Bounding the boundary by the minimum and maximum degree

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Abstract

A vertex v of a graph G is a *boundary vertex* if there exists a vertex u such that the distance in G from u to v is at least the distance from u to any neighbour of v. We give the best possible lower bound, up to a constant factor, on the number of boundary vertices of a graph in terms of its minimum degree (or maximum degree). This settles a problem introduced by Hasegawa and Saito.

1 Introduction

Let $G = (V, E)$ be a graph. For every vertex $v \in V$, let $N(v) := \{u \in V : uv \in E\}$ be the *neighbourhood* of v. A vertex $v \in V$ is a *boundary vertex of* G if there exists a vertex $u \in V$ such that $\mathrm{dist}(u, v) \geq \mathrm{dist}(u, w)$ for every $w \in N(v)$. Such a vertex u is a *witness* for v. The *boundary of* G is the set $\mathcal{B}(G)$ of boundary vertices of G.

The notion of boundary was introduced by Chartrand *et al.* [1, 2] and further studied by Hasegawa and Saito [3]. They proved that for every graph G,

$$\delta(G) \leq r\left(\binom{|\mathcal{B}(G)|-1}{2} * 4\right), \tag{1}$$

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where $\delta(G)$ is the minimum degree of G. The right-hand side of (1) is the (multicoloured) Ramsey number $r\left(\binom{|\mathcal{B}(G)|-1}{2} * 4\right)$, that is the smallest integer n such that each colouring of the edges of K_n with $\binom{|\mathcal{B}(G)|-1}{2}$ colours yields a monochromatic copy of K_4. As shown by Xiaodong *et al.* [4], the Ramsey number $r(k * 4)$ is $\Omega\left(5^k\right)$. Therefore, the lower bound on the number of boundary vertices yielded by (1) cannot be better than

$$|\mathcal{B}(G)| = \Omega(\sqrt{\log(\delta(G))}).$$

The following result is a significant improvement.

Theorem 1. *For every graph G of maximum degree Δ,*

$$|\mathcal{B}(G)| \geq \log_2(\Delta + 2).$$

The bound provided by Theorem 1 is sharp up to a multiplicative factor smaller than $3\log_3(2) \approx 1.89$.

Theorem 2. *For every positive integer n, there exists a graph G_n of minimum degree $\delta_n := 3^{n-1}$ and maximum degree $\Delta_n := 3^n + n - 1$ with $|\mathcal{B}(G_n)| = 3n$. Thus, $|\mathcal{B}(G_n)| = 3\left(\log_3(\delta_n) + 1\right) < 3\log_3(\Delta_n)$.*

In particular, we deduce that the lower bound on the size of the boundary in terms of the *minimum* degree implied by Theorem 1 is essentially best possible, which answers a question of Hasegawa and Saito [3].

As it happens the vertex-connectivity of the graph G_n in Theorem 2 is also 3^{n-1}, which shows that being highly vertex-connected is not a sufficient condition for having a large boundary.

2 Upper bound on the maximum degree

Throughout this section, let $G = (V, E)$ be a graph. The *endvertices* of a path P of G are the two vertices of degree 1 in P. A *shortest path of G* is a path whose length is precisely the distance in G between its endvertices. Given a shortest path P, an *extension* of P is a shortest path Q containing P. If Q is an extension of P, we say that P *extends* to Q. The proof of Theorem 1 relies on the following observation.

Lemma 3. *Each shortest path of G extends to a shortest path between two boundary vertices.*

Proof. Let P be a shortest path of G. If one of its endvertices is not a boundary vertex, then P extends to a longer path (which is also a shortest path between its endvertices), by the definition of a boundary vertex. As the graph G is finite, we eventually obtain an extension of P whose endvertices are boundary vertices. $\square$

For every vertex $v \in V$, let $C_v : N(v) \to 2^{\mathcal{B}(G)}$ be the mapping defined by

$$C_v(u) := \{b \in \mathcal{B}(G) : \mathrm{dist}(b, u) < \mathrm{dist}(b, v)\}.$$

The proof of the following lemma relies on Lemma 3.

Lemma 4. *Let $v \in V$. For each pair (u, u') of neighbours of v, $C_v(u) \neq C_v(u')$. Moreover, $C_v(u)$ is neither empty nor the whole set $\mathcal{B}(G)$.*

Proof. By Lemma 3 there exists a path P containing the vertices u and u' that is a shortest path between two boundary vertices b and b'. We may assume that u is closer to b than u'. Let $r := \mathrm{dist}(u, b)$ and $s := \mathrm{dist}(u', b')$.

First suppose that $uu' \notin E$. In this case we may assume that v belongs to P. Since P is a shortest path, $\mathrm{dist}(v, b) = r+1 = \mathrm{dist}(u', b)-1$. Consequently, $b \in C_v(u) \setminus C_v(u')$.

Assume now that $uu' \in E$. Since $\mathrm{dist}(v, b) + \mathrm{dist}(v, b') \geq \mathrm{dist}(b, b') = r+s+1$, it follows that $\mathrm{dist}(v, b) \geq r+1$ or $\mathrm{dist}(v, b') \geq s+1$. By symmetry, assume that $\mathrm{dist}(v, b) = r + 1$. Since P is a shortest path between b and b', we deduce that $\mathrm{dist}(u', b) = r + 1$, and therefore $b \in C_v(u) \setminus C_v(u')$.

Since the edge uv extends to a shortest path between two boundary vertices, we infer that $C_v(u)$ is neither empty nor the whole set $\mathcal{B}(G)$, which concludes the proof. $\square$

Proof of Theorem 1. Let v be a vertex of G of degree at least 2. By Lemma 4, there exists an injective mapping from $N(v)$ to $2^{\mathcal{B}(G)} \setminus \{\emptyset, \mathcal{B}(G)\}$. Therefore the degree of v is at most $2^{|\mathcal{B}(G)|} - 2$, which yields the desired result. $\square$

3 Construction of the graph G_n

Fix a positive integer n. Let the vertex-set of the graph G_n be $V := A \cup B$ where

$$A := \{0, 1, 2\}^n, \quad B := \{b_j^i : j \in \{1, \ldots, n\} \text{ and } i \in \{0, 1, 2\}\}.$$

Let the edge-set of the graph G_n be

$$E := \{uv : u, v \in A\} \cup \bigcup_{\substack{j \in \{1, \ldots, n\} \\ i \in \{0,1,2\}}} \{vb_j^i : v \in A \text{ and } (v)_j = i\}.$$

The vertex b_j^i is joined to exactly those vertices $v \in A$ whose j-th coordinate is i. Notice that the vertices of A have degree $3^n + n - 1$, and those of B have degree 3^{n-1}. So it only remains to establish that $\mathcal{B}(G_n) = B$.

Note that the diameter of G is 3. For every two indices $i \neq i'$, and every $j \in \{1, 2, \ldots, n\}$, the path $b_j^i v w b_j^{i'}$ is a shortest path of length 3, where $v, w \in A$ with $(v)_j = i$ and $(w)_j = i'$. Since every pair of vertices that are not both in B lie on such a shortest path, it follows that $\mathcal{B}(G) = B$.

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