Ideal gas approximation for a two-dimensional rarefied gas under Kawasaki dynamics

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Abstract

In this paper we consider a two-dimensional lattice gas under Kawasaki dynamics, i.e., particles hop around randomly subject to hard-core repulsion and nearest-neighbor attraction. We show that, at fixed temperature and in the limit as the particle density tends to zero, such a gas evolves in a way that is close to an ideal gas, where particles have no interaction. In particular, we prove three theorems showing that particle trajectories are non-superdiffusive and have a diffusive spread-out property. We also consider the situation where the temperature and the particle density tend to zero simultaneously and focus on three regimes corresponding to the stable, the metastable and the unstable gas, respectively.

Our results are formulated in the more general context of systems of "quasi random walks", of which we show that the lattice gas under Kawasaki dynamics is an example. We are able to deal with a large class of initial conditions having no anomalous concentration of particles and with time horizons that are much larger than the typical particle collision time. The results will be used in two forthcoming papers, dealing with metastable behavior of the two-dimensional lattice gas in large volumes at low temperature and low density.

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Key words and phrases. Lattice gas; Kawasaki dynamics; stable, metastable and unstable gas; independent random walks; quasi random walks; non-superdiffusivity; diffusive spread-out property; large deviations.

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1 Introduction

intro

1.1 Rarefied gas

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In this paper we consider a two-dimensional lattice gas at low density evolving under Kawasaki dynamics: particles hop around randomly subject to hard-core repulsion and nearest-neighbor attraction. Our goal is to prove an *ideal gas approximation*, i.e., we want to show that the dynamics is well approximated by a process of independent random walks (IRW's). Indeed, if the lattice gas is sufficiently rarefied, then each particle spends most of its time moving like a random walk. When the particles are in neighboring sites, the binding energy inhibits their random walk motion, and these pauses are long when the temperature is low. However, if the time intervals in which a particle is interacting with the other particles are short compared to the time intervals in which it is free, then we may hope to represent the interaction as a small perturbation of a free motion. We prove that this hope is justified in the low density limit $\rho \downarrow 0$, for any binding energy $-U \leq 0$ and any inverse temperature $\beta \geq 0$.

The situation is more interesting when $\beta \to \infty$ and $\rho \downarrow 0$ *simultaneously*, linked as $\rho := e^{-\beta\Delta}$ with $0 < \Delta < \infty$ an activity parameter. We will consider three regimes: $\Delta \in (2U, \infty)$ (stable gas), $\Delta \in (U, 2U)$ (metastable gas), $\Delta \in (0, U)$ (unstable gas). We will obtain results that hold up to long, moderate and short time horizons, respectively. The metastable gas is the most interesting. It is this regime that motivated the present paper and that we will address in two forthcoming papers (Gaudilliére, den Hollander, Nardi, Olivieri and Scoppola [13], [14]), both of which will rely on the results presented below. Note that the low temperature limit corresponds to a strong interaction regime, so that the ideal gas approximation is far from trivial.

In den Hollander, Olivieri and Scoppola [8], Bovier, den Hollander and Nardi [10], and Gaudillière, Olivieri and Scoppola [11], a *local version* of the model was considered in which the gas outside a finite box Λ_0 is replaced by creation and annihilation of particles at the boundary $\partial\Lambda_0$, at rates $e^{-\Delta\beta}$ and 1, respectively. This boundary condition replaces a *gas reservoir* surrounding Λ_0, with density $\rho = e^{-\Delta\beta}$. For this simplified model, the metastable behavior could be described in full detail. In [8], an extension of the local model was considered in which the gas reservoir consists of IRW's. It was shown that, for $\beta \to \infty$, this extension is well approximated by the local model, as far as metastability is concerned. Note that in the extended model, even though the system is an "ideal gas" outside Λ_0, it influences the Kawasaki gas inside Λ_0, and vice versa. The idea of QRW's was introduced in [8], to describe this mutual influence: the gas particles perform random walks interspersed with pause intervals, corresponding to the time lapses spent in interaction with the other particles, and interspersed with jumps, corresponding to the difference between the position of the particle at the end and

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at the beginning of a pause interval. Due to the fact that Λ_0 is finite, the jumps are small w.r.t. the displacement of the random walks on time scales that are exponentially large in β. Moreover, the number of pause interval is controlled by the rare returns of the random walk to Λ_0. These two ingredients – few pause intervals and small jumps – were sufficient to control the dynamics.

In the present paper, we consider the Kawasaki dynamics in an exponentially large box, at density $\rho = e^{-\Delta\beta}$ and before the formation of large clusters. We expect that the QRW-approximation continues to hold. Indeed, as long as the clusters are small, we expect small jumps, at most of the order of the size of the clusters. However, the crucial obstacle in approximating the gas particles by QRW's comes from the fact that the interaction acts everywhere, so that we need to replace the rare returns of a random walk to a fixed finite box by a control on the number of collisions particle-particle and particle-cluster. This is the crucial point developed in Gaudillière [12], and is the main tool in our analysis of the rarefied Kawasaki gas in the present paper. We quantitatively justify the approximation "rarefied gas ≈ ideal gas" via a precise concept of QRW's, in order to be able to extend the analysis in [8], [10] and [11] to the non-local model. The latter extension will be carried out in two forthcoming papers: [13], [14]. We note that the range of application of the results presented here is much larger than the regime of metastability.

Throughout the paper, 'cst' will denote a positive constant independent of the model parameters, the value of which may change from line to line. By "ideal gas approximation" we mean extending to the Kawasaki dynamics the following well-known properties of a system of N continuous-time independent random walk trajectories observed over a time T,

$$\zeta_i : t \in [0,T] \mapsto \zeta_i(t), \quad i \in \{1,\dots,N\},\, T \geq 2. \tag{1.1}$$

Three properties of IRW. Uniformly in N and T, the following properties hold:

(i) *Non-superdiffusivity:*

$$\forall i \in \{1,\dots,N\},\, \forall \delta > 0\colon\; \lim_{\beta\to\infty} \frac{1}{\beta} \ln P\left(\exists t \in [0,T) : \|\zeta_i(t) - \zeta_i(0)\|_2 > \sqrt{T} e^{\delta\beta}\right) = -\infty. \tag{1.2}$$

(ii) *Spread-out property, upper bound:*

$$\forall I \subset \{1,\dots,N\},\, \forall (z_i)_{i\in I} \in (\mathbb{Z}^2)^I : \qquad P\left(\forall i \in I : \zeta_i(T) = z_i\right) \leq \left(\frac{\text{cst}}{T}\right)^{|I|}. \tag{1.3}$$

(iii) *Spread-out property, lower bound:*

$$\forall I \subset \{1,\dots,N\},\, \forall (z_i)_{i\in I} \in (\mathbb{Z}^2)^I :$$
$$\begin{cases} \left(\forall i \in I : 0 \leq \|z_i - \zeta_i(0)\|_2 \leq \sqrt{T}\right) \Rightarrow P\left(\forall i \in I : \zeta_i(T) = z_i\right) \geq \left(\dfrac{\text{cst}}{T}\right)^{|I|}, \\ \left(\forall i \in I : 0 < \|z_i - \zeta_i(0)\|_2 \leq \sqrt{T}\right) \Rightarrow P\left(\forall i \in I : \lfloor \tau_{z_i}(\zeta_i) \rfloor = T\right) \geq \left(\dfrac{\text{cst}}{T \ln^2 T}\right)^{|I|}, \end{cases} \tag{1.4}$$

with

$$\tau_{z_i}(\zeta_i) := \inf\left\{t > 0\colon\; \zeta_i(t) = z_i\right\}. \tag{1.5}$$

(For proofs of these properties, see e.g. Jain and Pruitt [1] and Révèsz [7].)

We will generalize (1.1–1.5) to what we call a process of *Quasi Random Walks* (QRW's) and we will show that the low density Kawasaki dynamics with labelled particles is a QRW-process. Roughly speaking, particles that evolve according to a QRW-process move like an IRW-process except in occasional time intervals – called *pause intervals* – in which they remain confined to small domains. While the non-superdiffusivity will be proven for all particles, the spread-out property will be proven for "non-sleeping" particles only, i.e., those particles for which the pause intervals are not too long. However, we will see that in the stable regime ($\Delta > 2U$) with very large probability there are no sleeping particles, while for the metastable and unstable regimes ($\Delta < 2U$) the situation is more complex.

To show that Kawasaki dynamics is a QRW-process, we will couple it to an IRW-process and keep track of the distance between the two processes. This is different from the approach followed by Kipnis and Varadhan [6] to analyze the trajectory of a tagged particle in reversible interacting particle systems. Using martingale arguments, they proved that in *infinite* volume at *any* density and starting from *equilibrium*, if $X(t)$ denotes the position at time t of the tagged particle, then the process $(\sqrt{\epsilon}X(t/\epsilon))_{t\geq 0}$ converges to a rescaled Brownian motion $(D_{\mathrm{self}}B(t))_{t\geq 0}$ in the limit as $\epsilon \downarrow 0$. This is an invariance principle, where "cumulative chaos" leads to Gaussian behavior. Our approach is in some sense complementary, because we use the *low* density limit to view Kawasaki dynamics as a small perturbation of an IRW-process and prove large deviation and local occupation bounds, and this perturbation also works *away from equilibrium*. It will lead us to introduce a time scale beyond which our results no longer apply. This time scale will be much longer than the typical particle collision time, namely, it will be of the order of the *minimum* of the *square* of the typical particle collision time and the time of *first anomalous concentration* of particles.

We mention other papers where a coupling between the one-dimensional simple exclusion process (for which $D_{\mathrm{self}} = 0$ – see [3]) and an IRW-process was constructed. In Ferrari, Galves and Presutti [2] and in De Masi, Ianiro, Pellegrinotti and Presutti [5], Chapter 3, a hierarchy on the particles is introduced, which leads to a coupling with strong symmetry properties. This hierarchy is used to prove non-superdiffusivity. Unfortunately, in higher dimensions and as soon as $U > 0$, these symmetry properties are lost.

1.2 Kawasaki dynamics

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Let $\beta > 0$ be the inverse temperature and let $\Lambda_\beta \subset \mathbb{Z}^2$ be a large square box, centered at the origin and with periodic boundary conditions. With each $x \in \Lambda_\beta$ we associate an occupation variable $\eta(x)$ assuming the values 0 or 1. A lattice gas configuration is denoted by $\eta \in \mathcal{X} = \{0,1\}^{\Lambda_\beta}$. We frequently identify a configuration $\eta \in \mathcal{X}$ with its support, i.e., with the set $\{x \in \mathbb{Z}^d \colon \eta(x) = 1\}$.

Fix the number of particles in Λ_β at

$$N := \sum_{x\in\Lambda_\beta} \eta(x) = \rho|\Lambda_\beta| \quad \text{with } \rho := e^{-\Delta\beta}, \tag{1.6}$$

where $\Delta > 0$ is an activity parameter, ρ is the particle density and $|\Lambda_\beta|$ is the cardinality of Λ_β. (Here and in what follows we round off large integers, in order to avoid a plethora of brackets like $\lceil\cdot\rceil$.) We assume that Λ_β is exponentially large in β:

$$|\Lambda_\beta| =: e^{\Theta\beta} \quad \text{for some } \Theta > \Delta. \tag{1.7}$$

With each configuration η we associate an energy given by the Hamiltonian

$$\mathrm{H}(\eta) := -U \sum_{\langle x,y\rangle\in\Lambda_\beta^*} \eta(x)\eta(y), \tag{1.8}$$

where Λ_β^* denotes the set of nearest-neighbor non-oriented bonds in Λ_β, and $-U \leq 0$ is the binding energy felt by neighboring occupied sites. On the set of configurations with N particles, written

$$\mathcal{X}_N := \Big\{\eta \in \mathcal{X}\colon \sum_{x\in\Lambda_\beta} \eta(x) = N\Big\}, \tag{1.9}$$

we define the *canonical Gibbs measure* as

$$\nu_N(\eta) := \frac{e^{-\beta\mathrm{H}(\eta)}1_{\mathcal{X}_N}(\eta)}{Z_N}, \qquad \eta \in \mathcal{X}, \tag{1.10}$$

where Z_N is the normalizing partition sum.

Kawasaki dynamics is the continuous-time Markov chain $(\eta_t)_{t\geq 0}$ with state space $\mathcal{X}_N$ and generator

$$(\mathcal{L}f)(\eta) := \sum_{\langle x,y\rangle\in\Lambda_\beta^*} c(\langle x,y\rangle,\eta)[f(\eta^{\langle x,y\rangle}) - f(\eta)], \qquad \eta \in \mathcal{X}, \tag{1.11}$$

where

$$\eta^{\langle x,y\rangle}(z) := \begin{cases} \eta(z) & \text{if } z \neq x, y, \\ \eta(x) & \text{if } z = y, \\ \eta(y) & \text{if } z = x, \end{cases} \tag{1.12}$$

and

$$c(\langle x,y\rangle,\eta) := \frac{1}{4}e^{-\beta[\mathrm{H}(\eta^{\langle x,y\rangle})-\mathrm{H}(\eta)]_+}. \tag{1.13}$$

This is the standard Metropolis dynamics associated with H. The factor 1/4 is optional. For the coupling with the IRW-process it is convenient. It is easily verified that ν_N is the reversible equilibrium of the Metropolis dynamics:

$$\forall\eta \in \mathcal{X}_N,\ \forall\langle x,y\rangle \in \Lambda_\beta^*:\ \nu_N(\eta)c(\langle x,y\rangle,\eta) = \nu_N(\eta^{\langle x,y\rangle})c(\langle x,y\rangle,\eta^{\langle x,y\rangle}). \tag{1.14}$$

Kawasaki dynamics is a "dynamics of configurations", in the sense that it describes the evolution of a set of occupied sites rather than of individual particles occupying these sites. In Section 2 we will construct a process $\hat\eta = (\hat\eta_1, \ldots, \hat\eta_N)$ with state space

$$\hat{\mathcal{X}}_N := \big\{(z_1, \ldots, z_N) \in \Lambda_\beta^N\colon\ z_i \neq z_j\ \forall i, j \in \{1, \ldots, N\},\ i \neq j\big\} \tag{1.15}$$

that describes the trajectories $\hat\eta_i\colon\ t \mapsto \hat\eta_i(t)$ of N particles such that the Kawasaki dynamics is recovered by setting

$$(\eta_t)_{t\geq 0} := (\mathcal{U}(\hat\eta(t)))_{t\geq 0} \tag{1.16}$$

with $\mathcal{U}$ the natural *unlabelling application* that sends $\hat{\mathcal{X}}_N$ onto $\mathcal{X}_N$. We will couple $\hat\eta$ with an IRW-process $\zeta = (\zeta_1, \ldots, \zeta_N)$ on Λ_β^N by starting from ζ and building $\hat\eta$ out of ζ via random labels (see Section 2.1).

1.3 Three regimes

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There is a natural time scale on which we may expect the gas to behave like a gas of IRW's:

$$e^{\Delta\beta} = \left[\left(\frac{1}{\rho} \right)^{1/2} \right]^2. \tag{1.17}$$

Indeed, $(1/\rho)^{1/2}$ represents the average interparticle distance and its square is the corresponding average particle collision time. We have to compare this time with the pauses caused by the binding energy. We distinguish three cases.

(1) If $\Delta > 2U$ (stable gas), then the pauses are typically much shorter than $e^{\Delta\beta}$. On this time scale the gas will essentially behave like a gas of IRW's, i.e., the probabilities at time T to find a given set of particles in a given set of sites are similar to those for IRW's. We will be able to prove that this is true up to time scale $e^{2\Delta\beta}$, provided the gas starts from equilibrium, or up to time scale $e^{3\Delta\beta/2} \wedge e^{(2\Delta-2U)\beta}$ for a much wider class of starting configurations, namely, those that exclude anomalous concentrations of particles.

(2) If $\Delta < U$ (unstable gas), then the pauses are typically much longer than $e^{\Delta\beta}$. For this case we will only have very weak results, limited to time scale $e^{\Delta\beta}$.

(3) If $U < \Delta < 2U$ (metastable gas), then typically some pauses are much shorter than $e^{\Delta\beta}$ while others are much longer. For $D \in (U, \Delta)$, as close to U as we want, we will say that a particle "falls asleep" when it makes a pause longer than $e^{D\beta}$. We will say that non-sleeping particle are *active* and we will be able to obtain results for active particles up to time scale $e^{2\Delta\beta}$, provided the system starts from a "metastable equilibrium" and Θ is not too large.

In what follows, we will deal simultaneously with these three regimes. To that end, we introduce a constant $D \in (0, \Delta)$, as close to 0, U, $2U$ as we want in the unstable, metastable and stable regime, respectively. The different regimes will be discussed separately in Section 6 only.

Note that the simple exclusion process ($U = 0$) is a particular case of the stable regime, for which we have the strongest results. As mentioned before, these results can also be extended to the case of a rarefied gas evolving at fixed positive temperature under the Kawasaki dynamics.

1.4 Notation

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1. Apart from of the model parameters (U, Δ, Θ, β), we need three further parameters: $D \in (0, \Delta)$ (see above), $0 < \alpha \ll 1$, and $\beta \mapsto \lambda(\beta)$, a slowly increasing unbounded function that satisfies

$$\lambda(\beta) \ln \lambda(\beta) = o(\ln \beta), \tag{1.18}$$

e.g. $\lambda(\beta) = \sqrt{\ln \beta}$. Given $\alpha > 0$, we define a *reference time* almost of order $e^{\Delta\beta}$

$$T_\alpha := e^{(\Delta-\alpha)\beta}, \tag{1.19}$$

and we assume that α is small enough so that $T_\alpha > e^{D\beta}$.

2. For $\Lambda \subset \Lambda_\beta$, we write $\Lambda \sqsubset \Lambda_\beta$ if Λ is a square box, i.e., there are $a, b, c \in \mathbb{R}$ such that

$$\Lambda = ([a, a+c] \times [b, b+c]) \cap \Lambda_\beta. \tag{1.20}$$

For $\Lambda \subset \Lambda_\beta$ and $\eta \in \{0,1\}^{\Lambda_\beta}$, we denote by $\eta|_\Lambda$ the restriction of η to Λ, and put

$$\Big|\eta|_\Lambda\Big| := \sum_{x\in\Lambda} \eta(x). \tag{1.21}$$

We denote by $\mathcal{T}_{\alpha,\lambda}$ the *first time of anomalous concentration*:

$$\mathcal{T}_{\alpha,\lambda} := \inf\left\{t \geq 0\colon \Big|\eta_t|_\Lambda\Big| \geq \frac{\lambda(\beta)}{4} \text{ for some } \Lambda \sqsubset \Lambda_\beta \text{ with } |\Lambda| \leq e^{\left(\Delta - \frac{\alpha}{4}\right)\beta}\right\}. \tag{1.22}$$

3. For $p \geq 1$, the p-norm on $\mathbb{R}^2$ is

$$\|\cdot\|_p\colon\ (x,y) \in \mathbb{R}^2 \mapsto \begin{cases} (|x|^p + |y|^p)^{1/p} & \text{if } p < \infty, \\ |x| \vee |y| & \text{if } p = \infty. \end{cases} \tag{1.23}$$

We denote by $B_p(z,r)$, $z \in \mathbb{R}^2$, $r > 0$, the open ball with center z and radius r in the p-norm. The closure of $A \subset \mathbb{R}^2$ is denoted by $\overline{A}$.

4. For $\eta \in \mathcal{X}$, we denote by η^{cl} the *clusterized part of* η:

$$\eta^{cl} := \{z \in \eta\colon\ \|z - z'\|_1 = 1 \text{ for some } z' \in \eta\}. \tag{1.24}$$

We call *clusters of* η the connected components of the graph drawn on η^{cl} obtained by connecting nearest-neighbor sites. For $A \subset \mathbb{Z}^2$, we denote by ∂A its external border, i.e.,

$$\partial A := \left\{z \in \mathbb{Z}^2 \setminus A\colon\ \|z - z'\|_1 = 1 \text{ for some } z' \in A\right\}. \tag{1.25}$$

For $r > 0$, we put

$$[A]_r := \bigcup_{z\in A} \overline{B_\infty(z,r)} \cap \mathbb{Z}^2. \tag{1.26}$$

We say that A is a rectangle on $\mathbb{Z}^2$ if there are $a, b, c, d \in \mathbb{R}$ such that

$$A = [a,b] \times [c,d] \cap \mathbb{Z}^2. \tag{1.27}$$

We write $\mathrm{RC}(A)$, called the *circumscribed rectangle* of A, to denote the intersection of all the rectangles on $\mathbb{Z}^2$ containing A.

5. The hitting time of A for a generic random process ξ_0 is denoted by

$$\tau_A(\xi_0) := \inf\{t \geq 0\colon\ \xi_0(t) \in A\}. \tag{1.28}$$

6. A function $\beta \mapsto f(\beta)$ is called superexponentially small (SES) if

$$\lim_{\beta\to\infty} \frac{1}{\beta} \ln f(\beta) = -\infty. \tag{1.29}$$

If $(A_j)_{j\in J}$ is a family of events, then we say that "A_j occurs with probability $1 -$ SES uniformly in j" when there is an SES-function f independent of j such that

$$P(A_j^c) \leq f \qquad \forall j \in J. \tag{1.30}$$

For example, by Brownian approximation and scaling, for ζ_0 a simple random walk in continuous time and $\delta > 0$, we have

$$P\left(\exists t \in [0, m+1]\colon\ \|\zeta_0(t) - \zeta_0(0)\|_2 > e^{\delta\beta}\sqrt{m}\right) \leq \text{SES} \qquad \text{uniformly in } m \in \mathbb{N}. \tag{1.31}$$

1.5 Outline

In Section 2 we couple Kawasaki dynamics with labelled particles $\hat{\eta}$ to an IRW-process ζ. In Section 3 we give our main results, built on the notion of Quasi Random Walk (QRW). In Section 4 the non-superdiffusivity and the spread-out property are proved for QRW's. In Section 5 we prove that the low-density limit of Kawasaki dynamics with labelled particles is a QRW-process and prove some stronger estimates for the lower bound of the spread-out property as well. In Section 6 these results are applied to the three different regimes of Section 1.3. Some of the proofs in this paper do rely on the notion of QRW-process, and therefore are placed in Appendix A and B.

2 Kawasaki dynamics with labelled particles

kdlp

In Section 2.1 we couple Kawasaki dynamics to an IRW-process. In Section 2.2 we introduce free, active and sleeping particles. In Section 2.3 we introduce a special permutation rule for particles, as part of the coupling.

2.1 Coupling with Independent Random Walks

cplng

Given N Poisson processes $\theta_1, \ldots, \theta_N$ of intensity 1 and N families

$$(e_{1,k})_{k\in\mathbb{N}}, (e_{2,k})_{k\in\mathbb{N}}, \ldots, (e_{N,k})_{k\in\mathbb{N}} \tag{2.1}$$

of independent unit random vectors equally distributed in the four directions (north, south, east, west), all mutually independent, we define a process $\zeta = (\zeta_1, \ldots, \zeta_N)$ of N IRW's starting from $z = (z_1, \ldots, z_N) \in \Lambda_\beta^N$ by putting

$$\zeta_i(t) := z_i + \sum_{k=1}^{\theta_i(t)} e_{i,k}, \qquad i \in \{1, \ldots, N\},\, t \geq 0. \tag{2.2}$$

Suppose that $\zeta(0) = z \in \hat{\mathcal{X}}_N$ (recall (1.15)). To build a Kawasaki dynamics with labelled particles $\hat{\eta} = (\hat{\eta}_1, \ldots, \hat{\eta}_N)$ starting from z, we introduce N families

$$(U_{1,k})_{k\in\mathbb{N}}, (U_{2,k})_{k\in\mathbb{N}}, \ldots, (U_{N,k})_{k\in\mathbb{N}} \tag{2.3}$$

of independent marks, uniformly distributed in $[0, 1]$, mutually independent and independent of the families in (2.1), and apply the following three-step updating rule each time the process ζ changes position, i.e., each t with $\zeta(t_-) \neq \zeta(t)$:

1. Define a first candidate $\hat{\eta}'$ for the new configuration:

$$\hat{\eta}' := \hat{\eta}(t_-) + \zeta(t) - \zeta(t_-) \in \Lambda_\beta^N. \tag{2.4}$$

2. Test $\hat{\eta}'$ to define a second candidate $\hat{\eta}''$ as follows:
   - If $\hat{\eta}' \notin \hat{\mathcal{X}}_N$, then $\hat{\eta}'' := \hat{\eta}(t_-)$.

- If $\hat\eta' \in \hat{\mathcal{X}}_N$ and for some $i \in \{1,\dots,N\}$

$$\exp\Big[-\beta\Big(\mathrm{H}(\mathcal{U}(\hat\eta')) - \mathrm{H}(\mathcal{U}(\hat\eta(t_-)))\Big)\Big] \geq U_{i,\theta_i(t)} \quad \text{and} \quad \theta_i(t) \neq \theta_i(t_-), \tag{2.5}$$

  then $\hat\eta'' := \hat\eta'$.
- If $\hat\eta' \in \hat{\mathcal{X}}_N$ and for all $i \in \{1,\dots,N\}$

$$\exp\Big[-\beta\Big(\mathrm{H}(\mathcal{U}(\hat\eta')) - \mathrm{H}(\mathcal{U}(\hat\eta(t_-)))\Big)\Big] < U_{i,\theta_i(t)} \quad \text{or} \quad \theta_i(t) = \theta_i(t_-), \tag{2.6}$$

  then $\hat\eta'' := \hat\eta(t_-)$.

3. Define $\hat\eta(t)$ as the configuration obtained from $\hat\eta''$ by an appropriate local permutation (see below) of the positions of the particles (so that $\mathcal{U}(\hat\eta(t)) = \mathcal{U}(\hat\eta'')$).

Definition 2.1.1 *Associate with each $\hat\eta \in \hat{\mathcal{X}}_N$ the cluster partition on $\{1,\dots,N\}$ induced by the following equivalence relation: two particles labelled i and j are equivalent when they are in the same cluster of $\mathcal{U}(\hat\eta)^{cl}$.*

We will assume:

Local permutation: *The permutation performed at the third step of the updating rule respects the cluster partition of $\hat\eta(t_-)$.*

It is easy to check that the generator of the process

$$(\eta_t)_{t\geq 0} := (\mathcal{U}(\hat\eta(t)))_{t\geq 0} \tag{2.7}$$

is the same as (1.11) and is independent of the type of permutation performed at the third step of the updating rule.

2.2 Free, active and sleeping particles

fasp

Given $\eta \in \mathcal{X}_N$, we say that $z \in \Lambda_\beta$ is occupied by a *free particle* if for some $i \in \{1,\dots,N\}$ and some $\hat\eta(0) \in \mathcal{X}_N$ such that $\hat\eta_i(0) = z$ there is a trajectory up to some time T,

$$\hat\eta : t \in [0,T] \longmapsto \hat\eta(t) \in \mathcal{X}_N, \tag{2.8}$$

that respects the rules allowed by the process $\hat\eta$ and satisfies

- $\|\hat\eta_i(T) - \hat\eta_i(0)\|_2 > \sqrt{T_\alpha}$,
- $\forall s \in [0,T]\colon\ \mathcal{U}(\hat\eta(s))^{cl} = \eta^{cl}$.

Note that for $t < \mathcal{T}_{\alpha,\lambda}$ (i.e., prior to the first anomalous concentration; recall (1.22)) the clusterized part of η_t can be described as a collection of small islands (the clusters of η_t) surrounded by a sea (the single connected component of $\Lambda_\beta \setminus \eta_t^{cl}$ that wraps around the torus), and the free particles at time t can go anywhere in this sea without attaching themselves. The

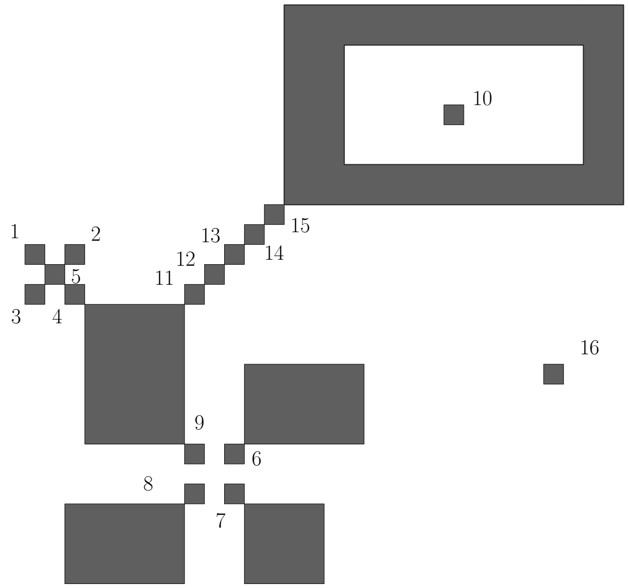

Figure 1: In this figure each particle is represented by a unit square. Particles 1–5 and 16 are free, particles 6–9, 10 and 11–15 are not free, while the other particles are clusterized.

set of sites occupied at time t by the free particles will be denoted by η_t^f. We have $\eta_t^f \subset \eta_t \setminus \eta_t^{cl}$, in some cases with strict inclusion (see figure 1).

We next define a new process $Z = (Z_1, \dots, Z_N)$ on Λ_β^N, coupled to $\hat{\eta}$ and ζ. To do so, we start from $Z(0) := \hat{\eta}(0) = \zeta(0)$ and apply the following rule each time the process ζ changes position:

$$\forall i \in \{1, \dots, N\} : \quad Z_i(t) := \begin{cases} Z_i(t_-) + \zeta_i(t) - \zeta_i(t_-) & \text{if } i \text{ was free at time } t_-, \\ Z_i(t_-) & \text{if } i \text{ was not free at time } t_-. \end{cases} \tag{2.9}$$

Then Z is a process of "random walks with pauses" according to the following definition.

prwp

Definition 2.2.1 *A process $Z = (Z_1, \dots, Z_N)$ on Λ_β^N is called a random walk with pauses (*RWP*) associated with the stopping times*

$$0 = \sigma_{i,0} = \tau_{i,0} \le \sigma_{i,1} \le \tau_{i,1} \le \sigma_{i,2} \le \tau_{i,2} \le \dots, \qquad i \in \{1, \dots, N\}, \tag{2.10}$$

if, for any $i \in \{1, \dots, N\}$, Z_i is constant on all intervals $[\sigma_{i,k}, \tau_{i,k}]$, $k \in \mathbb{N}_0$, and if the process $\tilde{Z} = (\tilde{Z}_1, \dots, \tilde{Z}_N)$ obtained by cutting off, for each i, these pause intervals, i.e.,

$$\begin{aligned} \tilde{Z}_i(s) &:= Z_i\left(s + \sum_{k<j_i(s)} (\tau_{i,k} - \sigma_{i,k})\right) \\ \textit{with} \quad j_i(s) &:= \inf\left\{ j \in \mathbb{N} \colon \; s + \sum_{k<j} (\tau_{i,k} - \sigma_{i,k}) \le \sigma_{i,j} \right\}, \end{aligned} \tag{2.11}$$

is an IRW*-process in law.*

Indeed, for fixed $i \in \{1, \dots, N\}$, define by induction the sequence of stopping times

$$0 = \sigma_{i,0} = \tau_{i,0} \le \sigma_{i,1} \le \tau_{i,1} \le \sigma_{i,2} \le \tau_{i,2} \le \dots \tag{2.12}$$

with

$$\forall k \in \mathbb{N}\colon \begin{cases} \sigma_{i,k} := \inf\{t > \tau_{i,k-1}\colon \ i \text{ is not free at time } t\}, \\ \tau_{i,k} := \inf\{t > \sigma_{i,k}\colon \ i \text{ is free at time } t\}. \end{cases} \tag{2.13}$$

Then Z_i is a Markov process that does not move inside the intervals $[\sigma_{i,k}, \tau_{i,k}]$, $k \in \mathbb{N}_0$ (these are the pause intervals), and outside these intervals moves exactly like a simple random walk in continuous time. $\tilde{Z}$ is an IRW-process as a consequence of the independence of the Poisson processes $\theta_1, \dots, \theta_N$ and the increments $(e_{i,k})_{i\in\{1,\dots,N\},k\in\mathbb{N}}$ in (2.1). Note that, for the same reasons, $Z - \zeta$ is a process of random walks with pauses in the intervals $[\tau_{i,k}, \sigma_{i,k+1}]$, $k \in \mathbb{N}_0$. Note also that on any of these intervals $\hat{\eta}_i$, Z_i and ζ_i evolve jointly, i.e., the pair differences are constant.

To prove our ideal gas approximation, we need to control two quantities:

- The number of pauses of the processes Z_i prior to time T.
- The distance between the processes $\hat{\eta}$ and Z.

The smaller these are, the closer are $\hat{\eta}$ and ζ. This is the idea that will lead us to introduce the concept of Quasi Random Walk (QRW) in Section 3. There is a third quantity that plays an important role in estimating $\|\hat{\eta} - \zeta\|_2$: the lengths of the pause intervals. That is why we introduce the notion of sleeping and active particles.

as

Definition 2.2.2 *For $t > e^{D\beta}$, we say that a particle is sleeping at time t if it was not free at any time $s \in [t - e^{D\beta}, t]$. We call a non-sleeping particle active. By convention, we will say that prior to time $e^{D\beta}$ all particles are active. With particle i we associate, at any time t, its wake-up time*

$$w_i(t) := \inf\{s \in [0, t)\colon \ i \text{ is active on the whole interval } [s, t]\}. \tag{2.14}$$

By convention for a sleeping particle at time t, we fix $w_i(t) = \inf \emptyset = +\infty$.

2.3 A special permutation rule

smr

We close this section by giving examples of local permutation rules. The first example is of particular interest in the study of the metastable regime. At each time t, we define a hierarchy on the particles in all clusters C of η_t: the later the particles lost their freedom, the higher they are in the hierarchy. With the notation of Section 2.1, this permutation rule is as follows:

Special permutation rule: *If some particles were in some cluster C at time t_- and were free in $\hat{\eta}''$, then $\hat{\eta}(t)$ is obtained from $\hat{\eta}''$ by exchanging randomly their positions with those of the higher particles in the hierarchy of C at time t_-.*

Other permutation rules are often considered in literature:

- $\hat{\eta}(t) := \hat{\eta}''$ (no permutation at all);

- if the first candidate did not violate the exclusion (i.e., if $\hat{\eta}' \in \hat{\mathcal{X}}_N$), then $\hat{\eta}(t) := \hat{\eta}''$, while if $\hat{\eta}' \notin \hat{\mathcal{X}}_N$, then $\hat{\eta}(t)$ is obtained from $\hat{\eta}'' = \hat{\eta}(t_-)$ by exchanging the position of the particles responsible for the violation of the exclusion.

The latter mixing rule is often used when the dynamics is built from Poisson processes associated with bonds rather than sites. In Ferrari, Galves and Presutti [2], the usual permutation rules are combined on the basis of the particle hierarchy. All these permutation rules satisfy our local permutation hypothesis.

3 Quasi Random Walks and Main results

rslts

In Section 3.1 we define the QRW-process. In Section 3.3 we state three propositions showing in what way (1.1–1.5) carry over to QRW-processes. In Section 3.4 we sharpen the lower bound in the spread-out property for Kawasaki dynamics.

3.1 Definition of QRW

qrw

Definition 3.1.1 *We say that a process $\xi = (\xi_1, \ldots, \xi_N)$ on Λ_β^N is a Quasi Random Walk (QRW) with parameter $\alpha > 0$ up to stopping time $\mathcal{T}$, written QRW$(\alpha, \mathcal{T})$, if there exists a coupling between ξ and an RWP-process Z associated with the stopping times* dfnqrw

$$0 = \sigma_{i,0} = \tau_{i,0} \le \sigma_{i,1} \le \tau_{i,1} \le \sigma_{i,2} \le \tau_{i,2} \le \ldots, \qquad i \in \{1, \ldots, N\}, \tag{3.1}$$

such that: $\xi(0) = Z(0)$, for any $i \in \{1, \ldots, N\}$, ξ_i and Z_i evolve jointly ($\xi_i - Z_i$ is constant) outside the pause intervals $[\sigma_{i,k}, \tau_{i,k}]$, $k \in \mathbb{N}_0$, and for any $t_0 \ge 0$ the following events occur with probability $1 -$ SES uniformly in i and t_0:

$$\begin{aligned} F_i(t_0) &:= \Big\{ \Big| \{k \in \mathbb{N} \colon \sigma_{i,k} \in [t_0 \wedge \mathcal{T}, (t_0 + T_\alpha) \wedge \mathcal{T}]\} \Big| \le l(\beta) \Big\} \\ G_i(t_0) &:= \Big\{ \forall\, k \in \mathbb{N}, \forall\, t \ge t_0 \colon \sigma_{i,k} \in [t_0 \wedge \mathcal{T}, (t_0 + T_\alpha) \wedge \mathcal{T}] \\ &\qquad\qquad \Rightarrow \|\xi(t \wedge \tau_{i,k} \wedge \mathcal{T}) - \xi(\sigma_{i,k})\|_2 \le l(\beta) \Big\} \end{aligned} \tag{3.2}$$

for some $\beta \mapsto l(\beta)$ satisfying

$$\lim_{\beta \to \infty} \frac{1}{\beta} \ln l(\beta) = 0. \tag{3.3}$$

Remarks:

1. In words, ξ is a QRW$(\alpha, \mathcal{T})$-process if "up to time $\mathcal{T}$" it can be coupled to an RWP-process Z (Definition 2.2.1) with few pause intervals on time scale T_α and in each of these pause intervals ξ has a small variation. More precisely, both the number of intervals and the variation of ξ are bounded by the same quantity $l(\beta)$, which by (3.3) is exponentially negligible. Outside these pause intervals ξ behaves like an IRW-process.

2. The parameter α determines the reference time T_α, which has to be thought of as a time smaller than but close to $1/\rho$ (recall (1.17–1.19)).

3. We used the expression "up to time $\mathcal{T}$" because the QRW-property does not imply anything about the process after time $\mathcal{T}$. If $t_0 \geq \mathcal{T}$, then the events described in (3.2) are trivially verified.

4. Any RWP-process is a QRW(α, ∞)-process provided the pauses are few. For example, a system of random walks in a random environment with local traps, where the particles get stuck during random times, is a QRW(α, ∞)-process as soon as the traps are sufficiently sparse (typically with density $\leq e^{-\Delta\beta}$).

3.2 Kawasaki dynamics and QRW

kqrw

Theorem 3.2.1 *For any increasing unbounded function λ satisfying* (1.18) *and any $\alpha \in (0, \Delta)$, $\hat{\eta}$ is a* QRW$(\alpha, \mathcal{T}_{\alpha,\lambda})$*-process. Moreover, the associated function l can be taken to be*

$$l(\beta) := (\Delta\beta)^{\mathrm{cst}\,\lambda(\beta)^8}. \tag{3.4}$$

Remarks:

1. As will become clear from the proof of Theorem 3.2.1 in Section 5.2, the role of the random time $\mathcal{T}_{\alpha,\lambda}$ is crucial. The fact that the QRW-property holds only up to this time is not a technical restriction: we are describing the Kawasaki dynamics prior to anomalous concentration and we may expect that its behavior changes beyond $\mathcal{T}_{\alpha,\lambda}$, for instance when the dynamics has grown a large cluster. In Section 6 we will give estimates on $\mathcal{T}_{\alpha,\lambda}$ in the three regimes (stable, metastable and unstable). In particular, we will see that in the stable regime the QRW-property itself in some sense preserves the absence of anomalous concentration. This is because an IRW-process produces anomalous concentration with a small probability, and hence so does a QRW-process in the stable regime.

2. To prove Theorem 3.2.1, we will show that Z (the RWP-process we constructed in Section 2.2) fits with Definition 3.1. Actually, $\hat{\eta}$ was not only coupled to Z, it was also coupled to an IRW-process $\zeta = (\zeta_1, \ldots, \zeta_N)$ such that $\zeta(0) = Z(0)$ and, for any $i \in \{1, \ldots, N\}$, ζ_i evolves jointly with Z_i (and $\hat{\eta}_i$) outside the pause intervals $[\sigma_{i,k}, \tau_{i,k}]$, $k \in \mathbb{N}_0$. It is easy to show that *any* RWP-process Z can be coupled to an IRW-process ζ that has such properties. This implies, in particular, that $Z - \zeta$ is an RWP-process with pauses in the intervals $[\tau_{i,k}, \sigma_{i,k+1}]$, $k \in \mathbb{N}_0$. In the sequel we will assume that a generic QRW$(\alpha, \mathcal{T})$-process ξ is not only coupled to an RWP-process Z associated with the stopping times

$$0 = \sigma_{i,0} = \tau_{i,0} \leq \sigma_{i,1} \leq \tau_{i,1} \leq \sigma_{i,2} \leq \tau_{i,2} \leq \ldots, \qquad i \in \{1, \ldots, N\}, \tag{3.5}$$

but also to such an IRW-process ζ. In addition, for any QRW$(\alpha, \mathcal{T})$-process ξ there is a natural generalization of the concepts of free, active and sleeping particles. We say that particle i is free outside the pause intervals of the coupled process Z_i, and define sleeping and active particles as in Definition 2.2.2.

3.3 Consequences of the QRW-property: Main results

We can now generalize the non-superdiffusivity and the spread-out property stated in (1.1–1.5) to general QRW-processes. To that end, we introduce a *standard behavior event* $\Omega^{(\delta)}$ of probability $1-$SES and we prove these properties with respect to $P^{(\delta)}$, the conditional probability given $\Omega^{(\delta)}$ defined by

$$P^{(\delta)}(\cdot) := P(\,\cdot\,|\Omega^{(\delta)}). \tag{3.6}$$

We recall that a generic QRW-process ξ is assumed to be coupled to an RWP-process Z, but also to an IRW-process ζ.

Definition 3.3.1 (Standard behavior event) *For $\delta > 0$, let*

$$\Omega^{(\delta)} := \bigcap_{i=1}^{N}\left(\bigcap_{k=1}^{T_\alpha} F_i(kT_\alpha) \cap G_i(kT_\alpha)\right) \cap \left(\bigcap_{m=1}^{T_\alpha^2} J^1_{i,m} \cap J^2_{i,m}\right), \tag{3.7}$$

where $F_i(t_0)$ and $G_i(t_0)$ are defined in Definition 3.1.1 *and*

$$\begin{aligned} J^1_{i,m} &:= \left\{\forall t \in [0, m+1]\colon\ \|Z_i(t) - Z_i(0)\|_2 \le e^{\frac{\delta}{10}\beta}\sqrt{m}\right\} \\ J^2_{i,m} &:= \Bigg\{\forall t \in [0, m+1]\colon\ \|(Z_i - \zeta_i)(t) - (Z_i - \zeta_i)(0)\|_2 \\ &\qquad\qquad \le e^{\frac{\delta}{10}\beta}\sqrt{\sum_{\sigma_{i,k}\le m} \mathcal{T}\wedge\tau_{i,k}\wedge m - \mathcal{T}\wedge\sigma_{i,k}}\Bigg\}. \end{aligned} \tag{3.8}$$

In words, $\Omega^{(\delta)}$ is the event that excludes: (1) number of pauses larger than l for any particle in any time interval $[kT_\alpha, (k+1)T_\alpha]$ before time $\mathcal{T}$; (2) trajectories longer than l for any unfree particle before time $\mathcal{T}$; (3) superdiffusive behavior for the RWP-processes Z and $Z-\zeta$. (Since, for any i, $Z_i - \zeta_i$ takes its pauses when Z_i does not, the sum that appears in the definition of $J^2_{i,m}$ is the difference between m and the total length of the pause intervals of $Z_i - \zeta_i$ up to time m.)

Proposition 3.3.2 *For any $\delta > 0$, $P(\Omega^{(\delta)}) \ge 1 -$ SES uniformly in $\hat\eta(0)$.*

Proof. Note that $\Omega^{(\delta)}$ is the intersection of an exponential number of events that each occur with probability $1-$SES uniformly in i, k and m. As far as the events $F_i(kT_\alpha)$ and $G_i(kT_\alpha)$ are concerned, this is a consequence of Definition 3.1.1. Since Z and $Z-\zeta$ are RWP-processes, the events $J^1_{i,m}$ and $J^2_{i,m}$ occur with probability $1-$SES, uniformly in i and m, as a consequence of the obvious extension of (1.31) to RWP-processes. □

Theorems 3.3.3–3.3.5 below are our main results and will be proven in Section 4. First we generalize the non-superdiffusivity.

Theorem 3.3.3 (Non-superdiffusivity) *Let ξ be a* QRW$(\alpha,\mathcal{T})$*-process and $\delta > 0$. Then there exists a $\beta_0 > 0$ such that, for all $T = T(\beta) \in [2, T_\alpha^2]$ and all $i \in \{1,\ldots,N\}$,*

$$\forall\beta > \beta_0, \quad P^{(\delta)}\left(\mathcal{T} > T \text{ and } \exists t \in [0,T)\colon\ \|\xi_i(t) - \xi_i(0)\|_2 > e^{\delta\beta}\sqrt{T}\right) = 0. \tag{3.9}$$

Consequently,

$$P\left(\mathcal{T} > T \text{ and } \exists t \in [0,T)\colon \ \|\xi_i(t) - \xi_i(0)\|_2 > e^{\delta\beta}\sqrt{T}\right) \leq \mathrm{SES} \tag{3.10}$$

uniformly in $\hat{\eta}(0)$*,* $i \in \{1, \ldots, N\}$ *and* $T = T(\beta) \in [2, T_\alpha^2]$*.*

We can generalize the spread-out property for the particles that are active on the whole interval $[0, T]$ those for which $w_i(T) = 0$. Because we control the number of pause intervals and the behavior of the QRW$(\alpha, \mathcal{T})$-process in these pauses intervals on the reference scale T_α, we must distinguish two cases: (1) $T \leq T_\alpha$; (2) $T > T_\alpha$. In case (1), we will extend the spread-out property "at resolution $1 : e^{D\beta}$", i.e., instead of considering the probability to be at a site z at time T we consider the probability to be in a square box with volume of order $e^{D\beta}$ at time T, which is the volume typically visited by a free particle in a time equal to the upper bound on the length of the pause intervals for active particles. In case (2), we have a result at lower resolution. In both cases, the time $\mathcal{T} \wedge T_\alpha^2$ again is a threshold beyond which we do not have any result.

Theorem 3.3.4 (Spread out property, upper bound) *Let* ξ *be a* QRW$(\alpha, \mathcal{T})$*-process and* $\delta > 0$*. Then there exists a* $\beta_0 > 0$ *such that, for all* $T = T(\beta) \in [2, T_\alpha^2]$ *and all* $I \subset \{1, \ldots, N\}$*, if* $(\Lambda_i)_{i \in I}$ *is a family of square boxes contained in* Λ_β *such that*

$$\forall i \in I\colon \ |\Lambda_i| \geq \left\lceil \frac{T}{T_\alpha} \right\rceil \left(\left\lceil \frac{T}{T_\alpha} \right\rceil \vee e^{D\beta} \right), \tag{3.11}$$

then

$$\forall \beta > \beta_0\colon \qquad P^{(\delta)}\left(\mathcal{T} > T \text{ and } \forall i \in I\colon \ \xi_i(T) \in \Lambda_i \text{ and } w_i(T) = 0\right) \leq \prod_{i \in I} \left(\frac{|\Lambda_i| e^{\delta\beta}}{T} \right). \tag{3.12}$$

Remarks:

1. If $T \leq T_\alpha$, then condition (3.11) reads $(\forall i \in I\colon \ |\Lambda_i| \geq e^{D\beta})$ and we have a result "at resolution $1 : e^{D\beta}$", as explained before.

2. The condition $T \leq T_\alpha^2$ is necessary for the relevance of the result, not for its validity. When this condition is violated, (3.11) implies $|\Lambda_i| \geq T$ and the probability in (3.12) is estimated from above by a number larger than 1.

3. For an active particle at time T, with $w_i(T) > 0$, by time translation, we get estimate in $\frac{1}{T - w_i(T)}$.

Theorem 3.3.5 (Spread-out property, lower bound) *Let* ξ *be a* QRW$(\alpha, \mathcal{T})$*-process,* $\delta > 0$*, and* I *a finite subset of* $\mathbb{N}$*. Then there exists a* $\beta_0 > 0$ *such that the following holds for any* $T = T(\beta) \in [2, T_\alpha^2]$ *and any family* $(\Lambda_i)_{i \in I}$ *of square boxes contained in* Λ_β*:*

(i) If cndt>

$$\forall i \in I: \quad |\Lambda_i| \geq e^{\delta\beta} \left\lceil \frac{T}{T_\alpha} \right\rceil \left(\left\lceil \frac{T}{T_\alpha} \right\rceil \vee e^{D\beta} \right) \quad \textit{and} \quad \Lambda_i \subset B_2\left(\xi_i(0), \sqrt{T}\right), \tag{3.13}$$

then

$$\forall \beta > \beta_0: \quad P^{(\delta)}\left(\mathcal{T} \geq \mathcal{T} \ \textit{or} \ \forall i \in I\colon \xi_i(T) \in \Lambda_i \ \textit{or} \ w_i(T) > 0\right) \geq \prod_{i \in I} \left(\frac{\mathrm{cst}\, |\Lambda_i|}{T} \right). \tag{3.14}$$

(ii) If, in addition,

$$\epsilon := \sup_{i \in I} \frac{4|\Lambda_i|}{T} \leq 1 \quad \textit{and} \quad \forall i \in I\colon \xi_i(0) \notin [\Lambda_i]_{\sqrt{|\Lambda_i|}}, \tag{3.15}$$

then eqtau>

$$\forall \beta > \beta_0\colon \ P^{(\delta)} \left(\begin{array}{l} T + \epsilon T \geq \mathcal{T} \\ \textit{or} \ \forall i \in I\colon \ \tau_{\Lambda_i}(\xi_i) \in [T, T + \epsilon T] \ \textit{or} \ w_i(T + \epsilon T) > 0 \end{array} \right) \geq \prod_{i \in I} \left(\frac{|\Lambda_i|}{T e^{\delta\beta}} \right). \tag{3.16}$$

Remarks:

1. If $T \leq T_\alpha$, then condition (3.13) reads ($\forall i \in I\colon |\Lambda_i| \geq e^{(D+\delta)\beta}$ and $\Lambda_i \subset B_2(\xi_i(0), \sqrt{T})$), and we have, once again, a result "at resolution $1 : e^{D\beta}$".

2. In (3.16) the quantity ϵT plays the role of a temporal indetermination on $\tau_{\Lambda_i}(\xi_i)$. This temporal indetermination is of order $\sup_{i \in I} |\Lambda_i|$: the temporal and spatial resolutions are of the same order.

3. The condition $T \leq T_\alpha^2$ is necessary for the relevance of the result, not for its validity: when this condition is violated there are no boxes $(\Lambda_i)_{i \in I}$ that satisfy (3.13) for large β.

4. As before we have estimates in $\frac{1}{T - w_i(T)}$ for any active particle.

5. In Theorem 3.3.4, $|I|$ may grow with β, while β_0 is independent of I. In Theorem 3.3.5, $|I|$ is a finite number independent of β, while β_0 depends on I. If we would be able to prove (3.14) and (3.16) for any set of indices I such that $|I|$ is an increasing unbounded function of β, then we would have SES *lower* bounds for a conditional probability given an event of probability $1 - \text{SES}$: estimates with a limited relevance. This is not the case for the SES *upper* bounds given for such sets I in Theorem 3.3.4. We will make use of these bounds in Section 6.

3.4 Stronger lower bounds for Kawasaki dynamics

As far as Kawasaki dynamics is concerned, for further application to the study of metastability we need some lower bounds to get a spread-out property at higher resolution – typically at resolution of order $1 : 1$ or $1 : \lambda$. In Section 5 we will prove the following.

Theorem 3.4.1 *Let I be a finite subset of $\mathbb{N}$, $\hat{\eta}(0) \in \hat{\mathcal{X}}_N$ such that $\mathcal{T}_{\alpha,\lambda} > 0$, $T = e^{C\beta}$ for some $C > 0$ different from U and $2U$, and $(z_i)_{i\in I} \in (\Lambda_\beta)^{|I|}$ such that, for all i in I, $\|z_i - \hat{\eta}_i(0)\|_2 \leq \frac{1}{2}\sqrt{T}$.*

(i) If $T \leq T_\alpha$, all the particles with label $i \in I$ are free at time $t = 0$ and posz

$$\forall i \in I\colon \ \inf_{1\leq j\leq N} \|z_i - \hat{\eta}_j(0)\|_1 > 13\lambda \ \ \textit{and} \ \ \inf_{j\in I, j\neq i} \|z_i - z_j\|_1 > 11, \tag{3.17}$$

then, for any $\delta > 0$,

$$P\left(\forall i \in I\colon \ \left\lfloor \tau_{\{z_i\}}(\hat{\eta}_i)\right\rfloor = \lfloor T \rfloor\right) \geq \left(\frac{1}{Te^{\delta\beta}}\right)^{|I|} - \mathrm{SES} \tag{3.18}$$

uniformly in $\hat{\eta}(0)$, T and $(z_i)_{i\in I}$.

(ii) If $T \leq T_\alpha$, $T > e^{D\beta}$ and (3.17) is satisfied, then, for any $\delta > 0$,

$$P\left(\forall i \in I\colon \ \left\lfloor \tau_{\{z_i\}}(\hat{\eta}_i)\right\rfloor = \lfloor T \rfloor \ \textit{or} \ w_i(T) > 0\right) \geq \left(\frac{1}{Te^{\delta\beta}}\right)^{|I|} - \mathrm{SES} \tag{3.19}$$

uniformly in $\hat{\eta}(0)$, T and $(z_i)_{i\in I}$.

(iii) If $T_\alpha < T < T_\alpha^2(T_\alpha^{-1/2} \wedge e^{-D\beta})$ and posz2

$$\forall i \in I\colon \ \inf_{1\leq j\leq N} \|z_i - \hat{\eta}_j(0)\|_1 > 17\lambda \ \ \textit{and} \ \ \inf_{j\in I, j\neq i} \|z_i - z_j\|_1 > 9\lambda, \tag{3.20}$$

then, for any $\delta > 0$,

$$P\left(T > \mathcal{T}_{\alpha,\lambda} \ \textit{or} \ \forall i \in I\colon \ \left\lfloor \tau_{[z_i]_{4\lambda}}(\hat{\eta}_i)\right\rfloor = \lfloor T \rfloor \ \textit{or} \ w_i(T) > 0\right) \geq \left(\frac{1}{Te^{\delta\beta}}\right)^{|I|} - \mathrm{SES} \tag{3.21}$$

uniformly in $\hat{\eta}(0)$, T and $(z_i)_{i\in I}$.

Remark: The condition $C \neq U, 2U$ is not actually necessary. In order to remove it, some of the estimates in Section 5.3 (e.g. the last estimate of Lemma 5.3.2) would need to be derived at a higher order of precision. We will not insist on this point.

4 Consequences of QRW-property: Proofs

prfsqrw

4.1 Non-superdiffusivity

Proof of Theorem 3.3.3: Fix $\delta > 0$ and $i \in \{1, \ldots, N\}$. By (3.7), on $\Omega^{(\delta)}$, Z_i will not have more than $\lceil T/T_\alpha \rceil l$ pauses up to time $T \wedge \mathcal{T}$, and during each of these pauses the distance between Z_i and ξ_i will not increase by more than l. Consequently (recall that $T \leq T_\alpha^2$)

$$\sup_{t\leq T\wedge\mathcal{T}} \|\xi_i(t) - Z_i(t)\|_2 \leq \left\lceil \frac{T}{T_\alpha} \right\rceil l^2 \leq e^{\frac{\delta}{10}\beta}\sqrt{T} \qquad \text{on } \Omega^{(\delta)} \tag{4.1}$$

for all $\beta \geq \beta_1(l,\delta)$. In addition,

$$\sup_{t\leq T} \|Z_i(t) - Z_i(0)\|_2 \leq e^{\frac{\delta}{10}\beta}\sqrt{T} \qquad \text{on } \Omega^{(\delta)}. \tag{4.2}$$

Consequently (by the triangular inequality),

$$P^{(\delta)}\left(\exists t \in [0,T]\colon\ \|\xi_i(t) - \xi_i(0)\|_2 > e^{\delta\beta}\sqrt{T}\right) = 0 \tag{4.3}$$

for all $\beta \geq \beta_0(l,\delta)$. □

4.2 Spread-out property, upper bound

Proof of Theorem 3.3.4: On $\Omega^{(\delta)}$, for any $i \in I$, $\|\xi_i - Z_i\|_2$ can be estimated from above as in Section 4.1, to get

$$\sup_{t\leq T\wedge\mathcal{T}} \|\xi_i(t) - Z_i(t)\|_2 \leq \left\lceil \frac{T}{T_\alpha} \right\rceil l^2 \leq e^{\frac{\delta}{9}\beta}\sqrt{|\Lambda_i|} \qquad \text{on } \Omega^{(\delta)} \tag{4.4}$$

for all $\beta \geq \beta_1(l,\delta)$. In addition, if i never falls asleep in the whole interval $[0, T\wedge\mathcal{T}]$, then, by using that

$$\Omega^{(\delta)} \subset \bigcap_{i=1}^{N} \bigcap_{m=1}^{T_\alpha^2} J^2_{i,m}, \tag{4.5}$$

we also get

$$\sup_{t\leq T\wedge\mathcal{T}} \|Z_i(t) - \zeta_i(t)\|_2 \leq e^{\frac{\delta}{10}\beta}\sqrt{\left\lceil \frac{T}{T_\alpha} \right\rceil l e^{D\beta}} \leq e^{\frac{\delta}{9}\beta}\sqrt{|\Lambda_i|} \qquad \text{on } \Omega^{(\delta)} \tag{4.6}$$

for all $\beta \geq \beta_2(l,\delta) > \beta_1(l,\delta)$. Consequently (by the triangular inequality),

$$\xi_i(T) \in \Lambda_i \Rightarrow \zeta_i(T) \in [\Lambda_i]_{e^{\frac{\delta}{8}\beta}\sqrt{|\Lambda_i|}} \tag{4.7}$$

for all $\beta \geq \beta_3(l,\delta) > \beta_2(l,\delta)$. If we choose β_3 large enough so that also

$$\left|[\Lambda_i]_{e^{\frac{\delta}{8}\beta}\sqrt{|\Lambda_i|}}\right| \leq |\Lambda_i| e^{\frac{\delta}{2}\beta} \qquad \text{and} \qquad P(\Omega^{(\delta)}) \geq \frac{1}{2}, \tag{4.8}$$

then it follows, for all $\beta \geq \beta_3(l,\delta)$ and by the spread-out property for the IRW-proces, that

$$\begin{aligned}
&P^{(\delta)}\left(\mathcal{T} > T \text{ and } \forall i \in I\colon\ \xi_i(T) \in \Lambda_i \text{ and } w_i(T) = 0\right) \\
&\quad\leq 2P\left(\Omega^{(\delta)} \cap \{\mathcal{T} > T \text{ and } \forall i \in I\colon\ \xi_i(T) \in \Lambda_i \text{ and } w_i(T) = 0\}\right) \\
&\quad\leq 2P\left(\forall i \in I\colon\ \zeta_i(T) \in [\Lambda_i]_{e^{\frac{\delta}{8}\beta}\sqrt{|\Lambda_i|}}\right) \\
&\quad\leq \prod_{i\in I}\left(\text{cst}\,\frac{|\Lambda_i| e^{\frac{\delta}{2}\beta}}{T}\right),
\end{aligned} \tag{4.9}$$

so that we get (3.12) for some $\beta_0 \geq \beta_3$ large enough to make $e^{\delta\beta}$ an upper bound for the factors $\text{cst}\, e^{\frac{\delta}{2}\beta}$ of the latter product. □

4.3 Spread-out property, lower bound

Proof of Theorem 3.3.5: Let

$$q := \frac{1}{8}\inf_{i\in I}\sqrt{|\Lambda_i|}, \tag{4.10}$$

and observe that

$$\left\lceil\frac{T}{T_\alpha}\right\rceil l^2c + e^{\frac{\delta}{10}\beta}\sqrt{\left\lceil\frac{T}{T_\alpha}\right\rceil le^{D\beta}} \le q \tag{4.11}$$

for all $\beta \ge \beta_1(l,\delta)$, so that, as in Section 4.2, for the particles $i\in I$ that never fall asleep in the whole time $[0, T\wedge\mathcal{T}]$,

$$\sup_{t\le T\wedge\mathcal{T}}\|\xi_i(t)-\zeta_i(t)\|_2 \le q \qquad \text{on } \Omega^{(\delta)}. \tag{4.12}$$

(i) For $i\in I$, let Λ_i' be the largest square box in Λ_β such that

$$[\Lambda_i']_q \subset [\Lambda_i]\,. \tag{4.13}$$

On the one hand, we have, for all $\beta\ge\beta_1$,

$$P\left(\forall i\in I\colon\ \zeta_i(T)\in\Lambda_i'\right) \ \le\ P^{(\delta)}\left(\mathcal{T}\le T \text{ or } \forall i\in I\colon\ \xi_i(T)\in\Lambda_i \text{ or } w_i(T)>0\right) + (1-P(\Omega^{(\delta)})). \tag{4.14}$$

On the other hand, by the spread-out property for the IRW-process, we have

$$P\left(\forall i\in I\colon\ \zeta_i(T)\in\Lambda_i'\right) \ \ge\ \prod_{i\in I}\frac{\text{cst}\,|\Lambda_i'|}{T} \ \ge\ \prod_{i\in I}\frac{\text{cst}\,\left(|\Lambda_i|-4q\sqrt{|\Lambda_i|}\right)}{T} \ \ge\ \prod_{i\in I}\frac{\text{cst}\,|\Lambda_i|}{T}. \tag{4.15}$$

Since $|I|$ is finite, does not depend on β, and $T\le T_\alpha^2$, the latter product is not SES. Consequently,

$$1-P(\Omega^{(\delta)}) \le \frac{1}{2}P\left(\forall i\in I\colon\ \zeta_i(T)\in\Lambda_i'\right) \tag{4.16}$$

for all $\beta\ge\beta_2(l,\delta)>\beta_1(l,\delta)$ that depend on the law of ξ only. This proves (3.14) for all $\beta_0\ge\beta_2$.

(ii) Assume now that $\xi_i(0)\notin[\Lambda_i]_{\sqrt{|\Lambda_i|}}$ for all $i\in I$ and define, for any $i\in I$,

$$\Lambda_i'' := [\Lambda_i]_q\,. \tag{4.17}$$

On the one hand, by Brownian approximation and scaling property, we have

$$P\left(\forall i\in I\colon\ \tau_{\Lambda_i''}(\zeta_i)\in[T, T+|\Lambda_i|e^{-\frac{\delta}{20}\beta}]\right) \ge \prod_{i\in I}\frac{\text{cst}\,|\Lambda_i|}{Te^{\frac{\delta}{10}\beta}}. \tag{4.18}$$

On the other hand, for all $\beta\ge\beta_3(l,\delta)$ that depends on the law of ξ only, we can show as previously that

$$\begin{aligned}
&P\Big(\forall i\in I\colon\ \tau_{\Lambda_i''}(\zeta_i)\in[T, T+|\Lambda_i|e^{-\frac{\delta}{20}\beta}]\Big)\\
&\quad\le P^{(\delta)}\left(\mathcal{T}\le T \text{ or } \forall i\in I\colon\ w_i(T)>0 \text{ or } \left\{\begin{array}{l}\tau_{\Lambda_i}(\xi_i)>T\\ \Lambda_i\subset B_2\left(\xi_i(T), 2\sqrt{|\Lambda_i|}\right)\end{array}\right.\right)\\
&\qquad+\frac{1}{2}P\Big(\forall i\in I\colon\ \tau_{\Lambda_i''}(\zeta_i)\in[T, T+|\Lambda_i|e^{-\frac{\delta}{20}\beta}]\Big).
\end{aligned} \tag{4.19}$$

Since

$$|\Lambda_i| \geq e^{\delta\beta} \left\lceil \frac{T}{T_\alpha} \right\rceil \left(\left\lceil \frac{T}{T_\alpha} \right\rceil \vee e^{D\beta} \right) \qquad \forall i \in I, \tag{4.20}$$

we also have

$$|\Lambda_i| \geq e^{\delta\beta} \left\lceil \frac{\epsilon T}{T_\alpha} \right\rceil \left(\left\lceil \frac{\epsilon T}{T_\alpha} \right\rceil \vee e^{D\beta} \right) \qquad \forall i \in I, \tag{4.21}$$

provided that

$$\epsilon := \sup_{i \in I} \frac{4|\Lambda_i|}{T} \leq 1. \tag{4.22}$$

We may now conclude the proof by using (4.18–4.19), the Markov property at time T, and (3.14) with ϵT instead of T. □

5 Back to Kawasaki dynamics

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We prove in this section Theorems 3.2.1 and 3.4.1. Section 5.1 recalls an estimate in Gaudilliére [12] on the non-collision probability for a system of random walks with obstacles. In Section 5.2 this estimate is used to prove the QRW-property for the Kawasaki dynamics with labelled particles stated in Theorem 3.2.1. This in turn is used in Section 5.3 to prove the stronger lower bounds stated in Theorem 3.4.1.

5.1 Preliminaries

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Let $\mathcal{R}$ be the collection of all finite sets of rectangles on $\mathbb{Z}^2$. We begin by defining a family of transformation $(g_r)_{r\geq 0}$ on $\mathcal{R}$ grouping into single rectangles those rectangles that have a distance smaller than r between them. To do so, with $r \geq 0$ and

$$\underline{S} = \left\{ R_1, R_2, \ldots, R_{|\underline{S}|} \right\} \in \mathcal{R} \tag{5.1}$$

we associate a graph $G = (V, E)$ with vertex set

$$V := \{1, 2, \ldots, |\underline{S}|\} \tag{5.2}$$

and edge set

$$E := \left\{ \{i, j\} \subset V : \ i \neq j \text{ and } \inf_{s \in R_i} \inf_{s' \in R_j} \|s - s'\|_\infty \leq r \right\}. \tag{5.3}$$

Calling C the set of the connected components of G, we define

$$\bar{g}_r : \underline{S} \in \mathcal{R} \longmapsto \left\{ \mathrm{RC} \left(\bigcup_{i \in c} R_i \right) \right\}_{c \in C} \in \mathcal{R}, \tag{5.4}$$

where RC denotes the circumscribed rectangle, and $g_r(\underline{S}) \in \mathcal{R}$ is defined as the limit set of the iterates of $\underline{S}$ under $\bar{g}_r$ (which clearly exists because $|\underline{S}|$ is finite). Note that $g_r(\underline{S}) = \underline{S}$ means that $\|R - R'\|_\infty > r$ for all $R, R' \in \underline{S}$ that are distinct.

We associate with $\underline{S} \in \mathcal{R}$ its *perimeter*

$$prm(\underline{S}) := \sum_{R \in \underline{S}} |\partial R| \tag{5.5}$$

and we use the notation

$$S := \operatorname{supp} \underline{S} := \bigcup_{R \in \underline{S}} R \subset \mathbb{Z}^2. \tag{5.6}$$

For $\underline{S} \in \mathcal{R}$, $n \in \mathbb{N}$ and $\zeta = (\zeta_1, \ldots, \zeta_n)$ an IRW-process on $(\mathbb{Z}^2)^n$, we define the *first collision time*

$$\mathcal{T}_c := \inf \left\{ t \geq 0 : \; \exists R \in \underline{S}, \exists (i,j) \in \{1, \ldots, n\}^2, \inf_{s \in R} \|\zeta_i(t) - s\|_1 = 1 \text{ or } \|\zeta_i(t) - \zeta_j(t)\|_1 = 1 \right\}. \tag{5.7}$$

Proposition 5.1.1 (Gaudillière [12]) *There exists a constant $c_0 \in (0, \infty)$ such that, for all $n \geq 2$ and $p \geq 2$ the following holds. If $\underline{S} \in \mathcal{R}$ is such that* condts

$$\begin{cases} g_3(\underline{S}) = \underline{S}, \\ prm(\underline{S}) \leq p, \end{cases} \tag{5.8}$$

and $\zeta(0) \in (\mathbb{Z}^2)^n$ is such that condtz0

$$\begin{cases} \inf_{i \neq j} \|\zeta_i(0) - \zeta_j(0)\|_1 > 1, \\ \inf_i \inf_{s \in S} \|\zeta_i(0) - s\|_\infty > 3, \end{cases} \tag{5.9}$$

then, for the IRW*-process on $(\mathbb{Z}^2)^n$ that starts from $\zeta(0)$,*

$$\forall T \geq T_0, \quad P\left(\mathcal{T}_c > T\right) \geq \frac{1}{(\ln T)^\nu} \tag{5.10}$$

with dfnnu

$$\begin{cases} \nu := c_0 n^4 p^2 \ln p, \\ T_0 := \exp\{\nu^2\}. \end{cases} \tag{5.11}$$

We will need two other results derived in [12], namely, the estimate estpg

$$\forall \underline{S} \in \mathcal{R}, \, \forall r \geq 0 : \; prm(g_r(\underline{S})) \leq prm(\underline{S}) + 4r(|\underline{S}| - |g_r(\underline{S})|), \tag{5.12}$$

and the following corollary of Propositon 5.1.1:

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Proposition 5.1.2 *There is a constant $c_0' \in (0, \infty)$ such that, if $n \geq 2$, $S \in \mathcal{R}$ ζ an* IRW*-process on $(\mathbb{Z}^2)^n$ verifying (5.8) and (5.9) for some $p \geq 2$, $z \in (\mathbb{Z}^2)^n$ and $T > 0$ satisfy* condtzt

$$\begin{cases} \inf_{i \neq j} \|z_i - z_j\|_1 > 1, \\ \inf_i \inf_{s \in S} \|z_i - s\|_\infty > 3, \\ \sup_i \|z_i - \zeta_i(0)\|_2 \leq \sqrt{T}, \\ T \geq \exp\left\{c_0' \nu^2\right\}, \end{cases} \tag{5.13}$$

with ν defined in (5.11) then the following holds:

$$P\left(\mathcal{T}_c > T \text{ and } \forall i \in \{1, \ldots, n\}, \, \zeta_i(T) = z_i\right) \geq \frac{1}{(\ln T)^{c_0' \nu^3}} \left(\frac{1}{T}\right)^n. \tag{5.14}$$

5.2 QRW-property

Proof of Theorem 3.2.1: We give the proof by showing that the RWP-process Z constructed in Section 2.2 fits with Definition 3.1.1. If $\hat{\eta}(0)$ is such that $\mathcal{T}_{\alpha,\lambda} = 0$, then there is nothing to prove. We therefore assume $\mathcal{T}_{\alpha,\lambda} > 0$.

We associate with each particle i a ball centered at its initial position with radius

$$r := e^{\frac{\alpha}{4}\beta}\sqrt{T_\alpha}, \tag{5.15}$$

and we call B_0 their union:

$$B_0 := \bigcup_{i=1}^{N} B_2\left(\hat{\eta}_i(0), r\right). \tag{5.16}$$

Since r is much larger than the diffusive distance associated with time T_α, this suggests a partition of $\{1, \dots, N\}$ into *clouds of potentially interacting particles on time scale* T_α. We say that two particles are in the same cloud when they belong to the same connected component of B_0. We call τ_e the first time when one of the particles leaves B_0 (B_0 is fixed and does not change with time) and observe that before τ_e each cloud evolves independently of the others. With these definitions we can divide the proof into 4 steps:

Step 1. We estimate from above the number of particles in each cloud using $\mathcal{T}_{\alpha,\lambda} > 0$.

Step 2. Given $i \in \{1, \dots, N\}$, for any $s \leq T_\alpha$ and conditionally on $\{s < \tau_e\}$, we estimate from below the probability that in the cloud to which i belongs and in the time interval $[s, T_\alpha \wedge \tau_e]$ no particle looses its freedom. After step 1, this can be done by using the estimates on the collision probability (Proposition 5.1.1).

Step 3. We deduce from the previous estimates that, with

$$\mathcal{T} := T_\alpha \wedge \tau_e, \tag{5.17}$$

$\hat{\eta}$ is a QRW$(\alpha, \mathcal{T})$-process associated with the function l defined in (3.4).

Step 4. We use the non-superdiffusivity of the QRW-process (Theorem 3.3.3) to get first that $\hat{\eta}$ is a QRW(α, T_α)-process and second that it is a QRW$(\alpha, \mathcal{T}_{\alpha,\lambda})$-process associated with the same function l.

Step 1. We divide Λ_β into $|\Lambda_\beta|/V$ square cells of volume

$$V := e^{(\Delta - \frac{\alpha}{4})\beta}. \tag{5.18}$$

It follows from $\mathcal{T}_{\alpha,\lambda} > 0$ that no cell contains more than $\lambda/4$ particles at time $t = 0$. Since

$$\frac{\sqrt{V}}{r} = e^{\frac{\alpha}{8}\beta}, \tag{5.19}$$

no connected component of B_0 can move from one side to the opposite side in any domino made of two contiguous cells (for β large enough). Consequently, each of these connected components is contained in a union of four cells, and each cloud contains at most λ particles.

Step 2. Given $i \in \{1, \ldots, N\}$ and $s \leq T_\alpha$, we call $\mathcal{C}_0$ the family of the clusters of $\hat{\eta}(s)$ that contain (at time s) some particle of the cloud (defined at time $t = 0$) to which i belongs. We define

$$\begin{aligned} \underline{S}_0 &:= \{\mathrm{RC}(c)\}_{c \in \mathcal{C}_0} \quad \in \mathcal{R} \\ \underline{S} &:= g_5(\underline{S}_0) \quad \in \mathcal{R} \\ [\underline{S}]_1 &:= \{[R]_1\}_{R \in \underline{S}} \quad \in \mathcal{R} \end{aligned} \tag{5.20}$$

We note that $g_3([\underline{S}]_1) = [\underline{S}]_1$, and that at time s the gas surrounding $[S]_1$ is made of free particles only.

Definition 5.2.1 (Enrichment and collision times) *Given $\underline{S} \in \mathcal{R}$, we say that the gas surrounding $[S]_1$ is* enriched *each time a particle arrives into ∂S from S, and we say that a* collision *occurs each time two particles collide outside $[S]_1$ or one particle arrives in $\partial [S]_1$ from $\Lambda_\beta \setminus ([S]_1 \cup \partial [S]_1)$.*

For the system restricted to the cloud to which i belongs (recall that before τ_e each cloud evolves independently of the other ones), we call $A(s)$ the sequence of the following events:

A_1**:** Before η^{cl} changes, outside $[[S]_1]_3$ all the free particles move without collision. Note that, after A_1 is completed, S contains all the unfree particles and $\partial [S]_1$, and $[S]_1 \setminus S$, does not contain particles.

A_2**:** The gas surrounding $[S]_1$ evolves without collision up to the first of the following three stopping times: the next enrichment time, τ_e and T_α.

A_3**:** After each enrichment, the particle responsible for the enrichment moves outside $[[S]_1]_3$ without collision and before $\eta|_S$ changes. Subsequently, the gas surrounding $[S]_1$ evolves without collision up to the next enrichment, and so on up to time $T_\alpha \wedge \tau_e$.

Note that $A(s)$ implies that there is no loss of freedom of particles in the cloud to which i belongs in the time interval $[s, T_\alpha \wedge \tau_e]$.

To estimate $P\big(A(s)\big|s < \tau_e\big)$ from below, we need some estimates on $|\partial[S]_1|$. Since there are no more than λ particles in the cloud, we have

$$prm\,(\underline{S}_0)) \leq 4\lambda \quad \text{and} \quad |\underline{S}_0| \leq \lambda, \tag{5.21}$$

so that, via (5.12),

$$prm(\underline{S}) \leq 24\lambda \quad \text{and} \quad |\underline{S}| \leq \lambda. \tag{5.22}$$

Moreover,

$$|\,\partial[S]_1\,| \;\leq\; prm\,(\underline{S}) + 8\,|\underline{S}| \;\leq\; 32\lambda. \tag{5.23}$$

It is then easy to get

$$P\left(A_1\middle|s < \tau_e\right) \geq \left(\frac{1}{4\lambda}\right)^{\mathrm{cst}\,\lambda}. \tag{5.24}$$

In addition, if $A(s)$ occurs, then no particle that exits S can come back. Consequently, there cannot be more than λ enrichments and we find, using the strong Markov property and Proposition 5.1.1, that typest1

$$P\left(A(s)\Big|s<\tau_e\right) \geq \left[\left(\frac{1}{4\lambda}\right)^{\mathrm{cst}\,\lambda}\left(\frac{1}{\ln T_\alpha}\right)^{\mathrm{cst}\,\lambda^6\ln\lambda}\right]^\lambda \geq \left(\frac{1}{\ln T_\alpha}\right)^{\mathrm{cst}\,\lambda^7\ln\lambda} \tag{5.25}$$

as soon as

$$T_\alpha = e^{(\Delta-\alpha)\beta} \geq \exp\left\{\mathrm{cst}\,(\lambda^6\ln\lambda)^2\right\}, \tag{5.26}$$

i.e., β larger than some β_0 that depends on Δ, α and λ only.

Step 3. We denote by $(\tau_m)_{m\geq 1}$ the increasing sequence of stopping times when a particle looses its freedom in the cloud i it belongs to. By the strong Markov property and the previous estimate, we have, for $\beta \geq \beta_0$ and any a, ubwa

$$P\Big(\left|\{m\geq 1:\ \tau_m \leq \mathcal{T}\}\right| \geq a\Big) \quad\leq\quad \left[1-\left(\frac{1}{\ln T_\alpha}\right)^{\mathrm{cst}\,\lambda^7\ln\lambda}\right]^a. \tag{5.27}$$

We also have (recall the definition of the $\sigma_{i,k}$, $\tau_{i,k}$ in Section 2.2)

$$\left|\{k\in\mathbb{N}:\ \sigma_{i,k}\in[0,\mathcal{T}]\}\right| \quad\leq\quad 1+\left|\{m\geq 1:\ \tau_m\leq\mathcal{T}\}\right|, \tag{5.28}$$

and it is easy to see that, under our local permutation hypothesis (see Section 2.1), for all $k\in\mathbb{N}$ and $t\geq 0$,

$$\left|\hat\eta_i(t\wedge\tau_{i,k}\wedge\mathcal{T})-\hat\eta_i(t\wedge\sigma_{i,k}\wedge\mathcal{T})\right| \quad\leq\quad 24\lambda\left(1+\left|\{m\geq 1:\ \tau_m\leq\mathcal{T}\}\right|\right). \tag{5.29}$$

(Define S_m at time τ_m like S at time s, observe that any unfree particle is contained in S_m up to time $\tau_{m+1}\wedge\mathcal{T}$, and recall that $|\partial S_m|\leq 24\lambda$if $\tau_m\leq\tau_e$.) Finally, we choose

$$a := \frac{(\ln T_\alpha)^{\lambda^8}-1}{24\lambda} \tag{5.30}$$

in (5.27) to get that $\hat\eta$ is a QRW$(\alpha,\mathcal{T})$-process for which the function l of Definition 3.1.1 can be taken as in (3.4). Note that if (1.18) holds, then (3.3) follows.

Step 4. By Theorem 3.3.3, the particles are non-superdiffusive on time scale T_α and up to time $\mathcal{T}$. This gives

$$P\left(T_\alpha = T_\alpha\wedge\tau_e\right) = 1-\mathrm{SES} \tag{5.31}$$

and implies that $\hat\eta$ is a QRW(α,T_α)-process associated with the same function l.

To prove that $\hat\eta$ is a QRW$(\alpha,\mathcal{T}_{\alpha,\lambda})$-process associated with the RWP-process Z, it suffices to prove that, for any $i\in\{1,\ldots,N\}$ and $t_0\geq 0$, and *conditionally on* $\{\mathcal{T}_{\alpha,\lambda}>t_0\}$, the inequalities that appear in Definition 3.1.1 hold with probability $1-$SES uniformly in i and t_0. Since, on the one hand, η and Z are Markov processes and, on the other hand, $Z-Z(t_0)+\hat\eta(t_0)$ and Z have the same pause intervals and evolve jointly, this is a direct consequence of the fact that $\hat\eta$ is a QRW(α,T_α)-process. □

Remark: As a byproduct of this proof we get the following.

Proposition 5.2.2 *If λ satisfies* (1.18)*, $\alpha<\Delta$, and $\hat\eta(0)$ is such that $\mathcal{T}_{\alpha,\lambda}>0$, then $\hat\eta$ is a* QRW(α,T_α)*-process.*

5.3 Stronger lower bounds for the spread out property

In this section we prove Theorem 3.4.1 and we use as a key estimate Proposition 5.1.2. But, like in Section 5.2, it cannot be applied directly because of the gas enrichment phenomena. There, we dealt with this difficulty by observing that, without collisions for the gas surrounding some $[\underline{S}]_1 \in \mathcal{R}$, there were at most λ effective enrichments in each cloud of potentially interacting particles. Here, we need more information on the enrichment phenomena. To get this information, we extend to our situation a few easy results on the local Kawasaki model in den Hollander, Olivieri and Scoppola [8] that come from the standard cycles and cycle-paths theory introduced by Freidlin and Wentzel [4] (see also Olivieri and Vares [9]). We need to extend the standard theory because the latter only applies to a finite state space, while we have to deal with a state space of cardinality of order $\lambda^{2\kappa\lambda}$, with λ our growing unbounded function of the large parameter β and κ some positive number. However, since λ is only *slowly* growing, the situation we face is not qualitatively different from the standard one. In addition, we do not generalize the full theory to our different context: we only give the definitions and prove the lemmas that we need to complete the proof of Theorem 3.4.1. The further study of metastability will require a much more complete analysis of cycles and cycle paths; this will be the object of [13].

For $\underline{S} \in \mathcal{R}$ such that $g_5(\underline{S}) = \underline{S}$, we define the associated *local Hamiltonian* $\mathrm{H}_{\underline{S}}$ by (recall (1.8))

$$\mathrm{H}_{\underline{S}}(\eta) = \sum_{R \in \underline{S}} \mathrm{H}\Big(\eta|_R\Big) + \Delta\Big|\eta|_{R \cup \partial R}\Big| \qquad \text{for all } \eta \in \mathcal{X} = \{0,1\}^{\Lambda_\beta}. \tag{5.32}$$

rdn

Definition 5.3.1 *Given $\underline{S} \in \mathcal{R}$ with $g_5(\underline{S}) = \underline{S}$ and $k \in \{0,1,2\}$ such that $kU < \Delta$, we say that a configuration $\eta \in \mathcal{X}$ is kU-reducible if there is a sequence of configurations $\eta = \eta_0$, $\eta_1, \dots, \eta_n$ in $\mathcal{X}$, each of them obtained from the previous one by a displacement of a single particle to a nearest-neighbor vacant site, such that*

$$\begin{cases} \mathrm{H}_{\underline{S}}(\eta_n) < \mathrm{H}_{\underline{S}}(\eta), \\ \sup_j \mathrm{H}_{\underline{S}}(\eta_j) \le \mathrm{H}_{\underline{S}}(\eta) + kU. \end{cases} \tag{5.33}$$

We say that a labelled configuration $\hat{\eta}$ is kU-reducible when $\mathcal{U}(\hat{\eta})$ is (recall (1.16)).

Remark: If $2U < \Delta$, then the only $2U$-irreducible configurations are the configurations without particles inside S. Indeed, any cluster carries at least four particles that can only be separated at cost $2U$.

Lemma 5.3.2 *Let $\lambda = \lambda(\beta)$ satisfy (1.18), $\kappa > 0$, $\underline{S} \in \mathcal{R}$ such that $g_5(\underline{S}) = \underline{S}$ and $prm(\underline{S}) \le \lambda^\kappa$, and let the initial labelled configuration $\hat{\eta}(0)$ be such that at time $t = 0$ there are no particles inside $\partial\,[S]_1$, no particles inside $[S]_1 \setminus S$, and no more than λ particles inside S. Let τ_c be the first collision time for the gas surrounding $[S]_1$ and τ_+ its first enrichment time.*

(1) If $\hat{\eta}(0)$ is kU-reducible, then, for any $\delta > 0$,

$$P\left(\exists t \le e^{(kU+\delta)\beta},\ \hat{\eta}(t) \text{ is } kU\text{-irreducible or } \tau_+ = t \,\Big|\, \tau_c > e^{(kU+\delta)\beta} \wedge \tau_+\right) \ge 1 - \mathrm{SES} \tag{5.34}$$

uniformly in $\underline{S}$ and $\hat{\eta}(0)$.

(2) If $\hat\eta(0)$ is kU-irreducible, then, for any $\delta > \delta' > 0$,

$$P\left(\tau_+ \leq e^{((k+1)U-\delta)\beta} \;\middle|\; \tau_c > e^{((k+1)U-\delta)\beta} \wedge \tau_+\right) \leq e^{-\delta'\beta} + \mathrm{SES} \tag{5.35}$$

uniformly in $\underline{S}$ and $\hat\eta(0)$.

Proof: See Appendix A. □

We are now ready to prove (i), (ii) and (iii) of Theorem 3.4.1.

Proof of Theorem 3.4.1:

Proof of (i):

Let $\hat\eta(0)$, I, T and $(z_i)_{i\in I}$ satisfy the required hypotheses. We define the clouds of potentially interacting particles on time scale T_α as in Section 5.2. By Theorem 3.3.3 and Proposition 5.2.2, with probability $1 - \mathrm{SES}$, uniformly in $\hat\eta(0)$ and I, each cloud evolves independently of the others up to time T_α. Consequently, it suffices to prove the result when all the particles $i \in I$ belong to the same cloud. We have seen in Section 5.2 that each cloud contains at most λ particles and we can now restrict ourselves to considering a single cloud of $n \leq \lambda$ particles.

We call $\underline{S}_0$ the set of the circumscribed rectangles of the clusters of the initial configuration, and we define

$$\begin{aligned} \underline{S}' &:= g_5(\underline{S}_0),\\ \underline{S} &:= \underline{S}' \cup \Big\{\{z_i\} \,:\, i \in I\Big\}. \end{aligned} \tag{5.36}$$

As seen in the previous subsection,

$$prm(\underline{S}') \leq 24\lambda, \tag{5.37}$$

and it follows from (3.17) that $g_5(\underline{S}) = \underline{S}$. In addition, for β large enough, we have

$$|I| \leq \lambda \quad \text{and} \quad |\underline{S}'| \leq \lambda, \tag{5.38}$$

and so

$$\begin{aligned} prm(\underline{S}) &\leq 24\lambda + 4\lambda = 28\lambda,\\ |\partial\, [S]_1\, | &\leq 28\lambda + 8(\lambda+\lambda) = 44\lambda. \end{aligned} \tag{5.39}$$

For the largest $k \in \{0, 1, 2\}$ that satisfies $kU < C$, we call τ_r the first time when $\hat\eta$ is not kU-reducible with respect to $\underline{S}$, and we consider, for the system restricted to the cloud to which each $i \in I$ belongs, the following sequence of events:

A_1**:** Before η^{cl} changes, all the free particles move without collision outside $[[S]_1]_3$. Note that, after A_1 is completed, S contains all the unfree particles (not more than λ) and $\partial\, [S]_1$, like $[S]_1 \setminus S$, do not contain particles.

A_2**:** The gas surrounding $[S]_1$ evolves without collision up to the first of the following three stopping times: the next enrichment time, τ_r and T_α.

A_3: After each enrichment, the particle responsible for the enrichment moves outside $[[S]_1]_3$ without collision and before $\eta|_S$ changes. Subsequently, the gas surrounding $[S]_1$ evolves without collision up to the next enrichment, and so on up to time $T_\alpha \wedge \tau_r$.

As Section 5.2, the probability of this sequence of events can be estimated from below by a non-exponentially small quantity p_1: typest2

$$p_1 \geq \left(\frac{1}{\ln T_\alpha}\right)^{\mathrm{cst}\,\lambda^7 \ln\lambda}. \tag{5.40}$$

By Lemma 5.3.2, we make only an SES-error by assuming that the time between each of the enrichments in this sequence of events and $\tau_r \wedge T_\alpha$, or the successive enrichment, is smaller than $e^{(kU+\delta_0/2)\beta}$ with $\delta_0 > 0$ such that

$$kU + \delta_0 < C. \tag{5.41}$$

Since in such a sequence of events there cannot be more than λ enrichments, we get, in particular, for $\delta > 0$ and β large enough,

$$P\left(\tau_r < e^{(kU+\delta_0)\beta} \text{ and } \tau_r < \tau_c\right) \geq e^{-\frac{\delta}{3}\beta} \tag{5.42}$$

with τ_c the first collision time in the gas surrounding $[S]_1$ for the system restricted to the cloud we consider.

We next choose $|I|$ distinct and non-nearest-neighbor sites $(z'_i)_{i\in I}$ such that

$$\forall i \in I,\ \|z'_i - [\{z_i\}]_1\|_\infty = \|z'_i - [S]_1\|_\infty = 4. \tag{5.43}$$

Condition (3.17) ensures that we can find such a family $(z'_i)_{i\in I}$. We claim esttm4

$$P\left(\tau_c \geq T-4 \text{ and } \forall i\in I,\, \hat\eta_i(T-4) = z'_i\right) \geq \frac{e^{-\frac{\delta}{2}\beta}}{T^{|I|}}. \tag{5.44}$$

To prove this estimate, we distinguish two cases: $k = 2$ and $k < 2$.

$k = 2$: At time τ_r, S does not contain particles. The estimate is then a consequence of the Markov property applied at time τ_r, the estimate (5.42), and Proposition 5.1.2.

$k < 2$: There exists some $\delta_1 > 0$ such that $(k+1)U - \delta_1 = C$. The probability that, for all $i \in I$, $\hat\eta_i(T-4) = z'_i$ without collision or enrichment for the gas surrounding $[S]_1$ between the times τ_r and $T-4$, can be estimated from below by Proposition 5.1.2 and Lemma 5.3.2. Once again (5.44) follows from the Markov property applied at time τ_r.

Finally, using the Markov property at time $T-4$ and "driving by hand" the particles after $T-4$, we obtain

$$P\left(\forall i\in I,\, \lfloor \tau_{\{z_i\}}(\hat\eta_i)\rfloor = \lfloor T\rfloor\right) \geq \frac{e^{-\delta\beta}}{T^{|I|}} \tag{5.45}$$

for the restricted system. □

Proof of (ii):

We can follow the proof of (i) up to (5.44), which we change into sesttm4

$$P\left(\tau_c \geq T-4 \text{ and } \forall i \in I,\, \hat{\eta}_i(T-4) = z'_i \text{ or } w_i(T-4) > 0\right) \geq \frac{e^{-\frac{2\delta}{3}\beta}}{T^{|I|}}. \tag{5.46}$$

$k = 2$: We still have (5.44), which implies (5.46).

$k < 2$: The previous arguments no longer give (5.44), because there can be some particles $i \in I$ in S at time τ_r. The arguments now give

$$P\left(\tau_c \geq T-4 \;\text{ and }\; \forall i \in I,\, \hat{\eta}_i(T-4) = z'_i \;\text{ or }\; \hat{\eta}_i(t) \in S, \forall t \in [0, T-4]\right) \geq \frac{e^{-\frac{\delta}{2}\beta}}{T^{|I|}}. \tag{5.47}$$

But a particle i that remains confined to S up to time $T - 4 > e^{(D+\delta_2)\beta}$ for $\delta_2 > 0$ and β large enough, fell asleep before $T-4$ with probability $1 - \text{SES}$. This can be seen as an application of Theorem 3.3.4: assume that $w_i(T-4) = 0$, choose a square box Λ_i of volume $e^{D\beta}$ that contains the connected component of S to which i remains confined, divide the time interval $[0, e^{(D+\delta_2)\beta}]$ into $e^{\delta_2\beta/2}$ intervals of length $e^{(D+\delta_2/2)\beta}$, and apply $e^{\delta_2\beta/2}$ times the proposition with $\delta = \delta_2/3$. Consequently, we get (5.46) for β large enough, and we again conclude with the Markov property applied at time $T-4$. □

Proof of (iii):

Let $\hat{\eta}(0)$, I, T and $(z_i)_{i\in I}$ satisfy the required hypotheses. We will work on two time scales: T_α, which allows for "high resolution estimates" (because on this time scale the cloud of potentially interacting particles contains a small number of particles), and T, for which we can use the lower resolution estimates. We will use different tools to deal with different time scales. The proof will be divided into five steps: the first two steps are relevant only for the starting configurations in which the initial positions $\hat{\eta}_i(0)$ of some particles $i \in I$ are "close" to their associated targets $[z_i]_{4\lambda}$.

Step 1. We begin by estimating from below the probability that none of the particles $i \in I$ enters $[z_i]_{4\lambda}$ before time T_α. To do so, we consider the clouds of potentially interacting particles on time scale T_α, we call, for $i \in I$, $\underline{S}''_{0,i}$ the set of the circumscribed rectangles of the clusters of $\hat{\eta}(0)$ made of particles contained in the cloud to which i belongs, and we define

$$\begin{aligned} \underline{S}'_{0,i} &:= g_5(\underline{S}''_{0,i}), \\ \underline{S}_{0,i} &:= \underline{S}'_{0,i} \cup \Big\{ [z_j]_{4\lambda} :\; j \in I \Big\}. \end{aligned} \tag{5.48}$$

Observe that, like previously, $prm(\underline{S}'_{0,i}) \leq 24\lambda$, and note that, by (3.20), $g_5(\underline{S}_{0,i}) = \underline{S}_{0,i}$. In addition, $|\underline{S}'_{0,i}| \leq \lambda$ and $|I| \leq \lambda$, so that

$$prm(\underline{S}_{0,i}) \leq 24\lambda + \lambda \times 33\lambda \leq 34\lambda^2 \tag{5.49}$$

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$$|\partial\,[S_{0,i}]_1\,| \le 34\lambda^2 + 8(\lambda+\lambda) \le 35\lambda^2 \tag{5.50}$$

for β large enough.

Let $\tau_{c,0,i}$ be the collision time associated with $[S_{0,i}]_1$ for the system restricted to the cloud that contains i, using the fact that, with probability $1-\text{SES}$, the various clouds do not interact with each other up to time T_α, following the arguments that led to (5.25) or (5.40), and taking into account (5.50), we conclude that for any $\delta > 0$,

$$\begin{aligned} P\left(\forall i \in I : \tau_{[z_i]_{4\lambda}}(\hat{\eta}_i) > T_\alpha\right) &\ge \prod_i P\left(\tau_{c,0,i} > T_\alpha\right) - \text{SES} \\ &\ge \prod_i \left(\frac{1}{\ln T_\alpha}\right)^{\text{cst}\,\lambda^9 \ln\lambda} - \text{SES} \\ &\ge (e^{-\delta\beta})^{|I|} - \text{SES} \end{aligned} \tag{5.51}$$

uniformly in $\hat{\eta}(0)$, T and $(z_i)_{i\in I}$.

Step 2. We deduce from this last estimate a lower bound for the probability that the first time τ_1 when all the particles $i \in I$ that never fell asleep are outside $[z_i]_{3e^{-\delta\beta}\sqrt{T_\alpha}}$ is such that

$$\tau_1 \le T_\alpha \wedge \inf\left\{\tau_{[z_i]_{4\lambda}}(\hat{\eta}_i) :\; i\in I, w_i(T_\alpha) > 0\right\}. \tag{5.52}$$

To do so, we assume without loss of generality for our final result that $e^{-2\delta\beta}T_\alpha$ is larger than $e^{D\beta}$ and we divide the time interval $[0, T_\alpha]$ into $e^{\delta\beta/2}$ subintervals of length $e^{-\delta\beta/2}T_\alpha$. By Theorem 3.3.4 applied at the end of each of these subintervals, a particle i that does not fall asleep in the interval $[0, T_\alpha]$ is in $[z_i]_{3e^{-\delta\beta}\sqrt{T_\alpha}}$ with a probability smaller than $e^{-\delta\beta}$, so that, by the Markov property applied at the end of each of the subintervals,

$$\begin{aligned} &P\left(\forall i \in I, w_i(T_\alpha) > 0 \text{ or } \tau_1 < \tau_{[z_i]_{4\lambda}}(\hat{\eta}_i)\wedge T_\alpha\right) \\ &\quad\ge\quad P\left(\forall i\in I, \tau_{[z_i]_{4\lambda}}(\hat{\eta}_i) > T_\alpha\right) - \left(|I|e^{-\delta\beta}\right)^{e^{\delta\beta/2}} \\ &\quad\ge\quad (e^{-\delta\beta})^{|I|} - \text{SES}. \end{aligned} \tag{5.53}$$

Using the non-superdiffusivity property (Theorems 3.2.1 and 3.3.3) and the fact that $\|z_i - \hat{\eta}_i(0)\|_2 \le \frac{1}{2}\sqrt{T}$, we have also the stronger result

$$\begin{aligned} &P\left(\forall i \in I : \begin{array}{c} w_i(T_\alpha) > 0 \text{ or } \tau_1 < \tau_{[z_i]_{4\lambda}}(\hat{\eta}_i)\wedge T_\alpha \\ \text{and}\quad [z_i]_{e^{-\delta\beta}\sqrt{T_\alpha}} \subset B_2\left(\hat{\eta}_i(\tau_1), \frac{3}{4}\sqrt{T}\right) \end{array}\right) \\ &\ge\quad (e^{-\delta\beta})^{|I|} - \text{SES} \end{aligned} \tag{5.54}$$

uniformly in $\hat{\eta}(0)$, T and $(z_i)_{i\in I}$.

Step 3. We next give a lower bound for the probability that all the particles $i \in I$ that never fell asleep are in $[z_i]_{e^{-\delta\beta/2}\sqrt{T_\alpha}}$ at some time T_2 smaller than $\tau_{[z_i]_{4\lambda}}$ and are contained in $[T - 2e^{-\delta\beta/2}T_\alpha, T - e^{-\delta\beta/2}T_\alpha]$, provided that $\mathcal{T}_{\alpha,\lambda} > T$. To do so, we will use the Markov property at time τ_1 and Theorem 3.3.5 with

$$\begin{aligned} \delta' &:= \delta, \\ T' &:= T - e^{-\delta\beta/2}T_\alpha - 16e^{-2\delta\beta}T_\alpha - \tau_1, \\ (\Lambda_i') &:= \left([z_i]_{e^{-\delta\beta}\sqrt{T_\alpha}}\right), \end{aligned} \tag{5.55}$$

instead of δ, T and (Λ_i). Conditionally on

$$A := \left\{ \forall i \in I : \ [z_i]_{e^{-\delta\beta}\sqrt{T_\alpha}} \subset B_2\left(\hat{\eta}_i(\tau_1), \frac{3}{4}\sqrt{T}\right) \right\} \tag{5.56}$$

and for δ small enough, the hypotheses (3.13) and (3.15) are easily verified at time τ_1 in place of 0 and we get, with θ_{τ_1} the usual shift on the trajectories of the Markov process,

$$\begin{aligned} &P\left(T - e^{-\delta\beta/2} T_\alpha > \mathcal{T}_{\alpha,\lambda} \text{ or } \forall i \in I : \ \begin{array}{l} w_i(T - e^{-\delta\beta/2} T_\alpha) > 0 \quad \text{or} \\ \tau_1 + \tau_{\Lambda'_i}(\hat{\eta}_i) \circ \theta_{\tau_1} \in \left[T' + \tau_1, T - e^{-\delta\beta/2} T_\alpha \right] \end{array} \middle| A \right) \\ &\geq \quad \left(\frac{4T_\alpha}{e^{3\delta\beta} T} \right)^{|I|} - \text{SES} \end{aligned} \tag{5.57}$$

uniformly in $\hat{\eta}(0)$, T and $(z_i)_{i \in I}$. We then define τ_2 as the first time after time τ_1 when one of the particles $i \in I$ that never fell asleep reaches $[z_i]_{e^{-\delta\beta}\sqrt{T_\alpha}}$, and we use the non-superdiffusivity property to get

$$\begin{aligned} &P\left(\tau_2 > \mathcal{T}_{\alpha,\lambda} \text{ or } \forall i \in I : \ \begin{array}{l} w_i(\tau_2) > 0 \quad \text{or} \\ \left\{ \begin{array}{l} \tau_2 \in \left[T - 2e^{-\delta\beta/2} T_\alpha, T - e^{-\delta\beta/2} T_\alpha \right] \\ \tau_2 < \tau_1 + \tau_{[z_i]_{4\lambda}}(\hat{\eta}_i) \circ \theta_{\tau_1} \\ \hat{\eta}_i(\tau_2) \in [z_i]_{e^{-\delta\beta/2}\sqrt{T_\alpha}} \end{array} \right. \end{array} \middle| A \right) \\ &\geq \quad \left(\frac{4T_\alpha}{e^{3\delta\beta} T} \right)^{|I|} - \text{SES}. \end{aligned} \tag{5.58}$$

Together with (5.54) and the Markov property at time τ_1, this gives

$$\begin{aligned} &P\left(\tau_2 > \mathcal{T}_{\alpha,\lambda} \text{ or } \forall i \in I : \ \begin{array}{l} w_i(\tau_2) > 0 \quad \text{or} \\ \left\{ \begin{array}{l} \tau_2 \in \left[T - 2e^{-\delta\beta/2} T_\alpha, T - e^{-\delta\beta/2} T_\alpha \right] \\ \tau_2 < \tau_{[z_i]_{4\lambda}}(\hat{\eta}_i) \\ \hat{\eta}_i(\tau_2) \in [z_i]_{e^{-\delta\beta/2}\sqrt{T_\alpha}} \end{array} \right. \end{array} \right) \\ &\geq \quad \left(\frac{4T_\alpha}{e^{4\delta\beta} T} \right)^{|I|} - \text{SES} \end{aligned} \tag{5.59}$$

uniformly in $\hat{\eta}(0)$, T and $(z_i)_{i \in I}$. This means that, with a probability of order $e^{-4\delta\beta} T_\alpha / T$, at time τ_2 the particles i that never fell asleep are at a diffusive distance (on a time scale of order T_α) of their targets $[z_i]_{4\lambda}$, never reached these targets before and have still ahead a time of order T_α until time T.

Step 4. We will be working once again on time scale T_α. We define at time τ_2 the clouds of potentially interacting particles on time scale T_α, we call, for $i \in I$, $S''_{2,i}$ the set of the circumscribed rectangles of the clusters of $\hat{\eta}(\tau_2)$ made of particles contained in the cloud to which i belongs, and we set:

$$\begin{aligned} \underline{S}'_{2,i} &:= \underline{S}''_{2,i} \cup \left\{ [z_j]_{4\lambda} : \ j \in I \right\}, \\ \underline{S}_{2,i} &:= g_5(\underline{S}'_{0,i}). \end{aligned} \tag{5.60}$$

Here, the union with the targets is made *before* applying the operator g_5, which is different from what was done previously, for example, in Step 1. Provided $\mathcal{T}_{\alpha,\lambda} > \tau_2$, we have

$$|\underline{S}_{2,i}| \leq 2\lambda \quad \text{and} \quad prm(\underline{S}'_{2,i}) \leq 4\lambda + \lambda \times 33\lambda \leq 34\lambda^2, \tag{5.61}$$

so that, by (5.12),

$$prm(\underline{S}_{2,i}) \leq 34\lambda^2 + 4 \times 5 \times 2\lambda \leq 35\lambda^2 \tag{5.62}$$

and

$$|\partial\, [S_{2,i}]_1\,| \leq 35\lambda^2 + 8 \times 2\lambda \leq 36\lambda^2 \tag{5.63}$$

for β large enough. We can then choose $|I|$ sites $(z'_i)_{i\in I}$ such that

$$\begin{cases} \inf_{i\in I} \inf_{s\in S_{2,i}} \|z'_i - S_{2,i}\|_\infty > 3, \\ \inf_{i\neq j} \|z'_i - z'_j\|_1 > 1, \\ \sup_{i\in I} \|z_i - z'_i\|_\infty \leq 19\lambda^2, \end{cases} \tag{5.64}$$

use the Markov property at time τ_2, and follow the arguments that led to (5.44) and (5.46), to get

$$\begin{aligned} &P\left(T - 19\lambda^2 > \mathcal{T}_{\alpha,\lambda} \text{ or } \forall i \in I: \begin{array}{c} w_i(T-19\lambda^2) > 0 \quad \text{or} \\ \begin{cases} T - 19\lambda^2 < \tau_{[z_i]_{4\lambda}}(\hat{\eta}_i) \\ \hat{\eta}_i(T-19\lambda^2) = z'_i \end{cases} \end{array} \right) \\ &\geq \quad \left(\frac{1}{e^{5\delta\beta}T}\right)^{|I|} - \text{SES} \end{aligned} \tag{5.65}$$

uniformly in $\hat{\eta}(0)$, T and $(z_i)_{i\in I}$.

Step 5. Finally, consider the clouds of potentially interacting particles defined at time $T_3 := T - 19\lambda^2$, call, for $i \in I$, $S'_{3,i}$ the set of the circumscribed rectangles of the clusters made of particles that are in the cloud containing i, and define $\underline{S}_{3,i} := g_5(\underline{S}'_{3,i})$. Since $prm(\underline{S}_{3,i}) \leq 24\lambda$ (provided $\mathcal{T}_{\alpha,\lambda} > T_3$) and $|\partial\, [z_i]_{4\lambda}\,| \geq 32\lambda$, the rectangles in $\underline{S}_{3,i}$ cannot cover $[z_i]_{4\lambda}$. Consequently, the particles i in z'_i at time T_3 can bypass these separated rectangles to reach their targets $[z_i]_{4\lambda}$ at time T with a non-exponentially small probability. Together with (5.65) and the Markov property at time T_3, this implies that, uniformly in $\hat{\eta}(0)$, T and $(z_i)_{i\in I}$,

$$P\left(T > \mathcal{T}_{\alpha,\lambda} \text{ or } \forall i \in I\colon\; \left\lfloor \tau_{[z_i]_{4\lambda}}(\hat{\eta}_i)\right\rfloor = \lfloor T \rfloor \text{ or } w_i(T) > 0\right) \geq \left(\frac{1}{e^{6\delta\beta}T}\right)^{|I|} - \text{SES} \tag{5.66}$$

and concludes the proof. □

6 Application to the three regimes

appl

The absence of superdiffusivity for Kawasaki dynamics has been established up to time $\mathcal{T}_{\alpha,\lambda} \wedge T^2_\alpha$ in Theorems 3.2.1 and 3.3.3. As far as the spread-out estimates are concerned, Theorems 3.3.4, 3.3.5 and 3.4.1 can be applied only to active particles up to time $\mathcal{T}_{\alpha,\lambda}$. Indeed, we derived upper (lower) estimates of intersections (unions) of events involving anomalous concentration, activity and localization of particles. Since activity is a notion that depends on the parameter D, which assumes different values in the three different regimes (as explained in Section1.3), we need to discuss the applicability and the consequences of our results in each of these regimes. This is done in Sections 6.1–6.3.

6.1 Stable gas

When $\Delta > 2U$ we choose $D \in (2U, \Delta)$. The simple exclusion process ($U = 0$) is part of this regime.

Proposition 6.1.1 *For $t > 0$, let $\mathcal{A}(t)$ be the event that all particles are active up to time t. Then*

$$P(\mathcal{A}(t) \text{ or } \mathcal{T}_{\alpha,\lambda} < t) = 1 - \text{SES}. \tag{6.1}$$

Proof: By Definition 2.2.2, prior to time $e^{D\beta}$ all particles are active. Assume now that some particle i looses its freedom at some time $t < \mathcal{T}_{\alpha,\lambda}$. Then we have to show that i will recover its freedom with probability $1 - \text{SES}$ before time $t + e^{D\beta}$. By the Markov property, we can restrict ourselves to the special case $t = 0$. By Proposition 5.2.2, which states that $\hat{\eta}$ is a $\text{QRW}(\alpha, T_\alpha)$-process, and by the non-superdiffusivity property, we can further restrict ourselves to considering the system reduced to the cloud of potentially interacting particles on time scale T_α to which i belongs.

Pick $\delta, \delta_0 > 0$ such that $D - (2U + \delta_0) = \delta$, set $t_n := ne^{(2U+\delta_0)\beta}$ and $0 \le n \le e^{\delta\beta} - 1$. Consider, at any time t_n, the set $\underline{S}'_n$ of the circumscribed rectangles of the clusters of the cloud, define $\underline{S}_n := g_5(\underline{S}'_n)$, let $\tau_{r,n}$ be the first time after time t_n when $\underline{S}_n$ does not contain any particles, i.e., the first time after time t_n when η is not $2U$-reducible with respect to $\underline{S}_n$, and denote by $\tau_{c,n}$ the associated collision time. Then, by (5.42) (established uniformly in the initial configuration for the special case $t_n = t_0 = 0$, but valid for any t_n by the Markov property),

$$P\left(\tau_{r,n} < t_n + e^{(2U+\delta_0)\beta} \text{ and } \tau_{r,n} < \tau_{c,n}\right) \ge e^{-\delta\beta/3}, \tag{6.2}$$

so, by the Markov property applied at t_0, t_1, t_2,..., we obtain

$$P\left(i \text{ is not free on the whole interval } [0, e^{D\beta}]\right) \le \left(1 - e^{-\delta\beta/3}\right)^{e^{\delta\beta}} = \text{SES}. \tag{6.3}$$

$\square$

Proposition 6.1.1 implies that, in the stable regime, our spread-out estimates can be stated in a stronger version: the intersection with $\{w_i(T) = 0\}$ can be removed from the statement of Theorem 3.3.4 and the unions with $\{w_i(T) > 0\}$ can be removed from the statements of Theorems 3.3.5 and 3.4.1.

Next, applying the spread-out estimates, we can control the first time of anomalous concentration that limits the strength of our results. We denote by $\mathcal{X}_N(\alpha, \lambda)$ the set of configurations without α-anomalous concentration, so that $\mathcal{T}_{\alpha,\lambda}$ is the hitting time of the complement of the set $\mathcal{X}_N(\alpha, \lambda)$.

Proposition 6.1.2 *If $\hat{\eta}(0) \in \mathcal{X}_N(\frac{\alpha}{5}, \lambda)$, then*

$$P\left(\mathcal{T}_{\alpha,\lambda} \ge T_\alpha^2 (T_\alpha^{-1/2} \wedge e^{-D\beta})\right) = 1 - \text{SES}. \tag{6.4}$$

Proof: $\hat{\eta}(0) \in \mathcal{X}_n(\frac{\alpha}{5}, \lambda)$ implies that $|\mathcal{U}(\hat{\eta}(0))|_\Lambda| < \frac{\lambda}{4}$ for any box Λ with $|\Lambda| < e^{(\Delta - \frac{\alpha}{20})\beta}$, and that $\mathcal{T}_{\alpha,\lambda} > \mathcal{T}_{\frac{\alpha}{5},\lambda} > 0$. Consequently,

$$P\left(\mathcal{T}_{\alpha,\lambda} < T_{\frac{\alpha}{19}}\right) = \text{SES}, \tag{6.5}$$

since such an event implies that there is at least one particle with superdiffusive behavior before $\mathcal{T}_{\alpha,\lambda}$.

For larger T such that

$$T \leq T_\alpha^2(T_\alpha^{-1/2} \wedge e^{-D\beta}), \tag{6.6}$$

the event $\{\lfloor \mathcal{T}_{\alpha,\lambda} \rfloor = \lfloor T \rfloor\}$ has probability SES. This follows from the upper bound of the spread-out property in Theorem 3.3.4 applied to a single box $|\Lambda| = e^{(\Delta - \frac{\alpha}{4})\beta}$. Indeed, the event $\{\lfloor \mathcal{T}_{\alpha,\lambda} \rfloor = \lfloor T \rfloor\}$ implies that with probability $1 -$ SES at time $T-1$ there are $\frac{\lambda}{4}$ particles in the box $[\Lambda]_\lambda$. On the one hand, the n particles that have a non-SES probability to be in $[\Lambda]_\lambda$ at time $T-1$ are contained in a box $[\Lambda]_{\sqrt{T}e^{\delta\beta}}$, for δ arbitrarily small, and so they are at most

$$n \leq \frac{\lambda}{4} \left\lceil \frac{T e^{\delta\beta}}{e^{(\Delta - \frac{\alpha}{20})\beta}} \right\rceil, \tag{6.7}$$

since $\hat{\eta}(0) \in \mathcal{X}_n(\frac{\alpha}{5}, \lambda)$. On the other hand, the probability p that $\frac{\lambda}{4}$ given particles are all in $[\Lambda]_\lambda$ at time $T-1$ is estimated, via Theorem 3.3.4 for $T > T_{\frac{\alpha}{5}}$, by

$$p \leq \left(\frac{|\Lambda| e^{\delta\beta}}{T} \right)^{\frac{\lambda}{4}}, \tag{6.8}$$

since $T \leq T_\alpha^2(T_\alpha^{-1/2} \wedge e^{-D\beta})$ implies that condition (3.11) is satisfied for $|\Lambda| = e^{(\Delta - \frac{\alpha}{4})\beta}$. We have

$$\binom{n}{\frac{\lambda}{4}} \left(\frac{|\Lambda| e^{\delta\beta}}{T} \right)^{\frac{\lambda}{4}} \leq \left(\frac{\lambda}{4} \left\lceil \frac{T e^{\delta\beta}}{e^{(\Delta - \frac{\alpha}{20})\beta}} \right\rceil \frac{|\Lambda| e^{\delta\beta}}{T} \right)^{\frac{\lambda}{4}} = \text{SES}, \tag{6.9}$$

and so we conclude that

$$P\left(T_{\alpha/5} \leq \mathcal{T}_{\alpha,\lambda} \leq T_\alpha^2(T_\alpha^{-1/2} \wedge e^{-D\beta})\right) = \text{SES}, \tag{6.10}$$

since Λ_β is only exponentially large in β. Together with (6.5) and the fact that $T_{\frac{\alpha}{5}} < T_{\frac{\alpha}{19}}$, this completes the proof. □

Finally, if the starting configuration is chosen according to the equilibrium measure ν_N defined in (1.10), then the first time of anomalous concentration is larger than any exponential in β:

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Proposition 6.1.3 *For all $C > 0$,*

$$P_{\nu_N}(\mathcal{T}_{\alpha,\lambda} \geq e^{C\beta}) = 1 - \text{SES}. \tag{6.11}$$

Proof: See Appendix B. □

There are interesting problems for the stable gas regime that are not in the range of application of our results. An example is the evolution of configurations with anomalous concentration, such as the evaporation of a macroscopic droplet.

6.2 Unstable gas

When $\Delta < U$ we choose $D \in (0, U)$. This is the regime in which the density is so high that the condensation starts immediately and all the clusterized particles fall asleep. We expect to see in a time $e^{(\Delta+\delta)\beta}$ an anomalous concentration, after which our claims are empty. Actually, our estimates only describe the gas in this initial transient period, i.e., in the short time of initial condensation. However, we note that starting from any configuration without anomalous concentration (i.e., such that $\mathcal{T}_{\alpha,\lambda} > 0$), our spread-out estimates hold up to time T_α (for active particles), as so does the non-superdiffusivity, by Proposition 5.2.2.

6.3 Metastable gas

When $\Delta \in (U, 2U)$ we choose $D \in (U, \Delta)$. This is the more interesting regime where active and sleeping particles are both present. As mentioned in Section 1, the study of this regime actually was the main motivation behind the present paper. This regime will be developed in detail in [13], [14], in order to analyze the escape time from metastability. To do so, we will combine the analysis of the sleeping particles developed for the local interaction model in den Hollander, Olivieri and Scoppola [8] with the estimates obtained in the present paper, to describe the active particles of the gas.

In Section 2.3 we introduced the special permutation rule in order to minimize the number of sleeping particles: once again, the fewer they are, the stronger are our results. As far as the first time of anomalous concentration is concerned, we will prove an *a priori* estimate analogous to (6.11): starting from the "metastable equilibrium" it will be possible to evaluate any anomalous concentration up to an exponential time that is larger than T_α^2 and that coincides with the typical exit time from metastability.

With this combination we will extend the notion of kU-reduction introduced in Section 5.3 to Δ-reduction. We will prove a recurrence property of the system in a time of order $e^{\Delta\beta}$ on the set of Δ-irreducible configurations. The clusterized part of these configurations are squares or quasi-squares, i.e., rectangles with sidelengths differing by at most 1. These squares and quasi-squares move slowly in the gas of the active particles, and they can be seen as heavy particles that evolves in this gas of active particles. For details we refer to [13], [14].

A Freidlin-Wentzel theory for a slowly growing state space

fw

In this appendix we prove Lemma 5.3.2.

Proof: We observe that up to time $\tau_+ \wedge \tau_c$ the evolution of the system inside S is independent of its evolution outside S. We distinguish between the cases $k = 0, 1, 2$.

• Case $k = 0$. If $\hat{\eta}(0)$ is 0-reducible, then there is a sequence of configurations $\mathcal{U}(\hat{\eta}) = \eta_0$, $\eta_1, \dots, \eta_n$ in $\mathcal{X}$, each of them obtained from the previous one by a displacement of a single particle to a nearest-neighbor vacant site, such that

$$\begin{cases} \mathrm{H}_{\underline{S}}(\eta_n) < \mathrm{H}_{\underline{S}}(\eta), \\ \sup_j \mathrm{H}_{\underline{S}}(\eta_j) \leq \mathrm{H}_{\underline{S}}(\eta). \end{cases} \tag{A.1}$$

Without loss of generality we may assume that $\mathrm{H}_{\underline{S}}(\eta_{j+1}) \leq \mathrm{H}_{\underline{S}}(\eta_j)$ for all $j < n$. This implies that the number of particles inside S does not increase along this sequence. We may further assume that η_n is the first configuration along this sequence where a 0-irreducible configuration is reached or the gas surrounding $[S]_1$ is enriched. This implies that the number of particles inside S is a constant $a \leq \lambda$ from η_0 to η_n. Finally, we may assume that n is smaller than or equal to the total number of configurations with $a \leq \lambda$ particles inside S, so that, using the isoperimetric inequality, we get

$$n \leq \binom{|S|}{a} \leq \binom{\lambda^{2\kappa}}{a} \leq \binom{\lambda^{2\kappa}}{\lambda} \leq \lambda^{2\kappa\lambda}. \tag{A.2}$$

Under these assumptions, the probability that the conditioned process restricted to S follows this sequence in a time $e^{\frac{\delta}{2}\beta}$ is larger than or equal to (recall (1.18))

$$\exp\left\{-\mathrm{cst}\,\lambda^{2\kappa\lambda}\right\} \geq e^{-\frac{\delta}{4}\beta} - \mathrm{SES} \tag{A.3}$$

uniformly in $\underline{S}$ and $\hat{\eta}(0)$. We now divide the time interval $[0, e^{\delta\beta}]$ into $e^{\frac{\delta}{2}\beta}$ intervals of length $e^{\frac{\delta}{2}\beta}$. By the Markov property, we get

$$\begin{aligned} &P\left(\forall t \leq e^{\delta\beta},\ \hat{\eta}(t) \text{ is 0-reducible and } \tau_+ \neq t \,\middle|\, \tau_c > e^{\delta\beta} \wedge \tau_+\right) \\ &\leq \left(1 - e^{-\frac{\delta}{4}\beta}\right)^{e^{\frac{\delta}{2}\beta}} + \mathrm{SES} \leq \mathrm{SES} \end{aligned} \tag{A.4}$$

uniformly in $\underline{S}$ and $\hat{\eta}(0)$.

If $\hat{\eta}(0)$ is 0-irreducible and $\tau_+ > 0$, then, conditionally on $\{\tau_c \geq \tau_+\}$, the system has to perform a move inside S of cost at least U to enrich the gas surrounding $[S]_1$. Since, up to time τ_c, the particles inside S cannot be more than λ, this move occurs within a time $e^{(U-\delta)\beta}$ with probability larger than or equal to

$$\lambda e^{-\delta\beta} \leq e^{-\delta'\beta} + \mathrm{SES} \tag{A.5}$$

uniformly in $\underline{S}$ and $\hat{\eta}(0)$. □

Before proceeding with the proof for the cases $k = 1$ and $k = 2$, we define the U-cycles associated with $\underline{S}$ and prove one of their properties.

uccls

Definition A.0.1 *Let $\underline{S} \in \mathcal{R}$ be such that $g_5(\underline{S}) = \underline{S}$. Suppose that C is a set of configurations satisfying*

(i) each η in C is 0-irreducible;

(ii) with positive probability C can be completely visited by the process $\mathcal{U}(\hat{\eta})$ without it going outside C;

(iii) there is a configuration η' outside C that can be obtained from some configuration in C by a displacement at cost U of a single particle inside S to a nearest-neighbor site;

(iv) any set of configurations that contains C and satisfies (i)–(iii) is equal to C.

Then we say that C is a U-cycle with exit η'.

Remark: We will not actually need the maximality property (iv). We added it here to recover, in our special case, the analogue of the cycles for the local model of den Hollander, Olivieri and Scoppola [8].

Lemma A.0.2 *Let $\underline{S} \in \mathcal{R}$ be such that $g_5(\underline{S}) = \underline{S}$ and $prm(\underline{S}) \leq \lambda^\kappa$, and let C be a U-cycle with exit η'. Then, for any $\hat{\eta}(0)$ in C such that there are no particles inside $\partial\,[S]_1$, no particles inside $[S]_1 \setminus S$, and no more than λ particles inside S at time $t = 0$, and for any $\delta > 0$,*

$$
\begin{aligned}
& P\left(\eta|_S(\tau) = \eta'|_S,\, \tau \leq e^{(U+\delta)\beta} \,\middle|\, \tau_c \geq e^{(U+\delta)\beta} \wedge \tau_+\right) \\
\geq \quad & \exp\left\{-\mathrm{cst}\,\lambda^{2\kappa\lambda} \ln \lambda\right\} - \mathrm{SES}
\end{aligned}
\tag{A.6}
$$

uniformly in $\hat{\eta}(0)$, $\underline{S}$, C and η', with τ the exit time from C.

Proof: On the one hand, conditionally on $\{\tau_c \geq e^{(U+\delta)\beta} \wedge \tau_+\}$, the probability of the event $\{\tau \leq e^{\frac{\delta}{2}\beta}\}$ is larger than or equal to

$$
\left(\frac{\mathrm{cst}}{4\lambda}\right)^{\lambda^{2\kappa\lambda}} e^{-U\beta} \geq e^{-(U+\frac{\delta}{3})\beta} - \mathrm{SES} \tag{A.7}
$$

uniformly in $\hat{\eta}(0)$ $\underline{S}$, C and η'. Hence, dividing the time interval $[0, e^{(U+\delta)\beta}]$ into $e^{(U+\frac{\delta}{2})\beta}$ intervals of length $e^{\frac{\delta}{2}\beta}$ and using the Markov property, we get

$$
P\left(\tau > e^{(U+\delta)\beta} \,\middle|\, \tau_c > e^{(U+\delta)\beta} \wedge \tau_+\right) \;\leq\; \left(1 - e^{-(U+\frac{\delta}{3})\beta}\right)^{e^{(U+\frac{\delta}{2})\beta}} \;\leq\; \mathrm{SES}. \tag{A.8}
$$

On the other hand, conditionally on $\{\tau_c \geq \tau_+\}$, the probability of the event $\{\eta|_S(\tau) = \eta'|_S\}$ is larger than or equal to

$$
\sum_{j\geq 0} \left(1 - \lambda e^{-U\beta}\right)^j \left(\frac{\mathrm{cst}}{4\lambda}\right)^{\lambda^{2\kappa\lambda}} e^{-U\beta} \;\geq\; \exp\left\{-\mathrm{cst}\,\lambda^{2\kappa\lambda} \ln \lambda\right\} - \mathrm{SES} \tag{A.9}
$$

uniformly in $\hat{\eta}(0)$, $\underline{S}$, C and η'. □

We are now ready to conclude the proof of Lemma 5.3.2 for the cases $k = 1$ and $k = 2$.

• Case $k = 1$. If $\hat{\eta}(0)$ is U-reducible, then it is easy to see that there exists a sequence of no more than $\lambda^{2\kappa\lambda}$ U-cycles and configurations such that

(1) $H_{\underline{S}}$ does not increase between two successive configurations;

(2) each cycle C is preceeded by a configuration it contains and followed by an exit configuration η;

(3) the sequence ends in a configuration η_n that is the first along this sequence where a U-irreducible configuration is reached or the gas surrounding $[S]_1$ is enriched.

Using Lemma A.0.2, we can estimate the probability that the conditioned process restricted to S follows this sequence in time $e^{(U+\frac{\delta}{2})\beta}$ from below by (recall (1.18))

$$\exp\left\{-\text{cst}\,\lambda^{4\kappa\lambda}\ln\lambda\right\} \geq e^{-\frac{\delta}{4}\beta} - \text{SES} \tag{A.10}$$

uniformly in $\underline{S}$ and $\hat{\eta}(0)$. After dividing the time interval $[0, e^{(U+\delta)\beta}]$ into $e^{\frac{\delta}{2}\beta}$ intervals of length $e^{(U+\frac{\delta}{2})\beta}$ and using the Markov property, we get

$$\begin{aligned} &P\left(\forall t \leq e^{(U+\delta)\beta},\ \hat{\eta}(t) \text{ is } U\text{-reducible and } \tau_+ \neq t \;\middle|\; \tau_c > e^{(U+\delta)\beta} \wedge \tau_+\right) \\ &\leq \left(1 - e^{\frac{\delta}{4}\beta}\right)^{e^{\frac{\delta}{2}\beta}} \leq \text{SES} \end{aligned} \tag{A.11}$$

uniformly in $\underline{S}$ and $\hat{\eta}(0)$.

If $\hat{\eta}(0)$ is U-irreducible, then, conditionally on $\{\tau_c \geq \tau_+\}$, to enrich the gas surrounding $[S]_1$ the system has to perform a move inside S of cost at least $2U$, or (by the result for the case $k = 0$) two moves of cost U in a time smaller than $e^{\delta''\beta}$ for a given $\delta'' > 0$. Since, up to time τ_c, the particles inside S cannot be more than λ, this occurs within time $e^{(2U-\delta)\beta}$ with probability less than or equal to

$$\lambda e^{-(\delta-\delta'')\beta} \leq e^{-\delta'\beta} + \text{SES} \tag{A.12}$$

uniformly in S and $\hat{\eta}(0)$, provided we choose δ'' such that $\delta - \delta'' > \delta'$. $\square$

• Case $k = 2$. Using the fact that any cluster carries at least four particles that can only be separated at cost $2U$, the first probability estimate is once again obtained after dividing the time interval $[0, e^{(2U+\delta)\beta}]$ into $e^{(2U+\frac{\delta}{2})\beta}$ intervals of length $e^{\frac{\delta}{2}\beta}$ and using the Markov property. The second probability estimate is obvious: the gas cannot be enriched if there are no particles inside S. $\square$

This completes the proof of Lemma 5.3.2. $\square$

B An estimate on the canonical Gibbs measure

can

In this appendix we prove here Proposition 6.1.3.

Proof: Since ν_N is the invariant measure of the dynamics, and Λ_β is only exponentially large in β, it is enough to prove

$$nu_N\left(\eta \in \mathcal{X}_N :\ \left|\eta|_\Lambda\right| = a\right) \leq \text{SES} \tag{B.1}$$

uniformly in $a \geq \lambda$ and for Λ a square box of volume $|\Lambda| = e^{(\Delta-\frac{\alpha}{4})\beta}$.

Pick such a and Λ. For any $\eta \in \mathcal{X} = \{0,1\}^{\Lambda_\beta}$ and x in η, we let $\text{cc}(x)$ be the connected component of x in η, i.e., either the cluster of η that contains x if $x \in \eta^{cl}$ or the singleton $\{x\}$ if $x \in \eta \setminus \eta^{cl}$. Let

$$A(\eta) := \{x \in \eta :\ \text{cc}(x) \cap \Lambda \neq \emptyset\}\,. \tag{B.2}$$

We will show that, uniformly in a and Λ,

$$\nu_N\left(\eta \in \mathcal{X}_N :\ |A(\eta)| = a\right) \leq \text{SES}, \tag{B.3}$$

which implies (B.1).

To prove (B.3), define

$$\begin{aligned}
Z_N &:= \sum_{|\eta|=N} \exp\{-\beta \mathrm{H}(\eta)\}, \\
Z_{out} &:= \sum_{|\eta|=N-a} \exp\{-\beta \mathrm{H}(\eta)\} \, \mathbb{1}_{\{|A|=0\}}(\eta), \\
Z_{in} &:= \sum_{|\eta|=a} \exp\{-\beta \mathrm{H}(\eta)\} \, \mathbb{1}_{\{|A|=a\}}(\eta).
\end{aligned} \tag{B.4}$$

Then, clearly,

$$\nu_N\left(\eta \in \mathcal{X}_N : \, |A(\eta)| = a\right) \leq \frac{Z_{out} Z_{in}}{Z_N} \tag{B.5}$$

and

$$\begin{aligned}
Z_N &\geq \frac{Z_{out}\,(N-a)!\,(|\Lambda_\beta| - (N-a)) \times \cdots \times (|\Lambda_\beta| - (N-a) - (a-1))}{N!} \\
&\geq Z_{out} \left(\frac{|\Lambda_\beta| - N}{N} \right)^a,
\end{aligned} \tag{B.6}$$

i.e.,

$$Z_N \geq Z_{out} \left(\frac{1 - e^{-\Delta\beta}}{e^{-\Delta\beta}} \right)^a. \tag{B.7}$$

Next, we derive an upper bound on Z_{in} by making the following observations:

- given a positive integer n_i, any cluster of n_i particles that intersects Λ is covered by a tree with one leaf in Λ and with n_i vertices that are connected by edges linking nearest-neighbor occupied sites;
- a random walker on such a tree can visit the whole tree, starting from that leaf, in at most $3n_i - 3$ steps;
- there are $4^{3(n_i-1)}$ random walks of length $3n_i - 3$ on $\mathbb{Z}^2$ that start from a given point;
- a single cluster η of volume n_i has an energy $-2U|\eta| + \frac{U}{2}|\partial\eta| \geq -2U(n_i - 1)$;
- there are $\binom{a-1}{k-1}$ ways of writing $a - k$ as a sum of k integers.

In view of these observations, we choose U' such that $2U < \Delta - \frac{\alpha}{4} < 2U' < \Delta$, so that for β

large enough,

$$
\begin{aligned}
Z_{in} &\leq \sum_{k=1}^{a} \sum_{\substack{n_1+\cdots+n_k=a \\ n_1,\ldots,n_k\geq 1}} |\Lambda|^k \prod_i 4^{3(n_i-1)} \exp\{2U(n_i-1)\beta\} \\
&\leq \sum_{k=1}^{a} \sum_{\substack{n_1+\cdots+n_k=a-k \\ n_1,\ldots,n_k\geq 0}} \exp\left\{k\left(\Delta-\frac{\alpha}{4}\right)\beta+(a-k)\left(\frac{3\ln 4}{\beta}+2U\right)\beta\right\} \\
&\leq \sum_{k=1}^{a} \binom{a-1}{k-1} \exp\left\{\left(\Delta-\frac{\alpha}{4}\right)a\beta\right\} \\
&\leq 2^{(a-1)} \exp\left\{\left(\Delta-\frac{\alpha}{4}\right)a\beta\right\} \\
&\leq \exp\left\{2U'a\beta\right\}.
\end{aligned}
\tag{B.8}
$$

Together with (B.5) and (B.7), this last estimate gives

$$
\nu_N\left(\eta\in\mathcal{X}_N:\ |A(\eta)|=a\right) \leq \left(\frac{\exp\{(2U'-\Delta)\beta\}}{1-e^{-\Delta\beta}}\right)^{\lambda} = \mathrm{SES} \tag{B.9}
$$

and concludes the proof. □

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