \leq \gamma N_{P'}-\epsilon n)\right]^{1/n} \leq \frac{\left(1-\frac{\epsilon}{1+\gamma}\right)^{1+\gamma-\epsilon}}{\left(1-\frac{\epsilon}{\gamma}\right)^{\gamma-\epsilon}} < 1. \tag{3.11}$$

This bound is better than the bound in (3.1) for small values of γ, but worse than (3.1) for large values of γ.

Proof of Theorem 1.9. The proof of Theorem 1.9 is similar to the proof of Theorem 1.8 above. We define the set of relevant configurations by

$$S_n = \{\omega\in\Omega : |C| = n, N_P - \gamma N_{P'} \geq \epsilon n^\alpha\}. \tag{3.12}$$

For every $\omega \in S_n$, we can choose $\delta\,\epsilon n^\alpha$ occurrences of P on C at sites of V and turn them into occurrences of P', where $0<\delta<1$ is a number that we will fix later. This will change the size of C to $n+\delta\,\epsilon n^\alpha\,\delta_c$, since $\delta_c = c_{P'} - c_P$ is the change in $|C|$ if we replace a single occurrence of P on C by an occurrence of P'.

For given N_P and $N_{P'}$, the ratio of the number of generated configurations in which there are $N_P - \delta\,\epsilon n^\alpha$ occurrences of P on C at sites of V and $N_{P'}+\delta\,\epsilon n^\alpha$ occurrences of P' on C at sites of V to the number of configurations in S_n from which they can be generated, is, similarly to (3.3), given by

$$\begin{aligned}\binom{N_P + N_{P'}}{N_P - \delta\,\epsilon n^\alpha}&\binom{N_P + N_{P'}}{N_P}^{-1}\\ &= \frac{N_P!}{(N_P - \delta\,\epsilon n^\alpha)!}\,\frac{N_{P'}!}{(N_{P'} + \delta\,\epsilon n^\alpha)}\\ &\geq \frac{(\gamma N_{P'} + \epsilon n^\alpha)!}{(\gamma N_{P'} + (1-\delta)\epsilon n^\alpha)!}\,\frac{N_{P'}!}{(N_{P'} + \delta\,\epsilon n^\alpha)!}\\ &\geq \frac{(\gamma n + \epsilon n^\alpha)!}{(n - \delta\,\epsilon n^\alpha)!}\,\frac{n!}{(n + \delta\,\epsilon n^\alpha)!}.\end{aligned} \tag{3.13}$$

Here, the first inequality follows from $N_P - \gamma N_{P'} \geq \epsilon n^\alpha$, and in the second inequality we used monotonicity of the expression in $N_{P'}$, which can be shown by an argument similar to the one below (3.3), together with the fact that $N_{P'} \leq n$. We note that we have assumed that $0 < \delta \leq \gamma^{-1}$ to obtain the second inequality.

Because for every configuration generated from $\omega \in S_n$, we have that $|C| = n + \delta\,\epsilon n^\alpha\,\delta_c$, we can write

$$\begin{aligned}\frac{c_{n+\delta\,\epsilon n^\alpha\,\delta_c}}{c_n} \geq \left(\frac{\mathbb{P}(\square P')}{\mathbb{P}(\square P)}\right)^{\delta\,\epsilon n^\alpha} &\frac{(\gamma n + \epsilon n^\alpha)!}{(n - \delta\,\epsilon n^\alpha)!}\,\frac{n!}{(n + \delta\,\epsilon n^\alpha)!}\\ &\times \mathbb{P}_n(N_P - \gamma N_{P'} \geq \epsilon n^\alpha).\end{aligned} \tag{3.14}$$

Using Stirling's formula and substituting (1.10) for $\gamma = \gamma_{PP'}$, this expression can be rewritten as

$$\begin{aligned}\frac{c_{n+\delta\,\epsilon n^\alpha\,\delta_c}}{c_n} \geq \mu^{\delta\,\epsilon n^\alpha\,\delta_c}\,&e^{(\gamma^{-1}\delta - \frac{1}{2}\gamma^{-1}\delta^2 - \frac{1}{2}\delta^2)\epsilon n^{2\alpha-1} + o(n^{2\alpha-1})}\\ &\times \mathbb{P}_n(N_P - \gamma N_{P'} \geq \epsilon n^\alpha).\end{aligned} \tag{3.15}$$

We now choose $\delta = (1+\gamma)^{-1}$, since this maximizes the middle stretched exponential term on the right-hand side. Raising to the power $n^{1-2\alpha}$ (here we assume $\alpha > \frac{1}{2}$) and taking the $\limsup_{n\to\infty}$ then leads to

$$\begin{aligned}e^{-\frac{1}{2}(\gamma+\gamma^2)^{-1}\,\epsilon}\,\limsup_{n\to\infty}&\left[\frac{c_{n+\delta\,\epsilon n^\alpha\,\delta_c}}{c_n}\,\mu^{-\delta\,\epsilon n^\alpha\,\delta_c}\right]^{1/n^{2\alpha-1}}\\ &\geq \limsup_{n\to\infty}\left[\mathbb{P}_n(N_P - \gamma N_{P'} \geq \epsilon n^\alpha)\right]^{1/n^{2\alpha-1}}.\end{aligned} \tag{3.16}$$

Observe that the left-hand side in (3.16) is smaller than 1 if $\delta_c = 0$, which immediately proves the first claim of the theorem.

For $\delta_c \neq 0$ we need a bound on the ratio of cluster-size probabilities appearing on the left-hand side of (3.16). The exponential decay (1.3) does not lead to useful bounds, but in the supercritical case, where $\mu = 1$, the stretched exponential decay (1.4) tells us that

$$\limsup_{n\to\infty} \left[\frac{c_{n+\delta\,\epsilon n^\alpha\,\delta_c}}{c_n} \right]^{1/n^\beta} = 1. \tag{3.17}$$

In (3.16), this makes the left-hand side smaller than 1 if we take $2\alpha - 1 \geq \beta$, that is, $\alpha \geq \frac{1}{2}(1+\beta)$. This completes the proof of Theorem 1.9. □

To conclude this section, we strengthen the above bound in the case $\delta_c < 0$ under the condition (1.5) of supermultiplicativity. This gives

$$\frac{c_{n-\delta\,\epsilon n^\alpha\,|\delta_c|}}{c_n} \leq \frac{1}{A}\,\frac{\delta\,\epsilon n^\alpha\,|\delta_c|}{c_{\delta\,\epsilon n^\alpha\,|\delta_c|}}. \tag{3.18}$$

Using the stretched exponential decay (1.4), we obtain from this inequality

$$\limsup_{n\to\infty} \left[\frac{c_{n-\delta\,\epsilon n^\alpha\,|\delta_c|}}{c_n} \right]^{1/n^{\alpha\beta}} \leq e^{\xi(\delta\,\epsilon\,|\delta_c|)^\beta} \tag{3.19}$$

for some $0 < \xi < \infty$. We note that $\alpha\beta = 2\alpha - 1$ if we take $\alpha = (2-\beta)^{-1}$. Then, if we insert the previous expression into (3.16), we see that the left-hand side becomes smaller than 1 if we take ϵ larger than some $a_0 > 0$, which proves the statement in the Remark following Theorem 1.9.

4 Proofs of the ratio limit theorems

We shall now show how the pattern theorems proved above can be combined to derive the ratio limit theorems (Corollaries 1.10 and 1.11). For this, we shall take a fixed cube diameter $r = 3$, and consider two specific patterns P and P': P is the pattern such that the site $(1, 1, \ldots, 1)$ is occupied and all other sites of Q are vacant, and P' is the pattern such that the origin is occupied and all other sites of Q are vacant, see Figure 2. For these two patterns and integers $i, j \geq 0$, we introduce the notation

$$c_n(i,j) = \mathbb{P}_p(|C| = n, N_P = i, N_{P'} = j). \tag{4.1}$$

Observe that in our earlier notation used to formulate the pattern theorems, $\delta_c = c_{P'} - c_P = 1$. For percolation, the patterns P and P' are chosen such that whenever we change an occurrence of P on C into an occurrence of P' on C, we do not change the probability of the configuration. This is, however, not generally the case for a Markovian random field.

Our proofs of the ratio limit theorems are based on the following observation. Let $S_n(i,j)$ be the collection of configurations such that $|C| = n$,

Figure 2: The patterns P and P' in dimension $d = 2$. White sites are vacant, black sites are occupied.

$N_P = i$ and $N_{P'} = j$. Then, for every $\omega \in S_n(i,j)$, there are i ways of changing one occurrence of P on C at a site of V into an occurrence of P' on C, leading to an $\omega' \in S_{n+1}(i-1, j+1)$. For each configuration that we generate in this way, there are j other configurations in the set $S_n(i,j)$ from which we could have obtained the same configuration by changing one occurrence of P on C at a site of V into an occurrence of P' on C. Therefore, for $i \geq 1$ and $j \geq 0$,

$$c_{n+1}(i-1, j+1) = \frac{i}{j+1} \frac{\mathbb{P}(\square P')}{\mathbb{P}(\square P)} c_n(i,j), \tag{4.2}$$

where we have again used the boundary-1 Markov property. From this equality one sees that bounds on the ratio $N_P/N_{P'}$ will lead directly to bounds on c_{n+1}/c_n. We will now use this to prove Corollary 1.10 for random fields satisfying the boundary-1 Markov property.

Proof of Corollaries 1.10 and 1.12. First let us show that if c_{n+1}/c_n converges to μ and $p_n = \sum_{m=n}^{\infty} c_m$, then p_{n+1}/p_n converges to μ as well. To show this, fix $0 < \epsilon < \mu$. Since c_{n+1}/c_n converges to μ, there exists an integer $N_\epsilon > 0$ such that

$$\left| \frac{c_{n+1}}{c_n} - \mu \right| \leq \epsilon \tag{4.3}$$

for all $n > N_\epsilon$. Observing that

$$\frac{c_n}{p_n} = \left[\sum_{m=0}^{\infty} \frac{c_{n+m}}{c_n} \right]^{-1} = \left[1 + \sum_{m=1}^{\infty} \prod_{k=1}^{m} \frac{c_{n+k}}{c_{n+k-1}} \right]^{-1}, \tag{4.4}$$

it follows that for $n > N_\epsilon$, if $\mu < 1$,

$$\frac{c_n}{p_n} \leq \left[\sum_{m=0}^{\infty} (\mu - \epsilon)^m \right]^{-1} = 1 - \mu + \epsilon, \tag{4.5}$$

$$\frac{c_n}{p_n} \geq \left[\sum_{m=0}^{\infty} (\mu + \epsilon)^m \right]^{-1} = 1 - \mu - \epsilon. \tag{4.6}$$

Since $p_{n+1}/p_n = 1 - c_n/p_n$, this implies $\lim_{n\to\infty} p_{n+1}/p_n = \mu$ for every $0 < \mu < 1$. Now note that (4.5) also applies if $\mu = 1$. Since $c_n/p_n \geq 0$ holds trivially, we also obtain $\lim_{n\to\infty} p_{n+1}/p_n = \mu$ when $\mu = 1$. Thus, to prove Corollary 1.10, it remains to show that c_{n+1}/c_n converges to μ.

Now let us write $c_n^* = \mathbb{P}\big(|C_{\text{le}}(0)| = n\big)$ and $p_n^* = \sum_{m=n}^{\infty} c_m^*$. We can repeat the argument above (with c_n^*, p_n^* in place of c_n, p_n) to see that $c_{n+1}^*/c_n^* \to \mu$ implies $p_{n+1}^*/p_n^* \to \mu$. However, for translation-invariant $\mathbb{P}$, we have that $c_n = n\, c_n^*$ for all $n \geq 1$ (this is Lemma 4.1 in [9]), so that $c_{n+1}^*/c_n^* \to \mu$ is implied by $c_{n+1}/c_n \to \mu$. We conclude that establishing that c_{n+1}/c_n converges to μ suffices to prove not only Corollary 1.10 but also Corollary 1.12.

We will show that c_{n+1}/c_n converges to μ for a random field satisfying the boundary-1 Markov property by using Theorems 1.7 and 1.8. Let the patterns P and P' be as above, and let $a > 0$ be the constant appearing in Theorem 1.7 for the pattern P'. Using the notation (4.1) introduced above, we can write

$$\frac{c_{n+1}}{c_n} = \sum_{j=-1}^{n} \sum_{i=1}^{n} \frac{c_{n+1}(i-1, j+1)}{c_n}. \tag{4.7}$$

We may use Theorems 1.7 and 1.8 together with (1.3), to restrict the sums in (4.7) at the cost of introducing an exponentially small error term. For convenience, we write $\gamma = \gamma_{PP'}$ for the constant defined in (1.10). Applying our observation (4.2), we obtain

$$\begin{aligned}
\frac{c_{n+1}}{c_n} &= \sum_{j=an}^{n} \sum_{i=\gamma j - o(n)}^{\gamma j + o(n)} \frac{c_{n+1}(i-1, j+1)}{c_n} + o(1) \\
&= \sum_{j=an}^{n} \sum_{i=\gamma j - o(n)}^{\gamma j + o(n)} \frac{i}{j+1} \frac{\mathbb{P}(\Box P')}{\mathbb{P}(\Box P)} \frac{c_n(i,j)}{c_n} + o(1) \\
&= \sum_{j=an}^{n} \sum_{i=\gamma j - o(n)}^{\gamma j + o(n)} \big(\gamma + o(1)\big) \frac{\mathbb{P}(\Box P')}{\mathbb{P}(\Box P)} \frac{c_n(i,j)}{c_n} + o(1) \\
&= \big(\mu + o(1)\big) \sum_{j=0}^{n} \sum_{i=0}^{n} \frac{c_n(i,j)}{c_n} + o(1) \\
&= \mu + o(1).
\end{aligned} \tag{4.8}$$

This proves that the ratio c_{n+1}/c_n converges to μ, which completes the proof of Corollaries 1.10 and 1.12. □

Proof of Corollary 1.11. Using the same notations as above, by Theorem 1.9

we can write

$$\frac{c_{n+1}}{c_n} = \sum_{j=an}^{n} \sum_{i}{}' \frac{i}{j+1} \frac{\mathbb{P}(\Box P')}{\mathbb{P}(\Box P)} \frac{c_n(i,j)}{c_n} + o\big(n^{-(1-\beta)/2}\big), \tag{4.9}$$

where the prime on the second sum means that i is restricted to run from $\gamma j - \epsilon n^{(1+\beta)/2}$ to $\gamma j + \epsilon n^{(1+\beta)/2}$, with arbitrary $\epsilon > 0$. Proceeding as in the proof of Corollary 1.10, this implies that

$$\left| \frac{c_{n+1}}{c_n} - 1 \right| \leq \frac{2\epsilon}{an^{(1-\beta)/2}} \tag{4.10}$$

for sufficiently large n, as required. This is turn implies

$$\frac{c_{n+\lfloor xn^\alpha \rfloor}}{c_n} = \prod_{k=1}^{\lfloor xn^\alpha \rfloor} \frac{c_{n+k}}{c_{n+k-1}} = \Big[1 + o\big(n^{-(1-\beta)/2}\big)\Big]^{\lfloor xn^\alpha \rfloor} = 1 + o(1) \tag{4.11}$$

for all $x > 0$ and $0 < \alpha \leq (1-\beta)/2$, establishing Corollary 1.11. □

We close off this section by proving the statement in the Remark below Corollary 1.11. By the Remark following Theorem 1.9, under (1.5) we may restrict i in the primed sum in (4.9) to the range $[\gamma j - a_0 n^{1/(2-\beta)}, n]$. The same reasoning as before then leads to the desired result

$$\frac{c_{n+1}}{c_n} \geq 1 - \frac{2a_0}{an^{(1-\beta)/(2-\beta)}}. \tag{4.12}$$

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