COMPARING SOME ALTERNATIVE LÉVY BASE CORRELATION MODELS FOR PRICING AND HEDGING CDO TRANCHES

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Abstract: In this paper we investigate alternative Lévy base correlation models that arise from the Gamma, Inverse Gaussian and CMY distribution classes. We compare these models to the basic (exponential) Lévy base correlation model and the classical Gaussian base correlation model. For all the investigated models, the Lévy base correlation curve is significantly flatter than the corresponding Gaussian one, which indicates better correspondence of the Lévy models with reality. Furthermore, we present the results of pricing bespoke tranchlets and comparing deltas of both standard and custom-made tranches under all the considered models. We focus on deltas with respect to CDS index and individual CDS, and the hedge ratio for hedging the equity tranche with the junior mezzanine.

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1 Introduction

We consider a collateralized debt obligation (CDO) with standard credit default swap (CDS) indices as the reference portfolio. Such a CDO is referred to as synthetic CDO, and it is designed to transfer the credit risk on a reference portfolio of assets between parties. CDOs have recently become very popular credit instruments. A standard feature of a CDO structure is the tranching of credit risk, i.e., creating multiple tranches of securities which have varying degrees of seniority and risk exposure: the equity tranche is the first to be affected by losses in the event of one or more defaults in the portfolio. If losses exceed the value of this tranche, they are absorbed by the mezzanine tranche(s). Losses that have not been absorbed by the other tranches are sustained by the senior tranche and finally by the super-senior tranche. In such a way, each tranche protects the ones senior to it from the risk of loss on the underlying portfolio. The CDO investors take on exposure to a particular tranche, effectively selling credit protection to the CDO issuer, and in turn collecting premiums (spreads).

In order to price a synthetic CDO one needs a model that captures the dependency structure in the underlying portfolio and gives a good fit to the market prices of different tranches simultaneously. The standard model for pricing CDOs established in the market is the Gaussian Copula model (see e.g. Vasicek [12]). It is basically a one-factor model with an underlying multivariate normal distribution. Actually, a very simple multivariate normal distribution is employed: all correlation between different components are taken equal. The one-factor Gaussian copula model is well-known not to provide an adequate solution for pricing simultaneously various tranches of a CDO. In order to deal with this problem, the base correlation concept was initiated (see e.g. O'Kane and Livasey [10]). Similarly to implied volatility in an equity setting, one uses a different base correlation for each tranche to be priced. Due to the construction, base correlation is quite adapted to interpolation for nonstandard tranches. One of the prime applications of base correlation is thus pricing bespoke tranches. The application of the Gaussian base correlation may, however, lead to arbitrage opportunities, providing higher prices for tranchlets with higher seniority. Another weakness of the Gaussian base correlation is that it significantly depends on the interpolation scheme.

A set of other one-factor models has recently been proposed in the literature. Moosbrucker [9] used a one-factor Variance Gamma model, Kalemanova et al. [8] and Guégan and Houdain [6] worked with a NIG factor model and Baxter [3] introduced the B-VG model. Most of these models are special cases of the generic one-factor Lévy model of Albrecher, Ladoucette, and Schoutens [2]. Lévy models bring more flexibility into the dependence structure and allow tail dependence.

Several Lévy models that extend the classical Gaussian copula model were investigated and compared in Garcia, Goossens, Masol, and Schoutens [5]. The proposed models are tractable and perform significantly better than the Gaussian copula model.

Furthermore, also the concept of Lévy base correlation was introduced and developed for the shifted Gamma model in [5]. This model basically replaced the Gaussian distribution with a distribution with a more fatter exponential tail. The use of the Lévy base correlation is completely analogous as in the Gaussian case. An example is the pricing of tranchlets (i.e., very thin tranches) by interpolation on the correlation curve. Historical studies show that the Lévy base correlation curve is always much flatter than the Gaussian counterpart. Related to this, is the fact that the pricing of tranchlets is less sensitive to the interpolation scheme. This indicates that the Lévy models do fit the observed data much better and are much more reliable for pricing bespoke tranches.

In this paper, we work out base correlation concept for two more Lévy models; further, we apply a number of base correlation models to price and investigate delta-hedge parameters for both standard and custom-made tranches. The remaining of the paper is arranged as follows. The generic one-factor Lévy model is briefly presented in Section 2. Some examples of Lévy models are given in Section 3. In Section 4 we do a historical study of Lévy base correlation for different models and compare tranchlets prices obtained under a number of base correlation models. Lévy prices turn out to be less sensitive to the interpolation technique used to interpolate base correlation, while the Gaussian do. Moreover, typical arbitrage opportunities for bespoke thin tranches under the Gaussian models, are no longer present under the Lévy models.

In Section 5, we compare delta hedge parameters of the different models. We focus on three common approaches to delta-hedge a standard CDO tranche: first, hedging a tranche with the index; second, hedging a tranche with a single name CDS, and, finally, hedging the equity tranche with the junior mezzanine tranche. The dynamics of the deltas with respect to the index is similar under all models. The difference is only in scale: equity deltas under the Lévy models are approximately 25% higher than equity deltas under the Gaussian; junior mezzanine delta is 15% lower and deltas of other tranches are 40% lower under the Lévy models than the corresponding Gaussian deltas. As a consequence the hedge ratios of Equity versus Mezzanine deltas over time of the Lévy models are approximately 50% higher than those under the Gaussian. We also consider deltas of bespoke tranches with respect to the CDS index.

2 Generic One-Factor Lévy Model

We are going to model a portfolio of n obligors such that all of them have equal weights in the portfolio. We will assume for simplicity that each obligor i, $i \in \{1, 2, \dots, n\}$, has the same recovery rate R in case of default, the same notional amount equal to $1/n$ of the total portfolio notional, and some individual default probability term structure $p_i(t)$, $t \geq 0$, which is the probability that obligor i will default before time t.

The one-factor Lévy model was introduced in Albrecher, Ladoucette, and Schoutens [2]. For the survey of the Lévy-based models in finance we refer to Schoutens [11]. We will briefly present the model below for ease of reading the sequel.

Let $X = \{X_t, t \in [0, 1]\}$ be a Lévy process based on an infinitely divisible distribution L, i.e. X_1 follows the law L, such that $\mathbb{E}[X_1] = 0$ and $Var[X_1] = 1$. Denote the cumulative distribution function of X_t by H_t, $t \in [0, 1]$, and assume it is continuous. It may be shown that $Var[X_t] = t$. Note that we will only work with Lévy processes with time running over the unit interval. Let $X^{(i)} = \{X_t^{(i)}, t \in [0, 1]\}$, $i = 1, 2, \dots, n$, and X be independent and identically distributed Lévy processes (i.e., all processes are independent of each other and are based on the same mother infinitely divisible distribution L).

Let us first fix a time horizon T. Further, let $\rho \in (0, 1)$ be the prescribed correlation between the defaults. We assume the asset value of obligor i is

$$A_i = X_\rho + X_{1-\rho}^{(i)}, \quad \rho \in (0,1).$$

By the stationary and independent increments property of Lévy processes, A_i has the same distribution as X_1, i.e., $A_i \simeq L$. Hence $\mathbb{E}A_i = 0$, $VarA_i = 1$, and

$$Corr\left[A_i, A_j\right] = \frac{\mathbb{E}\left[A_i A_j\right] - \mathbb{E}A_i \mathbb{E}A_j}{\sqrt{VarA_i}\sqrt{VarA_j}} = \rho, \quad i \neq j.$$

So, starting from any mother standardized infinitely divisible law, we can set up a one-factor model with the required correlation.
Let us now derive default probabilities under the Lévy model. We say the ith obligor *defaults at time* t, $0 \leq t \leq T$, if its asset value falls below a preset barrier $K_i(t)$, $A_i \leq K_i(t)$. Let $p_i(t)$ denote the default probabilities observed in the market. We set

$$K_i(t) = H_1^{[-1]}(p_i(t))$$

to match $p_i(t)$ to the default probabilities under the model, indeed

$$p_i(t) = \mathbb{P}\{A_i \leq K_i(t)\} = H_1(K_i(t)).$$

Conditional on the common factor X_ρ, the default events are independent. Denote by $p_i(y;t)$ the conditional probability that the ith firm defaults before time t, given $X_\rho = y$,

$$p_i(y;t) = \mathbb{P}\{A_i \leq K_i(t) | X_\rho = y\}$$

$$= \mathbb{P}\{X_\rho + X_{1-\rho}^{(i)} \leq K_i(t) | X_\rho = y\} = H_{1-\rho}(K_i(t) - y).$$

Denote by $\Pi_{n,y}^k(t)$ the conditional probability to have k out of n defaults before time t, given $X_\rho = y$, $k = 0, 1, \ldots, n$. It can be calculated recursively by n,

$$\begin{aligned}
&\Pi_{n,y}^0(t) = \Pi_{n-1,y}^0(t)\left(1 - p_n(y;t)\right), \quad \Pi_{0,y}^0(t) \equiv 1;\\
&\Pi_{n,y}^k(t) = \Pi_{n-1,y}^k(t)\left(1 - p_n(y;t)\right) + \Pi_{n-1,y}^{k-1}(t) p_n(y;t), \quad k = 1, ..., n-1;\\
&\Pi_{n,y}^n(t) = \Pi_{n-1,y}^{n-1}(t) p_n(y;t).
\end{aligned}$$

Let $D_{t,n}$ be the number of defaults in the portfolio. The unconditional probability of exactly k defaults out of n firms is

$$\begin{aligned}
\Pi_n^k(t) := \mathbb{P}\{D_{t,n} = k\} &= \int_{-\infty}^{\infty} \mathbb{P}\{D_{t,n} = k | X_\rho = y\} dH_\rho(y)\\
&= \int_{-\infty}^{\infty} \Pi_{n,y}^k(t) dH_\rho(y).
\end{aligned}$$

The expected percentage loss $L_{t,n}$ on the portfolio notional at time t is

$$\mathbb{E}[L_{t,n}] = \frac{(1-R)}{n} \sum_{k=1}^{n} k \cdot \Pi_n^k(t);$$

and the expected percentage loss on the CDO tranche $[K_1\% - K_2\%]$ is

$$\mathbb{E}\left[L_{t,n}^{K_1,K_2}\right] = \frac{\mathbb{E}\left[\min\{L_{t,n}, K_2\}\right] - \mathbb{E}\left[\min\{L_{t,n}, K_1\}\right]}{K_2 - K_1}.$$

The fair premium for the tranche $[K_1\% - K_2\%]$ can then be calculated as

$$s = \frac{\sum_j \left\{ \mathbb{E}\left[L_{t_j,n}^{K_1,K_2}\right] - \mathbb{E}\left[L_{t_{j-1},n}^{K_1,K_2}\right] \right\} D(0,t_j)}{\sum_j \left\{ 1 - \mathbb{E}\left[L_{t_j,n}^{K_1,K_2}\right] \right\} (t_j - t_{j-1}) D(0,t_j)},$$

where both summations are taken over the all payment dates, $D(0,t)$ is the discount factor from time t to time 0. The quantity in the denominator is referred to as the *risky annuity* (RA) and equals to the expected present value of 1 bp paid in premium until default or maturity, whichever is sooner. For the discussion about the difference between a risky annuity and risky duration we refer to [4].

3 Examples of Lévy Models

- **Shifted Gamma**

The characteristic function of the Gamma distribution Gamma(a,b), $a,b > 0$, is given by

$$\phi_{Gamma}(u; a, b) = (1 - \mathrm{i}u/b)^{-a}, \quad u \in \mathbb{R}.$$

Clearly, this characteristic function is infinitely divisible. The Gamma-process $X^{(G)} = \{X_t^{(G)}, t \geq 0\}$ with parameters $a, b > 0$ is defined as the stochastic process which starts at zero and has stationary, independent Gamma-distributed increments. More precisely, the time enters in the first parameter: $X_t^{(G)}$ follows a Gamma(at, b) distribution.

Let us start with a unit-variance Gamma-process $G = \{G_t, t \geq 0\}$ with parameters $a > 0$ and $b = \sqrt{a}$ such that $\mu := \mathbb{E}G_1 = \sqrt{a}$, $VarG_1 = 1$. As driving Lévy process, we then take the *Shifted Gamma process*

$$X_t = \mu t - G_t, \quad t \in [0,1].$$

The interpretation in terms of firm value is that there is a deterministic up-trend $\sqrt{a}t$ and random downward shocks $\{G_t\}$.

The one-factor Shifted Gamma-Lévy Model is

$$A_i = X_\rho + X_{1-\rho}^{(i)}, \quad i = 1, 2, \ldots, n,$$

where X, $\{X^{(i)}\}_{i=1}^n$ are independent standardized Shifted Gamma processes. Hereafter we will refer to the shifted Gamma-Lévy model with parameters $a > 0$ and $b = \sqrt{a}$ as *Gamma(a)*.

The unconditional probability of exactly k defaults out of n firms becomes

$$\Pi_n^k(t) \quad = \quad \int_0^{+\infty} \Pi_{n,(\sqrt{a}\rho - u/b)}^k(t) \frac{1}{\Gamma(a\rho)} u^{a\rho-1} \exp(-u) \mathrm{d}u,$$

where the last integral can be calculated by applying Gauss-Laguerre quadrature.

- **Shifted Inverse Gaussian**

The Inverse Gaussian IG(a,b) law with parameters $a>0$ and $b>0$ has characteristic function

$$\phi_{IG}(u;a,b)=\exp\left(-a(\sqrt{-2\mathrm{i}u+b^2}-b)\right),\quad u\in\mathbb{R}.$$

The IG-distribution is infinitely divisible and we define the IG-process $I=\{I_t,t\geq 0\}$ with parameters $a,b>0$ as the process which starts at zero, has independent and stationary IG-distributed increments, and such that

$$E[\exp(\mathrm{i}uI_t)]=\phi_{IG}(u;at,b)=\exp\left(-at(\sqrt{-2\mathrm{i}u+b^2}-b)\right),\quad u\in\mathbb{R},$$

meaning that I_t follows an IG(at,b) distribution.

Let us start with a unit variance IG-process $I=\{I_t,t\geq 0\}$ with parameters $a>0$ and $b=a^{1/3}$ such that $\mu:=\mathbb{E}I_1=a^{2/3}$, $VarI_1=1$. In our model, we then take

$$X_t=\mu t-I_t,\quad t\in[0,1].$$

The one-factor shifted IG-Lévy model, hereafter referred to as the *IG(a)* model, is

$$A_i=X_\rho+X^{(i)}_{1-\rho},$$

where $X,\{X^{(i)}\}_{i=1}^n$ are independent shifted IG-processes. In order to compute the unconditional probabilities Π_n^k one can rely on numerical integration schemes using the density of the IG processes or apply Laplace transform inversion methods starting from the characteristic function.

- **Shifted CMY**

The CMY(C,M,Y) distribution with parameters $C>0$, $M>0$, and $Y<2$ has characteristic function

$$\phi_{CMY}(u;C,M,Y)=\exp\left\{C\Gamma(-Y)\left[(M-iu)^Y-M^Y\right]\right\},\quad u\in\mathbb{R}.$$

The CMY distribution is infinitely divisible and we can define the CMY Lévy process $X^{(CMY)}=\{X_t^{(CMY)},t\geq 0\}$ that starts at zero and has stationary and independent CMY-distributed increments, i.e., $X_t^{(CMY)}$ follows a CMY(Ct,M,Y) distribution. Note that CMY(C, M, Y) reduces to Gamma(C, M) when $Y=0$.

Let us start with a unit CMY-process $C=\{C_t,t\geq 0\}$ with parameters $C>0$, $Y<2$, and $M=(C\Gamma(2-Y))^{\frac{1}{2-Y}}$, so that the mean of the process is $\mu:=\left(C\Gamma(1-Y)(1-Y)^{Y-1}\right)^{\frac{1}{2-Y}}$ and the variance is equal to one. As driving Lévy process we take

$$X_t=\mu t-C_t,\quad t\in[0,1].$$

The one-factor shifted CMY-Lévy model, hereafter referred to as the *CMY(C; Y)* model, is

$$A_i=X_\rho+X^{(i)}_{1-\rho},$$

where $X,\{X^{(i)}\}_{i=1}^n$ are independent shifted CMY-processes.

Since the cumulative distribution function $H_{CMY}(x; C, M, Y)$ of a CMY distribution can not be derived in a closed form, we numerically invert its Laplace transform, given by

$$\hat{H}_{CMY}(w; C, M, Y) = \frac{\exp\left\{C\Gamma(-Y)\left[(M + w)^Y - M^Y\right]\right\}}{w},$$

in order to calculate values of $H_{CMY}(x; C, M, Y)$. In particular, we employ the numerical inversion procedure described in Abate and Whitt [1].

4 Lévy Base Correlation

The concept of Lévy base correlation was introduced and illustrated with the Gamma model in [5]. The procedure of bootstrapping base correlations in Lévy case is exactly the same as in the Gaussian. The only difference is that in Lévy models we have additional distribution parameters. There are two alternative ways to define the distribution parameters. First is to take the parameters coming out from the global historical calibration, another is to set them equal to some values. The pros and cons of these two approaches were discussed in detail in [5], and the second way was chosen as the one being more stable in time, providing a faster base correlation bootstrapping, and still giving a good fit to the market data.

Following the methodology of base correlation contraction introduced for the Gamma(1) model, we fix the distribution parameters of the other Lévy models and set them equal to some values. In order to motivate our choice of these parameter values we have done historical global calibration and study of the models, and compared their performance for different parameter settings. We choose the parameter values from the estimated range in such a way that the underlying distributions have either slightly lighter or slightly heavier tail than the basic Gamma(1,1) (i.e. skewness and kurtosis are slightly smaller or larger than the Gamma's). In particular, we will consider Gamma(1), IG(1.5), IG(2), CMY(0.5, 0.6), CMY(0.6, 0.6), and CMY(0.7, 0.7) Lévy base correlation models (LBC). Table 1 summaries the properties of the chosen models (note that all the distributions have variance 1).

	Gamma(1)	IG(1.5)	IG(2)	CMY(0.5 0.6)	CMY(0.6 0.6)	CMY(0.7 0.7)
mean	1	1.3	1.6	1.4	1.6	1.1
skewness	2	2.3	1.9	2.5	2.2	1.9
kurtosis	9	12	8.9	13.7	11.3	9.1

Table 1: Properties of Selected Lévy Base Correlation Models

In order to find the vector of base correlations, we will calibrate our models to the time-series of daily iTraxx Europe Main 5Y data (Series 3) from March 21 to September 20, 2005. We note, that the equity tranche is traded with an upfront payment and a 500bp running spread. Without an upfront payment, the equivalent running spread may be expressed in terms of the quoted upfront as

$$s_{equity} = \frac{Upfront}{RA_{equity}} \cdot 100 + 500,$$

where RA_{equity} is the risky annuity of the [0%—3%] equity tranche.

The evolution of the base correlation over time for [0%–3%] and [12%–22%] tranches is presented in Figure 1. One can see that the dynamics of the base correlation is very similar for all the models. We have also plotted base correlation curve for the Gamma(a) model, where the gamma parameter a is free, in order to show that fixing the distribution parameter does not make a big change in base correlation values.

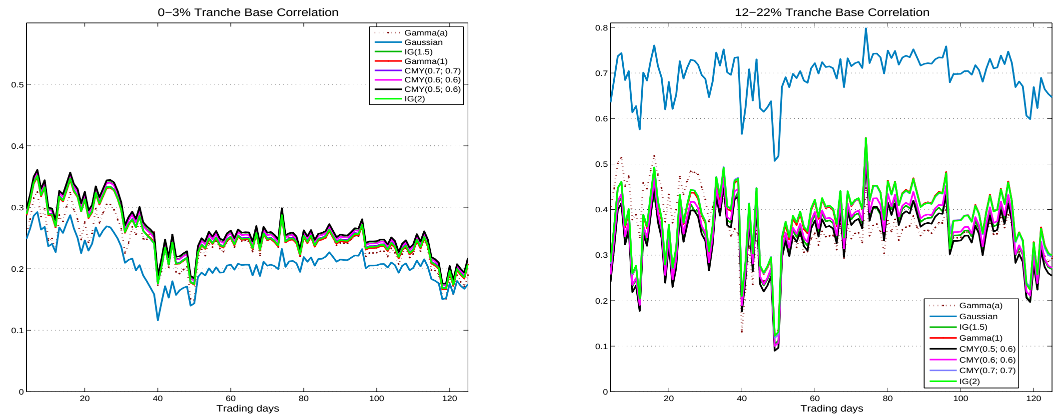


Figure 1: Base Correlation of 0%–3% and 12%–22% Tranches - iTraxx data 21-03-2005 – 20-09-2005

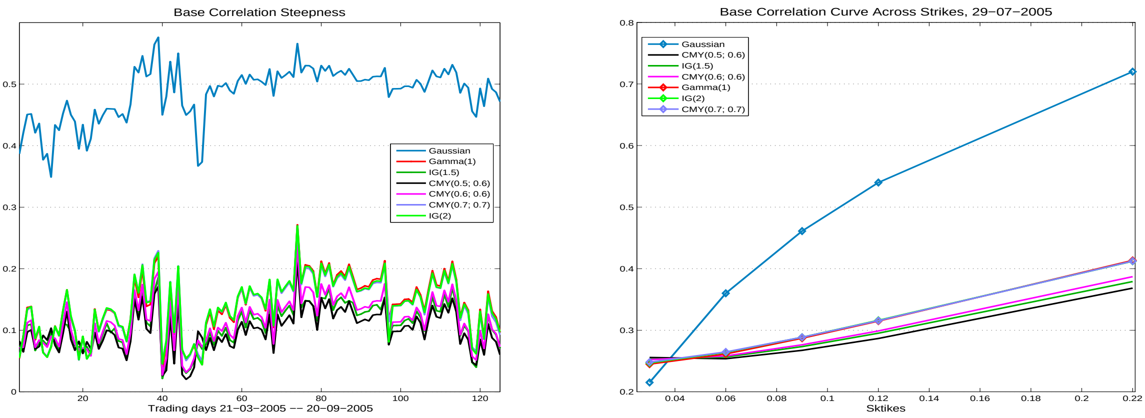
Figure 2(a) shows the steepness of the base correlation under the different models. The curve is plotted as the difference between the maximal and minimal base correlation values. All the Lévy curves are obviously flatter than the Gaussian one which indicates Lévy models provide better fit to the market. The Gaussian curve is in average 4 times steeper than the Lévy curves. The graph also shows that the steepness of the base correlation increases insignificantly when we move from Gamma(a) to Gamma(1): steepness of the Gamma(1) base correlation curve is in average 1.2 times higher than that of the Gamma(a).

Another observation is that the lighter the tail of the underlying distribution the steeper the base correlation curve; "overestimating" the tail of the underlying distribution does not however lead to a completely flat curve but to base correlation "smile" (see also Figure 2(b)).

4.1 Pricing bespoke tranches

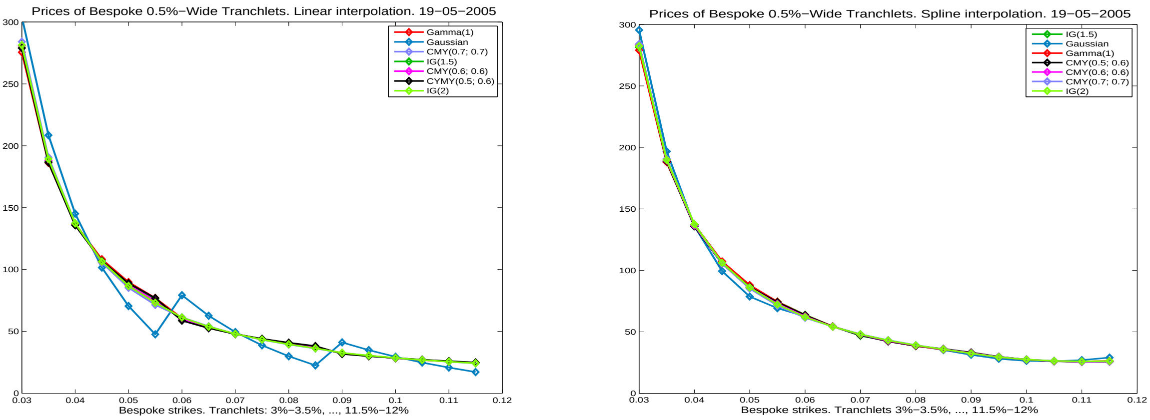
The base correlation construction is quite adapted to interpolation for non-standard strikes on the standard indices. One of the prime applications of base correlation is thus pricing non-standard tranches. In particular, tranchlets (i.e., very thin tranches) with width close to the expected loss point of the entire portfolio are most active.

Let us take tranchlets of width 0.5% and price them under all of the selected models. We calculate tranchlet prices using two different interpolation methods: (a) the simplest linear interpolation, and (b) more advanced spline interpolation. Figure 3 shows that pricing with Lévy base correlation is invariable the interpolation technique while prices under the Gaussian model significantly depend on the interpolation technique we use. The underlying reason is that the Lévy base correlation is much flatter than the Gaussian base correlation curve and thus much less sensitive to interpolation errors.



(a) Base correlation steepness, 21-03-2005 – 20-04-2005 (b) Base correlation over strikes, 29-07-2005

Figure 2: Base correlation under different models



(a) Linear interpolation (b) Spline interpolation

Figure 3: Lévy vs Gaussian Prices of Bespoke 0.5%-Wide Tranchlets - iTraxx data 19-05-2005

Moreover, Lévy base correlation, linear-interpolated on bespoke attachment points, typically does not generate arbitrage opportunities when its Gaussian counterpart does. Figure 3(a) illustrates such a situation. Under the Gaussian pricing, one can for example buy protection for the [5.5% - 6%] tranche for 50 bp and sell protection for the tranche [6% - 6.5%] with higher seniority for 80 bp.

5 Delta-Hedging CDO tranches

A tranche investor often hedges its position (dynamically) using a technique called "delta-hedge". Delta-hedging involves offsetting the impact of changing spread levels on the tranche value by buying protection in CDS index or a single-name CDS in an appropriate fraction of the tranche's notional amount. This specific fraction is called "delta". As spreads fluctuate, deltas also change, and the hedge must be frequently adjusted.

There are three common approaches to hedge a CDO tranche. First, to hedge a tranche with the index; second, to hedge a tranche using a single name CDS and finally to hedge tranche with another tranche (for example to hedge a long position in the equity tranche and by a short position in the junior mezzanine). Calculation of deltas, and hence implementation of a hedging strategy, is entirely model-dependent. In this section we study and compare hedge parameters of the four different base correlation models: Gaussian, Gamma, IG, and CMY.

In order to determine the deltas, we need the *risky annuity* (RA) and *mark-to-market* (MTM) concepts. As mentioned above, the risky annuity of a tranche is the present value of 1 bp of spread paid over the life of the contract. The mark-to-market for a long risk tranche trade is expressed as

$$MTM_{current} = (s_{initial} - s_{current}) \cdot RA_{current}.$$

5.1 Hedging with CDS Index

In order to delta-hedge a tranche with the CDS index, we need to calculate a delta for the tranche. Theoretical delta for the tranche determines the size of the hedge required and is calculated as a ratio of the tranche's mark-to-market change to that of the CDS index position, given a 1 bp parallel shift in the average of all CDS spreads in the reference pool,

$$\Delta_{index} = \frac{MTM^{Tranche}_{indexShift} - MTM^{Tranche}_{current}}{MTM^{index}_{indexShift} - MTM^{index}_{current}}.$$

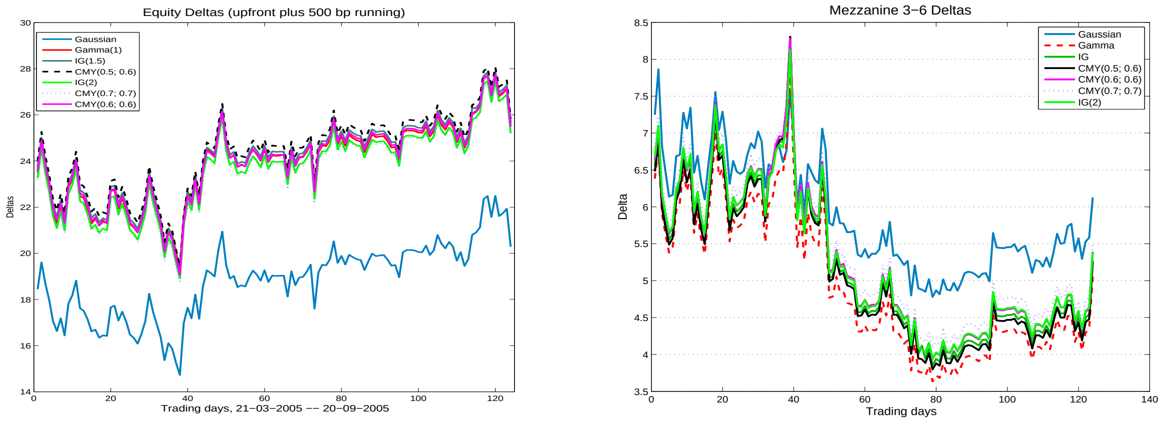
In practice, one can take a 5 bp or a 10 bp proportional shift, which is in line to the market. We use a 5 bp proportional shift in the examples below.

Figure 4 shows variation of the equity and junior mezzanine deltas over time. One can see that deltas obtained under the introduced Lévy base correlation models are completely consistent with Gaussian deltas, i.e. delta curves are roughly speaking parallel. In comparison to the Gaussian model, all the Lévy models split out higher deltas for the]0%–3%] equity tranche and lower deltas for all the other tranches. The estimates of how much higher/lower Lévy delta are under each of the models are given in Tabel 2. In average, Lévy equity deltas are 25% higher than the Gaussian; junior mezzanine deltas are 15% lower and senior mezzanine as well as senior deltas are 40% lower than their Gaussian counterparts.

Let us look closer on the delta behavior during the auto crisis in May 2005 (corresponds to trading days 35-45 in Figure 4). During this period the default correlation is very high and, consequently, more of the risk is shifted to the mezzanine and senior tranches as higher correlations mean that there is higher probability of joint defaults in the reference pool. Therefore, the mezzanine and senior deltas increase while the equity tranche delta decreases. Figure 4 confirms that all the models capture the general movement: equity deltas suddenly decline while mezzanine deltas increase. The highest peak on Figure 4(b) corresponds to the delta on May 19. One can see on the graph that Gaussian model underestimates mezzanine delta on this day, providing the lowest delta among all the models.

5.2 Delta-Hedging with a Single Name CDS

In order to delta-hedge a tranche with a single-name CDS, we need to calculate a delta for the tranche as the ratio of the tranche's mark-to-market change to that of the single-name CDS,



(a) Equity Deltas

(b) Junior Mezzanine Deltas

Figure 4: Equity and Junior Mezzanine Deltas for Different Models - iTraxx data 21-03-2005 – 20-09-2005

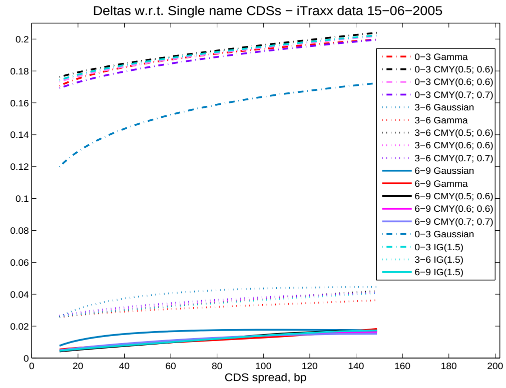
given a 1 bp parallel shift in the underlying spread curve of the CDS,

$$\Delta_{CDS} = \frac{MTM^{Tranche}_{CDSshift} - MTM^{Tranche}_{current}}{MTM^{CDS}_{CDSshift} - MTM^{CDS}_{current}}$$

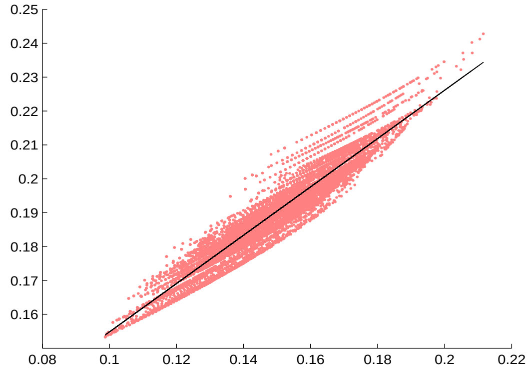
Figure 5(a) illustrates equity and mezzanine deltas with respect to single-name CDSs entering iTraxx index for all the models under investigation. One can see that, similar to deltas with respect to index, deltas with respect to single name CDSs under all the Lévy models are almost equal in values. Another similarity is that the difference between Lévy and Gaussian tranche deltas with respect to *single name CDSs* is of the same order of magnitude as the difference between Lévy and Gaussian tranche deltas with respect to the *index* (see Tabel 2). Note also that the percentage difference between equity deltas becomes exactly the same for index and single-name CDSs if we consider CDSs with spreads higher than 25 bp; the average index value for iTraxx Europe Main 5Y Series 3 is 40.

Since delta numbers split out by different Lévy models are very close, it is sufficient to take just one of the models in order to illustrate the difference between Lévy and Gaussian deltas with respect to a single-name CDS. In particular, we have plotted Gamma and Gaussian equity deltas for all the CDSs and all the considered dates in the scatter plot in Figure 5(b)) assigning Gaussian deltas to the horizontal axis and Gamma deltas to the vertical. Obviously, the data satisfies the assumption about linear dependence. Similar scatter diagrams can be plotted for all the models and other tranches; linear dependance weakens, however, as the seniority of tranches increases.

Tabel 2 summarises the magnitude of percent difference between Gaussian and Lévy deltas for all the tranches and for both index and single CDS hedging strategies. Plus sign in the second column indicates that Lévy deltas are higher and minus sign indicates that Lévy deltas are lower than the Gaussian. For example, [0%–3%] delta with respect to index under Gamma(1) model is 25% higher while senior [9%–12%] tranche delta with respect to index under CMY(0.7, 0.7) model is 40% lower than their Gaussian counterparts.



(a) 0%–3%, 3%–6%, 6%–9% Deltas, 15-06-2005



(b) Gamma vs Gaussian 0%–3% Deltas, iTraxx data 21-03-2005 – 20-09-2005

Figure 5: Deltas with respect to Single-Name CDSs

5.3 Mezz-equity hedging

Mezz-equity hedging is a hedging strategy which involves selling protection on the equity tranche and buying protection on the first mezzanine tranche or the other way around. The theoretical hedge ratio between two tranches can be expressed as

$$HedgeRatio_{mezz-equity} = \frac{\Delta^{equity}_{index}}{\Delta^{mezzanine}_{index}},$$

i.e. hedge ratio is the ratio of the equity tranche MTM change to that of the mezzanine, given 1 bp parallel shift in the underlying spread curve of the index.

Figure 6 shows the evolution of the hedge ratios over time. Lévy hedge ratios are in average 50% higher than the Gaussian. According to the Gaussian model, investors have to buy protection on the junior mezzanine tranche for the notional that is 3.3 times higher than the equity notional while according to the Lévy models junior mezzanine notional should be 5 times higher than the equity one.

5.4 Deltas of bespoke tranches with respect to Index

Now we can compute deltas of bespoke tranchlets applying spline interpolation of base correlation. First, we consider the sensitivity of deltas to the tranche width and seniority, and then relation between the deltas of standard tranches, e.g, $[3\% - 6\%]$, $[6\% - 9\%]$, and $[9\% - 12\%]$ tranches, and 0.5%-wide tranchlets constituting them.

Figure 7(a) illustrates the sensitivity of deltas to the tranche seniority and width. It is clear that equally wide tranches have different deltas depending on the seniority: the higher the seniority the lower the delta. Furthermore, among the tranches with equal attachment points, wider tranches have lower deltas while among the tranches with equal detachment points, wider tranches have higher deltas.

Figure 7(b) shows the evolution of deltas over time. In particular we plotted Gamma and Gaussian deltas of the standard $[3\% - 6\%]$ tranche and of the 1%-wide tranchlets constituting it.

	sign	Gamma(1)	IG(1.5)	IG(2)	CMY(0.5 0.6)	CMY(0.6 0.6)	CMY(0.7 0.7)
$\Delta^{[0-3]}_{index}$	+	0.25	0.25	0.23	0.26	0.25	0.23
$\Delta^{[0-3]}_{CDS}$	+	0.28	0.30	0.27	0.30	0.29	0.27
$\Delta^{[3-6]}_{index}$	-	0.17	0.13	0.11	0.12	0.11	0.10
$\Delta^{[3-6]}_{CDS}$	-	0.16	0.14	0.12	0.14	0.12	0.10
$\Delta^{[6-9]}_{index}$	-	0.39	0.39	0.35	0.4	0.38	0.34
$\Delta^{[6-9]}_{CDS}$	-	0.41	0.44	0.39	0.45	0.42	0.38
$\Delta^{[9-12]}_{index}$	-	0.39	0.44	0.40	0.45	0.43	0.40
$\Delta^{[9-12]}_{CDS}$	-	0.38	0.40	0.37	0.43	0.39	0.35
$\Delta^{[12-22]}_{index}$	-	0.45	0.40	0.37	0.47	0.45	0.41
$\Delta^{[12-22]}_{CDS}$	-	0.73	0.60	0.54	0.74	0.7	0.66

Table 2: Comparison of Lévy and Gaussian Deltas, iTraxx data 21-03-2005 – 20-09-2005

Deltas obtained from the other Lévy models considered in the paper have the same behaviour as the Gamma deltas.

Figure 8 shows the deltas of 0.5%-wide tranchlets along with the deltas of the standard tranches $[3\% - 6\%]$ and $[6\% - 9\%]$. Standard tranches deltas are plotted as straight lines on the levels corresponding to their values; blue lines correspond to Gaussian deltas and red lines to the Lévy. The trading days are chosen so that one of them is a "crisis" day and another is not. It is clear from the graph that in both cases delta of the standard tranche $[K1\% - K2\%]$ is approximately equal to the delta of the corresponding tranchlet $[(K1+1)\% - (K1+1.5)\%]$.

6 Conclusions

In this paper we have compared alternatives to the basic Lévy Base correlation model as introduced in [5] which follows the one-factor generic model introduced in [2]. All models under consideration are based on a (infinitely divisible) distribution which serves the role of a kind of firm's value indication. The basic Lévy case of [5] corresponds to the exponential distribution and the classical Base Correlation model in [10] to the Normal distribution. The alternatives investigated in this paper arise from the class of Gamma, the Inverse Gaussian distributions and the CMY class. All, these distribution do have (as the Exponential one) a much slower decaying tail behavior that the Gaussian distribution. Our results points out that also for the newly investigated models, the Lévy Base correlation curve is significantly flatter than the Gaussian Base Correlation model. As pointed out in earlier work, a flatter Base Correlation curves points to the fact that the model is more in correspondence with reality. Moreover, pricing tranchelets by

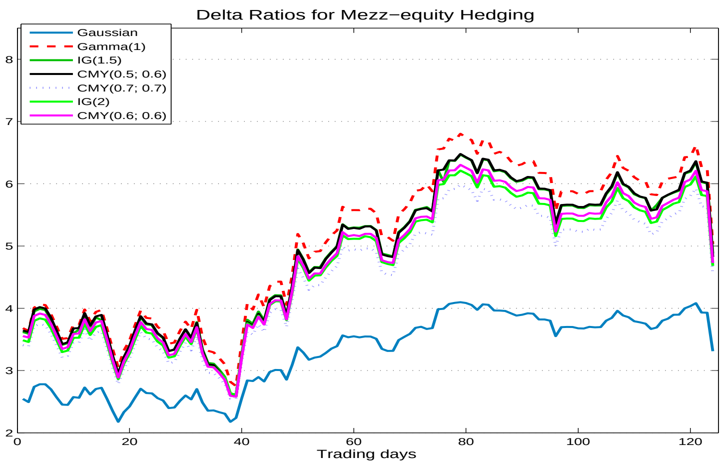


Figure 6: Hedge ratios for different models - iTraxx data 21-03-2005 – 20-09-2005

interpolation methods using a more flatter base correlation curve leads to much stable and more reliable prices. Additionally, under the Gaussian Base correlation this technique sometimes led to situations where the model was not arbitrage-free (more senior tranchelets had higher spreads than more junior ones); under the Lévy models these arbitrages are no longer observed in the cases considered.

The Lévy models (the Gamma, the IG and the CMY) all behave very similar to the basic (exponential) Lévy Base correlation model. More precisely, out of a calibration exercise on real market data on the iTraxx Series 3, the obtained Lévy base correlation curves are very close to each other and the hedging parameters (deltas and hedge ratio's) are also very similar. We compared deltas with respect to index and deltas with respect to single-name CDSs for all the models. In both cases we observed that the percentage difference between the deltas is of the same order of magnitude: Lévy deltas are approximately 25% higher than the Gaussian for the [0%–3%] tranche, 15% lower for the for [3%–6%] tranche, and 40% lower for the tranches with higher seniority.

Since from a numerical point of view the exponential distribution which lies at the heart of the basic case is much more tractable (inverse Fourier methods are needed for the other Lévy cases) and the fact that calculation times under basic case of tranche spread, delta's, etc. are in the same order of magnitude of the classical Gaussian model. One of the main conclusion is a recommendation of the basic Lévy base correlation model over the alternative Lévy base correlation models, and certainly over the Gaussian base correlation model.

7 Acknowledgments

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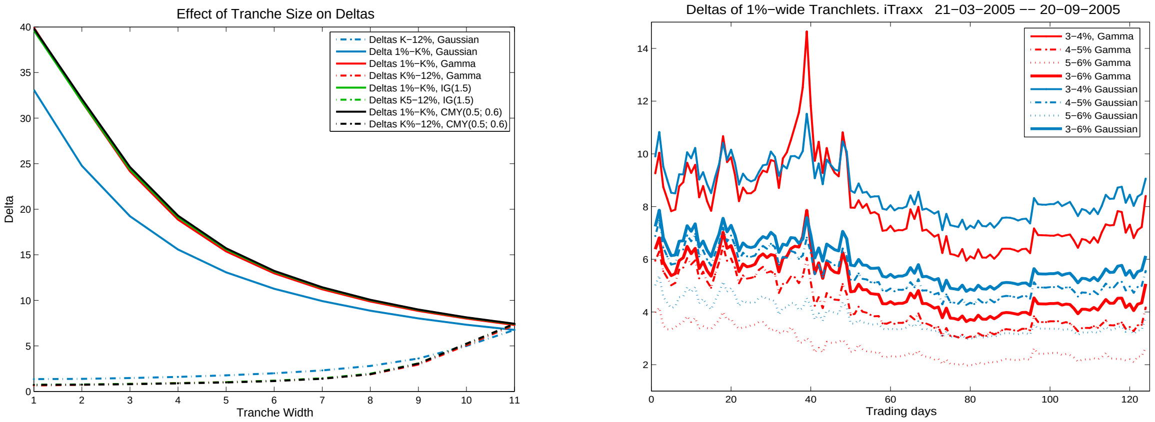
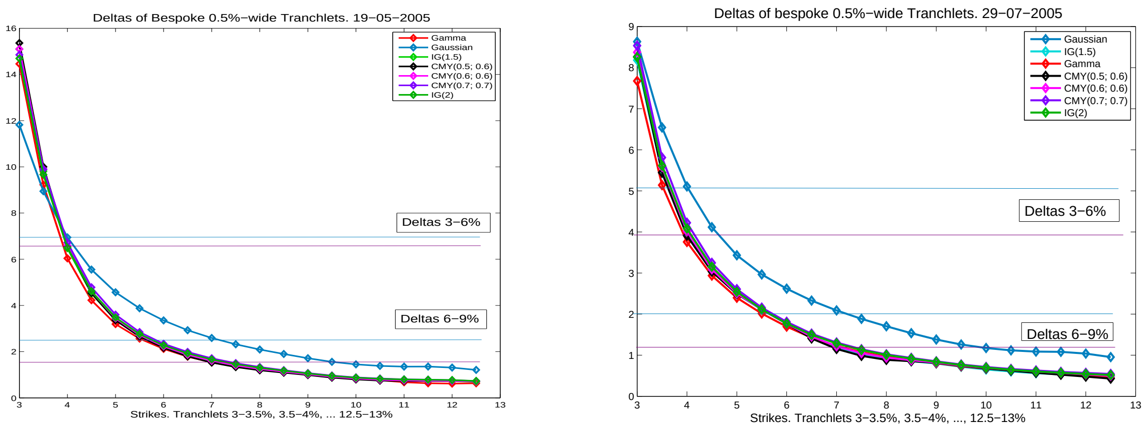


(a) Deltas sensitivity to Tranche Width and Seniority, iTraxx data 19-05-2005

(b) Deltas of Bespoke 1%-Wide Tranchlets

Figure 7: Deltas of Bespoke Tranchlets



(a) 19-05-2005. Auto crisis day

(b) 29-07-2005

Figure 8: Lévy vs Gaussian Deltas of Bespoke 0.5%-Wide Tranchlets - iTraxx data

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