

# Lévy Base Correlation

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**First Version: February 08, 2007**

**This Version: June 27, 2007**

**Abstract:** In this paper we investigate one factor models that extend the classical Gaussian copula model for pricing CDOs. The proposed models are very tractable and perform significantly better than the classical Gaussian copula model. Moreover, we introduce the concept of Lévy base correlation. The obtained Lévy base correlation curve is much flatter than the corresponding Gaussian one. This indicates that the models do fit the observed data much better. Additionally, flat base correlation curves are also much more reliable for pricing of bespoke tranches.

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# 1 Introduction

Collateralized Debt Obligations (CDOs) have become very popular credit instruments, which transfer the credit risk on a reference portfolio of assets to the protection sellers. A standard feature of a CDO structure is the *tranching* of credit risk. Credit tranching refers to creating multiple *tranches* of securities which have varying degrees of seniority and risk exposure: The *equity* tranche is the first to be affected by losses in the event of one or more defaults in the portfolio. If losses exceed the value of this tranche, they are absorbed by the *mezzanine* tranche(s). Losses that have not been absorbed by the other tranches are sustained by the *senior* tranche and finally by the *super-senior* tranche. In such a way, each tranche protects the ones senior to it from the risk of loss on the underlying portfolio. The CDO *investors* take on exposure to a particular tranche, effectively selling credit protection to the CDO issuer, and in turn collecting premiums (spreads).

The standard model for pricing CDOs is the Gaussian Copula model (see e.g. Vasicek [11]). It is basically a one-factor model with an underlying multivariate normal distribution. Actually, a very simple multivariate normal distribution is employed: all correlation between different components are taken equal. The one-factor Gaussian copula model is well-known not to fit simultaneously the prices of the different tranches of a CDO, leading to the correlation smile. In order to deal with this problem, the base correlation concept was initiated. Similarly, to implied volatility in an equity setting, one uses a different base correlation for each tranche to be priced.

Recently, other one-factor models based on different distributions have been proposed. Moosbrucker [7] used a one-factor Variance Gamma model, Kalemanova et al. [6] and Guégan and Houdain [5] worked with a NIG factor model and Baxter [3] introduced the B-VG model. A generic Lévy model approach was described in Albrecher, Ladoucette, and Schoutens [2]. These models bring more flexibility into the dependence structure and allow tail dependence.

In this paper we investigate and compare some of these Lévy models and moreover introduce the concept of Lévy base correlation. The use of this Lévy base correlation is completely analogous as in the Gaussian case. We illustrate this by pricing tranchlets. Additionally, we will illustrate by a historical study that the Lévy base correlation curve is always much flatter than the Gaussian counterpart. This indicates that indeed the Lévy models do much better from a fitting point of view. Moreover, because of a reduction of the interpolation error such a flatter base correlation curve leads also to a more stable pricing of bespoke tranches.

## 2 Generic One-Factor Lévy Model

### 2.1 Lévy Process

Suppose  $\phi(z)$  is the characteristic function of a distribution. If for every positive integer  $n$ ,  $\phi(z)$  is also the  $n$ th power of a characteristic function, we say that the distribution is *infinitely divisible*. One can define for any infinitely divisible distribution a stochastic process,  $X = \{X_t, t \geq 0\}$ , called a *Lévy process*, which starts at zero, has independent and stationary increments and such that the distribution of an increment over  $[s, s + t]$ ,  $s, t \geq 0$ , i.e.  $X_{t+s} - X_s$ , has  $(\phi(z))^t$  as characteristic function.

The function  $\psi(z) := \log \phi(z)$  is called the *characteristic exponent* and it satisfies the following

*Lévy-Khintchine formula* (see Bertoin [4]):

$$\psi(z) = i\gamma z - \frac{\zeta^2}{2} z^2 + \int_{-\infty}^{+\infty} (\exp(izx) - 1 - izx \mathbb{1}_{\{|x|<1\}}) \nu(dx), \quad z \in \mathbb{R}, \quad (1)$$

where  $\gamma \in \mathbb{R}$ ,  $\zeta^2 \geq 0$  and  $\nu$  is a measure on  $\mathbb{R} \setminus \{0\}$  with  $\int_{-\infty}^{+\infty} \min(1, x^2) \nu(dx) < \infty$ . From the Lévy-Khintchine formula, one sees that, in general, a Lévy process consists of three independent parts: a linear deterministic part, a Brownian part, and a pure jump part. We say that our infinitely divisible distribution has a triplet of Lévy characteristics  $[\gamma, \zeta^2, \nu(dx)]$ . The measure  $\nu(dx)$  is called the *Lévy measure* of  $X$  and it dictates how the jumps occur. Jumps of sizes in the set  $A$  occur according to a Poisson process with parameter  $\int_A \nu(dx)$ . If  $\zeta^2 = 0$  and  $\int_{-1}^{+1} |x| \nu(dx) < \infty$ , it follows from standard Lévy process theory (e.g., Bertoin [4], Sato [9]), that the process is of finite variation. For more details about the applications of Lévy processes in finance, we refer to Schoutens [10].

## 2.2 Generic One-Factor Lévy Model

We are going to model a portfolio of  $n$  obligors such that all of them have equal weights in the portfolio. We will work for simplicity with a situation where each obligor  $i$ ,  $i \in \{1, 2, \dots, n\}$ , has the same recovery rate  $R$  in case of default and some individual default probability term structure  $p_i(t)$ ,  $t \geq 0$ , which is the probability that obligor  $i$  will default before time  $t$ .

Let us first fix  $T$ . For the modeling, let us start with a mother infinitely divisible distribution  $L$ . Let  $X = \{X_t, t \in [0, 1]\}$  be a Lévy process based on that infinitely divisible distribution, such that  $X_1$  follows the law  $L$ . Note that we will only work with Lévy processes with time running over the unit interval. Denote the cumulative distribution function of  $X_t$  by  $H_t$ ,  $t \in [0, 1]$ , and assume it is continuous. Assume further that the distribution is standardized in the sense that  $\mathbb{E}[X_1] = 0$  and  $\text{Var}[X_1] = 1$ . In terms of  $\psi$  this means that  $\psi'(0) = 0$  and  $\psi''(0) = -1$ . Then, it is not that hard to prove that  $\text{Var}[X_t] = t$ .

Let  $X = \{X_t, t \in [0, 1]\}$  and  $X^{(i)} = \{X_t^{(i)}, t \in [0, 1]\}$ ,  $i = 1, 2, \dots, n$  be independent and identically distributed Lévy processes (so all processes are independent of each other and are based on the same mother infinitely divisible distribution  $L$ ).

Next, we propose the generic one-factor Lévy model. Let  $0 < \rho < 1$ . We assume that the asset value of obligor  $i$  is of the form

$$A_i = X_\rho + X_{1-\rho}^{(i)}, \quad i = 1, \dots, n. \quad (2)$$

Each  $A_i$  has by the stationary and independent increments property the same distribution as the mother distribution  $L$  with distribution function  $H_1$ . Indeed, adding an increment of the process over a time interval of length  $\rho$  and an independent increment over a time interval of length  $1 - \rho$  follows the distribution of an increment over an interval of unit length, i.e. is following the law  $L$ . As a consequence,  $\mathbb{E}[A_i] = 0$  and  $\text{Var}[A_i] = 1$ . Note that then for  $i \neq j$ , we have that  $\text{Corr}[A_i, A_j] = \rho$ . Indeed:

$$\text{Corr}[A_i, A_j] = \frac{\mathbb{E}[A_i A_j] - \mathbb{E}[A_i] \mathbb{E}[A_j]}{\sqrt{\text{Var}[A_i]} \sqrt{\text{Var}[A_j]}} = \mathbb{E}[A_i A_j] = \mathbb{E}[X_\rho^2] = \rho.$$

So, starting from any mother standardized infinitely divisible law, we can set up a one-factor model with the required correlation.

We say that the  $i$ th obligor *defaults at time  $T$*  if the firm value  $A_i(t)$  falls below some preset barrier  $K_i(t)$ :  $A_i(t) \leq K_i(t)$ . In order to match default probabilities under this model with default probabilities  $p_i(t)$  observed in the market, we have to set

$$K_i = K_i(t) := H_1^{[-1]}(p_i(t)).$$

Indeed, then  $P(A_i \leq K_i) = P(A_i \leq H_1^{[-1]}(p_i(t))) = H_1(H_1^{[-1]}(p_i(t))) = p_i(t)$ .

Note that conditional on the common factor  $X_\rho$ , the asset values and, hence, the default probabilities of different obligors are independent. Let us denote by  $p_i(y; t)$  the conditional probability that the firm's value  $A_i$  is below the barrier  $K_i(t)$ , given that the systematic factor  $X_\rho$  takes the value  $y$ . In other words  $p_i(y; t)$  is the conditional default probability of firm  $i$  on a realization of the common factor  $\{X_\rho = y\}$ .

Now:

$$\begin{aligned} p_i(y; t) &= \mathbb{P}(A_i \leq K | X_\rho = y) \\ &= \mathbb{P}(X_\rho + X_{1-\rho}^{(i)} \leq K_i(t) | X_\rho = y) \\ &= \mathbb{P}(X_{1-\rho}^{(i)} \leq K_i(t) - y) \\ &= H_{1-\rho}(K_i(t) - y). \end{aligned}$$

Further, let us denote with by  $\Pi_{n,y}^k(t)$  the probability to have  $k$  defaults out of group of  $n$  firms conditional on the market factor  $y$  at time  $t$ . Then we have the classical recursive loss distribution formula

$$\begin{aligned} \Pi_{n+1,y}^0(t) &= \Pi_{n,y}^0(t)(1 - p_{n+1}(y; t)) \\ \Pi_{n+1,y}^k(t) &= \Pi_{n,y}^k(t)(1 - p_{n+1}(y; t)) + \Pi_{n,y}^{k-1}(t)p_{n+1}(y; t), \quad k = 1, \dots, n, \\ \Pi_{n+1,y}^{n+1}(t) &= \Pi_{n,y}^n(t)p_{n+1}(y; t), \end{aligned}$$

together with the initial condition that  $\Pi_{0,y}^0(t) = 1$ .

Then finally, we have that the unconditional probability to have  $k$  defaults out of group of  $n$  firms, denoted by  $\Pi_n^k$  is given by

$$\Pi_n^k(t) = \int_{-\infty}^{+\infty} \Pi_{n,y}^k(t) dH_\rho(y), \quad k = 0, 1, \dots, n. \quad (3)$$

The expected value of the loss fraction  $L_{t,n}$  on the portfolio notional at time  $t$  is

$$\mathbb{E}[L_{t,n}] = \frac{(1-R)}{n} \sum_{k=1}^n k \cdot \Pi_n^k(t);$$

and the expected value of the loss fraction on the CDO tranche with attachment points  $K_1$  and  $K_2$  at time  $t$  is

$$\mathbb{E}[L_{t,n}(K_1, K_2)] = \frac{\mathbb{E}[\min\{L_{t,n}, K_2\}] - \mathbb{E}[\min\{L_{t,n}, K_1\}]}{K_2 - K_1}. \quad (4)$$

### 3 Examples

#### 3.1 The Shifted Gamma-Model

The density function of the Gamma distribution  $\text{Gamma}(a, b)$  with parameters  $a > 0$  and  $b > 0$  is given by

$$f_{\text{Gamma}}(x; a, b) = \frac{b^a}{\Gamma(a)} x^{a-1} \exp(-xb), \quad x > 0.$$

Let us denote the corresponding cumulative distribution function by  $\mathcal{H}_G(x; a, b)$ . The characteristic function is given by

$$\phi_{\text{Gamma}}(u; a, b) = (1 - iu/b)^{-a}, \quad u \in \mathbb{R}.$$

Clearly, this characteristic function is infinitely divisible. The Gamma-process  $X^{(G)} = \{X_t^{(G)}, t \geq 0\}$  with parameters  $a, b > 0$  is defined as the stochastic process which starts at zero and has stationary, independent Gamma-distributed increments. More precisely, the time enters in the first parameter:  $X_t^{(G)}$  follows a  $\text{Gamma}(at, b)$  distribution.

The properties of the  $\text{Gamma}(a, b)$  distribution given in Table 1 can easily be derived from the characteristic function.

	$\text{Gamma}(a, b)$	$\text{Gamma}(a, \sqrt{(a)})$
mean	$a/b$	$\sqrt{a}$
variance	$a/b^2$	1
skewness	$2/\sqrt{a}$	$2/\sqrt{a}$
kurtosis	$3(1 + 2/a)$	$3(1 + 2/a)$

Table 1: Mean, variance, skewness and kurtosis of the Gamma distribution.

Note also that we have the following scaling property: if  $X$  is  $\text{Gamma}(a, b)$  then for  $c > 0$ ,  $cX$  is  $\text{Gamma}(a, b/c)$ .

Let us start with a unit variance Gamma-process  $G = \{G_t, t \geq 0\}$  with parameters  $a > 0$  and  $b = \sqrt{a}$ . The mean of the process is then  $\sqrt{a}$ . As driving Lévy process, we then take

$$X_t = \sqrt{at} - G_t, \quad t \in [0, 1].$$

The interpretation in terms of firm value is that there is a deterministic up trend ( $\sqrt{at}$ ) with random downward shocks ( $G_t$ ).

The one-factor shifted Gamma-Lévy model is:

$$A_i = X_\rho + X_{1-\rho}^{(i)}.$$

Here  $X_\rho, X_{1-\rho}^{(i)}, i = 1, \dots, n$  are independent shifted Gamma-processes.

The cumulative distribution function  $H_t(x; a)$  of  $X_t, t \in [0, 1]$ , can easily be obtained from the Gamma cumulative distribution function

$$H_t(x; a) = \mathbb{P}(\sqrt{at} - G_t \leq x) = 1 - \mathbb{P}(G_t < \sqrt{at} - x) = 1 - \mathcal{H}_G(\sqrt{at} - x; at, \sqrt{a}), \quad x \in (-\infty, \sqrt{at}).$$

For the inverse function, we have the following relation for  $t \in [0, 1]$

$$H_t^{[-1]}(y; a) = \sqrt{at} - \mathcal{H}_G^{[-1]}(1 - y; at, \sqrt{a}), \quad y \in [0, 1].$$

Let us set the Gamma parameter  $b = \sqrt{a}$  and denote the mean by  $\mu = \sqrt{a}$ . In order to compute the integral in Equation 3, we can apply a Gauss-Laguerre scheme. Indeed, we actually have

$$\Pi_n^k = \int_{-\infty}^{\mu\rho} \Pi_{n,y}^k f_{SG}(y, a\rho, b, \mu\rho) dy,$$

where the density of the Shifted Gamma distribution is given by

$$f_{SG}(y, a, b, \mu) = f_{Gamma}(\mu - y, a, b) = \frac{b^a}{\Gamma(a)} (\mu - y)^{a-1} \exp(-(\mu - y)b), \quad y < \mu.$$

Hence we can write

$$\begin{aligned} \Pi_n^k &= \int_{-\infty}^{\mu\rho} \Pi_{n,y}^k \frac{b^{a\rho}}{\Gamma(a\rho)} (\mu\rho - y)^{a\rho-1} \exp(-(\mu\rho - y)b) dy, \\ &= \int_0^{+\infty} \Pi_{n,(\mu\rho-z)}^k \frac{b^{a\rho}}{\Gamma(a\rho)} (z)^{a\rho-1} \exp(-zb) dz, \\ &= \int_0^{+\infty} \Pi_{n,(\mu\rho-u/b)}^k \frac{1}{\Gamma(a\rho)} (u)^{a\rho-1} \exp(-u) du, \end{aligned}$$

where the last integral can be calculated by applying the Gauss-Laguerre quadrature.

### 3.2 The Shifted IG-Model

The Inverse Gaussian  $IG(a, b)$  law with parameters  $a > 0$  and  $b > 0$  has characteristic function

$$\phi_{IG}(u; a, b) = \exp\left(-a(\sqrt{-2iu + b^2} - b)\right), \quad u \in \mathbb{R}.$$

The IG-distribution is infinitely divisible and we define the IG-process  $I = \{I_t, t \geq 0\}$  with parameters  $a, b > 0$  as the process which starts at zero, has independent and stationary IG-distributed increments, and such that

$$\mathbb{E}[\exp(iuI_t)] = \phi_{IG}(u; at, b) = \exp\left(-at(\sqrt{-2iu + b^2} - b)\right), \quad u \in \mathbb{R},$$

meaning that  $I_t$  follows an  $IG(at, b)$  distribution.

The density function of the  $IG(a, b)$  law is explicitly known

$$f_{IG}(x; a, b) = \frac{a}{\sqrt{2\pi}} \exp(ab) x^{-3/2} \exp(-(a^2 x^{-1} + b^2 x)/2), \quad x > 0.$$

The characteristics given in Table 2 can easily be obtained from the characteristic function.

	$IG(a, b)$	$IG(a, a^{1/3})$
mean	$a/b$	$a^{2/3}$
variance	$a/b^3$	1
skewness	$3/\sqrt{ab}$	$3a^{-2/3}$
kurtosis	$3(1 + 5(ab)^{-1})$	$3(1 + 5a^{-4/3})$

Table 2: Mean, variance, skewness and kurtosis of the Inverse Gaussian distribution.

Let us start with a unit variance IG-process  $I = \{I_t, t \geq 0\}$  with parameters  $a > 0$  and  $b = a^{1/3}$ . In our model, we then take

$$X_t = \mu t - I_t, \quad t \in [0, 1],$$

where in this case  $\mu = a^{2/3}$ . We hence have again a deterministic up-trend and negative shock that now are coming from an inverse Gaussian process.

The one-factor shifted IG-Lévy model is:

$$A_i = X_\rho + X_{1-\rho}^{(i)}.$$

Here  $X_\rho, X_{1-\rho}^{(i)}, i = 1, \dots, n$  are independent shifted IG-processes. In order to compute the unconditional probabilities  $\Pi_n^k$  one can rely on numerical integration schemes using the density of the IG processes or apply Fourier inversion methods starting from the characteristic function.

### 3.3 The Shifted CMY Model

The CMY( $C, M, Y$ ) distribution with parameters  $C > 0$ ,  $M > 0$ , and  $Y < 2$  has characteristic function

$$\phi_{CMY}(u; C, M, Y) = \exp \{ C\Gamma(-Y) [(M - iu)^Y - M^Y] \}, \quad u \in \mathbb{R}.$$

The CMY distribution is infinitely divisible and we can define the CMY Lévy process  $X^{(CMY)} = \{X_t^{(CMY)}, t \geq 0\}$  that starts at zero and has stationary and independent CMY-distributed increments, i.e.,  $X_t^{(CMY)}$  follows a CMY( $Ct, M, Y$ ) distribution. The properties of the CMY( $C, M, Y$ ) distribution given in Table 3 can be derived from its characteristic function.

	CMY( $C, M, Y$ )	CMY( $C, (C\Gamma(2 - Y))^{\frac{1}{2-Y}}, Y$ )
mean	$CM^{Y-1}\Gamma(1 - Y)$	$(C\Gamma(1 - Y)(1 - Y)^{Y-1})^{\frac{1}{2-Y}}$
variance	$CM^{Y-2}\Gamma(2 - Y)$	1
skewness	$\frac{CM^{Y-3}\Gamma(3-Y)}{(CM^{Y-2}\Gamma(2-Y))^{3/2}}$	$(C\Gamma(3 - Y)(2 - Y)^{Y-3})^{\frac{1}{Y-2}}$
kurtosis	$3 + \frac{CM^{Y-4}\Gamma(4-Y)}{(CM^{Y-2}\Gamma(2-Y))^2}$	$\left[ (C\Gamma(4 - Y))^2 ((3 - Y)(2 - Y))^{Y-4} \right]^{\frac{1}{Y-2}}$

Table 3: Mean, variance, skewness and kurtosis of the CMY distribution.

Note that CMY( $C, M, Y$ ) reduces to Gamma( $C, M$ ) when  $Y = 0$ .

Let us start with a unit CMY-process  $C = \{C_t, t \geq 0\}$  with parameters  $C > 0$ ,  $Y < 2$ , and  $M = (C\Gamma(2 - Y))^{\frac{1}{2-Y}}$ , so that the mean of the process is  $\mu := (C\Gamma(1 - Y)(1 - Y)^{Y-1})^{\frac{1}{2-Y}}$  and the variance is equal to one. As driving Lévy process we take

$$X_t = \mu t - C_t, \quad t \in [0, 1].$$

The interpretation in terms of firm value is, again, that there is a deterministic up trend with random downward shocks coming now from a CMY process.

The one-factor shifted CMY-Lévy model is

$$A_i = X_\rho + X_{1-\rho}^{(i)},$$

where  $X_\rho, X_{1-\rho}^{(i)}, i = 1, \dots, n$ , are independent shifted CMY-processes.

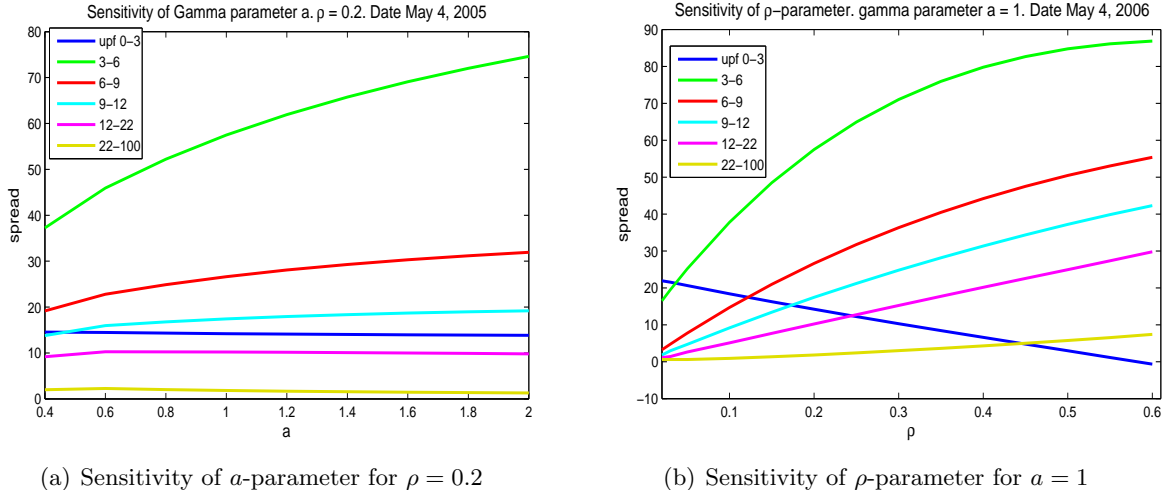


Figure 1: Sensitivity of Gamma model parameters on different tranche prices

Since the cumulative distribution function  $H_{CMY}(x; C, M, Y)$  of a CMY distribution can not be derived in a closed form, we numerically invert its Laplace transform, given by

$$\hat{H}_{CMY}(w; C, M, Y) = \frac{\exp \{C\Gamma(-Y) [(M + w)^Y - M^Y]\}}{w},$$

in order to calculate values of  $H_{CMY}(x; C, M, Y)$ . In particular, we employ the numerical inversion procedure described in Abate and Whitt [1], which uses the Bromwich integral, the Poisson summation formula, and Euler summation.

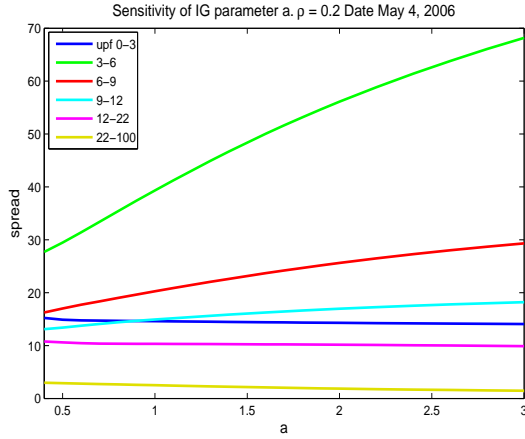
## 4 Parameters Sensitivity

Next, we investigate the sensitivity of different parameters of the above proposed models. The underlying default probabilities are taken from the CDSs of the iTraxx (Series 4) on May 4, 2006.

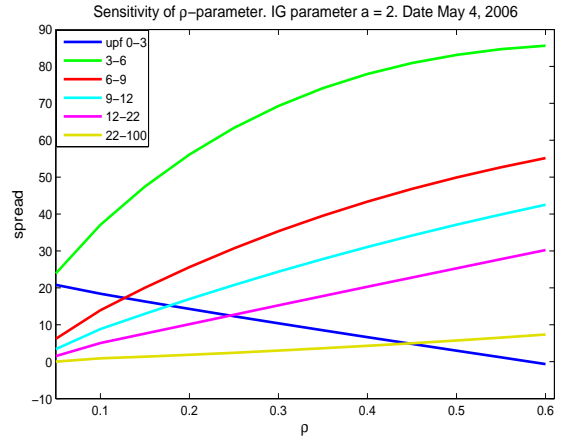
### 4.1 The Shifted Gamma Model

In this subsection, we investigate the sensitivity of different parameters of the Gamma model. From Figure 1(b), we clearly see that the protection seller of the equity tranche is long correlation. From Figure 1(a), we observe that varying the  $a$  parameter (for a fixed  $\rho$ ) has almost no effect on the equity-tranche price. This means that the equity tranche protection seller is neutral on  $a$  or in other words neutral to changing tail behavior of the firm. However, a seller of protection of a non-equity tranche is short  $a$ . Fatter tail behavior (higher kurtosis), decreases  $a$ , so one could say mezzanine and senior tranche protection sellers are long kurtosis. All this is quite useful, certainly in respect of situations like the May 2005 Credit crisis, where we have witnessed averse movements in equity and mezzanine tranches. One could say that this situation is corresponding to a joint movement in  $\rho$  and  $a$ , the move in  $\rho$  affects the equity tranche and the move in  $a$  affects all the tranches, but equity tranche can move in different directions than the other tranches.



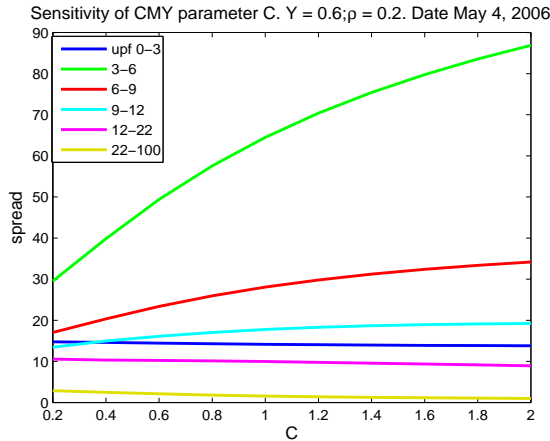


(a) Sensitivity of  $a$ -parameter for  $\rho = 0.2$

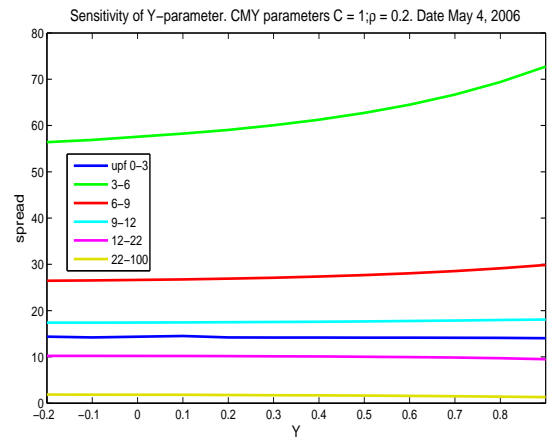


(b) Sensitivity of  $\rho$ -parameter for  $a = 2$

Figure 2: Sensitivity of Inverse Gaussian model parameters on different tranche prices



(a) Sensitivity of  $C$ -parameter for  $Y = 0.6$  and  $\rho = 0.2$



(b) Sensitivity of  $Y$ -parameter for  $C = 1$  and  $\rho = 0.2$

Figure 3: Sensitivity of CMY model parameters on different tranche prices

## 4.2 The Shifted IG

Next, we investigate the sensitivity of different parameters of the Inverse Gaussian Lévy model. In Figure 2 one can see an almost similar sensitivity as in the Gamma situation.

## 4.3 The Shifted CMY model

In this subsection, we investigate the sensitivity of different parameters of the CMY Lévy model. On Figure 3(a) we observe that varying the  $C$  parameter (for fixed values of the parameters  $Y$  and  $\rho$ ) has almost no effect on the equity-tranche price. This means that the equity tranche protection seller is neutral on  $C$  or in other words neutral to changing tail behavior of the firm. However, a seller of protection of a non-equity tranche is short  $C$ . Fatter tail behavior (higher kurtosis), decreases  $C$ , so one could say mezzanine and senior tranche protection sellers are long kurtosis. This situation is the same as for the Gamma parameter  $a$  since the CMY distribution is a generalization of the Gamma. On Figure 3(b) we can see that the investors into the mezzanine

tranches are short  $Y$  while equity and senior tranche protection sellers are neutral on  $Y$ .

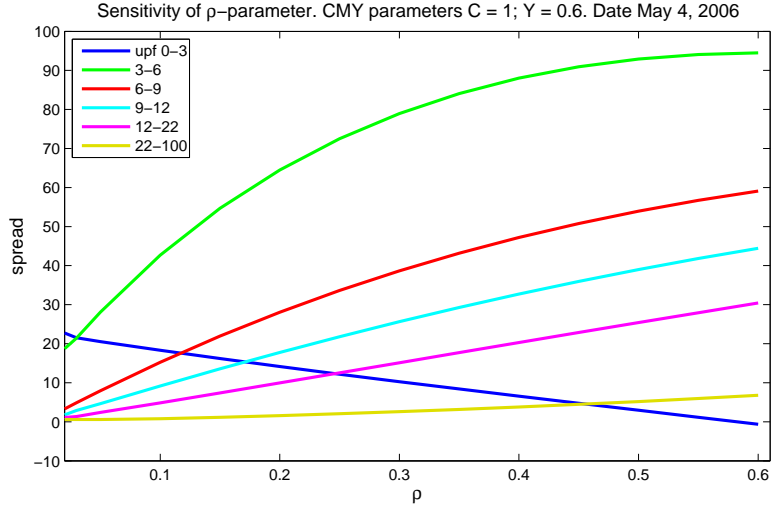


Figure 4: Sensitivity of  $\rho$ -parameter on different tranche prices for  $C = 1$ ,  $Y = 0.6$

As for the Gamma model, the protection seller of the equity tranche is long correlation while the protection sellers of other tranches are short (see Figure 4).

## 5 Global Calibration

We report on a calibration exercise of the newly proposed Gamma-model and IG-model. We calibrate the model to a time-series of weekly iTtraxx (series 4) data from the 10th of November 2005 until the 4th of May 2006. For a particular date we take into account the prices of the 0-3, 3-6, 6-9, 9-12, 12-22, 22-100 percent tranches. The quote of the equity 0-3 tranche is the upfront quote (in percentages) with 500 bp running.

The calibration was done under a weighted least-squared regime. First an optimal  $\rho$  parameter was determined based on the equity tranche. Second we determined on the equity tranche and the mezzanine 3 – 6-tranche the optimal  $a$  parameter in combination with the fixed  $\rho$  parameter coming from the equity tranche calibration. Finally, with these  $\rho$  and  $a$  parameters as starting values, we fine tune on all the tranches by a weighted least-squared direct search algorithm. The highest weights are given to the equity and the mezzanine tranche and are decreasing along seniority. The evolution of the total absolute calibration errors for the Lévy models is plotted in Figure 5.

We observe that the Lévy models clearly outperform the Gaussian model. The total bp error is reduced by more than a factor 3. Not really one particular Lévy model is outperforming then others. On the one hand, the CMY model, as a generalization of the Gamma model and IG mode, obviously always leads to a better fit. On the other hand, the CMY model requires more computational time, since one has to rely on a number of inversion procedures. One can advocate that the Shifted Gamma model is a good balance between accuracy and tractability.

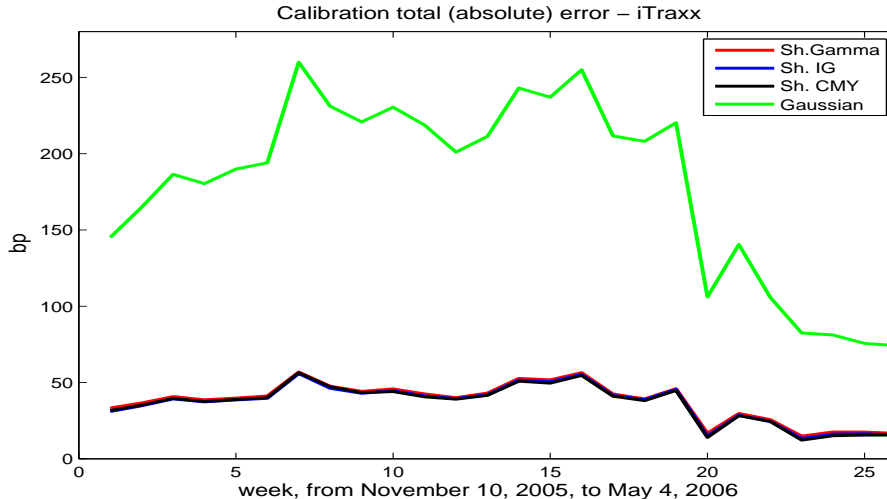


Figure 5: Calibration Error evolution - iTraxx data

## 6 Lévy Base Correlation

In this section, we introduce and illustrate the concept of Lévy Base Correlation. We work out the material for the Gamma model, for the other models the situation is completely analogous. For the definition and discussion of the Gaussian base correlation we refer to O’Kane and Livesey [8].

### 6.1 Bootstrapping Lévy Correlation Calibration

In order to define a Lévy base correlation curve, we need to fix first all the distribution parameters. In for example, the Shifted Gamma case, we need to fix first the  $a$ -parameter. One has several choices to fix this parameter.

One could for example, take the parameter coming out of the global calibration. The advantage of doing this is that typically in the Lévy models, the junior mezzanine tranche is under these parameters also perfectly matched with the market. This will lead later on to a flat base correlation curve at the mezzanine level. However, as market quotes vary from day to day, also these parameters will vary from day to day, leading to a different base correlation construction.

Alternatively, one can fix just the distributional parameters to some value once and leave these unchanged over time. We will then not have a flat curve at the junior tranches, but have a stable procedure over time. For the gamma case, we fix the parameter  $a$  and set it equal to one. We have several reasons to motivate this choice. A historical global calibration study shows that indeed the  $a$  parameter is fluctuating around this value. Secondly, the base-correlation curve is still much flatter than the Gaussian version. Equivalently, the fit of the model is still much better than the Gaussian model. Finally, the gamma distribution reduces for this value to an exponential distribution. We hence could speak about a pure exponential tail behavior. The logarithm of the density then decays linearly, in contrast with the Gaussian situation, where one has a quadratic decay. It was exactly the too light-tailed behavior that led to the inability of the Gaussian model to fit reality. In the sequel, we will follow this procedure and illustrate the *Exponential Lévy Base Correlation* concept.

Let us fix  $a = 1$  and construct the Lévy base correlation curve for this setting. We solve for the Lévy base correlation using a recursive technique called bootstrapping. We start with the equity tranche, the  $[0\% - 3\%]$  tranche and solve for the  $\rho$  parameter, such that the model prices matches the market quote. The  $\rho$  obtained, say  $\rho_{[0\% - 3\%]}$  is the so-called equity base correlation.

In the calculation of the fair spread of the  $[3\% - 6\%]$  tranche, we need to calculate the expected loss of the  $[3\% - 6\%]$  tranche. The expected loss of the  $[3\% - 6\%]$  tranche can be written as the expected loss of the  $[0\% - 6\%]$  tranche and the  $[0\% - 3\%]$  tranche, in a way similar to Equation (4). The expected loss of the  $[0\% - 3\%]$  tranche was calculated on the previous step when we found the equity base correlation. Now, we solve for the Lévy base correlation  $\rho_{[0\% - 6\%]}$  of the  $[0\% - 6\%]$  tranche, so that the  $[3\% - 6\%]$  tranche calculated under the model matches the market spread. Further, we proceed in the same manner through the higher tranches. In Figure 6, one sees a comparison of the Gaussian and the Lévy Base correlation curves.

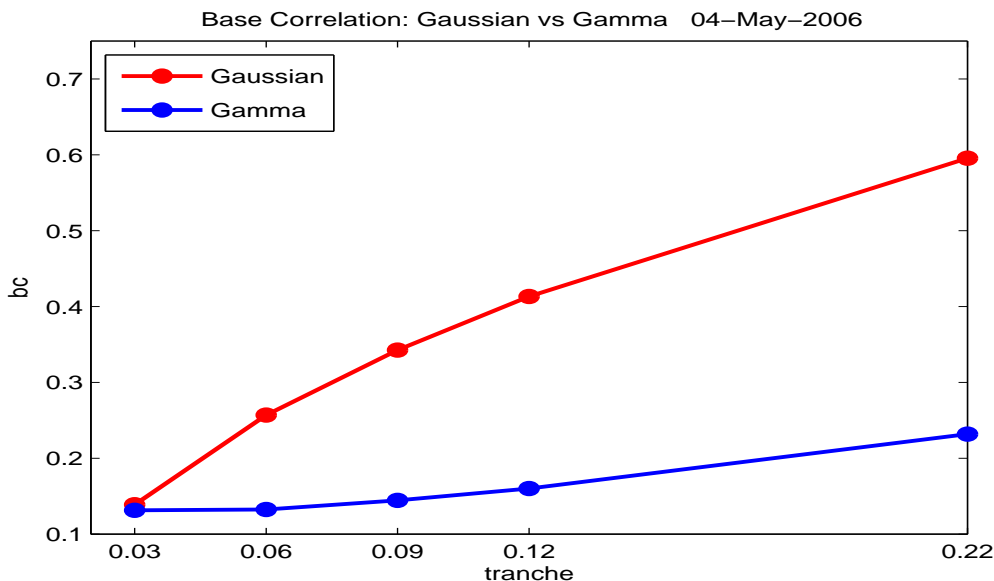


Figure 6: Gaussian versus Lévy base correlation - iTraxx data 2006-05-04

In Figure 7, one finds a comparison of the steepness of the Gaussian and the Lévy Base correlation curves. We plot the difference between the maximum and the minimum base correlation value. A low value means the curve is flat, the higher the difference between maximum and maximum, the more steep the curve is.

## 6.2 Pricing Non Standardized Tranches

Lévy base correlation enables a simple mechanism for pricing non-standard strikes on the standard indices which is more reliable than the Gaussian base correlation procedure. The underlying reason is that the Lévy base correlation is much flatter than the Gaussian base correlation curve and thus much less sensitive to interpolation errors.

Consider the following example. Suppose we want to price a  $[5\% - 10\%]$  tranche of the iTraxx portfolio. With the base correlation methodology, this requires a base correlation value for the  $[0\% - 10\%]$  tranche and for the  $[0\% - 5\%]$  tranche. However, the market information only gives

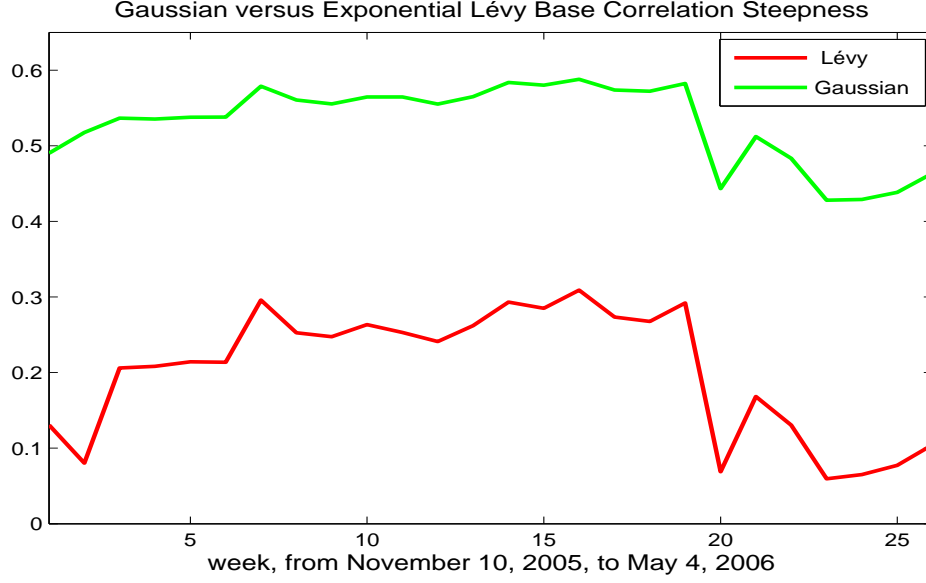


Figure 7: Gaussian versus Lévy base correlation - iTraxx data 2006-05-04

us the base correlations for the [0% – 3%], [0% – 6%], [0% – 9%], [0% – 12%] and [0% – 22%] tranches. For example, for 2006-05-04 the market implied base correlations are as in Table 4 and as in Figure 6.

model	[0% – 3%]	[0% – 6%]	[0% – 9%]	[0% – 12%]	[0% – 22%]
Gaussian	0.13883347	0.25701861	0.34281792	0.41341533	0.59564758
Lévy	0.13153939	0.13266463	0.14472385	0.16021431	0.23188058

Table 4: Gaussian base correlation versus Lévy base correlation - iTraxx data 2006-05-04

Then using linear interpolation between these values, the Gaussian base correlation for a [0% – 5%] tranche is

$$\rho_{[0\%-5\%]}^{(\text{Gaussian})} = \frac{1}{3} \times \rho_{[0\%-3\%]}^{(\text{Gaussian})} + \frac{2}{3} \times \rho_{[0\%-6\%]}^{(\text{Gaussian})} = \frac{1}{3} \times 0.13883347 + \frac{2}{3} \times 0.25701861 = 0.21762356.$$

However for the Lévy case we actually have

$$\rho_{[0\%-5\%]}^{(\text{Lévy})} = \frac{1}{3} \times \rho_{[0\%-3\%]}^{(\text{Lévy})} + \frac{2}{3} \times \rho_{[0\%-6\%]}^{(\text{Lévy})} = \frac{1}{3} \times 0.13153939 + \frac{2}{3} \times 0.13266463 = 0.13228955.$$

The base correlation for the [0% – 10%] tranche is for the Gaussian situation

$$\rho_{[0\%-10\%]}^{(\text{Gaussian})} = \frac{2}{3} \times \rho_{[0\%-9\%]}^{(\text{Gaussian})} + \frac{1}{3} \times \rho_{[0\%-12\%]}^{(\text{Gaussian})} = \frac{2}{3} \times 0.34281792 + \frac{1}{3} \times 0.41341533 = 0.36635039,$$

and for the Lévyian one

$$\rho_{[0\%-10\%]}^{(\text{Lévy})} = \frac{2}{3} \times \rho_{[0\%-9\%]}^{(\text{Lévy})} + \frac{1}{3} \times \rho_{[0\%-12\%]}^{(\text{Lévy})} = \frac{2}{3} \times 0.14472385 + \frac{1}{3} \times 0.16021431 = 0.149887336.$$

The Gaussian case leads to a price of 12.47 bp whereas the Lévy model prices its much higher at 14.74 bp.

Note that the procedure is also sensitive to the interpolation scheme. A spline interpolation, gives

$$\begin{aligned}\rho_{[0\%-5\%]}^{(\text{Gaussian})} &= 0.22221267 \\ \rho_{[0\%-5\%]}^{(\text{Lévy})} &= 0.13062478 \\ \rho_{[0\%-10\%]}^{(\text{Gaussian})} &= 0.36758164 \\ \rho_{[0\%-10\%]}^{(\text{Lévy})} &= 0.14965830\end{aligned}$$

This combination leads to a price of 14.00 bp and 14.13 bp under Gaussian and Lévy respectively. In Figure 8, we price tranchlets  $[K\% - (K + 1)\%]$ , for  $K = 3, 4, \dots, 12$ , and compare the prices obtained under Gaussian and Lévy base correlation procedure. We observe slight price difference.

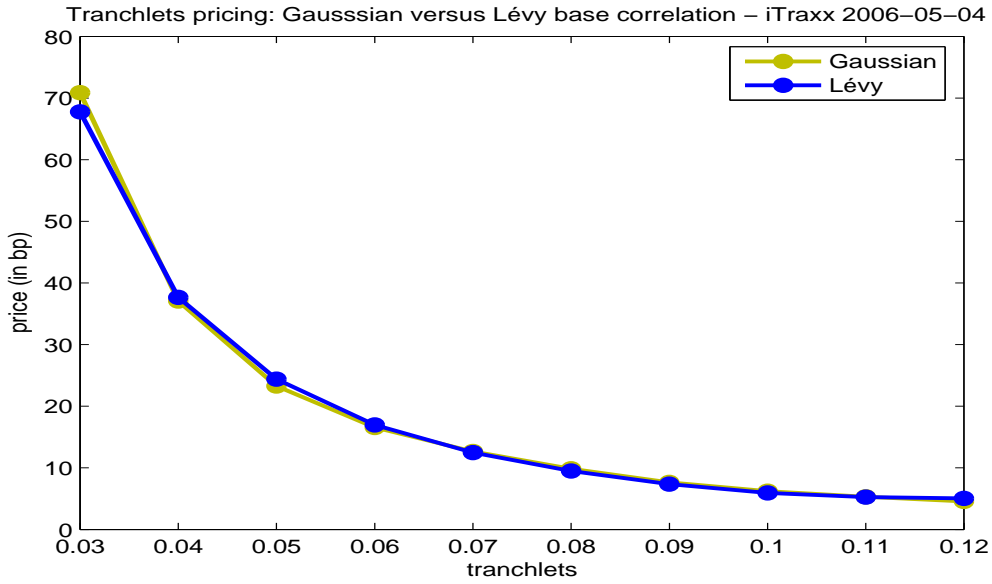


Figure 8: Tranchlets pricing: Gaussian versus Lévy base correlation - iTraxx data 2006-05-04

## 7 Conclusions

In this paper, we have investigated one-factor Lévy models for pricing CDOs. The study has shown evidence for improvements of Lévy models with respect to the classical Gaussian copula model, e.g. significant reduction of the total pricing error. No single Lévy model, however, outperforms the others. Hence we recommend the Gamma model as it is the most tractable one. We have also introduced the concept of Lévy base correlation and applied it to price non-standardized tranches of a CDO. The Lévy base correlation appears to be much flatter than the Gaussian, which once again seems indicative of the better performance for the Lévy models for pricing CDOs tranches, both standard and bespoke.

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