

# ASYMPTOTIC DECAY OF CORRELATIONS FOR A RANDOM WALK ON THE LATTICE $\mathbb{Z}^d$ IN INTERACTION WITH A MARKOV FIELD

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## Abstract

We consider a discrete-time random walk on  $\mathbb{Z}^d$ ,  $d = 1, 2, \dots$  in a random environment with Markov evolution in time. We complete and extend to all dimension  $d \geq 1$  the results obtained in [2] on the time decay of the correlations of the “environment from the point of view of the random walk”.

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## 1 Introduction. Description of the model and statement of the results.

Random walks on  $\mathbb{Z}^d$ ,  $d = 1, 2, \dots$ , in a time-dependent random environment with Markov evolution have been recently studied by several authors [1] [3] [4] [5] [8]. In the paper [2] we considered the time decay of the correlations of the jumps of the random walk in dimension  $d = 1$ . Under some standard assumptions we proved that if the random term

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is small enough the time correlations fall off as  $\frac{e^{-\kappa t}}{\sqrt{t}}$ , where  $\kappa$  is the spectral gap of the Markov process that describes the evolution of the environment ( $\kappa = -\ln |\mu_1|$ ,  $\mu_1$  being the first eigenvalue of the stochastic operator). This result was previously known only for dimension  $d \geq 3$ . In the paper [2] it is also shown that, under some particular circumstances, the fall-off can become of the type  $e^{-\kappa_1 t}$ , with  $0 < \kappa_1 < \kappa$ .

The aim of the present paper is to complete the results of [2] and to extend them to any dimension  $d > 1$ .

The model is a Markov chain consisting of a pair  $(\xi_t, X_t)$ ,  $t = 0, 1, \dots$ , where  $\xi_t$  is a random field  $\xi_t = \{\xi(t, x) : x \in \mathbb{Z}^d\}$ ,  $t = 0, 1, \dots$  and  $X_t$  is a random walk on the integer lattice  $\mathbb{Z}^d$ . The field at each site  $x \in \mathbb{Z}^d$  evolves as an independent copy of an ergodic Markov chain with finitely many states, and the conditional transition probabilities for the random walk depend locally on the values of the field.

For such models it was proved that if the stochastic term is small enough, then, in all dimension  $d \geq 1$ , the annealed random walk is diffusive, i.e., the Central Limit Theorem (CLT) holds as  $t \rightarrow \infty$  for the distribution of  $X_t$  induced by the field [3]. The “quenched” almost-sure CLT, i.e., the CLT for almost-all evolutions of the environment was proved for  $d \geq 3$  in [6], and, quite recently, for all dimensions [8].

The correlations of the jumps of the r.w. can be reduced, as in [2], to those of the “environment from the point of view of the r. w.”  $\eta_t$ ,  $t = 0, 1, \dots$  (to be defined below, see eq. (1.4)). In [5] it was shown in all dimension  $d \geq 1$  that, under some assumptions,  $\eta_t$  tends weakly, as  $t \rightarrow \infty$  to a stationary distribution which is absolutely continuous with respect to the equilibrium distribution of the product Markov chain  $\xi_t$ . It was also shown, only for  $d \geq 3$ , that the time correlations of  $\eta_t$  decay as  $t^{-\frac{d}{2}} e^{-\kappa t}$ , where  $\kappa$  is, as we said above, the spectral gap.

The case of odd dimension is a rather straightforward generalization of the results in [2], so we will mainly consider the even case  $d = 2s + 2$ ,  $s = 0, 1, \dots$ , which requires different estimates.

In this paper we assume for simplicity that the local field takes two values  $\xi(t, x) = \pm 1$ , and denote by  $\Omega = \{-1, 1\}^{\mathbb{Z}^d}$  the space of the environment configurations  $\xi_t$  at a given time  $t \geq 0$ . For the pair  $(\xi_t, X_t)$ ,  $t = 0, 1, \dots$  we take conditional independence, i.e., for any fixed choice of  $\xi_t \in \Omega$  and  $X_t \in \mathbb{Z}^d$ , the conditional distributions of  $X_{t+1}$  and  $\xi_{t+1}$  are independent, and given by the formulas

$$P(X_{t+1} = x + u | X_t = x, \xi_t = \bar{\xi}) = P(u) + ac(u)\bar{\xi}(x) \quad (1.1)$$

$$P(\xi(t+1, x) = s | \xi_t = \bar{\xi}) = q(\bar{\xi}(x), s), \quad x \in \mathbb{Z}^d, \bar{\xi} \in \Omega, s = \pm 1. \quad (1.2)$$

Here  $P$  is a non-degenerate random walk on  $\mathbb{Z}^d$ ,  $a \in (0, 1)$  is a fixed number such that  $P(u) \pm ac(u) \in [0, 1)$  for all  $u \in \mathbb{Z}^d$ , and  $\sum_u c(u) = 0$ .  $Q = \{q(s, s') : s, s' = \pm 1\}$ , is the transition matrix of an ergodic Markov chain (“local Markov chain”), which we suppose for simplicity to be symmetric, so that the invariant measure is  $\pi = (\frac{1}{2}, \frac{1}{2})$ . Taking the points of the state space as labels of the components, the eigenvectors are  $e_0(s) \equiv 1$ ,  $e_1(s) = s$ ,  $s = \pm 1$ , with corresponding eigenvalues  $\mu_0 = 1$  and  $\mu_1 \in (-1, 1)$ . We assume  $\mu_1 \neq 0$  to avoid trivialities.

The environment distribution at time  $t$  is a probability measure on the measurable space  $(\Omega, \mathcal{S})$ , where  $\mathcal{S}$  is the  $\sigma$ -algebra of subsets of  $\Omega$  generated by the cylinder sets. If  $\Pi_0$

is the initial distribution,  $\wp_{\Pi_0}$  will denote the corresponding distribution of the trajectories of the product Markov chain  $\{\xi_t : t \geq 0\}$  with initial measure  $\Pi_0$ .  $\wp_{\Pi_0}$  is a measure on  $\hat{\Omega} = \{-1, 1\}^{\mathbb{Z}_+^{d+1}}$ , where  $\mathbb{Z}_+^{d+1} = \{(t, x) \in \mathbb{Z}^{d+1} : t \geq 0\}$ , with the usual  $\sigma$ -algebra.

In the papers [3], [4], [5] the local Markov chain was a general ergodic Markov chain with finitely many states. The generalization of the results of the present paper in that sense is straightforward.

The pair  $(\xi_t, X_t)$ ,  $t \geq 0$ , with transition probabilities (1.1), (1.2) is also a Markov chain. We assume that the random walk starts at the origin:  $X_0 = 0$ . The probability measure on the trajectories of the chain with initial measure  $\delta_{X_0,0} \times \Pi_0$  is denoted  $\wp_{\Pi_0,0}$ .

For the transition probabilities we assume exponential decay: i.e.,  $P(u) + |c(u)| \leq C\bar{q}^{|u|}$ , for all  $u \in \mathbb{Z}^d$ , for some constants  $C > 0$ ,  $\bar{q} \in (0, 1)$ . Moreover we take  $P(u)$  even in  $u$  and  $c(u)$  either even or odd.

The Fourier transforms  $\tilde{p}_0(\lambda) = \sum_u P(u)e^{i\langle \lambda, u \rangle}$ ,  $\tilde{c}(\lambda) = \sum_u c(u)e^{i\langle \lambda, u \rangle}$  are analytic in the complex neighborhood  $W_{\bar{q}} = \{\lambda = \lambda^{(1)} + i\lambda^{(2)} : \lambda^{(1)} \in T^d, \lambda^{(2)} \in \mathbb{R}^d, |\lambda^{(2)}| < -\ln \bar{q}\}$  of the  $d$ -dimensional torus  $T^d$ , and, moreover  $\tilde{p}_0(\lambda)$  is real and even. (Here and in the following we use the norm  $|\lambda| = |\lambda_1| + |\lambda_2| + \dots + |\lambda_d|$  for  $\lambda \in \mathbb{R}^d$ .)

In analogy with condition VIII\* of [5] we need that  $\min\{\tilde{p}_0(\lambda) : \lambda \in T^d\} > 0$ , so that the Fourier coefficients  $r(u) = \int_T \frac{e^{-i\langle \lambda, u \rangle}}{\tilde{p}_0(\lambda)} dm(\lambda)$  exist. ( $dm(\lambda) = \frac{d\lambda}{(2\pi)^d}$  is the Haar measure on  $T^d$ .) We in fact assume the following condition (which implies  $\min_\lambda \tilde{p}_0(\lambda) > |\mu_1|$ ):

$$|\mu_1| \sum_u |r(u)| < 1. \quad (1.3)$$

The environment from the point of view of the particle (hereafter "e.p.v." for short) is the field  $\eta_t = \{\eta(t, x) : x \in \mathbb{Z}^d\} \in \Omega$  defined by the relation

$$\eta(t, x) = \xi(t, X(t) + x), \quad t = 0, 1, \dots \quad (1.4)$$

$\eta_t$  is a Markov chain, and its transition probabilities, for  $t \geq 1$ ,  $A \in \mathcal{S}$ ,  $\bar{\eta} \in \Omega$ , are

$$P(\eta_t \in A | \eta_{t-1} = \bar{\eta}) = \sum_y (P(y) + a c(y) \bar{\eta}(0)) P(\eta_t \in \tau_y A | \bar{\eta}) =: \hat{P}(A | \bar{\eta}). \quad (1.5)$$

Here  $\tau_y$  denotes the translation operator:  $(\tau_y \eta)(x) = \eta(x - y)$ , and  $\tau_y A = \{\tau_y \eta, \eta \in A\}$ .

As  $X_0 = 0$ , the initial measure for  $\eta_t$  and  $\xi_t$  are the same. The distribution on the space of the trajectories of  $\eta_t$  with initial measure  $\Pi_0$  is denoted  $\hat{\wp}_{\Pi_0}$ : it is a kind of projection of the full distribution  $\wp_{\Pi_0,0}$  under the application which maps the trajectories of the full Markov chain  $\{(\xi_t, X_t) : t \geq 0\}$  on the trajectories of the chain  $\{\eta_t : t \geq 0\}$ .

Let  $f, g$  be two functions on the state space  $\{-1, 1\}$  and  $x, y$  any two points on  $\mathbb{Z}^d$ . We consider the correlation

$$\langle f(\eta_t(x)), g(\eta_0(y)) \rangle_{\hat{\wp}_{\Pi_0}} = \langle f(\eta_t(x))g(\eta_0(y)) \rangle_{\hat{\wp}_{\Pi_0}} - \langle f(\eta_t(x)) \rangle_{\hat{\wp}_{\Pi_0}} \langle g(\eta_0(y)) \rangle_{\hat{\wp}_{\Pi_0}} \quad (1.6)$$

(The angular brackets  $\langle \cdot \rangle_m$  denote expectations with respect to the measure  $m$ .)  $\langle \cdot \rangle$ .

**Theorem 1.1** *Under the assumptions above, the correlation (1.6) has the following asymptotics, as  $t \rightarrow \infty$ .*

*i) If  $a < a_0$ , with  $a_0 > 0$  small enough, we have*

$$\begin{aligned} \langle f(\eta_t(x)), g(\eta_0(y)) \rangle_{\varphi_{\Pi_0,0}} &= C_0 \mu_1^t t^{-\frac{d}{2}} (1 + \mathcal{O}(\frac{\log t}{t})) \quad d \text{ even} \\ \langle f(\eta_t(x)), g(\eta_0(y)) \rangle_{\varphi_{\Pi_0,0}} &= C_0 \mu_1^t t^{-\frac{d}{2}} (1 + \mathcal{O}(\frac{1}{t})) \quad d \text{ odd.} \end{aligned} \quad (1.7)$$

*ii) if some conditions on the parameters, to be specified later, hold, then there is some  $\bar{a} > a_0$  and an open set  $\mathcal{I} \subset (a_0, \bar{a})$  such that for  $a \in \mathcal{I}$*

$$\langle f(\eta_t(x)), g(\eta_0(y)) \rangle_{\hat{\varphi}_{\Pi_0}} = e^{-\alpha_1 t} \sum_{k=1}^m p_k(t) \cos(\theta_k t) (1 + \mathcal{O}(e^{-\delta_1 t})). \quad (1.8)$$

Here  $0 < \alpha_1 < -\ln |\mu_1|$ ,  $\theta_k \in (0, \frac{\pi}{2})$ ,  $p_k(t)$  is a polynomial of order  $k = 0, \dots, m-1$ , and the constant  $\delta_1 > 0$ , and  $m \geq 0$  depend only on the transition probabilities (1.1), whereas the coefficients of the polynomials  $p_k(t)$  and the constant  $C_0$  depend also on  $f, g, x, y$ , and on the initial distribution  $\Pi_0$ .

Let  $\Delta_t = X_t - X_{t-1}$  denote the increment (jump) of the random walk at time  $t$ . If  $f_1, f_2$  are bounded functions on  $\mathbb{Z}^d$ , we consider the correlation

$$\langle f_2(\Delta_t), f_1(\Delta_1) \rangle_{\varphi_{\Pi_0,0}}. \quad (1.9)$$

We state the result on the correlations of the jumps (1.9) as a second theorem. We will however only prove Theorem 1.1, because it is not hard to see that the correlation of the jumps can be reduced to the correlation of the e.p.v. (1.6), and behave in the same way (see [2], end of §2).

**Theorem 1.2** *The asymptotics of the correlation (1.9) has again the form (1.7) (1.8), under the same conditions.*

*The constants  $\alpha_1, \delta_1, m, \theta_k, k = 0, \dots, m-1$  are the same. The constant  $C_0$  and the polynomials  $p_k(t)$  are replaced by a constant  $\bar{C}_0$  and polynomials  $q_k, k = 0, \dots, m-1$ , which, in analogy with the previous ones, also depend on  $\Pi_0$ , and the functions  $f_1, f_2$ .*

The starting point for the proofs is a representation of the correlations based on transfer matrix techniques. This part is independent of dimension and is carried out in detail in [2]. We repeat in §2 some of the results, without proofs, for convenience of the reader. The main part of the paper is the analysis of the final representation leading to the asymptotics, which depends on dimension, and is carried out in Sec. 3, 4.

## 2 Recalling some preliminary facts.

We start with some preliminary constructions which are carried out in detail in [2]. In what follows  $C$  and  $q$  may denote different positive constants, always with  $q \in (0, 1)$ .

Consider the Hilbert space  $\mathcal{H} = L_2(\Omega, \Pi)$ , where  $\Pi = \pi^{\mathbb{Z}^d}$  is the invariant measure of the product Markov chain  $\xi_t$ . The transfer matrix  $\mathcal{T}$  of the chain  $\{\eta_t : t = 0, 1, \dots\}$  is a linear operator on  $\mathcal{H}$ , which acts as

$$(\mathcal{T}\phi)(\bar{\eta}) = \int_{\Omega} \phi(\eta) \widehat{P}(d\eta|\bar{\eta}), \quad \bar{\eta} \in \Omega, \quad \phi \in \mathcal{H}, \quad (2.1)$$

where the measure  $\widehat{P}(\cdot|\bar{\eta})$  is given by (1.5).

As shown in [4, Th. 3.1],  $\mathcal{H}$  is decomposed into a direct sum of subspaces, which are invariant with respect to  $\mathcal{T}$ ,

$$\mathcal{H} = \mathcal{H}_0 + \mathcal{H}_1 + \mathcal{H}_2, \quad (2.2)$$

where  $\mathcal{H}_0$  is the one-dimensional space of the constants, and the properties of  $\mathcal{H}_i, i = 1, 2$  are described later. Correspondingly the correlation  $\langle \phi_1(\eta_t), \phi_2(\eta_0) \rangle_{\widehat{P}_{\Pi_0}}$  of any two functions  $\phi_1, \phi_2$  on  $\Omega$  can be represented as

$$\begin{aligned} \langle \phi_1(\eta_t), \phi_2(\eta_0) \rangle_{\widehat{P}_{\Pi_0}} &= \langle (\mathcal{T}^t \phi_1)(\eta_0), \phi_2(\eta_0) \rangle_{\Pi_0} = \langle (\mathcal{T}_1^t \phi_1^{(1)})(\eta_0), \phi_2(\eta_0) \rangle_{\Pi_0} \\ &\quad + \langle (\mathcal{T}_2^t \phi_1^{(2)})(\eta_0), \phi_2(\eta_0) \rangle_{\Pi_0} \end{aligned} \quad (2.3)$$

where  $\mathcal{T}_j = \mathcal{T}|_{\mathcal{H}_j}, j = 1, 2$ , and  $\phi_1^{(1)}, \phi_1^{(2)}$  are the components of  $\phi_1$  in the expansion  $\phi_1 = \phi_1^{(0)} + \phi_1^{(1)} + \phi_1^{(2)}$ , with  $\phi_1^{(0)} \in \mathcal{H}_0, \phi_1^{(1)} \in \mathcal{H}_1$ , and  $\phi_1^{(2)} \in \mathcal{H}_2$ . In [2] it is shown that if  $\phi$  is a function of the e.p.v. at some site  $x, \phi(\eta) = g(\eta(x)), \eta \in \Omega$ , then

$$\begin{aligned} \|\mathcal{T}_1^t \phi\|_{\mathcal{H}_1} &\geq K_1(\phi) (|\mu_1| \min_{\lambda} \tilde{p}_0(\lambda) + \mathcal{O}(a))^t, \\ \|\mathcal{T}_2^t \phi\|_{\mathcal{H}_2} &\leq K_2(\phi) (\mu_1^2 + \mathcal{O}(a))^t, \end{aligned} \quad (2.4)$$

for some constants  $K_1, K_2$  depending on  $\phi$ . So if  $\phi_1$  is such a function the first term on the right in (2.3) is the leading one, if  $a$  is small.

Furthermore in the space  $\mathcal{H}_1$  one can find a basis  $\{h_y : y \in \mathbb{Z}^d\}$  on which  $\mathcal{T}_1$  acts as

$$\mathcal{T}_1 h_z = \sum_y (\mathcal{T}_1)_{z,y} h_y \quad (\mathcal{T}_1)_{z,y} = \bar{p}(z-y) + S(z,y), \quad (2.5)$$

with  $\bar{p}(z) = \mu_1 P(z) + d(z)$ , satisfying, for some  $q \in (0, 1)$ , the inequalities

$$|S(z,y)| \leq C a^2 q^{|z|+|y|}, \quad |d(u)| \leq C a^2 q^{|u|}. \quad (2.6)$$

Moreover  $S$  and  $d$  are even:  $S(z,y) = S(-z,-y), d(-z) = d(z)$  and such that

$$\sum_z d(z) = 0, \quad \sum_z S(z,y) = 0 \quad \text{for all } y \in \mathbb{Z}^d. \quad (2.7)$$

Expanding  $\phi_1^{(1)}$  in the basis  $\{h_y\}$ , as  $\phi_1^{(1)} = \sum_y c_y h_y$ , by (2.5) one can see that

$$\langle \mathcal{T}_1^t \phi_1^{(1)}(\eta_0), \phi_2(\eta_0) \rangle_{\Pi_0} = \sum_y c_y (\mathcal{T}_1^t)_{y,z} \langle h_z, \phi_2 \rangle_{\Pi_0} = \sum_y c_y (\mathcal{T}_1^t)_{y,z} D_z \quad (2.8)$$

where  $D_z = \langle h_z, \phi_2 \rangle_{\Pi_0}$ . Moreover  $|c_y| < Cq^{|y|}$ ,  $|D_z| < Cq^{|z|}$ ,  $C > 0$ ,  $q \in (0, 1)$ . Therefore the Fourier transforms

$$\varphi(\lambda) = \sum_y c_y e^{i\langle \lambda, y \rangle}, \quad \psi(\lambda) = \sum_z D_z e^{i\langle \lambda, z \rangle}, \quad \lambda \in T^d \quad (2.9)$$

are analytic in the complex neighborhood  $W_q = \{\lambda = \lambda^{(1)} + i\lambda^{(2)} : \lambda^{(1)} \in T^d, \lambda^{(2)} \in \mathbb{R}^d, |\lambda^{(2)}| < -\ln q\}$  of the torus  $T^d$ . Moreover the right side of (2.8) has the form

$$\int_{T^d} \int_{T^d} \varphi(\lambda) \tilde{\mathcal{T}}_1^{(t)}(\lambda, \mu) \overline{\psi(\mu)} dm(\lambda) dm(\mu) \quad (2.10)$$

where  $\tilde{\mathcal{T}}^{(t)}(\lambda, \mu)$  is the Fourier transform of  $\mathcal{T}_{y,z}^{(t)}$ .

$\mathcal{T}_1^t$  is represented in terms of the resolvent  $\mathcal{R}_{\mathcal{T}_1}(z) = (\mathcal{T}_1 - zE)^{-1}$  as

$$\mathcal{T}_1^t = \frac{1}{2\pi i} \int_{\gamma} z^t \mathcal{R}_{\mathcal{T}_1}(z) dz, \quad (2.11)$$

where the integration is over a contour  $\gamma$  in the complex plane  $z$ , which goes around the spectrum of  $\mathcal{T}_1$ . The kernel  $R_z(\lambda, \mu)$  of  $\mathcal{R}_{\mathcal{T}_1}(z)$  is (see [5]).

$$R_z(\lambda, \mu) = \frac{\delta(\lambda - \mu)}{\tilde{p}(\lambda) - z} - \frac{\mathcal{D}(\lambda, \mu; z)}{\Delta(z)(\tilde{p}(\lambda) - z)(\tilde{p}(\mu) - z)}. \quad (2.12)$$

Here  $\delta$  is the Dirac  $\delta$ -function,  $\tilde{p}$  is the Fourier transform of  $\bar{p}$ , and  $\Delta(z)$  and  $\mathcal{D}(\lambda, \mu; z)$  are given by the series

$$\Delta(z) = 1 + \sum_{n=1}^{\infty} \frac{1}{n!} \int_{T^d} \cdots \int_{T^d} \frac{K_n(\lambda^{(1)}, \dots, \lambda^{(n)})}{\prod_{i=1}^n [\tilde{p}(\lambda^{(i)}) - z]} dm(\lambda^{(1)}) \dots dm(\lambda^{(n)}), \quad (2.13)$$

$$\mathcal{D}(\lambda, \mu; z) = \tilde{S}(\lambda, \mu) + \sum_{n=1}^{\infty} \frac{1}{n!} \int_{T^d} \cdots \int_{T^d} \frac{\tilde{K}_n(\lambda, \mu; \lambda^{(1)}, \dots, \lambda^{(n)})}{\prod_{i=1}^n [\tilde{p}(\lambda^{(i)}) - z]} dm(\lambda^{(1)}) \dots dm(\lambda^{(n)}), \quad (2.14)$$

with  $K_n(\lambda^{(1)}, \dots, \lambda^{(n)}) = \det \{\tilde{S}(\lambda^{(i)}; \lambda^{(j)})\}$ ,  $K_1(\lambda^{(1)}) = \tilde{S}(\lambda^{(1)}, \lambda^{(1)})$ , and

$$\tilde{K}_n(\lambda, \mu; \lambda^{(1)}, \dots, \lambda^{(n)}) = \begin{vmatrix} \tilde{S}(\lambda, \mu) & \tilde{S}(\lambda, \lambda^{(1)}) & \dots & \dots & \tilde{S}(\lambda, \lambda^{(n)}) \\ \tilde{S}(\lambda^{(1)}, \mu) & \tilde{S}(\lambda^{(1)}, \lambda^{(1)}) & \dots & \dots & \tilde{S}(\lambda^{(1)}, \lambda^{(n)}) \\ \dots & \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots \\ \tilde{S}(\lambda^{(n)}, \mu) & \tilde{S}(\lambda^{(n)}, \lambda^{(1)}) & \dots & \dots & \tilde{S}(\lambda^{(n)}, \lambda^{(n)}) \end{vmatrix} \quad (2.15)$$

$\tilde{S}(\lambda, \mu)$  is the Fourier transform of  $S$  in (2.5). As  $\bar{p}(y)$  and  $S(x, y)$  are even,  $\tilde{p}(\lambda)$  and  $\tilde{S}(\lambda, \mu)$  are real and even. Moreover, by (2.7),  $\tilde{S}(0, \mu) \equiv 0$ .

Therefore the main term in the correlation (2.3) is, by (2.12)

$$\int_{\gamma} z^t dz \left[ \int_{T^d} \frac{\varphi(\lambda) \overline{\psi(\lambda)}}{\tilde{p}(\lambda) - z} dm(\lambda) - \int_{T^d} \int_{T^d} \frac{\mathcal{D}(\lambda, \mu; z) \varphi(\lambda) \overline{\psi(\mu)}}{\Delta(z)(\tilde{p}(\lambda) - z)(\tilde{p}(\mu) - z)} dm(\lambda) dm(\mu) \right]. \quad (2.16)$$

The proof of Theorem 1.1 is based on the analysis of the integral (2.16), which we will carry out in the next paragraph.

### 3 Asymptotics of the integral (2.16).

By (2.5), we have  $\tilde{p}(\lambda) = \mu_1 \tilde{p}_0(\lambda) + \tilde{d}(\lambda)$ , where  $\tilde{d}(\lambda) = \sum_x d(x) e^{i(\lambda, x)}$ . From the preceding paragraph (see (2.6), (2.7)) we have,  $\tilde{d}(\lambda) = \mathcal{O}(a^2)$ ,  $\tilde{d}(0) = 0$  and  $\tilde{d}(\lambda) = \tilde{d}(-\lambda)$ . Hence if  $a$  is small enough, if  $\mu_1 > 0$ , then  $\kappa_0 := \max_\lambda \tilde{p}(\lambda) = \mu_1$  and

$$\kappa_1 := \min_\lambda \tilde{p}(\lambda) > \mu_1 \min \tilde{p}_0(\lambda) - \text{const } a^2$$

will also be positive for small  $a$ . For  $\mu_1 < 0$  the range of  $\tilde{p}(\lambda)$  is obtained by reflecting the corresponding range with the opposite eigenvalue  $-\mu_1 > 0$  and the same choice of  $P, c$  and  $a$ . We will only consider the case  $\mu_1 > 0$ .

Let  $\Sigma \subset \mathbb{C}$  denote the complex plane with a cut along the real interval  $[\kappa_1, \kappa_0]$ .  $\Delta(z)$ , and the two functions

$$\beta(z) = \int_{T^d} \frac{\varphi(\lambda) \bar{\psi}(\lambda)}{\tilde{p}(\lambda) - z} dm(\lambda), \quad \Gamma(z) = \int_{T^d} \int_{T^d} \frac{\mathcal{D}(\lambda, \mu; z) \varphi(\lambda) \bar{\psi}(\mu)}{(\tilde{p}(\lambda) - z)(\tilde{p}(\mu) - z)} dm(\lambda) dm(\mu) \quad (3.1)$$

are analytic in  $\Sigma$ . We now formulate a general result concerning the representation of the functions  $\Delta(z)$ ,  $\beta(z)$  and  $\Gamma(z)$  in a neighborhood of the point  $z = \kappa_0$  for even dimension  $d = 2s + 2$ ,  $s = 0, 1, \dots$ . The case of odd dimension is briefly discussed at the end of the paper.

Let  $\mathcal{A}_{\kappa_0}^\delta$  be the class of functions that are analytic in the region  $\mathcal{U}_\delta(\kappa_0) \cap \Sigma$ , where  $\mathcal{U}_\delta(\kappa_0)$  is a circle with center  $\kappa_0$  and some (appropriately small) radius  $\delta$ , and can be represented in the form

$$g(z) = \sum_{k=0}^{\infty} [(z - \kappa_0)^{s+1} \log(z - \kappa_0)]^k h_k(z) \quad (3.2)$$

where  $h_k(z)$ ,  $k = 0, 1, \dots$  are analytic bounded functions in  $\mathcal{U}_\delta(\kappa_0)$  and  $\log(z - \kappa_0)$  is the branch that is real on the real axis for  $z > \kappa_0$ .

The representation (3.2) for the functions of  $\mathcal{A}_{\kappa_0}^\delta$  is unique, and  $\mathcal{A}_{\kappa_0}^\delta$  is a Banach algebra of functions with respect to the usual addition and multiplication, and with norm

$$\|g\|_\delta = \sum_{k=0}^{\infty} \sup_{z \in \mathcal{U}_\delta(\kappa_0)} |h_k(z)|. \quad (3.3)$$

Moreover if  $\delta$  is small enough we have

$$\sup_{z \in \mathcal{U}_\delta(\kappa_0)} |g(z)| < \|g\|_\delta. \quad (3.4)$$

We recall that, by (2.6),  $\tilde{p}(\lambda)$  is an analytic function in  $W_q$ , is positive for real  $\lambda$ , and has a non-degenerate maximum  $\kappa_0 = \tilde{p}(0) \in (0, 1)$  at  $\lambda = 0$ .  $\tilde{S}(\lambda, \mu)$  is analytic in  $W_q \times W_q$ , and for real  $\lambda, \mu$  is real and satisfies a bound  $|\tilde{S}(\lambda, \mu)| \leq C_* a^2$  for some  $C_* > 0$ . Moreover the range  $[\kappa_1, \kappa_0]$  of  $\tilde{p}(\lambda)$  is contained in the positive real axis.

**Lemma 3.1** *Under the assumptions above, for  $d = 2s + 2$ ,  $s \geq 0$ , one can find  $\bar{a} > 0$  and  $0 < \delta < \kappa_0 - \kappa_1$ , such that for all  $a \in (0, \bar{a})$  the functions  $\beta(z)$ ,  $\Delta(z)$  and  $\Gamma(z)$  have the following representation for  $z \in \mathcal{U}_\delta(\kappa_0) \cap \Sigma$ :*

$$i) \quad \beta(z) = h(z)(z - \kappa_0)^s \log(z - \kappa_0) + H(z); \quad (3.5)$$

$$ii) \quad \Delta(z) = 1 + \psi(z), \quad \psi \in \mathcal{A}_{\kappa_0}^\delta; \quad (3.6)$$

$$iii) \quad \Gamma(z) = \tilde{H}(z) + \sum_{n=1}^{\infty} \frac{1}{n!} (z - \kappa_0)^{(ns+n-1)} [\log(z - \kappa_0)]^n \cdot \tilde{h}_n(z). \quad (3.7)$$

The functions  $h(z)$ ,  $H(z)$ ,  $\tilde{H}(z)$  and  $\tilde{h}_n(z)$  are analytic in  $\mathcal{U}_\delta(\kappa_0)$ .

iv) Moreover the zeroes of the function  $\Delta(z)$  are always in finite number, and if  $a$  is small enough, they cannot lie in the regions  $\operatorname{Re} z > \kappa_0 - \frac{\delta}{2}$  and  $\{z : \min_{\lambda \in T^d} |z - \tilde{p}(\lambda)| > \frac{\delta}{2}\}$ .

Lemma 3.1 is proved in the last section.

**Proof of Theorem 1.1** Assume first that  $\Delta(z) \neq 0$  for all  $z \in \Sigma$ . Then, if  $h(\kappa_0) \neq 0$ , where  $h$  appears in (3.5), by the results of Lemma 3.1, the function in the square brackets of (2.16), if  $\delta$  is small enough, has, in the region  $\mathcal{U}_\delta(\kappa_0) \cap \Sigma$  the following representation

$$F(z) := \beta(z) - \frac{\Gamma(z)}{\Delta(z)} = H^*(z) + \sum_{n=1}^{\infty} \frac{1}{n!} (z - \kappa_0)^{(ns+n-1)} [\log(z - \kappa_0)]^n \cdot h_n^*(z), \quad (3.8)$$

where  $H^*$  and  $h_n^*$  are analytic functions in  $\mathcal{U}_\delta(\kappa_0)$ . The representation (3.8) is obtained by expanding  $1/\Delta(z)$  in (3.6) in power series of  $\psi$ , which is possible for  $a$  small.

We write  $F(z)$  in the form

$$F(z) = H^*(z) + h_1^*(z)(z - \kappa_0)^s \log(z - \kappa_0) + (z - \kappa_0)^{2s+1} \log^2(z - \kappa_0) \widehat{\Phi}(z), \quad (3.9)$$

where  $\widehat{\Phi}(z) = \widehat{H}(z) + \sum_{k=1}^{\infty} [(z - \kappa_0)^{s+1} \log(z - \kappa_0)]^k \widehat{h}_k(z) \in \mathcal{A}_{\kappa_0}^\delta$  and  $\|\widehat{\Phi}\|_\delta \leq Ca^2$ .

As one can see from (2.12) and assertion iv) of Lemma 3.1, for small  $a$  the spectrum of  $\mathcal{T}_1$ , i.e., the singularities of the resolvent, consist in the cut  $[\kappa_1, \kappa_0]$  on the real axis and some possible poles in the region  $\operatorname{Re} z < \kappa_0 - \frac{\delta}{2}$ , at a distance smaller than  $\frac{\delta}{2}$  from the cut. Therefore we take the contour  $\gamma$  as made of two parts,  $\gamma = \gamma_1 \cup \gamma_2$ , where  $\gamma_1$  is “degenerate“ and made of two segments,  $[\kappa_0 - \frac{\delta}{2}, \kappa_0]$  above the cut and  $(\kappa_0 - \frac{\delta}{2}, \kappa_0)$  below the cut, and  $\gamma_2$  is a circle with center at the origin and radius  $r = \kappa_0 - \frac{\delta}{2}$  (see Fig. 1).

Clearly

$$\left| \int_{\gamma_2} z^t F(z) dz \right| \leq C (\kappa_0 - \frac{\delta}{2})^t.$$

As for the integral over  $\gamma_1$ , which will give the leading term, observe that the analytic function  $H^*$  in (3.9) gives no contribution, as its jump over the cut vanishes. For the second term, as the jump of the logarithm over the cut is  $2\pi i$ , the contribution is

$$(-1)^s \int_{\kappa_0 - \delta}^{\kappa_0} (\kappa_0 - z)^s h_1^*(z) z^t dz.$$



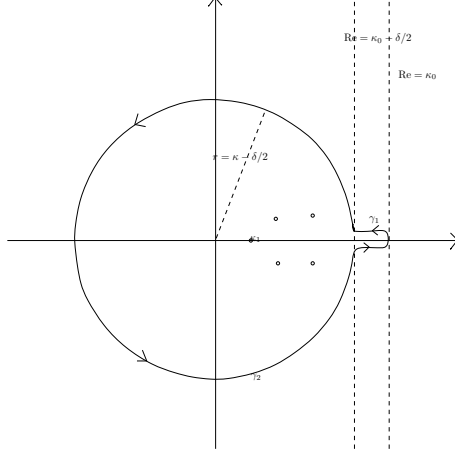


Figure 1: The contour  $\gamma$  is made of two parts,  $\gamma = \gamma_1 \cup \gamma_2$

We set  $z = \kappa_0(1 - \frac{w}{t})$  and consider the asymptotics of this integral as  $t \rightarrow \infty$ :

$$(-1)^s \frac{\kappa_0^t}{t^{s+1}} \kappa_0^{s+1} \int_0^{\frac{t\delta}{\kappa_0}} w^s (1 - \frac{w}{t})^t h^*(\kappa_0(1 - \frac{w}{t})) dw \sim C_0 \frac{\kappa_0^t}{t^{s+1}}$$

with  $C_0 = (-1)^s (\kappa_0)^{s+1} h^*(\kappa_0) \int_0^\infty w^s e^{-w} dw$ . As  $\kappa_0 = \mu_1$ , we get the leading term of the asymptotics (1.7).

For the next term in (3.9), denoting by  $\widehat{\Phi}_\pm(z)$  the values of the function  $\widehat{\Phi}$  computed above and below the cut, respectively (i.e., for  $\log(z - \kappa_0) = \log|z - \kappa_0| \pm i\pi$ ), we find that the difference across the cut is  $(z - \kappa_0)^{2s+1} \widehat{\Delta}(z)$  with

$$\widehat{\Delta}(z) = (\log^2 |z - \kappa_0| - \pi^2) (\widehat{\Phi}_+(z) - \widehat{\Phi}_-(z)) + 4i\pi \log |z - \kappa_0| (\widehat{\Phi}_+(z) + \widehat{\Phi}_-(z)).$$

Changing variables as before, the contribution of the integral over  $\gamma_1$  is

$$\frac{\kappa_0^d}{2\pi i} \frac{\kappa_0^t}{t^d} \int_0^{\frac{t\delta}{\kappa_0}} w^{2s+1} (1 - \frac{w}{t})^t \widehat{\Delta} \left( \kappa_0(1 - \frac{w}{t}) \right) dw.$$

Observe that  $z = \log u + i\pi = \rho e^{i\phi}$ , with  $\rho = \sqrt{\log^2 u + \pi^2}$  and  $\phi = \sin^{-1}(\frac{\pi}{\rho})$ , so that  $|\phi| \leq \frac{\pi^2}{2\rho}$  and we get  $|(\log u + i\pi)^k - (\log u - i\pi)^k| \leq \pi^2 k \rho^{k-1}$ . Therefore, setting  $\widehat{\Phi}_+(z) - \widehat{\Phi}_-(z) = 2ig(z)$ , we find, for some constant  $C > 0$

$$|g(z)| \leq Ca^2 |\kappa_0 - z|^{s+1} \sum_{k=1}^{\infty} k |h_k(z)| [|\log |\kappa_0 - z| + i\pi| (\kappa_0 - z)^{s+1}]^{k-1} \leq Ca^2 (z - \kappa_0)^{s+1}.$$

The series converges and is small for  $a$  and  $\delta$  small enough. Hence the contribution of the first term of  $\widehat{\Delta}$  to the integral over  $\gamma_1$  is bounded by  $C \frac{\kappa_0^t \log^2 t}{t^{\frac{3d}{2}}}$ , for some  $C > 0$ . In a similar way one can see that the second term of  $\widehat{\Delta}$  is of the order  $\kappa_0^t \frac{\log t}{t^d}$ .

This proves assertion i) of Theorem 1.1.

As for assertion ii), observe that for values of  $a \in (0, \bar{a})$ , for which the representations i), ii), iii) of Lemma 3.1 hold, a finite number of zeroes of  $\Delta(z)$  may appear in the half-plane  $\operatorname{Re} z > \kappa_0$ . Suppose that this is the case. Clearly only zeroes of  $\Delta$  which are in the spectrum of  $\mathcal{T}_1$  matter, and they are inside the unit sphere. Since moreover  $\Delta(\kappa_0) = 1$ , we can find  $\delta_1 > 0$  such that the contours  $\gamma_1$  and  $\gamma_2$  can be taken as before, except that  $\frac{\delta}{2}$  is replaced by  $\delta_1$ , and we have to add a third contour  $\gamma_3$  which goes around the zeroes of  $\Delta$  with  $\operatorname{Re} z > \kappa_0$ . Such zeroes can only be a finite number of complex conjugate pairs  $(z_1, \bar{z}_1), \dots, (z_m, \bar{z}_m)$ ,  $m \geq 1$ , with  $\operatorname{Im} z_j \geq 0$ , and  $\operatorname{Re} z_j = \alpha_j$ ,  $j = 1, \dots, m$ .

The integral over  $\gamma_3$  is a sum of contributions: each pair  $(z_k, \bar{z}_k)$ ,  $k = 1, \dots, m$ , contributes a quantity  $e^{-\alpha_k t} q_k(t) \cos(\theta_k t)$ , where  $\theta_k$  is the argument of  $z_k$  and the order of the polynomial  $q_k(t)$  is  $n_k - 1$ , where  $n_k$  is the order of the zero  $z_k$ . As the remaining part of the spectrum is to the left of  $\kappa_0 - \delta_1$ , we get the assertion (1.8).

Theorem 1.1 for  $d$  even is proved.

**Sketch of the proof of Theorem 1.1 for  $d$  odd.** If  $d = 2s + 1$ ,  $s > 0$ , the fundamental Lemma 3.1, as it is easy to check, following the proof in [2], is modified by replacing the assertions by the following.

- i')  $\beta(z) = h(z)(z - \kappa_0)^{s-\frac{1}{2}} + H(z)$ .
- ii')  $\Delta(z) = 1 + \bar{h}(z)(z - \kappa_0)^{s+\frac{1}{2}} + \bar{H}(z)$ .
- iii')  $\Gamma(z) = \tilde{H}(z) + (z - \kappa_0)^{s-\frac{1}{2}} \tilde{h}(z)$ .

The functions  $h, H, \bar{h}, \bar{H}, \tilde{h}, \tilde{H}$  are analytic in  $\mathcal{U}^\delta(\kappa_0)$ . Assertion iv) of Lemma 3.1 is unchanged, and the smallness of the term  $\bar{H}(z)$  for small  $a$  also holds.

For assertion i), following the steps of [2] we see that (3.8) is replaced by

$$F(z) = h(z)(z - \kappa_0)^{s-\frac{1}{2}} + H(z).$$

The proof then follows as in [2] (see also [7]). Assertion ii) is proved in the same way as for even dimension.

## 4 Proof of Lemma 3.1.

*Proof of i)* If  $f(\lambda)$  is analytic in  $W_q$  for some  $q \in (0, 1)$ , and such that  $f(0) \neq 0$ , then, following [7], we consider the integral

$$\begin{aligned} F(z) &:= \int_{T^d} \frac{f(\lambda)}{\tilde{p}(\lambda) - z} dm(\lambda) = -\frac{1}{\kappa_0} \int_{T^d} \frac{f(\lambda)}{\frac{z}{\kappa_0} - 1 - \frac{\tilde{p}(\lambda)}{\kappa_0} + 1} dm(\lambda) = \\ &= -\frac{1}{\kappa_0} \left( \int_{u^2(\lambda) < \delta} \frac{f(\lambda)}{\beta + u^2(\lambda)} dm(\lambda) + \int_{u^2(\lambda) \geq \delta} \frac{f(\lambda)}{\beta + u^2(\lambda)} dm(\lambda) \right). \end{aligned} \quad (4.1)$$

Here  $u^2(\lambda) = 1 - \frac{\tilde{p}(\lambda)}{\kappa_0}$  and  $\beta = \frac{z}{\kappa_0} - 1$ , and  $\delta$  is so small that in the region  $u^2(\lambda) < \delta$  we can apply the Morse Lemma. The second term is analytic for  $|\beta| < \delta$ , and bounded for  $|\beta| < \delta' < \delta$ . Following [7], the first integral in the second line of (4.1) is represented as

$$\int_{u^2(\lambda) < \delta} \frac{f(\lambda)}{\beta + u^2(\lambda)} dm(\lambda) = g(\beta)\beta^s \log \beta + G(\beta), \quad (4.2)$$

where  $g, G$  are analytic for  $|\beta| < \delta$ . The coefficients of the expansion of  $g$  at  $\beta = 0$  are a simple expression of the coefficients of the Taylor expansion of  $f$  and the jacobian  $J(u)$  of the transformation  $\lambda = (\lambda_1, \dots, \lambda_d) \rightarrow u = (u_1, \dots, u_d)$ , with the variables  $u_1, \dots, u_d$  being such that  $u^2 = u_1^2 + \dots + u_d^2 = u^2(\lambda)$ . In particular,  $g(0) = \frac{(-1)^{s+1} \omega_d}{2(2\pi)^d} f(0)J(0)$ , where  $\omega_d$  is the measure of the surface of the unit sphere in  $\mathbb{R}^d$ . It follows that

$$\int_{T^d} \frac{f(\lambda)}{\tilde{p}(\lambda) - z} dm(\lambda) = h(z)(z - \kappa_0)^s \log(z - \kappa_0) + H(z) \quad (4.3)$$

where  $h, H$  are analytic in  $\mathcal{U}_\delta(\kappa_0)$  for some  $\delta > 0$ , and  $h(\kappa_0) = (-1)^s \frac{f(0)J(0)}{2\kappa_0^{s+1}(2\pi)^d}$ , and, as shown in [7],  $J(0) = \sqrt{\det \mathcal{C}^{-1}}$ , where  $\mathcal{C}$  is the matrix of the second derivatives of the function  $\kappa_0^{-1} \tilde{p}(\lambda)$  at  $\lambda = 0$ , which is positive definite for small  $a$ .

Taking  $f(\lambda) = \phi(\lambda)\overline{\psi(\lambda)}$ , this proves assertion i).

**Remark.** If  $f(0) = 0$ , then, as shown in [7], we have, instead of (4.3),

$$F(z) = \int_{T^d} \frac{f(\lambda)}{\tilde{p}(\lambda) - z} dm(\lambda) = h(z)(z - \kappa_0)^{s+1} \log(z - \kappa_0) + H(z). \quad (4.4)$$

$h, H$  have the same properties as before, except for a different expression of  $h(0)$ , which now depends on the second derivatives of  $f$  at  $\lambda = 0$ . It is not hard to see that

$$\sup_{z \in \mathcal{U}_\delta(\kappa_0)} |h(z)| + \sup_{z \in \mathcal{U}_\delta(\kappa_0)} |H(z)| \leq R_\delta \sup_{\lambda \in W_q} |f(\lambda)|, \quad (4.5)$$

where  $R_\delta$  depends on  $\delta$  and  $\tilde{p}$  only. (A similar inequality holds for (4.3).) In fact such inequality is obvious for the second integral in the second line of (4.1). For the first integral we refer to our paper [7] in which the coefficients of the power series for  $g(\beta)$  and  $G(\beta)$  in (4.2) are given explicitly in terms of the derivatives of the function  $f(\lambda)$  at  $\lambda = 0$ . Such derivatives, as  $f(\lambda)$  is analytic, are estimated in terms of  $\sup_{\lambda \in W_q} |f(\lambda)|$ .

*Proof of ii).* As  $\tilde{S}(0, \lambda) \equiv 0$  the second term in the series (2.13) defining  $\Delta(z)$  is

$$\Phi_1(z) := \int_{T^d} \frac{\tilde{\mathcal{S}}(\lambda, \lambda)}{\tilde{p}(\lambda) - z} dm(\lambda) = h_1(z)(z - \kappa_0)^{s+1} \log(z - \kappa_0) + H_1(z) \in \mathcal{A}_{\kappa_0}^\delta, \quad (4.6)$$

and the norm, by (4.5), is  $\|\Phi_1(z)\|_\delta \leq R_\delta C_* a^2$ . For an estimate of the generic term it is convenient to consider different values of  $z$  for each variable, i.e., to study the function

$$\Phi_n(z_1, \dots, z_n) := \int_{T^d} \int_{T^d} \dots \int_{T^d} \frac{K(\lambda^{(1)}, \dots, \lambda^{(n)})}{\prod_{i=1}^n [\tilde{p}(\lambda^{(i)}) - z_i]} dm(\lambda^{(1)}) \dots dm(\lambda^{(n)}). \quad (4.7)$$

Clearly  $K(\lambda^{(1)}, \dots, \lambda^{(n)}) = 0$  whenever  $\lambda^{(j)} = 0$  for at least one  $j = 1, \dots, n$ , and, by the Hadamard lemma  $\sup_{(\lambda^{(1)}, \dots, \lambda^{(n)}) \in W_d^n} |K(\lambda^{(1)}, \dots, \lambda^{(n)})| \leq (C_* a^2)^n n^{\frac{n}{2}}$ . Therefore integrating over  $\lambda^{(1)}$  we see, by (4.5), (4.6), that

$$\Phi_n(z_1; \lambda^{(2)}, \dots, \lambda^{(n)}) := \int_{T^d} \frac{K(\lambda^{(1)}, \dots, \lambda^{(n)})}{[\tilde{p}(\lambda^{(1)}) - z_1]} dm(\lambda^{(1)}) \in \mathcal{A}_{\kappa_0}^\delta$$

and  $\|\Phi_n(z_1; \lambda^{(2)}, \dots, \lambda^{(n)})\|_\delta \leq R_\delta (C_* a^2)^n n^{\frac{n}{2}}$ . Integrating now over  $\lambda^{(2)}$  we get a function  $\Phi_n(z_1, z_2; \lambda^{(3)}, \dots, \lambda^{(n)})$  which belongs to the Banach algebra  $(\mathcal{A}_{\kappa_0}^\delta)^2$ , i.e., the algebra of the functions of  $(z_1, z_2) \in \mathcal{U}_\delta(\kappa_0) \times \mathcal{U}_\delta(\kappa_0)$  that can be written as series

$$g(z_1, z_2) = \sum_{k_1, k_2=0}^{\infty} [(z_1 - \kappa_0)^{s+1} \log(z_1 - \kappa_0)]^{k_1} [(z_2 - \kappa_0)^{(s+1)} \log(z_2 - \kappa_0)]^{k_2} h_{k_1, k_2}(z_1, z_2), \quad (4.8)$$

where  $h_{k_1, k_2}$  are analytic and bounded in  $\mathcal{U}_\delta^2(\kappa_0)$ , with norm

$$\|g\|_\delta^{(2)} = \sum_{k_1, k_2=0}^{\infty} \max_{(z_1, z_2) \in \mathcal{U}_\delta^2} |h_{k_1, k_2}(z_1, z_2)|.$$

For this one has to integrate over  $dm(\lambda^{(2)})$  the two terms which come out of the first integration. The norm is bounded by

$$\|\Phi_n(z_1, z_2; \lambda^{(3)}, \dots, \lambda^{(n)})\|_\delta^{(2)} \leq 2R_\delta^2 (C_* a^2)^n n^{\frac{n}{2}}. \quad (4.9)$$

Integrating over  $\lambda^{(3)}, \dots, \lambda^{(n)}$ , with obvious definitions of the Banach algebras  $(\mathcal{A}_{\kappa_0}^\delta)^j$ ,  $j = 1, 2, \dots, n$ , with norms  $\|\cdot\|_\delta^{(j)}$ , we get a function  $\Phi_n(z_1, \dots, z_n) \in (\mathcal{A}_{\kappa_0}^\delta)^n$  with norm

$$\|\Phi_n(z_1, \dots, z_n)\|_\delta^{(n)} \leq 2^{n-1} R_\delta^n (C_* a^2)^n n^{\frac{n}{2}}.$$

Clearly  $\Phi_n(z) := \Phi_n(z, \dots, z) \in \mathcal{A}_{\kappa_0}^\delta$  and  $\|\Phi_n(z)\|_\delta \leq \|\Phi_n(z_1, \dots, z_n)\|_\delta^{(n)}$ . Hence for  $a$  small the series (2.13) converges and ii) is proved.

*Proof of iii).* The proof follows immediately along the same lines as for the proof of assertion ii), with due care for the fact that  $\phi(0)$  and  $\psi(0)$  do not necessarily vanish.

*Proof of iv).* For  $a < \bar{a}$  we have by assertion ii) of Lemma 3.1 that

$$|\Delta(z) - \Delta(\kappa_0)| = |\Delta(z) - 1| \leq C_1 a^2 g(|z|), \quad g(r) = r \sqrt{\log^2 r + \pi^2}, \quad r > 0$$

where  $C_1$  is a constant independent of  $a$ . Hence if  $a$  is so small that  $C_1 a^2 g(\frac{\delta}{2}) < 1$ , then  $\Delta(z) \neq 0$  for all  $z \in \mathcal{K}_\delta$ , the circle with center  $\kappa_0$  and radius  $\frac{\delta}{2}$ . On the other hand if  $z \in \mathcal{E}_\delta := \{z : \min_{\lambda \in T} |\tilde{p}(\lambda) - z| > \frac{\delta}{2}\}$ , then, by the inequality  $|K_n(\lambda_1, \dots, \lambda_n)| < (C_* a^2)^n n^{\frac{n}{2}}$  we find that  $|\Delta(z) - 1| \leq \bar{C}_1 \frac{a^2}{\delta}$ , for some other constant  $\bar{C}_1$ . Hence if  $\bar{C}_1 \frac{a^2}{\delta} < 1$  there is no zero of  $\Delta(z)$  in the region  $\mathcal{E}_\delta$ . This implies that  $\Delta(z) \neq 0$  in the half-plane  $\operatorname{Re} z > \kappa_0 - \frac{\delta}{2}$ , and the first assertion of iv) is proved.

If  $a$  is large enough some zeroes of  $\Delta(z)$  which belong to the spectrum of  $\mathcal{T}_1$  may appear in the half-plane  $\operatorname{Re} z > \kappa_0$ . They are always in finite number, because analytic functions can only have a finite number of zeroes in a compact connected region, and such zeroes cannot lie, as we have seen, near  $\kappa_0$ , or outside the unit sphere.

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