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HITTING TIMES FOR MULTIPLICATIVE GROWTH-COLLAPSE PROCESSES

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Abstract: We consider a stochastic process $(X_t)_{t\geq 0}$ that grows linearly in time and experiences collapses at times governed by a Poisson process with rate λ. The collapses are modeled by multiplying the process level by a random variable supported on $[0,1)$. For the hitting time defined as $\tau_y = \inf\{t > 0|X_t = y\}$ we derive power series for the Laplace transform and all moments. We further discuss the asymptotic behavior of the mean of τ_y as y tends to infinity.

1. Introduction

We investigate a growth-collapse process $(X_t)_{t\geq 0}$ with deterministic growth and a multiplicative collapse structure. The process increases linearly with rate one and at the time T_i of the ith collapse, the process jumps down to $Q_i \cdot X_{T_i}$, where $Q, Q_1, Q_2, \ldots$ are i.i.d. random variables with distribution function F_Q supported on $[0,1)$.

The collapse times $(T_i)_{i\in\mathbb{N}}$ are governed by a Poisson process with rate λ. We define the hitting time of level y by

$$\tau_y = \inf\{t > 0|X_t = y\}. \tag{1}$$

We shall establish a formula for the Laplace transform of τ_y, as well as for its moments, and the asymptotic behavior of its mean.

Growth-collapse processes are real-valued processes that grow between random collapse times, at which they jump down according to some distribution depending on their current level. This evolutionary pattern is encountered in a large variety of physical phenomena, see [16], like build-up of friction, earthquakes, avalanches, neuron firing, and shot noise, as well as in other fields like insurance mathematics [29], queueing theory [7] and mathematical finance [9]. Some properties of Markovian growth collapse models have been studied in [12]. The case where Q is a constant (usually $Q = 1/2$) is used as a model for the Transmission Control Protocol (TCP), the dominant protocol for data transfer over the internet, cf. Section 7 and [4, 5, 8, 15, 19, 27]. Picking items on a circle [23, 24] and DNA replication [22] are further applications of this case. Moreover, as indicated in [26] (see also [19]), the process $(X_t)_{t\geq 0}$ is equivalent to an exponential functional ([9, 13]) of a Lévy process.

In the literature, emphasis lies on analyzing the stationary behavior of the growth-collapse processes. The results we present for the hitting times are some of the few results obtained on the transient behavior of growth-collapse processes.

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Another noteworthy contribution on transient behavior is [15] in which results are obtained for the mean hitting time of a growth-collapse process with Q a constant and the Poisson process having rate $\lambda \cdot X_t$ (level-dependent).

To obtain our results, we invoke techniques from the field of piecewise deterministic Markov processes (see [14]). We consider the infinitesimal generator of the strong Markov process $(X_t)_{t\geq 0}$ and apply the well known Dynkin formula. In Theorem 2 we find for τ_y, starting from level $x < y$, that its mean can be expressed as

$$E_x\tau_y = \sum_{n=1}^{\infty} \frac{y^n - x^n}{n!} \prod_{i=1}^{n-1} \theta_i$$

and its Laplace transform as

$$E_x e^{-s\tau_y} = \frac{1 + s\sum_{n=1}^{\infty} \frac{x^n}{n!} \prod_{i=1}^{n-1}(s+\theta_i)}{1 + s\sum_{n=1}^{\infty} \frac{y^n}{n!} \prod_{i=1}^{n-1}(s+\theta_i)},$$

where $\theta_i = \lambda\left(1 - E(Q^i)\right)$.

We also investigate the asymptotic behavior of $E_x\tau_y$ as $y \to \infty$. Clearly the asymptotic behavior will strongly depend on the behavior of $F_Q(x)$ as $x \to 1$. It turns out that the asymptotic relations for $E_x\tau_y$ all have as their leading behavior $e^{\lambda y}$ as $y \to \infty$, and so one might argue that the mean hitting times are almost exponential. In the context of Markov chains on finite state spaces there is a vast literature on almost exponential hitting times (see [1] for an overview). As advocated by Aldous [3], hitting times of rarely-visited sets are approximately exponentially distributed if stationarity is reached rapidly. In [26] the authors obtained expressions for the transient moments of $(X_t)_{t\geq 0}$. These moments were shown to converge to their stationary counterparts exponentially fast.

A hitting time is a valuable characteristic, as it indicates the ability of the process to recover after a collapse. We therefore introduce the closely related notion of *recovery time* σ_x, defined as the time to reach level x conditioned on the fact that a collapse took place when the process was exactly in x. As for the hitting time, we obtain for the recovery time explicit expressions for the Laplace transform and its mean. Moreover, we show that the recovery time and hitting time have the same asymptotic behavior.

We have structured the paper as follows. A formal introduction to the model is presented in Section 2, including the infinitesimal generator, Dynkin's formula and a specific class of test functions. In Section 3 we derive explicit expressions for the means and Laplace transforms of the hitting- and recovery times. Section 4 is concerned with the asymptotic behavior of the mean hitting time. The main theorem, covering various modes of asymptotics, is proved in Section 5. We conclude with two special choices of Q in Sections 6 and 7. In Section 6 we treat the case where Q follows a Beta distribution. Several characteristics are reformulated in terms of hypergeometric functions and exponential integrals. In Section 7 we consider the case where Q is some constant in $[0, 1)$, which gives rise to q-calculus.

In what follows let P_x denote as usual the conditional probability given that the process starts at $X_0 = x$. Moreover, let $P_{(x)}$ be the probability given that the process starts with a jump from x and let E_x and $E_{(x)}$ be the respective expectations. Note that $E_{(x)}$ is obtained from E_x by conditioning on the first jump,

$$E_{(x)}(\cdot) = \int_0^1 E_{xy}(\cdot)\, dF_Q(y). \tag{2}$$

For a probability distribution function F, let $\overline{F} = 1 - F$. For the asymptotic behavior of functions we sometimes write $f(x) \asymp g(x)$ if there is a constant $C \in (0, \infty)$ such that $\lim f(x)/g(x) = C$. In particular, if $C = 1$ we use the usual notation $f(x) \sim g(x)$ for asymptotic equivalence. The infimum over the empty set is defined to be infinite, empty sums are zero, empty products are one.

We already introduced the coefficients $\theta_a = \lambda\,(1 - E(Q^a))$. We further introduce $\pi_0(s) = 1$ and the product

$$\pi_k(s) = \prod_{i=1}^{k} (\theta_i + s).$$

2. The model

We thus consider the process $(X_t)_{t \geq 0}$, a Markov process with linear deterministic increase and multiplicative jumps. The process jumps down at times $(T_i)_{i \in \mathbb{N}}$ that are governed by a Poisson process $(N_t)_{t \geq 0}$ with intensity λ. At T_i the process jumps from $X_{T_i -}$ to $X_{T_i} = Q_i \cdot X_{T_i -}$ where $Q, Q_1, Q_2, \ldots$ are i.i.d. random variables with distribution function F_Q supported on $[0, 1)$. We exclude the case where $P(Q = 1) > 0$, although it could be included by changing the jump intensity from λ to $P(Q < 1) \cdot \lambda$.

The process is a special case of a piecewise deterministic Markov process introduced by Davis [14]. The state space $\mathcal{S}$ consists of all non-negative real numbers. The extended generator of the strong Markov process $(X_t)_{t \geq 0}$ is given by

$$\mathscr{A} f(x) = f'(x) - \lambda f(x) + \lambda \int_0^1 f(xy)\, dF_Q(y) \quad , \quad x \in \mathcal{S} = [0, \infty).$$

The domain of $\mathscr{A}$ contains absolutely continuous functions $f : [0, \infty) \to \mathbb{R}$ that are either locally bounded or for which $f(xy) \leq f(x) f(y)$ holds (see [26]). The defining property of the extended generator is that for all functions f in the domain of $\mathscr{A}$ the stochastic process $f(X_t) - \int_0^t \mathscr{A} f(X_s)\, ds$ is a martingale.

Our central observation is that for all a for which $E(Q^a) < \infty$ the application of the generator to power functions leads to a sum of two powers. That is,

$$\mathscr{A} x^a = a x^{a-1} - x^a \theta_a. \tag{3}$$

It has been shown in [26] that (3) can be used to obtain a formula for the transient moments of the process. In this paper we will demonstrate how (3) can be utilized to find expressions for the mean and the Laplace transform of the hitting- and recovery times. The preservation of powers property in (3) is crucial in proving Theorem 1, which gives a solution to the generator equation $\mathscr{A} f(x) = s f(x) + z(x)$, where z is some arbitrary function.

3. Hitting- and recovery times

We have defined the *hitting time* of the level $y \in \mathcal{S}$ as $\tau_y = \inf\{t > 0 | X_t = y\}$. We have furthermore defined the *recovery time* σ_x as the time the process needs to reach level x, conditionally on the fact that a jump takes place when the process is in level x. It has been shown in [12] and [26] that $E_x \tau_y < \infty$ for all $x, y \in \mathcal{S}$. Obviously, $E_{(x)} \sigma_x \leq E_0 \tau_x < \infty$ for all $x \in \mathcal{S}$.

To prepare the ground for our main result we provide a solution to the generator equation $\mathscr{A} f(x) = s f(x) + z(x)$. The result is more general than needed, but perhaps of some interest for future study of this model.

Theorem 1. *Let $z : \mathbb{R} \to \mathbb{R}$ be a function analytic in a neighborhood zero. A solution of $\mathscr{A} f(x) = sf(x) + z(x)$ with $f(0) = a_0 \in \mathbb{R}$ is given by*

$$f(x) \;=\; a_0 + \sum_{n=1}^{\infty} \left(sa_0 + \sum_{k=0}^{n-1} \frac{z^{(k)}(0)}{\pi_k(s)} \right) \pi_{n-1}(s) \frac{x^n}{n!}.$$

Proof. It is straightforward to show that for a function $h(x) = \sum_{n=0}^{\infty} a_n \pi_{n-1}(s) \frac{x^n}{n!}$ we get

$$\mathscr{A} h(x) \;=\; (1-s)h(0) + sh(x) + \sum_{n=1}^{\infty} \Delta a_n \pi_{n-1}(s) \frac{x^{n-1}}{(n-1)!},$$

where $\Delta a_n = a_n - a_{n-1}$. Thus the equation $\mathscr{A} f(x) = sf(x) + z(x)$ can be written as

$$(1-s)a_0 + \sum_{n=1}^{\infty} \Delta a_n \pi_{n-1}(s) \frac{x^{n-1}}{(n-1)!} = \sum_{n=1}^{\infty} z^{(n-1)}(0) \frac{x^{n-1}}{(n-1)!}.$$

It follows from comparison of the coefficients that $z(0) = (1-s)a_0 + (a_1 - a_0) = a_1 - sa_0$ and $\Delta a_n = z^{(n-1)}(0)$, and finally that $a_n = sa_0 + \sum_{k=1}^{n} \frac{z^{(k-1)}(0)}{\pi_{k-1}(s)}$ for $n \geq 1$. □

We now present our main result. Let the function $R_s(x)$ be given by

$$R_s(x) = \sum_{n=1}^{\infty} \pi_{n-1}(s) \frac{x^n}{n!}. \tag{4}$$

Theorem 2. *For $x \leq y$ the means and Laplace transforms of τ_y and σ_x are given by*

$$E_x \tau_y \;=\; R_0(y) - R_0(x), \tag{5}$$

$$E_x e^{-s\tau_y} \;=\; \frac{1 + sR_s(x)}{1 + sR_s(y)}, \tag{6}$$

$$E_{(x)} \sigma_x \;=\; \frac{1}{\lambda} \left(R_0'(x) - 1 \right), \tag{7}$$

$$E_{(x)} e^{-s\sigma_x} \;=\; 1 + \frac{1}{\lambda} \left(s - \frac{R_s'(x)}{1 + sR_s(x)} \right). \tag{8}$$

Proof. We utilize the fact that, if f is in the domain of $\mathscr{A}$,

$$f(X_t) - \int_0^t \mathscr{A} f(X_s)\, ds \tag{9}$$

and

$$f(X_t) \exp \left(- \int_0^t \mathscr{A} f(X_s)/f(X_s)\, ds \right) \tag{10}$$

are martingales (see e.g. [15, 17]). It follows from Theorem 1 that $R_s(x)$ is the solution of the equation $\mathscr{A} f(x) = sf(x) + 1$ with $f(0) = 0$. In particular, $\mathscr{A} R_0(x) = 1$, so that $R_0(X_t) - t$ is a martingale and optional stopping, if allowed, yields $E_x \tau_y = R_0(y) - R_0(x)$. On $\{t < \tau_y\}$ we have that $|R_0(X_t) - t| \leq R_0(y) + t = O(t)$ as $t \to \infty$. Since $E_x \tau_y < \infty$ it follows that

$$E_x(R_0(X_t) - t; t < \tau_y) = E_x(R_0(X_t) - t | t < \tau_y) o(1/t) \to 0$$

as $t \to \infty$ and thus the optional stopping theorem can be applied (see [17] for this criterion) and (5) follows.

Since $\mathscr{A}(1 + sR_s(x)) = s(1 + sR_s(x))$ it follows that $e^{-st}(1 + sR_s(X_t))$ is a martingale and optional stopping yields (6). Because $R_s(X_t)$ is bounded for $t < \tau_y$, optional stopping is allowed here (see also [21]).

We can find an expression for the mean of σ_x in terms of the function R_0 by applying the conditioning formula (2), i.e.,

$$E_{(x)}\sigma_x = \int_0^1 E_{xy}\tau_x \, dF_Q(y) = \int_0^1 \left(R_0(x) - R_0(xy)\right)\, dF_Q(y).$$

Since $\mathscr{A}R_0(x) = 1 = R_0'(x) - \lambda\int_0^1 \left(R_0(x) - R_0(xy)\right)\, dF_Q(y)$, it follows that

$$E_{(x)}\sigma_x = \frac{R_0'(x) - \mathscr{A}R_0(x)}{\lambda} = \frac{R_0'(x) - 1}{\lambda}.$$

A similar method works for the Laplace transform. Using (2) and (6) yields

$$E_{(x)}e^{-s\sigma_x} \quad = \quad E_{(x)}e^{-s\tau_x} = \int_0^1 \frac{1 + sR_s(xy)}{1 + sR_s(x)}\, dF_Q(y).$$

Since $\mathscr{A}\left(1 + sR_s(x)\right) = s\left(1 + sR_s(x)\right)$ it follows that

$$s = \frac{\mathscr{A}\left(1 + sR_s(x)\right)}{1 + sR_s(x)} = \frac{R_s'(x)}{1 + sR_s(x)} - \lambda + \lambda\int_0^1 \frac{1 + sR_s(xy)}{1 + sR_s(x)}\, dF_Q(y).$$

Consequently,

$$E_{(x)}e^{-s\sigma_x} \quad = \quad 1 + \frac{\mathscr{A}\left(1 + sR_s(x)\right) - R_s'(x)}{\lambda\left(1 + sR_s(x)\right)} = 1 + \frac{s}{\lambda} - \frac{R_s'(x)}{\lambda\left(1 + sR_s(x)\right)}.$$

This completes the proof. □

To derive formulas for the higher moments we define

$$\pi_k^{(m)}(s) = \sum_{\substack{K\subseteq\{1,\ldots,k\}\\ |K|=k-m}} \prod_{i\in K}(\theta_i + s),$$

and $\kappa_n(x) = \sum_{k=n}^\infty \frac{x^k}{k!}\pi_{k-1}^{(n-1)}(0)$. Note that $\pi_k(s) = \pi_k^{(0)}(s)$ and thus $\kappa_1(x) = R_0(x)$. The next lemma shows that these functions are solutions of the equation $\mathscr{A}^k\kappa_k(x) = 1$, where $\mathscr{A}^k$ denotes the repeated application of the operator $\mathscr{A}$. Moreover, although we will not use this fact in the sequel, it is shown that the κ_k are the coefficients of the power series expansion of R_s in s.

Lemma 3. *$\mathscr{A}\kappa_n(x) = \kappa_{n-1}(x)$ and $R_s(x) = \sum_{n=1}^\infty \kappa_n(x)s^{n-1}$.*

Proof. We have

$$\begin{aligned} R_s(x) \quad &= \quad \sum_{n=1}^\infty \prod_{k=1}^{n-1}(\theta_k + s)\frac{x^n}{n!} = \sum_{n=1}^\infty \left(\sum_{k=0}^{n-1}\pi_{n-1}^{(k)}(0)s^k\right)\frac{x^n}{n!} \\ &= \quad \sum_{n=1}^\infty s^{n-1}\sum_{k=n}^\infty \pi_{k-1}^{(n-1)}(0)\frac{x^k}{k!} = \sum_{n=1}^\infty s^{n-1}\kappa_n(x). \end{aligned}$$

That $\mathscr{A}\kappa_n(x) = \kappa_{n-1}(x)$ follows from $\mathscr{A}R_s(x) = 1 + sR_s(x)$. □

Theorem 4. *For $x \le y$ the higher moments of τ_y and σ_x are given via the recursions*

$$E_x\tau_y^n \quad = \quad n!\left(\kappa_n(x) - \kappa_n(y) - \sum_{k=1}^{n-1}(-1)^k\frac{\kappa_k(y)}{(n-k)!}E_x\tau_y^{n-k}\right), \tag{11}$$

$$E_{(x)}\sigma_x^n \quad = \quad (-1)^n n!\left(\frac{\kappa_{n-1}(x) - \kappa_n'(x)}{\lambda} - \sum_{k=1}^{n-1}(-1)^k\frac{\kappa_k(x)}{(n-k)!}E_{(x)}\sigma_x^{n-k}\right). \tag{12}$$

Proof. It is shown in [25] that (11) holds if $\mathscr{A}\kappa_n(x) = \kappa_{n-1}(x)$. Equation (12) is immediate if we use (2). We have

$$\begin{aligned} E_{(x)}\sigma_x^n &= (-1)^n n! \int_0^1 \left(\kappa_n(xu) - \kappa_n(x) - \sum_{k=1}^{n-1} (-1)^k \frac{\kappa_k(x)}{(n-k)!} E_{xu}\tau_x^{n-k} \right) dF_Q(u) \\ &= (-1)^n n! \left(\frac{\kappa_{n-1}(x) - \kappa_n'(x)}{\lambda} - \sum_{k=1}^{n-1} (-1)^k \frac{\kappa_k(x)}{(n-k)!} E_{(x)}\sigma_x^{n-k} \right). \quad \square \end{aligned}$$

4. Asymptotic behavior of the hitting time and recovery time

Our first asymptotic result gives an upper bound for the mean of τ_y and σ_y and states that as $y \to \infty$ both values grow equally fast in the sense of asymptotic equivalence (denoted by $\sim$). Both statements are not difficult to prove and we will see later that the actual growth may differ considerably from these bounds.

Proposition 5. *We have*

$$\frac{e^{\theta_1 y} - 1}{\theta_1} \le E_0\tau_y \le \frac{e^{\lambda y} - 1}{\lambda} \tag{13}$$

and $\frac{1}{\lambda}\left(e^{\theta_1 y} - 1\right) \le E_{(y)}\sigma_y \le \frac{1}{\lambda}\left(e^{\lambda y} - 1\right)$. *Moreover, as* $y \to \infty$,

$$E_{(y)}\sigma_y \sim E_x\tau_y. \tag{14}$$

Proof. It is immediate from the definition of R_s and $\theta_1 \le \theta_n \le \lambda$ that

$$\frac{e^{\theta_1 y} - 1}{\theta_1} = \sum_{n=1}^{\infty} \left(\prod_{k=1}^{n-1} \lambda(1 - EQ) \right) \frac{y^n}{n!} \le R_s(y) \le \sum_{n=1}^{\infty} \lambda^{n-1} \frac{y^n}{n!} = \frac{e^{\lambda y} - 1}{\lambda}.$$

Next, we have $E_{(y)}\sigma_y = \frac{1}{\lambda}\left(R_0'(y) - 1\right)$ and

$$e^{\theta_1 y} = \sum_{n=0}^{\infty} \left(\prod_{k=1}^{n} \theta_1 \right) \frac{y^n}{n!} \le R_0'(y) \le \sum_{n=0}^{\infty} \lambda^n \frac{y^n}{n!} = e^{\lambda y}.$$

Finally, with $\pi_n = \pi_n(0)$,

$$R_0'(y) = \sum_{n=0}^{\infty} \pi_n \frac{y^n}{n!} = \theta_1 + \sum_{n=1}^{\infty} \frac{\pi_n}{\pi_{n+1}} \pi_{n+1} \frac{y^n}{n!},$$

so that relation (14) follows from Lemma 8 and the fact that $\pi_{n+1}/\pi_n = \theta_{n+1}$ converges to λ as $n \to \infty$. $\square$

The next result shows that there is no smaller coefficient than λ in the linear term in the exponent.

Proposition 6. *For every* $\varepsilon > 0$,

$$e^{(1-\varepsilon)\lambda y} \le E_0\tau_y \le \frac{e^{\lambda y} - 1}{\lambda}, \tag{15}$$

for large enough values of y.

Proof. The upper bound in (15) is copied from (13). For an entire function $f(x) = \sum_{n=0}^{\infty} a_n x^n$ for which $e^{c(1-\varepsilon)x^\rho} \le f(x) \le e^{c(1+\varepsilon)x^\rho}$ holds ultimately for every $\varepsilon > 0$ the coefficients ρ and c are called *order* and *type* of f. One can calculate the order and type by using the formulas $\rho = -\lim_{n\to\infty} n\log n/\log|a_n|$ and $c = \lim_{n\to\infty} \frac{n}{e\rho}|a_n|^{\rho/n}$, respectively (see e.g. [11]). Stirling's formula and $\theta_n^{n-1} \le \pi_n \le \theta_1^{n-1}$ yield

$$\rho^{-1} = \lim_{n\to\infty} \frac{\log n! - \log \pi_{n-1}}{n\log n} = \lim_{n\to\infty}\left(1 - \frac{\log \pi_{n-1}}{n\log n}\right) = 1.$$

$$c = \lim_{n\to\infty} \frac{n}{e}\left(\pi_{n-1}/n!\right)^{1/n} = \lim_{n\to\infty} |\pi_{n-1}|^{1/n} = \lim_{n\to\infty} \frac{\sum_{i=1}^{n-1}\log\theta_i}{n} = \lambda.$$

Thus (15) follows. □

For $q \in [0,1]$, let $(q)_\infty = \prod_{k=1}^\infty (1-q^k)$ denote the infinite q-series (see Askey et al. [2]) and define $x_Q = \sup\{x \in [0,1] | F_Q(x) < 1\}$. A measurable function $L : [0,\infty) \to [0,\infty)$ is said to be *slowly* varying at 0 if $L(cx)/L(x) \to 1$ as $x \to 0$ for all $c > 0$ (cf. [10]). The central result of this section is the following theorem, which gives the asymptotic behavior of $E_x\tau_y$ as y tends to infinity in some important cases.

Theorem 7. *Let F be a probability distribution function on $[0,1]$ with*

$$\overline{F}(x) = (1-x)^\beta L(1-x)$$

and $L(x)$ slowly varying as $x \to 0$. The following asymptotic relations hold as $y \to \infty$.

1. *If either $x_Q < 1$ or $F_Q(x) \ge F(x)$, $\beta = 1$ and $\int_1^\infty L(1/u)/u\, du < \infty$ then*

$$E_x\tau_y \sim \frac{D}{\lambda}\cdot e^{\lambda y} \tag{16}$$

with $D = \prod_{k=1}^\infty \left(1 - E(Q^k)\right)$. Moreover, $(x_Q)_\infty \le D \le (EQ)_\infty$.

2. *If $F_Q(x) \sim F(x)$ as $x \to 1$ and $\beta \in (\frac{1}{2}, 1]$ then $E_x\tau_y \sim C\cdot\exp\left(\int_0^y \theta_{\lambda t}\, dt\right)$ with some $C > 0$. In particular,*
   (a) *if $\beta = 1$ then*

$$E_x\tau_y \sim \exp\left(\lambda y - \int_1^{\lambda y} \frac{L(1/t)}{t}\, dt\cdot(1+o(1))\right). \tag{17}$$

   (b) *if $\beta \in (\frac{1}{2}, 1)$ then*

$$E_x\tau_y \sim \exp\left(\lambda y - (\lambda y)^{1-\beta} L\left(1/(\lambda y)\right)\cdot\left(\frac{\Gamma(1+\beta)}{1-\beta} + o(1)\right)\right). \tag{18}$$

Some remarks are in order.

Firstly, for the third assertion in Theorem 7, we rely heavily on a depoissonization lemma of Foss & Korshunov [18] (see Lemma 11). Although this result may be used also for our $\beta \in (0, \frac{1}{2})$ case, the results in [18] are not easy to state in this case. We refer to the original paper for the interested reader. Further depoissonization methods can be found in [20, 28].

Secondly, note the appearance of the $o(1)$ term in (17) and (18). As $y \to \infty$ the quotients

$$E_x\tau_y/\exp\left(\lambda y - \int_1^{\lambda y} \frac{L(1/t)}{t}\, dt\right)$$

and

$$E_x\tau_y/\exp\left(\lambda y - (\lambda y)^{1-\beta} L\left(1/(\lambda y)\right)\cdot\left(\frac{\Gamma(1+\beta)}{1-\beta}\right)\right)$$

may considerably deviate from constants. It seems complicated though to derive more accurate results.

5. Proof of Theorem 7

We need several lemmas. The first lemma is well known from calculus and shows that functions expanded in power series with asymptotically equivalent coefficients are asymptotically equivalent.

Lemma 8. *Let (a_n) and (b_n) be two non-negative real sequences and let $A(x) = \sum_{k=1}^{\infty} a_k x^k$ and $B(x) = \sum_{k=1}^{\infty} b_k x^k$ be convergent for all x. If $a_k \sim c \cdot b_k$ as $k \to \infty$ then $A(x) \sim c \cdot B(x)$ as $x \to \infty$. In particular, if $a_k \asymp b_k$ then $A(x) \asymp B(x)$.*

Proof. First note that $A(x)$ and $B(x)$ tend to ∞ as $x \to \infty$. For all $\varepsilon > 0$ there is an $m \in \mathbb{N}$ such that $b_k(1-\varepsilon/2) \le a_k \le b_k(1+\varepsilon/2)$. Thus

$$A(x) \le \sum_{k=1}^{m-1} x^k a_k + \sum_{k=m}^{\infty} x^k b_k(1+\varepsilon/2)$$

Let $a'_m = \sum_{k=1}^{m-1} x^k a_k$ and $b'_m = \sum_{k=1}^{m-1} x^k b_k$. Then

$$\frac{A(x)}{B(x)} \le \frac{a'_m}{B(x)} + (1+\varepsilon/2)(1-\frac{b'_m}{B(x)}).$$

Thus for every $\varepsilon > 0$, $A(x)/B(x) \le (1+\varepsilon)$ ultimately as $x \to \infty$. It can be shown similarly that $A(x)/B(x) \ge (1-\varepsilon)$. □

The next result gives the asymptotics of the coefficients $\pi_k(s) = \prod_{i=1}^{k}(\theta_i + s)$ in terms of the moments $E(Q^k)$.

Lemma 9. *Let M be the smallest integer such that $\sum_{j=1}^{\infty} E(Q^j)^M < \infty$. Then*

$$\pi_k(s) \asymp (\lambda+s)^k \exp\left(-\sum_{i=1}^{M-1} \frac{1}{i}\left(\frac{\lambda}{\lambda+s}\right)^i \sum_{j=1}^{k} E(Q^j)^i\right). \tag{19}$$

Proof. For brevity we write $\sigma_k(s) = (\lambda+s)^{-k}\,\pi_k(s)$. Note that $\sigma_k(s) = \prod_{j=1}^{k}(1-\mu_j)$, with $\mu_t = \frac{\lambda}{\lambda+s}E(Q^t)$. By taking logarithms we obtain $\log \sigma_k(s) = \sum_{j=1}^{k} \log\left(1-\mu_j\right)$. We have $\mu_j \le c = \frac{\lambda}{\lambda+s}EQ < 1$ and from Taylor's formula $\log(1-x) = -\sum_{i=1}^{n} \frac{x^i}{i} - \frac{(x/z)^{n+1}}{n+1}$ for all $x \in [0,c)$ and some $z = z(x) \in [1-c, 1)$. Hence

$$\log \sigma_k(s) = -\sum_{i=1}^{M-1} \frac{1}{i} \sum_{j=1}^{k} \mu_j^i - d^M C$$

with $d \in [1, \frac{1}{1-c}]$ and $C = \frac{1}{M}\sum_{j=1}^{k} \mu_j^M < \infty$, and so (19) follows. □

The next result relates $F_Q(x)$ at $x = 1$ to the Mellin transform of Q at infinity.

Lemma 10. *Let $\beta > 0$. Then the following relations are equivalent:*

$$\overline{F}_Q(x) \sim (1-x)^\beta L(1-x) \qquad (x \to 1)$$

$$E(Q^t) \sim \Gamma(\beta+1)t^{-\beta}L(1/t) \qquad (t \to \infty)$$

Proof. Let $W(x) = \overline{F}_Q(e^{-x})$ be the probability distribution function of $W = -\log Q$. Since $(1-e^{-x})^\beta \sim x^\beta$ and

$$\frac{L(1-e^{-1/x})}{L(1/x)} = \frac{L(1/x + O(1/x^2))}{L(1/x)} \to 1, \tag{20}$$

we know that $W(x) \sim x^\beta L(1/x)$ as $x \to 0$. Therefore, by Karamata's Tauberian theorem (cf. [10])

$$E(Q^t) = \int_0^\infty w^t \, dF_Q(w) = \int_0^\infty e^{-tw} \, dW(w) \sim \Gamma(\beta+1) t^{-\beta} L(1/t),$$

which concludes the proof. $\square$

Lemma 11 (Foss & Korshunov [18]). *Let $(\xi_n)_{n\in\mathbb{N}}$ be a sequence of i.i.d. non-negative random variables with $E\xi_1^{1/\gamma} < \infty$ for some $\gamma \in (\frac{1}{2}, 1]$ and let $\lambda = E(\xi_1)$. Let N_t be the associated renewal counting process and let X be a random time with tail function $\overline{G}(t) = P(X > t)$. If $\overline{G}(n + h(n)) \sim \overline{G}(n)$ as $n \to \infty$ for all functions $h(n) = o(n^\gamma)$, then $P(X > \sum_{i=1}^n \xi_i) \sim \overline{G}(\lambda n)$.*

Lemma 12. *Let g be a non-increasing function such that $g(x) \to 0$ and $g(x+h(x)) \sim g(x)$ as $x \to \infty$ for all functions $h(n) = o(n^\gamma)$ for some $\gamma \in (\frac{1}{2}, 1]$. Then, as $x \to \infty$,*

$$\sum_{k=1}^\infty g(k) \frac{x^k}{k!} \sim e^x g(x).$$

Proof. We may assume that $g(0) = 1$ and set $\overline{G}(x) = P(X > x) = g(x)$ for some random variable X. Let $(N_t)_{t\geq 0}$ be a Poisson process with intensity one and let $\xi_i = N_i - N_{i-1}$. Then clearly $E\xi_i^a < \infty$ for all $a > 0$, and according to Lemma 11,

$$g(n) \sim P(X > \sum_{i=1}^n \xi_i) = P(X > N_n) = \sum_{k=1}^\infty e^{-n} g(k) \frac{n^k}{k!}.$$

Finally note that $P(N_{n+1} < X) \leq P(N_x < X) \leq P(N_n < X)$ for $x \in [n, n+1]$. It follows that $P(N_{n+1} < X)/P(N_n < X) \to 1$ and $g(n)/g(x) \to 1$ as $n, x \to \infty$. $\square$

Proof of Theorem 7. Clearly if $F_Q(x) \geq F(x)$ then $E_x\tau_y \geq E_x\tau'_y$, where τ'_y is the modified hitting time of a process where F_Q has been replaced by F. Since $\int_1^\infty L(u)/u \, du < \infty$ it follows from Lemma 10 that $\sum_{k=1}^\infty EQ^k < \infty$. In particular the variable M in Lemma 9 is equal to one and thus $\pi_k \asymp \lambda^k$. Applying Lemma 8 yields $E_x\tau_y \asymp e^{\lambda y}$. The inequalities for D are immediate from $x_Q^k \leq E(Q^k) \leq (EQ)^k$. This proves assertion 1.

To prove assertion 2 note that it follows from Lemma 10 that $E(Q^t) \sim \Gamma(\beta + 1)t^{-\beta}L(1/t)$, so that $\int_1^\infty E(Q^t) \, dt$ is finite and thus $\sum_{k=1}^\infty (EQ^k)^2$ is finite. Hence, application of Lemma 9 with $M = 2$ gives

$$\pi_k(s) \asymp \lambda^k \exp\left(-\sum_{j=1}^k E(Q^j) \right) \tag{21}$$

as $k \to \infty$. Since the function $t \mapsto EQ^t$ is decreasing it follows that

$$\int_1^k E(Q^t) \, dt \leq \sum_{j=1}^k E(Q^j) \leq \int_1^k E(Q^t) \, dt + EQ,$$

so that the sum in (21) can be approximated by an integral without changing the asymptotic behavior (w.r.t. $\asymp$). Let $h(x) = o(x^{\beta-\delta})$ with $\delta \in (0, \beta - \frac{1}{2})$. It follows from Lemma 10 and the mean value theorem that for x large enough there are constants $c, c' > 0$, such that

$$\begin{aligned}\int_x^{x+h(x)} E(Q^t)\,dt &\leq c\cdot\int_x^{x+h(x)}\frac{L(1/t)}{t^\beta}\,dt \leq c\cdot\int_x^{x+h(x)}\frac{1}{t^{\beta-\delta/2}}\,dt\\ &\leq c'\cdot h(x)\cdot x^{\delta/2-\beta} = o(x^{-\delta/2}).\end{aligned}$$

For $g(x) = \exp\left(-\int_1^x E(Q^t)\,dt\right)$ and $\gamma = \beta-\delta \in (\frac{1}{2}, 1]$, the conditions of Lemma 12 are fulfilled. The first part of assertion 2 follows from

$$\sum_{k=1}^\infty \pi_{k-1}(s)\frac{x^k}{k!} \asymp \sum_{k=1}^\infty \exp\left(-\int_1^k E(Q^t)\,dt\right)\frac{(\lambda x)^k}{k!} \sim \exp\left(\lambda x - \int_1^{\lambda x} E(Q^t)\,dt\right)$$

and

$$\lambda x - \int_1^{\lambda x} E(Q^t)\,dt = 1 + \frac{1}{\lambda}\int_1^{\lambda x}\lambda(1-E(Q^t))\,dt = 1 + \int_1^x \lambda(1-E(Q^{\lambda t}))\,dt.$$

Moreover, Lemma 10 yields

$$\int_1^{\lambda x} E(Q^t)\,dt \sim \Gamma(1+\beta)\int_1^x \frac{L(1/t)}{t^\beta}\,dt,$$

which proves assertion 2a in the $\beta = 1$ case. To see that 2b is true for $\beta < 1$, observe that

$$\Gamma(1+\beta)\int_1^x \frac{L(1/t)}{t^\beta}\,dt \sim \frac{\Gamma(1+\beta)}{1-\beta}\lambda^{1-\beta}x^{1-\beta}L(1/(\lambda x))$$

from Karamata's integration theorem (see [10]). □

6. Special case: beta distributed Q

We assume that Q has a beta distribution with density

$$f_Q(x) = \frac{x^{\alpha-1}(1-x)^{\beta-1}}{B(\alpha,\beta)},$$

for $x \in [0,1]$, where $B(\alpha,\beta) = \int_0^1 x^{\alpha-1}(1-x)^{\beta-1}\,dx = \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)}$ and $\alpha > 0$, $\beta > 0$. In this case we have

$$\theta_t = \lambda\left(1 - \frac{B(\alpha+t,\beta)}{B(\alpha,\beta)}\right) = \lambda\left(1-\frac{\Gamma(\alpha+t)\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\alpha+\beta+t)}\right).$$

Lemma 10 yields $E(Q^t) \sim \Gamma(\alpha+\beta)/\Gamma(\alpha)\cdot t^{-\beta}$ as $t\to\infty$. According to Theorem 7 we then have, for $\beta \in (\frac{1}{2}, 1)$,

$$E_x\tau_y = C\cdot\exp\left(\lambda y - (\lambda y)^{1-\beta}\frac{\Gamma(\alpha+\beta)}{(1-\beta)\Gamma(\alpha)} + o(y^{1-\beta})\right).$$

In case $\beta = 1$, when $Q \overset{d}{=} U^{1/\alpha}$, we can obtain more explicit results and sharper asymptotics.

Corollary 13. *For $x \leq y$ and the mean hitting- and recovery times are given by*

$$E_x\tau_y = \int_x^y \alpha\lambda^{-\alpha}u^{-\alpha}e^{\lambda u}\gamma(\alpha,\lambda u)\,du, \tag{22}$$

$$E_{(x)}\sigma_x = \alpha\lambda^{-\alpha-1}x^{-\alpha}e^{\lambda x}\gamma(\alpha,\lambda x) - \frac{1}{\lambda}, \tag{23}$$

where $\gamma(\alpha,\lambda u) = \int_0^{\lambda u} t^{\alpha-1}e^{-t}\,dt$.

Proof. Since $\theta_t = \frac{\lambda t}{\alpha + t}$ it follows that

$$\pi_k(s) = (\lambda + s)^k \, \frac{(\frac{\alpha s}{\lambda+s} + 1)_k}{(\alpha+1)_k}, \tag{24}$$

where $(x)_k = x \cdot (x+1) \cdots (x+k-1)$ is the Pochhammer symbol. We thus have

$$R_0'(x) = \sum_{k=1}^{\infty} \frac{(\lambda x)^{k-1}}{(k-1)!} \frac{(k-1)!}{(\alpha+1)_{k-1}} = \alpha \lambda^{-\alpha} x^{-\alpha} e^{\lambda x} \gamma(\alpha, \lambda x).$$

With $E_x \tau_y = R_0(y) - R_0(x)$ and $E_{(x)}\sigma_x = \frac{1}{\lambda}\left(R_0'(x) - 1\right)$ this gives the result. □

Expression (23) leads, using $\lim_{x\to\infty} \gamma(\alpha, \lambda x) = \Gamma(\alpha)$, to a sharp result for the asymptotics.

Corollary 14. *For $x \le y$ and $Q \stackrel{d}{=} U^{1/\alpha}$ we have the asymptotics*

$$E_{(y)}\sigma_y \;\sim\; E_x \tau_y \;\sim\; \frac{\Gamma(\alpha+1)}{\lambda^{\alpha+1} y^{\alpha}} e^{\lambda y}. \tag{25}$$

In the special case $Q \stackrel{d}{=} U$, i.e. if $\alpha = 1$, we obtain $R_0(x) = C + \frac{1}{\lambda}\left(\mathrm{Ei}(\lambda u) - \log u\right)$ with $\mathrm{Ei}(x) = -\int_{-x}^{\infty} \frac{1}{u} e^{-u}\, du$ the exponential integral. Consequently,

$$E_x \tau_y = \frac{\mathrm{Ei}(\lambda y) - \mathrm{Ei}(\lambda x) - \log(y/x)}{\lambda} \tag{26}$$

and

$$E_{(x)}\sigma_x = \frac{e^{\lambda x} - 1 - \lambda x}{\lambda^2 x}. \tag{27}$$

The asymptotic result $E_{(y)}\sigma_y \sim E_x \tau_y \sim \frac{1}{\lambda^2 y} e^{\lambda y}$ follows immediately from (25) with $\alpha = 1$. Alternatively, it follows from (26) and the fact that $\mathrm{Ei}(x) \sim \frac{1}{x} e^x$ (see [2], p. 231).

Let ${}_1F_1(a;b;x) = \sum_{k=0}^{\infty} \frac{(a)_k}{(b)_k} \frac{x^k}{k!}$ denote Kummer's hypergeometric function (see e.g. [6]).

Corollary 15. *For $x \le y$ and $Q \stackrel{d}{=} U^{1/\alpha}$ the Laplace transforms of the hitting- and recovery times are given by*

$$E_x e^{-s\tau_y} \;=\; \frac{{}_1F_1(\frac{\alpha s}{\lambda+s}; \alpha; (\lambda+s)y)}{{}_1F_1(\frac{\alpha s}{\lambda+s}; \alpha; (\lambda+s)x)},$$

$$E_x e^{-s\sigma_x} \;=\; 1 + \frac{s}{\lambda}\left(1 - \frac{{}_1F_1(\frac{\alpha s}{\lambda+s} + 1; \alpha+1; (\lambda+s)x)}{{}_1F_1(\frac{\alpha s}{\lambda+s}; \alpha; (\lambda+s)x)}\right).$$

Proof. We get from (4) and (24) that

$$\frac{1}{s} R_s'(x) \;=\; {}_1F_1\Big(\frac{\alpha s}{\lambda+s} + 1; \alpha+1; (\lambda+s)x\Big)$$

and since ${}_1F_1(a;b;x) = \frac{b-1}{a-1} \cdot \frac{d}{dx} {}_1F_1(a-1; b-1; x)$ and $R_s(0) = 1$,

$$1 + sR_s(x) \;=\; {}_1F_1\Big(\frac{\alpha s}{\lambda+s}; \alpha; (\lambda+s)x\Big).$$

Application of Theorem 2 then completes the proof. □

7. Special case: deterministic Q

In this section we discuss the case where Q is a constant, say $Q = q \in [0, 1)$. The case $q = 1/2$ corresponds to the standard model for TCP, which is the dominant protocol for data transmission over the internet. The case $q = 0$ describes the age process of renewal theory with exponential renewal epochs.

In general, we have $\theta_a = \lambda(1 - q^a)$ and

$$\prod_{j=1}^{k}(\theta_{a+j} + u) = (\lambda + u)^k \prod_{j=0}^{k-1}(1 - \frac{\lambda q^{a+1}}{\lambda + u} q^j) = (\lambda + u)^k (\frac{\lambda}{\lambda + u} q^{a+1}; q)_k,$$

where $(a; q)_k = \prod_{i=0}^{k-1}(1-aq^i)$ is the so called q-series (see Askey et al. [2]). Recall that $(q)_\infty = \lim_{k\to\infty}(q; q)_k$. Specializing Theorem 2 to the deterministic case immediately yields the following result.

Corollary 16. *For $x \le y$ and $Q = q \in [0, 1)$ the means and Laplace transforms of τ_y and σ_x are given by*

$$E_x\tau_y = \frac{1}{\lambda}\sum_{k=1}^{\infty}\frac{(\lambda y)^k - (\lambda x)^k}{k!}(q; q)_{k-1}, \tag{28}$$

$$E_x e^{-s\tau_y} = \frac{1 + \sum_{k=1}^{\infty}(x(\lambda + s))^k/k! \cdot (\frac{\lambda}{\lambda+s}; q)_k}{1 + \sum_{k=1}^{\infty}(y(\lambda + s))^k/k! \cdot (\frac{\lambda}{\lambda+s}; q)_k}, \tag{29}$$

$$E_{(x)}\sigma_x = \frac{1}{\lambda}\sum_{k=1}^{\infty}\frac{(\lambda x)^k}{k!}(q; q)_k, \tag{30}$$

$$E_{(x)}e^{-s\sigma_x} = 1 + \frac{s}{\lambda}\left(1 - \frac{1 + \sum_{k=1}^{\infty}((\lambda + s)x)^k/k!(\frac{\lambda}{\lambda+s}q; q)_k}{1 + \sum_{k=1}^{\infty}((\lambda + s)x)^k/k!(\frac{\lambda}{\lambda+s}; q)_k}\right). \tag{31}$$

The asymptotic result below is an immediate consequence of Theorem 7, part 1.

Corollary 17. *For $x \le y$ and $Q = q \in [0, 1)$ we have the asymptotics*

$$E_{(y)}\sigma_y \sim E_x\tau_y \sim \frac{(q)_\infty}{\lambda}e^{\lambda y}. \tag{32}$$

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