Gaussian expansions and bounds for the Poisson distribution applied to the Erlang B formula

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Abstract

This paper presents new Gaussian approximations for the cumulative distribution function \( P(A_\lambda \leq s) \) of a Poisson random variable \( A_\lambda \) with mean \( \lambda \). Using an integral transformation, we first bring the Poisson distribution into quasi-Gaussian form, which permits evaluation in terms of the normal distribution function \( \Phi \). The quasi-Gaussian form contains an implicitly defined function \( y \), which is closely related to the Lambert W function. A detailed analysis of \( y \) leads to a powerful asymptotic expansion and sharp bounds on \( P(A_\lambda \leq s) \).

The results for \( P(A_\lambda \leq s) \) differ from most classical results related to the central limit theorem in that the leading term \( \Phi(\beta) \), with \( \beta = (s - \lambda)/\sqrt{\lambda} \), is replaced by \( \Phi(\alpha) \), where \( \alpha \) is a simple function of \( s \) that converges to \( \beta \) as \( s \to \infty \). Changing \( \beta \) into \( \alpha \) turns out to increase precision for small and moderately large values of \( s \).

The results for \( P(A_\lambda \leq s) \) lead to similar results related to the Erlang B formula. The asymptotic expansion for Erlang’s B is shown to give rise to accurate approximations; the obtained bounds seem to be the sharpest in the literature thus far.

**Keywords:** Erlang B formula, Erlang loss model, Poisson distribution, Normal distribution, Gaussian integrals, Lambert W function, Ramanujan’s conjecture, asymptotic expansions, bounds.

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