

# Pricing Constant Maturity Credit Default Swaps Under Jump Dynamics

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### **Abstract**

In this paper we discuss the pricing of Constant Maturity Credit Default Swaps (CMCDS) under single sided jump models. The CMCDS offers default protection in exchange for a floating premium which is periodically reset and indexed to the market spread on a CDS with constant maturity tenor written on the same reference name. By setting up a firm value model based on single sided Lévy models we can generate dynamic spreads for the reference CDS. The valuation of the CMCDS can then easily be done by Monte Carlo simulation.

**Keywords:** Single sided Levy processes; Structural models; Credit risk; Default probability; Constant Maturity Credit Default Swaps; Monte Carlo methods

**JEL subject category:** C02, C15, C63, G12

# 1 Introduction

Constant Maturity Credit Default Swaps (CMCDS) are similar to the common Credit Default Swap (CDS), offering the investor protection in exchange of a periodically paid spread. In contrast to the CDS spread, which is fixed throughout the maturity of the CDS, the spread of a CMCDS is floating and is indexed to a reference CDS with a fixed time to maturity at reset dates. The floating spread is proportional to the constant maturity CDS market spread. The maturity of the CMCDS and of the reference CDS does not have to be the same.

The aim of this paper is to present a Monte Carlo method for estimating the participation rate based on single sided Lévy models. We set up a firm's value model where the value is driven by the exponential of a Lévy process with positive drift and only negative jumps. These single sided firm's value models allow us to calculate the default probabilities fast by a double Laplace inversion technique presented in Rogers (2000) and Madan and Schoutens (2007). The fast calculation of the default probabilities implies a fast calculation of CDS values which is important for calibration. The model's ability to calibrate on a CDS term structure has already been proven in Madan and Schoutens (2007). Based on the single sided firm's value model Jönsson and Schoutens (2007) present how a dynamic spread generator can be set up that allows pricing of exotic options on single name CDS by Monte Carlo simulations.

The paper is organized as follows. In the following section we present the mechanics and valuation of Constant Maturity Credit Default Swaps. In Section 3 the single sided firm's value model is introduced. The Monte Carlo algorithm and numerical results are given in Section 4. The paper ends with conclusions.

## 2 Constant Maturity Credit Default Swaps

### 2.1 The Mechanics of CMCDS

A single name Constant Maturity Credit Default Swap (CMCDS) has the same features as a standard single name CDS. It offers the protection buyer protection against loss at the event of a default of the reference credit in exchange for a periodically paid spread. The difference is that the spread paid is reset at pre-specified reset dates. At each reset date the CMCDS spread is set to a reference CDS market spread times a multiplier, the so called *participation rate*. The reference CDS has a constant maturity which is not necessarily the same as the

maturity of the CMCDS.

## 2.2 Valuation

We want to value a CMCDS with maturity  $\hat{T}$  and  $M$  reset dates  $0 = t_0 < t_1 < t_2 < \dots < t_{M-1} < \hat{T}$  with a reference CDS with constant maturity tenor  $T$ . To value the CMCDS is to find the participation rate, i.e. the factor we should multiply the reference spread of the CDS. Just as for the CDS we equate the present value of the loss leg and the payment leg. However, the loss leg of the CMCDS and the loss leg of a CDS written on the same reference name and with the same maturity  $\hat{T}$  are identical. This implies that the premium legs of the two contracts must be the same.

Denote by  $\tau$  the default time of the reference credit.

Let  $D(t_0, t)$  denote the  $t_0$ -value of a defaultable zero-coupon bond with maturity  $t$ . We will assume that the spot rate  $r = \{r_t, t \geq 0\}$  is deterministic so that the value of the defaultable zero-coupon bond is

$$\begin{aligned} D(t_0, t) &= \mathbb{E}_{\mathbb{Q}} \left[ \exp \left( - \int_{t_0}^t r_s ds \right) \mathbf{1}(\tau > t) \right] \\ &= \exp \left( - \int_{t_0}^t r_s ds \right) P_{\mathbb{Q}}(\tau > t), \end{aligned}$$

under a risk -neutral measure  $\mathbb{Q}$ .

Assuming a constant recovery rate  $R$  the fair spread of the reference CDS with constant maturity tenor  $T$  at time  $t$  is

$$S(t, t+T) = \frac{(1-R) \left( - \int_0^T d(t, t+s) dP_{\mathbb{Q}}(\tau > t+s | \tau > t) \right)}{\int_0^T d(t, t+s) P_{\mathbb{Q}}(\tau > t+s | \tau > t) ds}, \quad (1)$$

where  $d(t, t+s) = \exp \left( - \int_t^{t+s} r_u du \right)$  is the riskless discount factor and the probability of no default before time  $t+s$ ,  $s \geq 0$ , given that there was no default before time  $t$ , that is, the probability that the firm survives at least to time  $t+s$  given that it survived until  $t$ , is denoted by  $P_{\mathbb{Q}}(\tau > t+s | \tau > t)$ .

The value of the payment at time  $t_{m+1}$  is based on the floating spread reset at  $t_m$ , that is, for  $m = 0, 1, \dots, M-1$ ,

$$Z_{m+1}(t_{m+1}) = \Delta(t_m, t_{m+1}) S(t_m, t_m + T) \mathbf{1}(\tau > t_{m+1}),$$

with  $t_M = \hat{T}$  and where  $Z_{m+1}(t_{m+1})$  is the time  $t_{m+1}$  value of the payment

scheduled for  $t_{m+1}$ ,  $\Delta(t_m, t_{m+1})$  is the length of the period over which the spread is paid, expressed in the appropriate day-count convention,  $S(t_m, t_m + T)$  the market spread of the reference CDS at time  $t_m$ ,  $\tau$  is the default time and  $\mathbf{1}(\tau > t_{m+1})$  is the survival indicator until time  $t_{m+1}$ .

Using the defaultable zero-coupon bond with maturity  $t_{m+1}$  as the numeraire we can express the  $t_0$ -value of the payment scheduled for  $t_{m+1}$  as

$$Z_{m+1}(t_0) = D(t_0, t_{m+1})\Delta(t_m, t_{m+1})\mathbb{E}_{\mathbb{Q}_{m+1}}[S(t_m, t_m + T)],$$

for  $m = 0, 1, \dots, M - 1$ , where the expectation is taken with respect to the risk-neutral probability measure corresponding to the numeraire.

The  $t_0$ -value of the floating premium leg is thus

$$\text{FL}(t_0, \hat{T}, T) = \sum_{m=0}^{M-1} D(t_0, t_{m+1})\Delta(t_m, t_{m+1})\mathbb{E}_{\mathbb{Q}_{m+1}}[S(t_m, t_m + T)], \quad (2)$$

with  $t_M = \hat{T}$ . (We have omitted any premium accrued on default for ease of presentation.)

As mentioned before, the values at the valuation date  $t_0$  of the fee leg of the CMCDS and the fee leg of a CDS written on the same reference name and with the same maturity  $\hat{T}$  must be equal. Hence we should find a participation rate  $p(t_0, \hat{T}, T)$  such that the fee leg of the CMCDS equals the fee leg of a CDS written on the same reference name with maturity  $\hat{T}$  at time  $t_0$ , that is

$$p(t_0, \hat{T}, T)\text{FL}(t_0, \hat{T}, T) = S(t_0, \hat{T})\text{PV01}(t_0, \hat{T}), \quad (3)$$

where  $\text{PV01}(t_0, \hat{T})$  is the time  $t_0$  risky annuity of a CDS with the same maturity and written on the same reference credit as the CMCDS, that is, the  $t_0$ -value of the premium leg assuming a premium of 1 basis point

$$\text{PV01}(t_0, \hat{T}) = \int_{t_0}^{\hat{T}} \exp\left(-\int_{t_0}^s r_u du\right) \mathbb{P}_{\mathbb{Q}}(\tau > s) ds.$$

Thus, from (3) we have that the participation rate is

$$p(t_0, \hat{T}, T) = \frac{S(t_0, \hat{T}) \int_{t_0}^{\hat{T}} \exp\left(-\int_{t_0}^s r_u du\right) \mathbb{P}_{\mathbb{Q}}(\tau > s) ds}{\sum_{m=0}^{M-1} D(t_0, t_{m+1})\Delta(t_m, t_{m+1})\mathbb{E}_{\mathbb{Q}_{m+1}}[S(t_m, t_m + T)]}. \quad (4)$$

### 2.3 Caps and Floors

A natural extension of the floating premium CMCDS is to incorporate a cap. Following Pedersen and Sen (2004) we will assume that the cap acts directly on the reset spread. The cap is a portfolio of caplets. A caplet is an European call option and is used to limit the spread paid.

Denote by  $K_C$  the spread cap. The  $t_0$ -value of the caplet at reset date  $t_{m+1}$ , ignoring premium accrual on default, is

$$C_{m+1}(t_0) = D(t_0, t_{m+1})\Delta(t_m, t_{m+1})\mathbb{E}_{\mathbb{Q}_{m+1}}[(S(t_m, t_m + T) - K_C)^+].$$

The  $t_0$ -value of the cap is the sum of the  $t_0$ -values of the caplets

$$C(t_0) = \sum_{m=0}^{M-1} C_{m+1}(t_0).$$

Similarly, with  $K_F$  denoting the spread floor, the  $t_0$ -value of the floorlet at the reset date  $t_{m+1}$  is

$$F_{m+1}(t_0) = D(t_0, t_{m+1})\Delta(t_m, t_{m+1})\mathbb{E}_{\mathbb{Q}_{m+1}}[(K_F - S(t_m, t_m + T))^+],$$

and  $t_0$ -value of the floor, that is, the portfolio of floorlets,

$$F(t_0) = \sum_{m=0}^{M-1} F_{m+1}(t_0).$$

### 2.4 Mark-to-Market

The mark-to-market of a CMCDS is done by comparing the present value of the contract floating fee leg with the market value of the protection leg. The value of the protection leg at a time  $t$ ,  $t_0 \leq t \leq \hat{T}$ , is  $S(t, \hat{T})\text{PV01}(t, \hat{T})$ . The value of the floating fee leg is  $p_{t_0}\text{FL}(t, \hat{T}, T)$ . The mark-to-market for the protection seller is thus

$$\text{MTM}_{\text{CMCDS}}(t) = \left( \frac{p(t_0, \hat{T}, T)}{p(t, \hat{T}, T)} - 1 \right) S(t, \hat{T})\text{PV01}(t, \hat{T}),$$

since the floating fee leg at time  $t$  is equal to the protection leg at time  $t$  divided by the participation rate at  $t$ , i.e.,  $\text{FL}(t, \hat{T}, T) = S(t, \hat{T})\text{PV01}(t, \hat{T})/p(t, \hat{T}, T)$ .

The mark-to-market of a standard CDS with maturity  $\hat{T}$  is for the protection

seller

$$\text{MTM}_{\text{CDS}}(t) = \left( S(t_0, \hat{T}) - S(t, \hat{T}) \right) \text{PV01}(t, \hat{T}).$$

## 2.5 Valuation Using Forward Spreads and Convexity Adjustment

A first approximation to the value of the floating fee leg is to approximate the expected market spread at the reset dates with the forward spread at time  $t_0$ . The forward spread is the fair spread for a forward starting CDS. Denote by  $S(t_0, t, t + T)$  the forward spread at time  $t_0$  of a CDS starting at time  $t$  with maturity  $t + T$ . Its value is given by

$$S(t_0, t, t + T) = \frac{S(t_0, t + T)\text{PV01}(t_0, t + T) - S(t_0, t)\text{PV01}(t_0, t)}{\text{PV01}(t_0, t + T) - \text{PV01}(t_0, t)}.$$

Substituting the expected spreads in (2) with the forward spreads the  $t_0$ -value of the fee leg is approximated by

$$\text{FL}(t_0, \hat{T}, T) \approx \sum_{m=0}^{M-1} D(t_0, t_{m+1})\Delta(t_m, t_{m+1})S(t_0, t_m, t_m + T), \quad (5)$$

where  $S(t_0, t_0, t_0 + T) = S(t_0, t_0 + T)$ .

We need however to adjust this approximation since the realized spread at the reset dates are not equal to the forward spreads calculated at the valuation date  $t_0$ . The adjustment that has to be added to the fee leg is called the *convexity adjustment* and is given by

$$A(t_0; \hat{T}, T) = \sum_{m=1}^{M-1} D(t_0, t_{m+1})\Delta(t_m, t_{m+1})A(t_m, t_m + T), \quad (6)$$

where for  $m = 1, \dots, M - 1$

$$A(t_m, t_m + T) = \mathbb{E}_{\mathbb{Q}_{m+1}}[S(t_m, t_m + T)] - S(t_0, t_m, t_m + T).$$

## 3 Single Sided Firm's Value Model

Lévy models have proven their usefulness in financial modelling, such as in equity and fixed income settings, over the last decade, see e.g. Schoutens (2003), and has recently gained growing interest in credit risk modelling, see e.g. Cariboni

(2007) and Cariboni and Schoutens (2007).

We will in this section set up the single sided firm's value model presented in Madan and Schoutens (2007). We thus model the value of the reference entity of a CDS by exponential Lévy driven models with positive drift and only negative jumps. Following the same methodology as Black and Cox (1976) default is triggered the first time the firm's value is crossing a low barrier. The models were used to construct spread dynamics to price exotic credit default swaptions in Jönsson and Schoutens (2007).

### 3.1 Single Sided Lévy Processes

We first introduce some notation. Let  $Y = \{Y_t, t \geq 0\}$  be a pure jump Lévy process that has only negative jumps, that is,  $Y$  is spectrally negative, and let  $X = \{X_t, t \geq 0\}$  be given by

$$X_t = \mu t + Y_t, \quad t \geq 0,$$

where  $\mu$  is positive real number.

The Laplace transform of  $X_t$

$$E[\exp(zX_t)] = \exp(t\psi_X(z)),$$

where  $\psi_X(z)$  is the Lévy exponent, which by the Lévy-Khintchin representation has the form

$$\psi_X(z) = \mu z + \int_{-\infty}^0 (e^{zx} - 1 + z(|x| \wedge 1))\nu(dx).$$

The Lévy measure  $\nu(dx)$  satisfies the integrability condition

$$\int_{-\infty}^0 (|x| \wedge 1)\nu(dx) < \infty.$$

For the processes we consider in this paper the Lévy measure has a density and we can write  $\nu(dx) = m(x)dx$ , where  $m(x)$  is the density function. For the general theory of Lévy processes see, for example, Bertoin (1996) and Sato (2000).



### 3.2 Firm's Value Model

Let  $X = \{X_t, t \geq 0\}$  be a pure jump Lévyprocess. The risk neutral value of the firm at time  $t$  is then

$$V_t = V_0 \exp(X_t), \quad t \geq 0,$$

and we work under an admissible pricing measure  $\mathbb{Q}$ .

For a given recovery rate  $R$  default occurs the first time the firm's value is below the value  $RV_0$ . That is, the time of default is defined as

$$\tau := \inf\{t \geq 0 : V_t \leq RV_0\}.$$

Let us denote by  $P(t) := P_{\mathbb{Q}}(\tau > t)$  the risk-neutral *survival probability* between 0 and  $t$ :

$$\begin{aligned} P(t) &= P_{\mathbb{Q}}(X_s > \log R, \text{ for all } 0 \leq s \leq t) \\ &= P_{\mathbb{Q}}\left(\min_{0 \leq s \leq t} X_s > \log R\right) \\ &= \mathbb{E}_{\mathbb{Q}}\left[\mathbf{1}\left(\min_{0 \leq s \leq t} X_s > \log R\right)\right] \\ &= \mathbb{E}_{\mathbb{Q}}\left[\mathbf{1}\left(\min_{0 \leq s \leq t} V_s > RV_0\right)\right] \end{aligned}$$

where we used the indicator function  $\mathbf{1}(A)$ , which is equal to 1 if the event  $A$  is true and zero otherwise; the subindex  $\mathbb{Q}$  refers to the fact that we are working in a risk-neutral setting.

As can be seen in (1) and (4) the price of the CDS and the participation rate of the CMCDs, respectively, depends on the survival probability, or non-default probability, of the firm. In our case, where we work under single sided Lévy models with positive drift and only negative jumps, the default probabilities can be calculated by a double Laplace inversion based on the Wiener-Hopf factorization as presented in Madan and Schoutens (2007).

### 3.3 Example - The Shifted Gamma-Model

Three well known examples of single sided jump models with positive drift were presented in Madan and Schoutens (2007), namely: the Shifted Gamma, the Shifted Inverse Gaussian and the Shifted CMY model. We present here in detail only the Shifted Gamma model.

The density function of the Gamma distribution  $\text{Gamma}(a, b)$  with parameters  $a > 0$  and  $b > 0$  is given by

$$f_{\text{Gamma}}(x; a, b) = \frac{b^a}{\Gamma(a)} x^{a-1} \exp(-xb), \quad x > 0.$$

The characteristic function is given by

$$\phi_{\text{Gamma}}(u; a, b) = (1 - iu/b)^{-a}, \quad u \in \mathbb{R}.$$

Clearly, this characteristic function is infinitely divisible. The Gamma-process  $G = \{G_t, t \geq 0\}$  with parameters  $a, b > 0$  is defined as the stochastic process which starts at zero and has stationary, independent Gamma-distributed increments. More precisely, the time enters in the first parameter:  $G_t$  follows a  $\text{Gamma}(at, b)$  distribution.

The Lévy density of the Gamma process is given by

$$m(x) = a \exp(-bx)x^{-1}, \quad x > 0.$$

The properties of the  $\text{Gamma}(a, b)$  distribution given in Table 1 can easily be derived from the characteristic function.

	$\text{Gamma}(a, b)$
mean	$a/b$
variance	$a/b^2$
skewness	$2/\sqrt{a}$
kurtosis	$3(1 + 2/a)$

Table 1: Mean, variance, skewness and kurtosis of the Gamma distribution.

Note also that we have the following scaling property: if  $X$  is  $\text{Gamma}(a, b)$  then for  $c > 0$ ,  $cX$  is  $\text{Gamma}(a, b/c)$ .

Let us start with a unit variance Gamma-process  $G = \{G_t, t \geq 0\}$  with parameters  $a > 0$  and  $b > 0$ . As driving Lévy process (in a risk-neutral setting), we then take

$$X_t = \mu t - G_t, \quad t \geq 0,$$

where in this case  $\mu = r - \log(\phi(i)) = r + a \log(1 + b^{-1})$ . Thus, there is a deterministic up trend with random downward shocks coming from the Gamma process.

The characteristic exponent is in this case available in closed form

$$\psi(z) = \mu z - a \log(1 + zb^{-1}).$$

### 3.3.1 Calibration

We have calibrated the Shifted Gamma model to the term structure of ABN-AMRO CDSs minimizing the average absolute percentage error

$$\text{APE} = \frac{1}{\text{mean CDS spread}} \sum_{\text{CDS}} \frac{|\text{market CDS spread} - \text{model CDS spread}|}{\text{number of CDSs}}.$$

Calibrating the Shifted Gamma model to the term structure of ABN-AMRO on the 5th of January 2005 gives the parameters  $a = 0.74475$  and  $b = 6.59491$ . The fit of the Shifted Gamma model on the market CDSs is shown in Figure 1. The evolution of 1, 3, 5, 7 and 10 years par spreads of the ABN-AMRO CDSs from 5<sup>th</sup> January 2005 to 8<sup>th</sup> February 2006 is shown in Figure 2. The evolution over time of the parameters of the Shifted Gamma model calibrated on the term structure is shown in Figure 3.

An extensive calibration study was performed by Madan and Schoutens (2007) and the fitting error was typically around 1-2 basis points per quote.

## 4 A Monte Carlo Valuation Approach

As seen from (2) we need a model for the spread dynamics of the reference CDS with constant maturity. We will use the spread dynamics developed in Jönsson and Schoutens (2007).

The method is based on four steps. The first step is to calibrate the model on a given term structure of market spreads. The calibration gives us the model parameters that best matches the current market situation. Next we precalculate for a fine grid of firm values  $\{v_1, \dots, v_K\}$  the corresponding spread values  $\{S(v_i, R, r, T, \theta), i = 1, \dots, K\}$  by using the fast way of calculating the default probabilities presented by Madan and Schoutens (2007). The third step is to generate firm's value paths on a time gride. Finally, for every path and each value on the time grid the corresponding spread is obtained by interpolating the simulated firm value in  $\{v_1, \dots, v_K\}$  and its corresponding spread values  $\{S(v_i, R, r, T, \theta), i = 1, \dots, K\}$ .

For each reset date  $t_m, m = 0, 1, \dots, M - 1$ , estimate the expected value in

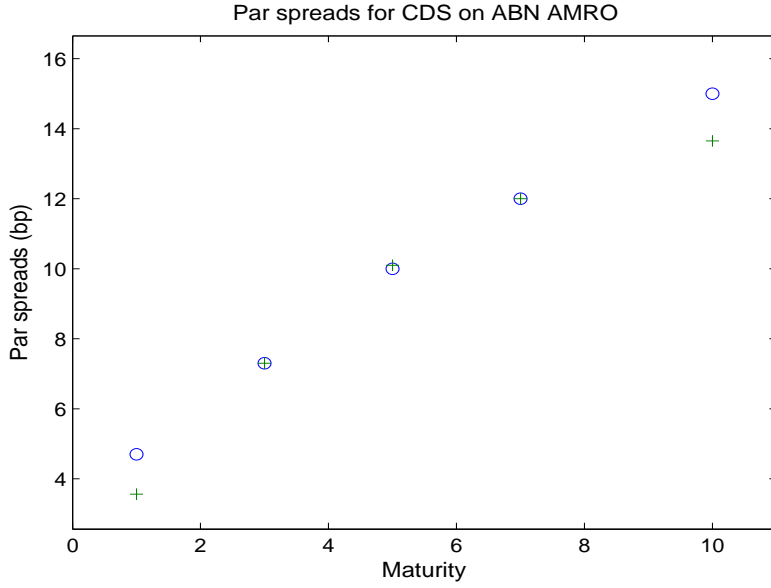


Figure 1: Calibration on ABN AMRO, January 5, 2005. Market spreads are marked with 'o' and model spreads are marked with '+'. Underlying model is the Shifted Gamma with  $a = 0.74475$  and  $b = 6.59491$ .

(2) by simulating  $N$  spread paths of the reference CDS with constant maturity

$$\mathbb{E}_{Q_{m+1}}[S(t_m, t_m+T)] \approx \hat{s}_m = \frac{1}{NP_{\mathbb{Q}}(\tau > t_{m+1})} \sum_{n=1}^N s^{(n)}(t_m, t_m+T) \mathbf{1}^{(n)}(\tau > t_{m+1}),$$

where  $s^{(n)}(t_m, t_m+T)$  is the spread at reset time  $t_m$  of the  $n$ :th path and  $\mathbf{1}^{(n)}(\tau > t_m)$  is the survival indicator function until  $t_m$  of the  $n$ :th path.

The participation rate is then

$$\hat{p}(t_0, \hat{T}, T) = \frac{s(t_0, \hat{T})PV01(t_0, \hat{T})}{\sum_{m=0}^{M-1} D(t_0, t_{m+1})\Delta(t_m, t_{m+1})\hat{s}_m}. \quad (7)$$

To be more precise, we estimate  $q = 1/p$  by

$$\hat{q} = \frac{\sum_{m=0}^{M-1} D(t_0, t_{m+1})\Delta(t_m, t_{m+1})\hat{s}_m}{s(t_0, \hat{T})PV01(t_0, \hat{T})}.$$

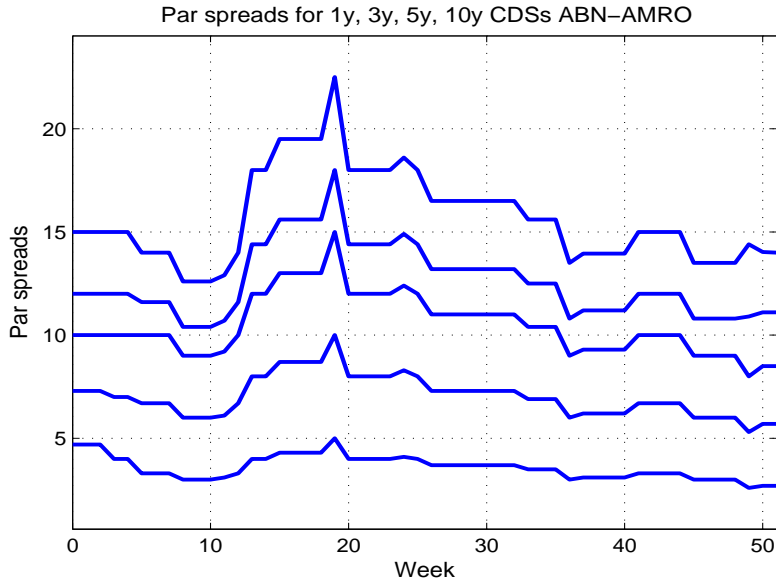


Figure 2: The par spreads of a 1, 3, 5, 7 and 10 year CDS spreads on ABN-AMRO, weekly data from 5<sup>th</sup> January 2005 to 8<sup>th</sup> February 2006.

The variance of this estimate is

$$\text{Var}(\hat{q}) = \frac{\sum_{m=0}^{M-1} (D(t_0, t_{m+1}) \Delta(t_m, t_{m+1}))^2 \text{Var}(\hat{s}_m)}{\left( s(t_0, \hat{T}) \text{PV01}(t_0, \hat{T}) \right)^2},$$

since we use independent paths to estimate each expected reset spread  $\hat{s}_m$ .

The variance of the estimate  $\hat{p}$  is given by noticing that

$$\mathbb{E}[(\hat{q} - q)^2] = \mathbb{E}\left[\left(\frac{1}{\hat{p}} - \frac{1}{p}\right)^2\right] \approx \frac{\mathbb{E}[(\hat{p} - p)^2]}{p^4},$$

if  $\hat{p}$  is close to its true value  $p$ . Thus,

$$\text{Var}(\hat{p}) = \mathbb{E}[(\hat{p} - p)^2] \approx p^4 \text{Var}(\hat{q}) \approx \hat{p}^4 \text{Var}(\hat{q}).$$

The standard deviation of our estimate is therefore

$$\sigma_{\hat{p}} \approx \hat{p}^2 \sqrt{\text{Var}(\hat{q})}.$$

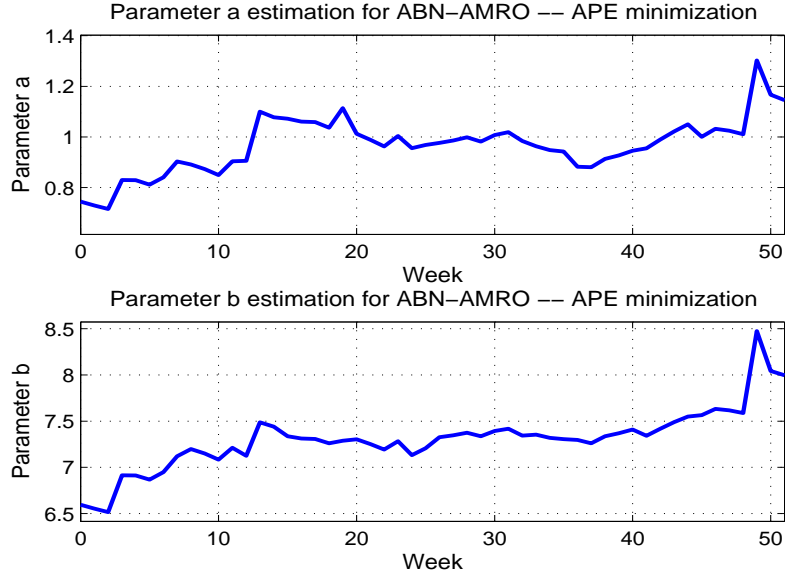


Figure 3: The parameters of Shifted Gamma model calibrated to the term structure of the CDSs on ABN-AMRO, weekly data from 5<sup>th</sup> January 2005 to 8<sup>th</sup> February 2006. The objective function to be minimized was the average absolute percentage error (APE).

The convexity adjustment for the premium leg between dates  $t_m$  and  $t_{m+1}$ , that is, the difference between the expected spread realized at  $t_m$  and the forward spread for the same period, is approximated by

$$\hat{A}(t_0; t_m, t_m + T) = \hat{s}_m - S(t_0; t_m, t_m + T),$$

where  $S(t_0; t_m, t_m + T)$  is the  $t_0$  forward spread on a CDS starting at time  $t_m$  and maturing at  $t_m + T$ . The convexity adjustment (6) is approximated by

$$\hat{A}(t_0; \hat{T}, T) = \sum_{m=1}^{M-1} D(t_0, t_m) \Delta(t_m, t_{m+1}) \hat{A}(t_m, t_m + T).$$

#### 4.1 Numerical Results

We calculated the participation rate of a 3 year CMCDs with a reference CDS written on ABN-AMRO using the proposed Monte Carlo approach with 100,000 paths for each reset date. Total time for calculating one participation rate is ap-

proximately 30 minutes. In Table 2 we present the participation rate with standard errors estimated using the single sided firm's value Monte Carlo approach, the participation rate calculated using the forward spreads without convexity adjustments, and the convexity adjustment for different constant maturities of the reference CDS. The participation rate calculated using the Monte Carlo approach and the forward rate approach (without convexity adjustment) are shown for different constant maturities in Figure 4.

From Table 2 and Figure 4 we can see that the Monte Carlo participation rate is always lower than the forward spread participation rate, which implies that the forward rates underestimate the expected future reset spreads in this case. The decrease of the participation rates is due to the fact that we have an upward sloping term structure of the reference CDS.

$T_{CDS}$	Part. rate MC	(s.e.)	Part. rate F	Conv. adj.	(s.e.)
1.0	0.834099	(0.004879)	0.840152	0.000018	(0.000015)
2.0	0.700115	(0.002919)	0.715743	0.000066	(0.000013)
3.0	0.617491	(0.002000)	0.634810	0.000093	(0.000011)
4.0	0.563900	(0.001520)	0.579728	0.000102	(0.000010)
5.0	0.524616	(0.001230)	0.541190	0.000123	(0.000009)
6.0	0.498530	(0.001034)	0.513811	0.000126	(0.000009)
7.0	0.478470	(0.000899)	0.493920	0.000138	(0.000008)
8.0	0.462596	(0.000804)	0.479082	0.000157	(0.000008)
9.0	0.452172	(0.000728)	0.467648	0.000155	(0.000008)
10.0	0.444485	(0.000667)	0.458462	0.000145	(0.000007)

Table 2: The constant maturity of the reference CDS ( $T_{CDS}$ ), participation rates using the Monte Carlo approach, standard errors, participation rate using forward spreads (without convexity adjustment), and convexity adjustment. The CMCDS has a 3 year maturity and resets quarterly, the term structure of interest rate is assumed to be flat at 3%, the underlying model is the Shifted Gamma with parameters  $a = 0.74475$  and  $b = 6.59491$ . Valuation date is 5<sup>th</sup> January 2005.

The size of the convexity adjustment versus constant maturity is given in Figure 5.

Valuation of a cap is done by valuating each individual caplet. A caplet is an option to buy protection for the strike spread  $K_C$  and is therefore similar to a payer on the reference constant maturity tenor CDS. Payers and receivers were valued in Jönsson and Schoutens (2007) using the same spread dynamics we use here.

$T_{CDS}$	Part. rate Cap	(s.e.)	Cap	(s.e.)
1.0	2.129509	(0.002020)	0.001423	(0.000015)
2.0	1.448004	(0.001181)	0.001411	(0.000012)
3.0	1.112441	(0.000843)	0.001340	(0.000011)
4.0	0.921242	(0.000671)	0.001236	(0.000009)
5.0	0.802036	(0.000568)	0.001144	(0.000009)
6.0	0.724393	(0.000500)	0.001046	(0.000008)
7.0	0.670125	(0.000452)	0.000964	(0.000007)
8.0	0.629794	(0.000419)	0.000892	(0.000007)
9.0	0.599598	(0.000398)	0.000804	(0.000006)
10.0	0.576215	(0.000384)	0.000714	(0.000006)

Table 3: Participation rate with cap estimated using the Monte Carlo approach for different constant maturity tenors of the reference CDS ( $T_{CDS}$ ). The CM-CDS has a 3 year maturity and resets quarterly, the term structure of interest rate is assumed to be flat at 3%, the underlying model is the Shifted Gamma with parameters  $a = 0.74475$  and  $b = 6.59491$ . The cap strike was set to three times the market CDS spread. Valuation date is 5<sup>th</sup> January 2005.

The impact of using a cap on the reset spread is clearly visible in Table 3 and Figure 6. If we have a cap on the reset spread the participation rate will, as expected, increase to compensate for the fact that the seller (buyer) of protection will not receive (pay) a higher spread than the cap strike. The cap strike was set to three times the market spread of the reference CDS, see Figure 1 for the 1, 3, 5, 7 and 10 years market spreads. It is more likely for the reset spread to be higher than the cap for the shorter maturities since the cap is a multiple of the initial market spread and the market spread is increasing with maturity. It is therefore not surprising that the cap plays a more significant role for the shorter maturities than for the longer. Comparing the "uncapped" participation rates given in Table 2 with the participation rates in Table 3 we see that the participation rate is significantly higher when a cap is present, especially for the shorter constant maturities. The increase is ranging from 30% for the longest maturity to 155% for the shortest maturity. The smaller standard error on the cap participation rate is a natural effect of the cap since we cut off high values of the reset spreads.

The participation rate with and without cap for different maturities of the CMCDS are shown in Table 4 together with the price of the corresponding caps. The cap strike (30 bp) was chosen to be three times the initial reset spread (10 bp). The increase is ranging from 48% for the shortest maturity up to 70% for



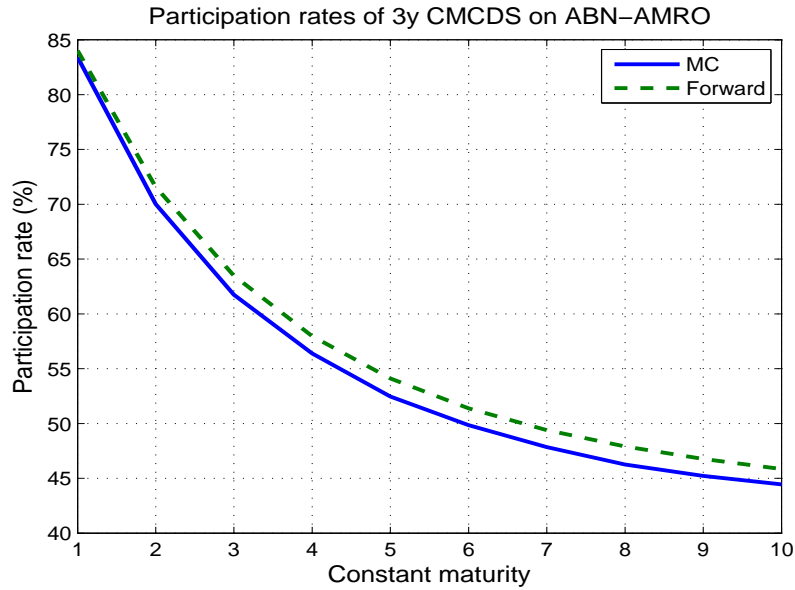


Figure 4: Participation rate of a 3 years CMCDS on ABN-AMRO for different constant maturities calculated using the Monte Carlo approach and forward spreads, respectively. The underlying model is the Shifted Gamma with parameters  $a = 0.74475$  and  $b = 6.59491$ . Valuation date 5<sup>th</sup> January 2005.

the longest maturity which is expected since it is more likely that the cap strike will be reached for longer CMCDS maturities. The smaller standard error on the cap participation rate is a natural effect of the cap since we cut off high values of the reset spreads. The prices of the cap is increasing since the number of reset dates, and hence the number of caplets, increases with the CMCDS maturity.

If all contract parameters are kept constant except the cap strike the participation rate and cap price are decreasing with increasing cap strike, as can be seen in Table 5.

In Figure 7 the participation rate and mark-to-market of a  $\hat{T} = 3$  year CMCDS with quarterly reset indexed to a 5 year CDS on ABN-AMRO over one year from the valuation date 5th of January 2005. At each mark-to-market date  $t$  the Shifted Gamma model is calibrated on the given term structure of the CDS on ABN-AMRO and the participation rate of a new CMCDS with the same characteristics and same maturity, that is, the time to maturity of the new

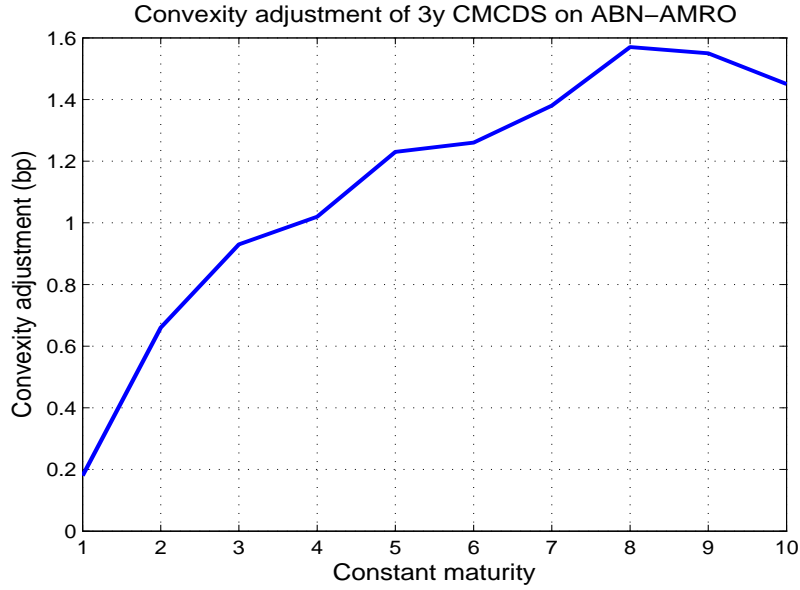


Figure 5: Convexity adjustment of a 3 years CMCDS on ABN-AMRO for different constant maturity tenors. The underlying model is the Shifted Gamma with parameters  $a = 0.74475$  and  $b = 6.59491$ . Valuation date  $5^{th}$  January 2005.

$T_{cmcds}$	Part.rate	(s.e.)	Part.rate Cap	(s.e.)	Cap	(s.e.)
1	0.313751	0.000829	0.465747	0.000440	0.000117	0.000003
2	0.430554	0.001116	0.626417	0.000493	0.000535	0.000006
3	0.523892	0.001232	0.801649	0.000567	0.001148	0.000009
4	0.592256	0.001287	0.963917	0.000633	0.001941	0.000011
5	0.647413	0.001299	1.106990	0.000684	0.002800	0.000014

Table 4: The participation rates with and without cap using the Monte Carlo approach. The CMCDS has a 1 to 5 years maturity and resets quarterly, the constant maturity of the reference CDS is 5 years, the term structure of interest rate is assumed to be flat at 3%, the initial reset spread is 10 bp and the cap strike is  $K_C = 30$  bp. The underlying model is the Shifted Gamma with parameters  $a = 0.74475$  and  $b = 6.59491$ . Valuation date is  $5^{th}$  January 2005.

CMCDS is  $\hat{T} - t$ , is calculated using these parameters. The notional is assumed to be 10 million. The mark-to-market is increasing because the participation rate is decreasing with decreasing time to maturity, which is as expected.

The behaviour of the participation rate and the mark-to-market depends of

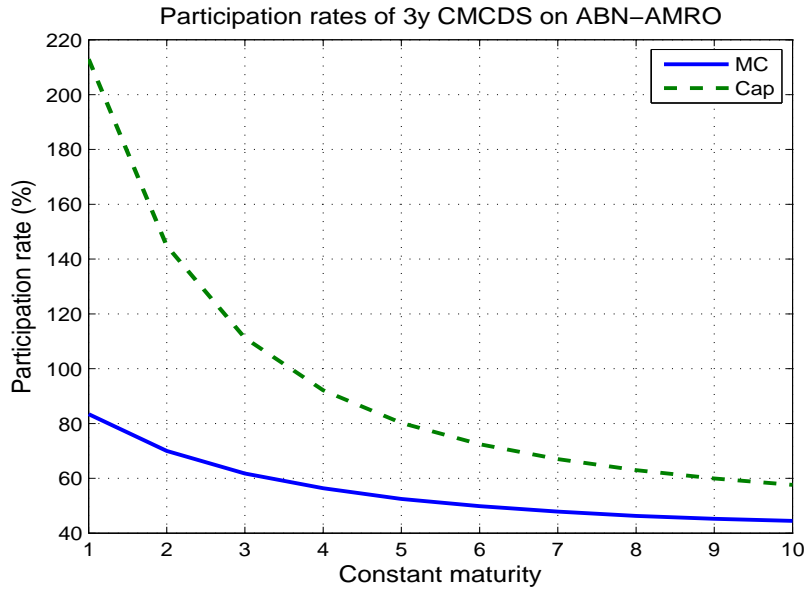


Figure 6: Participation rate of a 3 years CMCDS on ABN-AMRO for different constant maturities calculated using the Monte Carlo approach with and without a cap on the reset spread. The underlying model is the Shifted Gamma with parameters  $a = 0.74475$  and  $b = 6.59491$ . Valuation date 5<sup>th</sup> January 2005.

course on the CDS term structure and the model parameters estimated from it. There are three significant "jumps" in the participation rate in week 3, 14 and 49 that results in "jumps" in the mark-to-market. These jumps can be traced back to changes of the reference CDS' spread curve's slope and level, which forced the the calibrated model parameters to "jump", see Figure 2 and Figure 3. Further the changes of the spread curve's level and slope influence the market spread of the CDS with the same maturity as the CMCDS,  $\hat{T}$ , as can be seen in Figure 8. It is interesting to note that the change of the mark-to-market in week 19 and 36 is not found in the participation rate. Looking at the spread curve evolutions in Figure 2 we see a peak in week 19 and a downward movement in week 36 over all maturities, which of course influence the market spread of the CDS with maturity  $\hat{T}$ .

Cap strike (bp)	Part.rate Cap	(s.e.)	Cap price	(s.e.)
20	0.884944	0.000488	0.001386	0.000009
30	0.802223	0.000568	0.001140	0.000009
40	0.756298	0.000623	0.000980	0.000008
50	0.726193	0.000667	0.000864	0.000008
60	0.704539	0.000703	0.000775	0.000008
70	0.688038	0.000734	0.000703	0.000008
80	0.674905	0.000762	0.000643	0.000008
90	0.664163	0.000787	0.000593	0.000007
100	0.655167	0.000810	0.000549	0.000007
110	0.647498	0.000831	0.000511	0.000007
120	0.640876	0.000851	0.000477	0.000007
130	0.635081	0.000869	0.000447	0.000007
140	0.629958	0.000886	0.000420	0.000007
150	0.625369	0.000902	0.000396	0.000006

Table 5: The participation rates with cap using the Monte Carlo approach. The CMCDS has a 3 years maturity and resets quarterly, the constant maturity tenor of the reference CDS is 5 years, the term structure of interest rate is assumed to be flat at 3%, and the initial reset spread is 10 bp. The participation rate of the CMCDS without cap is 52.5% (standard error 0.124%). The underlying model is the Shifted Gamma with parameters  $a = 0.74475$  and  $b = 6.59491$ . Valuation date is 5<sup>th</sup> January 2005.

## 5 Conclusions

We have presented a Monte Carlo approach to value Constant Maturity Credit Default Swaps (CMCDS) based on a single sided Lévy firm's value model. A CMCDS is linked to a reference CDS with constant maturity tenor. At specified dates the spread of the CMCDS is reset to the par spread of the reference CDS. The pricing of a CMCDS is equivalent to determining the participation rate, which is the proportion of the reference par spread to be paid at the next payment. The estimation of the participation rate is done by estimating the expected par spread of the reference CDS on each reset date. To achieve this we set up a spread generator where, after the model has been calibrated on an appropriate CDS term structure, the generated firm's value paths are mapped into spread paths of the reference CDS. With the proposed Monte Carlo approach we have estimated the participation rate of a CMCDS for different constant maturity tenors and we have compared this with the participation rate based on the forward spreads without convexity adjustment. For both approaches the participation rate is decreasing with increasing constant maturity tenor. However, the

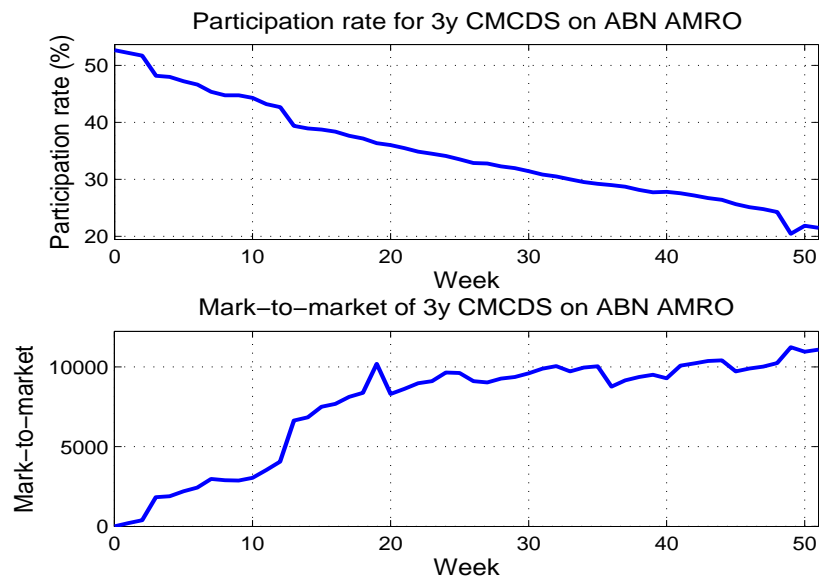


Figure 7: Participation rate and mark-to-market of a quarterly reset CMCDS indexed to a 5 year CDS on ABN-AMRO calculated on weekly data from 5<sup>th</sup> January 2005 to 8<sup>th</sup> February 2006. The CMCDS has a 3 years maturity at week 0. The underlying model is Shifted Gamma.

Monte Carlo estimate is always smaller than the non-adjusted forward spread participation rate. The convexity adjustment, that is, the difference between the expected reset spread and the forward spread, is increasing with increasing constant maturity tenor. Furthermore, we showed how to price caps and how the participation rate is affected if we introduce a cap on the reset spread. We finally showed the evolution of the participation rate and the mark-to-market of a CMCDS over time using weekly evolution of a 5 year CDS on ABN-AMRO as reference.

## Acknowledgements

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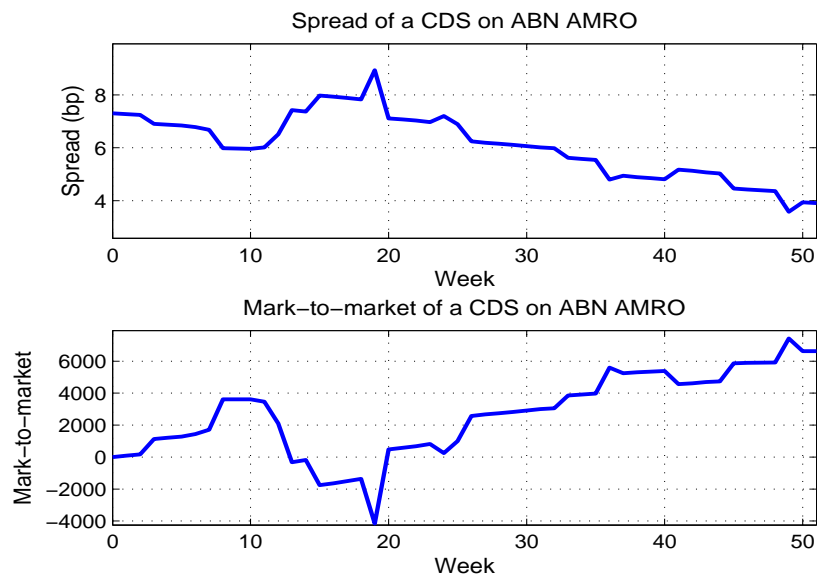


Figure 8: Par spread and markt-to-market of a CDS on ABN-AMRO with 3 years maturity at week 0 calculated on weekly data from 5<sup>th</sup> January 2005 to 8<sup>th</sup> February 2006. The spread is calculated using the Shifted Gamma model.

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