

Circular choosability is rational

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April 8, 2008

Abstract

The circular choosability or circular list chromatic number of a graph is a list-version of the circular chromatic number, introduced by Mohar [4] and studied in [17, 2, 5, 7, 8, 15] and [10]. One of the nice properties that the circular chromatic number enjoys is that it is a rational number for all finite graphs G (see for instance [16]), and a fundamental question, posed by Zhu [17] and reiterated in [2] and [5], is whether the same holds for the circular choosability. In this paper we show that this is indeed the case.

Keywords: list colouring, circular colouring, circular choosability.

1 Introduction and statement of result

For $r > 0$ we shall denote by $S(r)$ the interval $[0, r]$ with the endpoints $\{0, r\}$ identified. For $x \in [0, r]$ we shall denote $x_r := \min(x, r - x)$. Alternatively $S(r)$ can be viewed as a circle of circumference r , and x_r as the length of the shortest circular arc between 0 and x . An r -circular colouring of a graph G is a map $c : V(G) \rightarrow S(r)$ such that

$$|c(v) - c(u)|_r \geq 1$$

for any edge $uv \in E(G)$. The *circular chromatic number* of G is defined as:

$$\chi_c(G) := \inf\{r \geq 1 : \text{there exists an } r\text{-circular colouring } c : V(G) \rightarrow S(r)\}.$$

The circular chromatic number has received considerable attention since it was first introduced by Vince in 1988 [12]. It enjoys several nice properties, including that $\lceil \chi_c(G) \rceil = \chi(G)$ – so that it is a “refinement” of the ordinary chromatic number – and that $\chi_c(G)$ is a rational number for all finite G . For proofs of these facts and an overview of the most important properties of χ_c , see for instance [16].

The circular choosability is a “list version” of the circular chromatic number that was introduced by Mohar in 2002 [4]. One of several equivalent definitions is as follows. If G is a graph and $r > 0$ then an r -circular list assignment L assigns to each vertex $v \in V(G)$ a set $L(v) \subseteq S(r)$. We say that G is L -circular choosable if there exists an r -circular colouring $c : V(G) \rightarrow S(r)$ with $c(v) \in L(v)$ for all $v \in V$. For W a Lebesgue measurable subset of \mathbb{R} or $S(r)$, let $\mu(W)$ denote the Lebesgue measure (“length”) of W . If G is L -circular choosable for each $r \geq t$ and for each r -circular list assignment L with $L(v)$ Lebesgue measurable and $\mu(L(v)) \geq t$ for all v , then G is t -circular choosable. The *circular choosability* of G is defined as:

$$\text{cch}(G) := \inf\{t \geq 1 : G \text{ is } t\text{-circular choosable}\}.$$

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Since its definition by Mohar [4], circular choosability has been studied by several authors including Zhu [17], Havet et al. [2], Norin [5], Norin et al. [7], Norin and Zhu [8], Raspaud and Zhu [10] and Yu et al. [15].

A fundamental open question, posed by Zhu [17] and reiterated by Havet et al. [2] and Norine [5] is whether $\text{cch}(G)$ is a rational number for every finite graph G . Here we will answer this question in the affirmative.

Theorem 1. $\text{cch}(G) \in \mathbb{Q}$ for any finite graph G .

It should perhaps be mentioned that our definition of $\text{cch}(G)$ differs somewhat from the original definition of Mohar [4], who only allowed circular list assignments L for which $L(v)$ consists of finitely many open intervals (ie. under his definition G is t -circular choosable if it is L -circular choosable for all circular list assignments with $L(v)$ consisting of finitely many open intervals and $\mu(L(v)) \geq t$ for all v). We are on the other hand allowing the $L(v)$ to be arbitrary Lebesgue measurable sets. It is not immediately clear – at least to the authors – that both definitions necessarily give the same value. That this is the case is however an easy corollary to Proposition 8 below.

2 The proof

We first need to introduce some extra notation and definitions. Throughout this paper all graphs will be finite. Whenever $G = (V, E)$ is a graph, we shall denote $n := |V|$ and we will always assume that $V = \{1, \dots, n\}$. If L is an r -circular list assignment and $c : V \rightarrow S(r)$ a (valid) L -circular colouring if it is a valid circular colouring and $c(v) \in L(v)$ for all $v \in V$. From now on, when there is mention of a circular list assignment L we will always assume that $L(v)$ is Lebesgue measurable for all v . We will usually speak simply of a circular list assignment L and leave the circumference r of the circle $S(r)$ that the lists are defined on implicit. If we do need to refer to this circumference, we will denote it by $r(L)$. We will also denote

$$t(L) := \min_{v \in V(G)} \mu(L(v)).$$

The following observation is immediate from the definition of cch :

Lemma 2. $\text{cch}(G) = \sup\{t(L) : G \text{ is not } L\text{-circular choosable}\}$ provided G has at least one edge.

If G is L -circular choosable for any list assignment L with the property that $\mu(L(v)) \geq t$ and $L(v)$ consists of at most m disjoint intervals for all $v \in V$, then we will say that G is (t, m) -circular choosable. Let us define

$$\text{cch}_m(G) := \inf\{t \geq 1 : G \text{ is } (t, m)\text{-circular choosable}\}.$$

Clearly $\text{cch}(G) \geq \text{cch}_m(G)$ for all m . According to Proposition 8 below we even have that $\text{cch}(G) = \text{cch}_m(G)$ for sufficiently large m . The proof of this nontrivial fact accounts for the bulk of this paper.

Let us observe that when computing $\text{cch}_m(G)$ we can restrict attention to lists consisting of only open intervals or only closed intervals if that is more convenient; and we can also allow lists that consist of at most m intervals and singletons (“degenerate intervals”).

We will call a map $c : V(G) \rightarrow S(r)$ a *strict circular colouring* of G if $|c(v) - c(u)|_r > 1$ for all $uv \in E(G)$. If L is a circular list assignment, then we will say that G is *L -strict circular choosable* if there exists a strict circular colouring with $c(v) \in L(v)$ for all $v \in V$; and we will say that G is *t -strict circular choosable* if it is L -strict circular choosable for every list assignment with $\mu(L(v)) \geq t$ for all $v \in V$. If G is L -strict circular choosable for every list assignment with the property that $\mu(L(v)) \geq t$ and $L(v)$ consists of at most m disjoint intervals then we will say that G is *(t, m) -strict circular choosable*. For technical reasons we need to work with strict circular colourings rather than ordinary circular colourings in some of our proofs. The following lemma says that we can reformulate the definitions of $\text{cch}(G)$ and $\text{cch}_m(G)$ in terms of strict circular colourings.

Lemma 3. *The following are equivalent formulations of cch , resp. cch_m :*

- (i) $\text{cch}(G) = \inf\{t \geq 1 : G \text{ is } t\text{-strict circular choosable}\};$
- (ii) $\text{cch}_m(G) = \inf\{t \geq 1 : G \text{ is } (t, m)\text{-strict circular choosable}\}.$

Proof: We shall only give the proof of (i), because the proof of (ii) is completely analogous. Let $\tau(G)$ denote the infimum in the right-hand side of (i). Clearly $\text{cch}(G) \leq \tau(G)$, since strict circular colourings are also circular colourings.

Now let L be a list assignment that does not allow a strict circular colouring. Let $0 < \varepsilon < 1$ be arbitrary. Let us set $r := r(L)$, $r' := (1 - \varepsilon)r$ and define L' by setting $L'(v) = (1 - \varepsilon)L(v) \subseteq S(r')$. We claim that L' does not allow a (non-strict) circular colouring. From this it will follow that $\text{cch}(G) \geq t(L') = (1 - \varepsilon)t(L)$ and, since L, ε are arbitrary, it also follows that $\text{cch}(G) \geq \tau(G)$.

To prove the claim, suppose that $c' : V \rightarrow S((1 - \varepsilon)r)$ is a (non-strict) circular colouring with $c'(v) \in L'(v)$ and let $c : V \rightarrow S(r)$ be given by $c(v) := c'(v)/(1 - \varepsilon)$. If $uv \in E$ is an edge then

$$\begin{aligned} |c(v) - c(u)|_r &= \min(|c(v) - c(u)|, r - |c(v) - c(u)|) \\ &= \min(|c'(v) - c'(u)|, r' - |c'(v) - c'(u)|)/(1 - \varepsilon) \\ &= |c'(v) - c'(u)|_{r'}/(1 - \varepsilon) \geq 1/(1 - \varepsilon), \end{aligned}$$

using that $|\lambda x - \lambda y| = \lambda|x - y|$ for all $x, y \in \mathbb{R}$ and $\lambda \geq 0$. Thus c is a strict circular colouring with $c(v) \in L(v)$ for all v , contradicting the choice of L . So the claim holds indeed. \blacksquare

The next lemma shows that when determining $\text{cch}(G)$ we can restrict ourselves to circular list assignments with $r(L)$ not larger than $n \cdot t(L)$.

Lemma 4. *Let G be a graph, and let L be a circular list assignment such that G is not L -strict circular choosable. Then there exists a circular list assignment L' with $t(L') \geq t(L)$ and $r(L') \leq n \cdot t(L)$ such that G is also not L' -strict circular choosable.*

Proof: We may assume wlog. that $\mu(L(v)) = t(L)$ for all $v \in V$. Let us set $A := \bigcup_{v \in V} L(v)$ and $r' := \mu(A)$ (note that $r' \leq n \cdot t(L)$). In the remainder of the proof we will treat A and the lists $L(v)$ as subsets of $[0, r]$. Let us consider the map $\phi : [0, r] \rightarrow [0, r']$ given by $\phi(x) := \mu([0, x] \cap A)$. Notice that ϕ is continuous and non-decreasing. Let us now set

$$L'(v) := \text{cl}(\phi[L(v)]).$$

(Here $\text{cl}(\cdot)$ denotes topological closure.) We will show that if we interpret the $L'(v)$ as subsets of $S(r')$ then the r' -circular list assignment L' is as required by the lemma. First note that $L'(v)$ is Lebesgue measurable for all $v \in V$, since it is closed. We now need to check that $\mu(L'(v)) \geq t(L)$ for all $v \in V$ and that G is not L' -circular choosable.

To this end, we first claim that:

$$\mu(\phi^{-1}[I] \cap A) = \mu(I) \quad \text{for any interval } I \subseteq [0, r']. \quad (1)$$

For, if $a' < b' \in [0, r']$ are the endpoints of I then it follows from the fact that ϕ is continuous and nondecreasing that $\phi^{-1}[I]$ is also an interval and its endpoints $a < b \in [0, r]$ satisfy $\phi(a) = a', \phi(b) = b'$. By definition of ϕ we have $\mu(I) = b' - a' = \phi(b) - \phi(a) = \mu([a, b] \cap A) = \mu(\phi^{-1}[I] \cap A)$, proving the claim.

Next, recall that according to the outer-measure construction (see for instance [1], pages 14–21) for all Lebesgue measurable B it holds that $\mu(B) = \inf_{\mathcal{J}} \sum_{I \in \mathcal{J}} \mu(I)$, where the infimum is over all countable collections of open intervals that cover B . Thus, let I_1, I_2, \dots be countably many open intervals that cover $L'(v)$. Let us set $J_k := \phi^{-1}[I_k \cap [0, r']] \cap A$ for all $k \geq 1$. Then the J_k clearly cover $L(v)$, so that by (1) and countable subadditivity of the Lebesgue measure:

$$\sum_k \mu(I_k) \geq \sum_k \mu(I_k \cap [0, r']) = \sum_k \mu(J_k) \geq \mu(L(v)).$$

Since the I_k were an arbitrary collection of open intervals that cover $L'(v)$, it follows that $\mu(L'(v)) \geq \mu(L(v))$.

Finally suppose that there exist $c'(v) \in L'(v)$ so that $|c'(v) - c'(u)|_{r'} > 1$ for all $uv \in E$. We can assume wlog. that $c'(v) \in \phi(L(v))$ for all v (since $L'(v)$ is the closure of $\phi(L(v))$ and $(x_1, \dots, x_n) \mapsto \min_{uv \in E} |x_v - x_u|_{r'}$ is continuous). Let us thus assume $c'(v) \in \phi(L(v))$, and pick an arbitrary $c(v) \in \phi^{-1}(c'(v)) \cap L(v)$ for all v . Now let $uv \in E$ be arbitrary. We can assume wlog. $c(v) > c(u)$. By definition of ϕ :

$$\begin{aligned} |c(v) - c(u)|_r &= \min(|c(v) - c(u)|, r - |c(v) - c(u)|) \\ &= \min(\mu([c(u), c(v)]), \mu([0, c(u)]) + \mu([c(v), r])) \\ &\geq \min(\mu([c(u), c(v)] \cap A), \mu([0, c(u)] \cap A) + \mu([c(v), r] \cap A)) \\ &= \min(|c'(v) - c'(u)|, r' - |c'(v) - c'(u)|) \\ &= |c'(v) - c'(u)|_{r'} > 1. \end{aligned}$$

But this shows that c is an r -strict circular colouring with $c(v) \in L(v)$, which contradicts the choice of L . It follows that G is not L' -strict circular choosable as required. \blacksquare

We will also need the following fact:

Theorem 5 ([17]). $\text{cch}(G) \leq \Delta(G) + 1$ for all finite graphs G .

Here we should remark that although we have not yet proved that the definition used in this paper is equivalent to the original definition of Mohar [4] (which is used in the proof of Theorem 5 given in [17]), the proof given in [17] can easily be adapted to work for our definition as well.

Lemma 6. For each finite G there exists an $r = r(G)$ and an r -circular list assignment L with $\mu(L(v)) \geq \text{cch}(G)$ for all $v \in V$ such that G is not L -strict circular choosable.

Proof: By Lemma 2 there exists a sequence of (not necessarily distinct) circular list assignments L_1, L_2, \dots such that G is not L_m -strict circular choosable for all m and $t(L_m) \rightarrow \text{cch}(G)$. For convenience, let us set $t_m := t(L_m)$ and $r_m := r(L_m)$. By Lemma 4 and Theorem 5 we may assume that $r_m \leq n^2$ for all m , and consequently we can assume (restricting to a subsequence if necessary) that r_m tends to some limit $r \leq n^2$. In the remainder of the proof we will treat the lists $L_m(v)$ as subsets of $[0, n^2]$.

We shall inductively define a decreasing sequence of infinite subsets \mathcal{M}_k of \mathbb{N} and vectors $a^k \in \{0, 1\}^{V \times \{0, \dots, n^2 2^k - 1\}}$ for $k \in \mathbb{N}$ such that:

- (i) $\mathcal{M}_{k+1} \subseteq \mathcal{M}_k$;
- (ii) \mathcal{M}_k is infinite;
- (iii) If $a_{v,i}^k = 1$ then $L_m(v) \cap [\frac{i}{2^k}, \frac{i+1}{2^k}] \neq \emptyset$ for all $m \in \mathcal{M}_k$;
- (iv) If $a_{v,i}^k = 0$ then $L_m(v) \cap [\frac{i}{2^k}, \frac{i+1}{2^k}] = \emptyset$ for all $m \in \mathcal{M}_k$.

The construction goes as follows. For each $k \in \mathbb{N}$ and $a \in \{0, 1\}^{V \times \{0, \dots, n^2 2^k - 1\}}$, let \mathcal{M}_k^a denote the set of those $m \in \mathbb{N}$ for which $L_m(v) \cap [\frac{i}{2^k}, \frac{i+1}{2^k}] \neq \emptyset$ precisely when $a_{v,i} = 1$. Then \mathcal{M}_1^a must be infinite for at least one a , because the \mathcal{M}_1^a partition \mathbb{N} . Let $a^1 \in \{0, 1\}^{V \times \{0, \dots, n^2 - 1\}}$ be such an a and set $\mathcal{M}_1 := \mathcal{M}_1^{a^1}$. Similarly, given \mathcal{M}_{k-1} , for at least one $a \in \{0, 1\}^{V \times \{0, \dots, n^2 2^k - 1\}}$ the set $\mathcal{M}_{k-1} \cap \mathcal{M}_k^a$ must be infinite. Pick such an a and put $a^k := a, \mathcal{M}_k := \mathcal{M}_{k-1} \cap \mathcal{M}_k^a$. Clearly, the \mathcal{M}_k, a^k thus constructed satisfy the demands (i)-(iv).

For $v \in V, k \in \mathbb{N}$, let us denote $L^k(v) := \bigcup \{[\frac{i}{2^k}, \frac{i+1}{2^k}] : a_{v,i}^k = 1\}$. Note that $L_m(v) \subseteq L^k(v)$ for all $m \in \mathcal{M}_k, v \in V, k \in \mathbb{N}$. Thus:

$$\mu(L^k(v)) \geq \sup_{m \in \mathcal{M}_k} \mu(L_m(v)) \geq \text{cch}(G),$$

by choice of the initial sequence $(L_m)_m$. Next, let us define

$$L(v) := \bigcap_{k=1}^{\infty} L^k(v).$$

It remains to be seen that the $L(v)$ yield an r -circular list assignment that satisfies the requirements of lemma 6.

Clearly $L(v)$ is closed (and hence Lebesgue measurable) for all v . Observe that $L^{k+1}(v) \subseteq L^k(v)$ for all v and k by construction. By “continuity of measure” we therefore have:

$$\mu(L(v)) = \lim_{k \rightarrow \infty} \mu(L^k(v)) \geq \text{cch}(G),$$

for all $v \in V$.

Next note that $L(v) \subseteq [0, r]$ for all $v \in V$, because $L^k(v) \subseteq [0, r_m + \frac{1}{2^k}]$ for all $m \in \mathcal{M}_k$ by construction and $r_m \rightarrow r$.

Finally, suppose that there exists a function $c : V \rightarrow [0, r]$ with $c(v) \in L(v)$ for all $v \in V$ and $|c(v) - c(u)|_r > 1$ for all $uv \in E$. For each $k \in \mathbb{N}$, let us arbitrarily pick an $m_k \in \mathcal{M}_k$. By construction, for each $v \in V$ and $k \in \mathbb{N}$, we can pick a $c_k(v) \in L_{m_k}(v)$ such that $|c_k(v) - c(v)| < 2^{-k}$. But then it holds that

$$\begin{aligned} \lim_{k \rightarrow \infty} |c_k(v) - c_k(u)|_{r_{m_k}} &= \lim_{k \rightarrow \infty} \min(|c_k(v) - c_k(u)|, r_{m_k} - |c_k(v) - c_k(u)|) \\ &= \min(|c(v) - c(u)|, r - |c(v) - c(u)|) \\ &= |c(v) - c(u)|_r > 1, \end{aligned}$$

for all $uv \in E$. Hence G is L_{m_k} -circular choosable for k sufficiently large, contradicting our choice of the initial sequence $(L_m)_m$. So L is indeed as required. \blacksquare

It is perhaps interesting to remark that there is no nonstrict analogue of Lemma 6 (for instance: $\text{cch}(K_2) = 2$ and it is 2-circular choosable).

We will say that a set $I \subseteq S(r)$ is a *circular interval* if it is of the form $I = \{x \bmod r : x \in J\}$ for some interval $J \subseteq \mathbb{R}$.

Lemma 7. *For each $r > 0$ and $0 < \varepsilon < r$ there exists a finite set $A \subseteq S(r)$ such that:*

- (i) $I \cap A \neq \emptyset$ whenever $I \subseteq S(r)$ is a circular interval with $\mu(I) \geq \varepsilon$;
- (ii) $a - 1 \bmod r \in A$ for all but at most one $a \in A$;
- (iii) $a + 1 \bmod r \in A$ for all but at most one $a \in A$.

Proof: First suppose that $r = \frac{p}{q}$ is rational. Pick an integer N such that $\frac{1}{Nq} < \varepsilon$. It can be easily checked that the set $A := \{\frac{k}{Nq} : k = 0, \dots, Np\}$ is as required (and in fact there are no exceptions to demands (ii), (iii)).

Now suppose r is irrational. First note that the points $i \bmod r, i \in \mathbb{Z}$ are all distinct (if $i \bmod r = j \bmod r$ for integers $i < j$ then $j - i = kr$ for some positive integer k , which implies $r = (j - i)/k \in \mathbb{Q}$). We claim that for $M = M(r)$ a sufficiently large integer, the set

$$A := \{i \bmod r : i = 0, \dots, M\}$$

is as required. To see this first note that for $m > \lceil r/\varepsilon \rceil$, the set $\{i \bmod r : i = 0, \dots, m\}$ must contain two points at distance $< \varepsilon$. In fact, if $|i \bmod r - j \bmod r|_r < \varepsilon$ with $i < j$ then we also have $|(j - i) \bmod r|_r < \varepsilon$. Hence we can pick $m_0 \leq \lceil r/\varepsilon \rceil + 1$ such that $|m_0 \bmod r|_r < \varepsilon$. Let us assume $m_0 \bmod r \in [0, \varepsilon)$ (the case when $m_0 \bmod r \in (r - \varepsilon, r]$ is similar). Set $l := m_0 \bmod r$ and note that $2m_0 \bmod r = 2l, 3m_0 \bmod r = 3l$ etc. The set $\{i \cdot m_0 \bmod r : i = 0, 1, \dots, \lfloor r/l \rfloor\}$ thus already satisfies (i) and it follows that if we set $M := m_0 \cdot \lfloor r/l \rfloor$ then the set A is as required (the two exceptions to demands (ii) and (iii) being 0 and $M \bmod r$). \blacksquare

Proposition 8. *For each finite graph G , there exists an $m = m(G)$ such that $\text{cch}(G) = \text{cch}_m(G)$.*

Proof: First note that $\text{cch}(G) = 1$ iff G has no edges at all, and $\text{cch}(G) \geq 2$ otherwise. Furthermore, if G has at least one edge, $1 \leq t < 2$ and L is defined by $L(v) := S(t)$ for all $v \in V$, then G is not L -circular choosable. So if $\text{cch}(G)$ equals 1 or 2 then $\text{cch}(G) = \text{cch}_1(G)$. For the remainder of the proof, we can therefore assume that $\text{cch}(G) > 2$.

Let $r = r(G)$ and L be as provided by Lemma 6. Choose an $\varepsilon < \min((r-2)/2, \frac{r}{4n+2})$ and let $A = A(\varepsilon, r) \subseteq S(r)$ be as provided by Lemma 7. Set $M := |A|$ and $m := (2n+1) \cdot n! \cdot M^n$.

Pick an arbitrary vertex $v \in V$. The idea for the rest of the proof is to show there exists $L'(v) \supseteq L(v)$ that consists of at most m intervals and singletons, such that if we set $L'(u) := L(u)$ for all $u \neq v$ then G is not L' -strict circular choosable either. From this the proposition follows by induction on the number of vertices whose lists are not the union of at most m intervals and singletons.

Let us relabel A as $A = \{a_0, a_1, \dots, a_{M-1}\}$ where a_{i+1} is the point immediately clockwise from a_i . Here and in the rest of the proof addition of indices is always taken modulo M . Let a_{b_1}, a_{b_2} denote the ‘‘bad’’ points for which $a_{b_i} - 1 \pmod r$ or $a_{b_i} + 1 \pmod r$ is not in A . (If there are no bad points, or only one then we can arbitrarily pick two or one point from A and treat them as bad in the rest of the proof.) For convenience let us assume (wlog.) that $a_0 = 0$. Let I_i denote the interval $I_i := [a_i, a_{i+1})$ if $i < M-1$ and $I_{M-1} := [a_{M-1}, r)$.

For $k \in \{0, \dots, 2n\}$ let us write $a_i^k := a_i + 2k\varepsilon \pmod r$ and set $A^k := \{a_0^k, \dots, a_{M-1}^k\}$, and $I_i^k := \{(x + 2k\varepsilon) \pmod r : x \in I_i\}$. (Ie. the superscript k denotes that the whole construction has been shifted clockwise by $2k\varepsilon$.) Let us first observe that for any map $c : V \rightarrow S(r)$ there exists a $k \in \{0, \dots, 2n\}$ such that $c(u) \notin I_{b_1-1}^k \cup I_{b_1}^k \cup I_{b_2-1}^k \cup I_{b_2}^k$ for all $u \in V$. To see this, notice that the sets $I_{b_1-1}^k \cup I_{b_1}^k$ are disjoint for $k = 0, \dots, 2n$. Hence there are at least $n+1$ values of k for which $I_{b_1-1}^k \cup I_{b_1}^k$ does not contain any $c(u)$; and, since the same argument applies to the $I_{b_2-1}^k \cup I_{b_2}^k$, there must indeed be at least one value of k for which $I_{b_1-1}^k \cup I_{b_1}^k \cup I_{b_2-1}^k \cup I_{b_2}^k$ does not contain any $c(u)$.

For $k \in \{0, \dots, 2n\}, p : V \rightarrow \{0, \dots, M-1\} \setminus \{b_i - 1, b_i : i = 1, 2\}$ and σ a permutation of $V = \{1, \dots, n\}$, let $\mathcal{C}_{k,p,\sigma}$ denote the set of all maps $c : V \rightarrow S(r)$ for which

- (C-1) $c(i) \in I_{p(i)}^k$ for all $i \in V$;
- (C-2) $|c(i) - a_{p(i)}^k|_r \leq |c(j) - a_{p(j)}^k|_r$ if $\sigma(i) < \sigma(j)$;
- (C-3) $c(i) \in L(i)$ for all $i \neq v$.
- (C-4) c is a strict circular colouring.

Let us denote

$$O_{k,p,\sigma} := \{x \in S(r) : \text{there exists } c \in \mathcal{C}_{k,p,\sigma} \text{ with } c(v) = x\},$$

and observe that

$$\bigcup_{k,p,\sigma} O_{k,p,\sigma} = \{x \in S(r) : \exists \text{ a strict circular colouring } c \text{ with } c(v) = x \text{ and } c(u) \in L(u) \text{ for all } u \neq v\}.$$

We shall show that $O_{k,p,\sigma}$ is either the empty set, a singleton or an interval for each triple k, p, σ . This shows that $L'(v) := S(r) \setminus \bigcup_{k,p,\sigma} O_{k,p,\sigma}$ is a union of at most $m = (2n+1) \cdot n! \cdot M^n$ intervals and singletons. Since $L'(v)$ is precisely the set of all $x \in S(r)$ for which there is no strict circular colouring $c : V \rightarrow S(r)$ with $c(v) = x$ and $c(u) \in L(u)$ for all $u \neq v$, this choice of $L'(v)$ is as required.

Let us thus pick an arbitrary triple k, p, σ and consider $O_{k,p,\sigma}$. To ease the burden of notation we will assume wlog. that $k = 0$ and σ is the identity. Observe that for $x \in I_i$ we have $|x - a_i|_r = x - a_i$.

A key property of A is that if $a_i + 1 \pmod r = a_j$ then also $a_{i+1} + 1 \pmod r = a_{j+1}$, unless i or j is in $\{b_1 - 1, b_1, b_2 - 1, b_2\}$. By choice of ε, A and p the following thus hold for every pair of vertices $i, j \in V$:

- If $|a_{p(j)} - a_{p(i)}|_r > 1$ then $|x - y|_r > 1$ for all $x \in I_{p(i)}, y \in I_{p(j)}$;
- If $|a_{p(j)} - a_{p(i)}|_r < 1$ then $|x - y|_r < 1$ for all $x \in I_{p(i)}, y \in I_{p(j)}$;
- If $a_{p(j)} = a_{p(i)} + 1 \pmod r$ and $x \in I_{p(i)}, y \in I_{p(j)}$ then $|x - y|_r > 1$ iff $x - a_{p(i)} < y - a_{p(j)}$.

Let us say that an edge $ij \in E$ with $j < i$ is:

- *bad* if $|a_{p(j)} - a_{p(i)}|_r < 1$ or if $a_{p(j)} = a_{p(i)} + 1 \pmod r$;
- *relevant* if $a_{p(j)} = a_{p(i)} - 1 \pmod r$;
- *good* if it is not bad or relevant.

Observe that any map $c : V \rightarrow S(r)$ that satisfies **(C-1)** and **(C-2)** is a strict circular colouring iff there are no bad edges and the inequality in **(C-2)** is strict for all relevant edges. From now on we shall assume there are no bad edges (since otherwise $\mathcal{C}_{k,p,\sigma}$ and $O_{k,p,\sigma}$ are both empty and we are done).

Let us set $c_{\min}(1) = \inf I_{p(1)} \cap L(1)$ (we can assume that $I_{p(1)} \cap L(1) \neq \emptyset$ – otherwise $\mathcal{C}_{k,p,\sigma}$ and $O_{k,p,\sigma}$ are empty). If the infimum is genuine (ie. not a minimum) then we will say that $c_{\min}(1)$ is *dangerous*. Next, suppose that for some $i \leq v$ the values $c_{\min}(j)$ have been defined for all $j < i$. Let $X(i)$ denote the set of all $x \in I_{p(i)}$ such that:

- (X-1)** $x - a_{p(i)} \geq c_{\min}(j) - a_{p(j)}$ for all $j < i$;
- (X-2)** $x - a_{p(i)} > c_{\min}(j) - a_{p(j)}$ if $j < i$ and $c_{\min}(j)$ is dangerous;
- (X-3)** $x - a_{p(i)} > c_{\min}(j) - a_{p(j)}$ if $j < i$ and ij is a relevant edge.

Let us set $c_{\min}(i) := \inf X(i) \cap L(i)$ for $i < v$ and $c_{\min}(v) := \inf X(v)$. If the infimum in the definition of $c_{\min}(i)$ is genuine (ie. not a minimum) then we will say that $c_{\min}(i)$ is *dangerous*. By a straightforward inductive argument $c(i) \geq c_{\min}(i)$ for all $c \in \mathcal{C}_{k,p,\sigma}$ and all $i \leq v$; and if $X(i) \cap L(i) = \emptyset$ for some $i < v$ or $X(v) = \emptyset$ then $\mathcal{C}_{k,p,\sigma}$ and $O_{k,p,\sigma}$ are also empty – in which case we are done, so we shall assume this is not the case.

Similarly, let us put $c_{\max}(n) = \sup I_{p(n)} \cap L(n)$ (again we may assume $I_{p(n)} \cap L(n) \neq \emptyset$ – otherwise $\mathcal{C}_{k,p,\sigma}$ and $O_{k,p,\sigma}$ are both empty too). If the supremum is genuine (ie. not a maximum) then we will say that $c_{\max}(n)$ is *dangerous*. Suppose that for some $i \geq v$ the values $c_{\max}(j)$ have been defined for all $j > i$. Let $Y(i)$ denote the set of all $y \in I_{p(i)}$ such that:

- (Y-1)** $y - a_{p(i)} \leq c_{\max}(j) - a_{p(j)}$ for all $j > i$;
- (Y-2)** $y - a_{p(i)} < c_{\max}(j) - a_{p(j)}$ if $j > i$ and $c_{\max}(j)$ is dangerous;
- (Y-3)** $y - a_{p(i)} < c_{\max}(j) - a_{p(j)}$ if $j > i$ and ij is a relevant edge.

Let us set $c_{\max}(i) = \sup Y(i) \cap L(i)$ for $i > v$ and $c_{\max}(v) = \sup Y(v)$. We will call $c_{\max}(i)$ *dangerous* if the supremum in the definition is genuine. Again a straightforward inductive argument shows that $c(i) \leq c_{\max}(i)$ for all $c \in \mathcal{C}_{k,p,\sigma}$ and $i \geq v$; and that we can assume $Y(v) \neq \emptyset$ and $Y(i) \cap L(i) \neq \emptyset$ for all $i > v$.

We have seen that $c_{\min}(v) \leq c(v) \leq c_{\max}(v)$ for all $c \in \mathcal{C}_{k,p,\sigma}$. So if $c_{\min}(v) \geq c_{\max}(v)$ then $O_{k,p,\sigma}$ is either empty or a singleton and we are done. Let us therefore assume that $c_{\min}(v) < c_{\max}(v)$, and pick an arbitrary $c_{\min}(v) < x < c_{\max}(v)$. To finish the proof it suffices to construct a $c \in \mathcal{C}_{k,p,\sigma}$ with $c(v) = x$.

Claim 9. It is possible to pick $c(1), \dots, c(n)$ such that $c(v) = x, c(i) \in L(i) \cap I_{p(i)}$ for $i \neq v$ and:

- (c-1)** $c(i) = c_{\min}(i)$ if $i < v$ and $c_{\min}(i)$ is not dangerous;
- (c-2)** $\min_{i < j \leq v} c(j) - a_{p(j)} > c(i) - a_{p(i)} > c_{\min}(i) - a_{p(i)}$ if $i < v$ and $c_{\min}(i)$ is dangerous;

(c-3) $c(i) = c_{\max}(i)$ if $i > v$ and $c_{\max}(i)$ is not dangerous;

(c-4) $\max_{v \leq j < i} c(j) - a_{p(j)} < c(i) - a_{p(i)} < c_{\max}(i) - a_{p(i)}$ if $i > v$ and $c_{\max}(i)$ is dangerous.

Proof of Claim 9: Set $c(v) = x$. Let us first pick an $i < v$ and suppose that $c(j)$ has already been defined for all $i < j < v$ in such a way that (c-1) and (c-2) hold for all j with $i < j < v$ (this is certainly true when $i = v - 1$). If $c_{\min}(i)$ is not dangerous, then we can simply put $c(i) := c_{\min}(i)$. Now suppose that $c_{\min}(i)$ is dangerous. In this case $(c_{\min}(i), c_{\min}(i) + \varepsilon) \cap L(i)$ is nonempty for all $\varepsilon > 0$ (since the infimum in the definition of $c_{\min}(i)$ is genuine). In addition $c_{\min}(i) - a_{p(i)} < \min_{i < j \leq v} c(j) - a_{p(j)}$. To see this, suppose that $c_{\min}(i) - a_{p(i)} \geq c(j) - a_{p(j)}$ for some $i < j \leq v$. Because $c(j) - a_{p(j)} \geq c_{\min}(j) - a_{p(j)}$ (by (c-1), resp. (c-2)) and $c_{\min}(i) - a_{p(i)} \leq c_{\min}(j) - a_{p(j)}$ (by (X-1)), we necessarily have $c(j) - a_{p(j)} = c_{\min}(j) - a_{p(j)} = c_{\min}(i) - a_{p(i)}$. Then $c_{\min}(j)$ must also be dangerous, because $c_{\min}(i)$ is dangerous (cf. (X-2)). But this contradicts assumption (c-2). So indeed $c_{\min}(i) - a_{p(i)} < \min_{i < j \leq v} c(j) - a_{p(j)}$, and hence we can choose

$$c(i) \in (c_{\min}(i), a_{p(i)} + \min_{i < j \leq v} c(j) - a_{p(j)}) \cap L(i).$$

Thus we can indeed pick $c(i)$ satisfying (c-1) and (c-2) for all $1 \leq i < v$. The proof that we can also pick $c(i)$ satisfying (c-3) and (c-4) for all $v < i \leq n$ is completely analogous to the preceding argument. ■

It remains to be seen that $c \in \mathcal{C}_{k,p,\sigma}$. Clearly (C-1) and (C-3) hold. To see that (C-2) also holds, pick $1 \leq i < j \leq n$. First suppose that $i < j \leq v$. If $c_{\min}(i)$ is dangerous then $c(i) - a_{p(i)} \leq c(j) - a_{p(j)}$ by (c-2). If $c_{\min}(i)$ is not dangerous then $c(i) - a_{p(i)} = c_{\min}(i) - a_{p(i)} \leq c_{\min}(j) - a_{p(j)} \leq c(j) - a_{p(j)}$ (by (c-1), (c-2) and (X-1)). If $v \leq i < j$ then we also have $c(i) - a_{p(i)} \leq c(j) - a_{p(j)}$, by an analogous argument. Finally, if $i < v < j$ then $c(i) - a_{p(i)} \leq c(v) - a_{p(v)} \leq c(j) - a_{p(j)}$, so that (C-2) indeed holds.

To finish the proof we now only need to verify that c is a strict colouring. Let $ij \in E$ be a relevant edge. First suppose that $i < j \leq v$. If $c_{\min}(i)$ is dangerous then $c(i) - a_{p(i)} < c(j) - a_{p(j)}$ by (c-2). So suppose that $c_{\min}(i)$ is not dangerous. Then we have $c(i) = c_{\min}(i)$, and by (X-3) either $c_{\min}(j) - a_{p(j)} > c_{\min}(i) - a_{p(i)}$ or $c_{\min}(j) - a_{p(j)} = c_{\min}(i) - a_{p(i)}$ and $c_{\min}(j)$ is dangerous. In the first case it follows from $c(j) - a_{p(j)} \geq c_{\min}(j) - a_{p(j)}$ (by (c-1) and (c-2)) that $c(i) - a_{p(i)} = c_{\min}(i) - a_{p(i)} < c(j) - a_{p(j)}$. In the second case the same thing is immediate from (c-2).

A completely analogous argument shows that if $v \leq i < j$ then we also have $c(i) - a_{p(i)} < c(j) - a_{p(j)}$.

Let us thus suppose that $i < v < j$. If $c_{\min}(i)$ is dangerous then $c(i) - a_{p(i)} < c(v) - a_{p(v)} \leq c(j) - a_{p(j)}$ using (c-2) and (C-2). If $c_{\min}(i)$ is not dangerous, then $c(i) - a_{p(i)} = c_{\min}(i) - a_{p(i)} \leq c_{\min}(v) - a_{p(v)} < c(v) - a_{p(v)} \leq c(j) - a_{p(j)}$ by (c-1), (X-1), the choice of $x = c(v)$ and (C-2).

This shows that $|c(i) - c(j)|_r > 1$ for all edges $ij \in E$ with $i < j \leq v$ (there are no bad edges by assumption and we do not need to worry about good edges), which concludes the proof. ■

As an aside let us also remark that, as mentioned in the introduction, Proposition 8 shows that our definition indeed coincides with the original definition of Mohar [4]. Let us say that G is *t-finite open circular choosable* if it is L -circular choosable for any circular list assignment L with $\mu(L(v)) \geq t$ and $L(v)$ a union of finitely many open intervals. The definition of Mohar is:

$$\text{cch}^{\text{Mohar}}(G) := \inf\{t \geq 1 : G \text{ is } t\text{-finite open circular choosable}\}.$$

Observe that $\text{cch}^{\text{Mohar}}(G) = \sup_{m \geq 1} \text{cch}_m(G)$, since we can restrict attention to lists consisting of at most m open intervals when computing $\text{cch}_m(G)$. The following is now immediate from Proposition 8:

Corollary 10. $\text{cch}(G) = \text{cch}^{\text{Mohar}}(G)$ for all finite G .

For the proof of Theorem 1 we also need the following observation:

Lemma 11. *Let L be an circular list assignment where every list $L(v)$ consists of finitely many closed intervals. Let us write*

$$L(v) := \bigcup_{i=1}^{m(v)} [a_i(v), b_i(v)].$$

If G is L -circular choosable then there also exists a valid circular colouring c with

$$c(v) \in \{a_i(w) + k \bmod r(L) : w \in V, i = 1, \dots, m(w), k = -n, \dots, n\} \cap L(v). \quad (2)$$

for all $v \in V$.

Proof: Suppose that L is as above and G is L -circular choosable, but there is no circular colouring of the required form. For convenience let us write $r := r(L)$. For $c : V(G) \rightarrow S(r)$ a circular colouring, let H_c be the graph with vertex set $V(G)$ and an edge $uv \in E(H_c)$ iff $uv \in E(G)$ is an edge of G and $|c(u) - c(v)|_r = 1$. For each vertex $v \in V$, let $C_c(v)$ denote the component of H_c that contains v . Notice that $c(v)$ satisfies (2) iff $c(u)$ satisfies (2) for all $u \in C_c(v)$. Now pick a circular colouring $c : V(G) \rightarrow S(r)$ such that $c(u) \in L(u)$ for all $u \in V$, and the number of vertices $v \in V(G)$ with $c(v)$ of the form (2) is as large as possible and, subject to this, the number of components of H_c is as small as possible. Pick a vertex $v \in V(G)$ with $c(v)$ not of the form (2). For each $u \in C_c(v)$ there is an index $j(u)$ such that $c(u) \in (a_{j(u)}(u), b_{j(u)}(u)]$. For $x, y \in S(r)$, let $\text{cdist}(x, y)$ denote the *clockwise distance* from x to y , ie. if $0 \leq x \leq y \leq r$ then $\text{cdist}(x, y) = y - x$ and otherwise $\text{cdist}(x, y) = r - x + y$. Let us define:

$$\alpha := \min \left(\min_{u \in C_c(v)} c(u) - a_{j(u)}, \min_{\substack{u \in C_c(v), w \notin C_c(v), \\ uw \in E(G)}} \text{cdist}(c(w), c(u)) - 1 \right).$$

(Here we use the convention that the minimum of the empty set is $+\infty$.) Clearly $\alpha > 0$. Let us define a new colouring $c' : V \rightarrow S(r)$ by setting $c'(u) = c(u) - \alpha$ for $u \in C_c(v)$ and $c'(u) = c(u)$ for $u \notin C_c(v)$. By definition of α we still have $c'(u) \in L(u)$ for all $u \in V$ and c' is a valid circular colouring. Moreover, either $H_{c'}$ has fewer components than H_c , or $C_{c'}(v) = C_c(v)$ and $c'(v)$ satisfies (2). But this contradicts the choice of c . The lemma follows. \blacksquare

We are now in a position to finish the proof of Theorem 1.

Proof of Theorem 1: By Proposition 8 and Lemma 11 there exists an integer m such that $\text{cch}(G)$ is the supremum of $t(L)$ over all list assignments L of the form

$$L(v) = \bigcup_{i=1}^m [a_i(v), b_i(v)],$$

for which none of the maps $c : V \rightarrow \{a_i(w) + k \bmod r(L) : w \in V, i = 1, \dots, m, k = -n, \dots, n\}$ is a valid L -circular colouring. This allows us to write $\text{cch}(G)$ as an optimisation problem with finitely many variables, which we will now proceed to do. We begin with the following set of linear inequalities, which express that the variables $a_1(1), b_1(1), \dots, a_m(n), b_m(n)$ correspond an r -circular list assignment L with $t(L) \geq t$:

$$\begin{aligned} \sum_{i=1}^r b_i(v) - a_i(v) &\geq t, & (\forall v \in V), \\ 0 \leq a_1(v) \leq b_1(v) \leq a_2(v) \leq \dots \leq a_m(v) \leq b_m(v) \leq r, & (\forall v \in V), \\ r &\geq t \geq 1. \end{aligned} \quad (3)$$

Let us also set:

$$P := \{x = (t, r, a_1(1), b_1(1), \dots, a_m(n), b_m(n)) \in \mathbb{R}^{2+2nm} : x \text{ satisfies (3)}\}.$$

Notice that we can write $P = \{x \in \mathbb{R}^{2+2nm} : Ax \leq b\}$ for some matrix A and vector b with all entries of A, b integers (in fact only the values $-1, 0, 1$ appear in A, b).

We now wish to capture the fact that none of colourings of the special form provided by Lemma 11 is a valid L -circular colouring. Let Ψ denote the set of all mappings $V \rightarrow V \times \{1, \dots, m\} \times \{-n, \dots, n\}^2$. For notational convenience only we will introduce some additional auxiliary variables. For each $\psi \in \Psi, v \in V$ with $\psi(v) = (w, i, k, l)$ let us set

$$c^\psi(v) := a_i(w) + k + l \cdot r. \quad (4)$$

Notice that $c^\psi(v)$ equals an integer plus an integer linear combination of r and $a_i(w)$. Also note that if c is of the standard form provided by Lemma 11 then there is some $\psi \in \Psi$ such that $c^\psi(v) = c(v)$ for all $v \in V$ – but not every ψ will correspond to such a c . For $x \in P$ the corresponding circular list assignment L has a valid circular colouring iff $c^\psi(v) : v \in V$ is such a valid circular colouring for some $\psi \in \Psi$. Moreover, if G is L -circular choosable there must also exist a permutation $\sigma \in S_n$ such that

$$c^\psi(\sigma(1)) \leq \dots \leq c^\psi(\sigma(n)), \quad (5)$$

and c^ψ defines a valid L -circular colouring. For each pair (σ, ψ) let $E^{\sigma, \psi}$ be the set of constraints:

$$c^\psi(\sigma(i+1)) < c^\psi(\sigma(i)), \quad i = 1, \dots, n-1, \quad (6)$$

$$c^\psi(v) < a_1(v), \quad (v \in V), \quad (7)$$

$$c^\psi(v) > b_m(v), \quad (v \in V), \quad (8)$$

$$b_j(v) < c^\psi(v) < a_{j+1}(v), \quad (v \in V, j = 1, \dots, m-1), \quad (9)$$

$$c^\psi(v) - c^\psi(w) < 1, \quad (vw \in E(G), \sigma^{-1}(v) > \sigma^{-1}(w)), \quad (10)$$

$$c^\psi(v) - c^\psi(w) > r - 1, \quad (vw \in E(G), \sigma^{-1}(v) > \sigma^{-1}(w)). \quad (11)$$

Now note that (5) fails iff (6) holds for some i . Also note that $c^\psi(v) \notin L(v)$ iff one of (7), (8) or (9) holds for v (and some j). If c^ψ satisfies (5), then it is not a valid circular colouring iff either (10) or (11) holds for some $vw \in E$.

So in other words, c^ψ is a valid circular colouring that satisfies (5) and $c^\psi(v) \in L(v)$ for all $v \in V$ iff all the demands of $E^{\sigma, \psi}$ fail. In yet other words, G is not L -circular choosable iff for each pair σ, ψ one of the $M = 3n - 1 + n(m - 1) + 2|E(G)| \leq n(n + m + 1)$ constraints of $E^{\sigma, \psi}$ holds. Let us arbitrarily label the constraints in $E^{\sigma, \psi}$ as $E_i^{\sigma, \psi}, i = 1, \dots, M$. For each triple $\sigma \in S_n, \psi \in \Psi, i \in \{1, \dots, M\}$, let $R_i^{\sigma, \psi}$ denote

$$R_i^{\sigma, \psi} := \{x \in P : x \text{ satisfies } E_i^{\sigma, \psi}\}.$$

and for a map $f : S_n \times \Psi \rightarrow \{1, \dots, M\}$ let us set

$$R_f := \bigcap_{\sigma \in S_n, \psi \in \Psi} R_{f(\sigma, \psi)}^{\sigma, \psi}.$$

Here we should stress again that the variables $c^\psi(v)$ have been introduced for notational convenience only, and that (6)-(11) can be rewritten completely in terms of the variables $r, t, a_i(j), b_i(j)$ (in fact as linear inequalities with integer coefficients and constants). Thus we can express R_f as $R_f = \{x \in \mathbb{R}^{2+2nm} : Ax \leq b, A_f x < b_f\}$ with all entries of A, A_f, b, b_f integers. Now observe that, by the previous, the set of circular list assignments L of the required form for which G is not L -circular choosable corresponds precisely to:

$$R := \bigcup_f R_f$$

where the union is over all maps $f : S_n \times \Psi \rightarrow \{1, \dots, M\}$. So $\text{ch}(G)$ equals the supremum over all $x \in R$ of the first coordinate of x . Since $\text{ch}(G) \leq n$ by Theorem 5 this supremum is finite, and hence there must be an f such that $\text{ch}(G) = \max\{x_1 : x \in \text{cl}(R_f)\}$. Pick such an f and put $P_f := \text{cl}(R_f)$. We claim that:

Claim 12. $P_f = \{x \in \mathbb{R}^{2+2nm} : Ax \leq b, A_fx \leq b_f\}$.

Proof of Claim 12: First observe that P has nonempty interior (this can be seen by constructing an $x \in \mathbb{R}^{2+2nm}$ for which strict inequality holds in all the inequalities of (3)). Now recall that:

$$\text{cl}(\text{int}(C)) = \text{cl}(C) \quad \text{for all convex } C \subseteq \mathbb{R}^{2+2nm} \text{ with } \text{int}(C) \neq \emptyset. \quad (12)$$

(Here and in the sequel $\text{int}(\cdot)$ denotes topological interior.) Suppose that $\text{int}(P) \cap \{x : A_fx < b_f\} = \emptyset$. Because $\mathbb{R}^{2+2nm} \setminus \{x : A_fx < b_f\}$ is closed and contains $\text{int}(P)$ it then follows that $R_f = \text{cl}(\text{int}(P)) \cap \{x : A_fx < b_f\} = \emptyset$. But this contradicts the fact that R_f contains at least one point (namely a point whose first coordinate equals $\text{cch}(G)$). It follows that $\text{int}(R_f) = \text{int}(P) \cap \{x : A_fx < b_f\} \neq \emptyset$. By two applications of (12) we now find:

$$\begin{aligned} P_f &= \text{cl}(\text{int}(\{x : Ax \leq b, A_fx < b_f\})) = \text{cl}(\{x : Ax < b, A_fx < b_f\}) \\ &= \text{cl}(\text{int}(\{x : Ax \leq b, A_fx \leq b_f\})) = \{x : Ax \leq b, A_fx \leq b_f\}, \end{aligned}$$

proving the claim. ■

The value of $\text{cch}(G)$ thus corresponds to maximising a linear function (the first coordinate) over the polyhedron $P_f = \{x : Ax \leq b, A_fx \leq b_f\}$. It can be seen from (3) that P_f is *pointed*, ie. that $\{x : Ax = 0, A_fx = 0\} = \{0\}$. By considering the simplex method (see for instance [11], pages 129–131), we now see that there is some vertex v of P_f such that $\text{cch}(G)$ equals the first coordinate of v . Recall that a vertex of the polyhedron P_f is the unique solution of some subsystem $A'x = b'$ of $2 + 2nm$ linearly independent equalities taken from the system $Ax = b, A_fx = b_f$ (see for instance [11], page 104). Since all the entries of A, A_f, b, b_f are integers, it follows by considering Gaussian elimination that all coordinates of the vertex $v = (A')^{-1}b'$ are rational. In particular $\text{cch}(G)$, the first coordinate of v , is rational. ■

3 Discussion

In this paper we have shown that the circular choosability $\text{cch}(G)$ is a rational number for every finite graph G . Our proof does however not give any explicit information about the actual value of $\text{cch}(G)$. We know that $\text{cch}(G)$ is some rational $1 \leq a/b \leq n$ and it would be interesting to see what can be said about the size of the denominator b (after common factors have been divided out). A crude bound can be obtained from our proof as follows. Recall that $A^{-1} = \text{adj}(A)/\det(A)$ for invertible matrices A , where $\text{adj}(A)$ is a matrix whose (i, j) -entry equals $(-1)^{i+j}$ times the determinant of the matrix obtained from A by deleting the i -th row and j -th column. Thus, $\det(A')$ is a natural upper bound on the denominator b , where A' is as in the end of the proof Theorem 1. By (3), (4) and (6)-(11) we see that at most n of the rows of A' have $2m + 1$ nonzero entries (the rows corresponding to the first line of (3)) and all other rows have at most four nonzero entries. What is more, all entries of A' are in $\{-1, 0, 1\}$ except for those in the column corresponding to the coefficients of r , which are between $-n$ and n . The determinant formula $\det(A) = \sum_{\sigma \in S_n} (-1)^\sigma a_{1\sigma(1)} \dots a_{n\sigma(n)}$ thus gives that $\det(A') \leq n(2m+1)^n 4^{2+(2m-1)n} = \exp[\Theta(nm)]$. Although $m(G)$ is not given explicitly in Proposition 8 (in fact, since the proof of Lemma 6 is not constructive and $m(G)$ depends on $r(G)$, it is not quite clear how to get an upper bound on $m(G)$) it can be seen that the m given in the proof is at least super exponential in n . The crude reasoning we have just outlined thus gives an upper bound for the denominator of $\text{cch}(G)$ which is *the exponential of a super exponential function* of the number of vertices n . One might hope that some variation on our proof together with a more careful analysis will yield an upper bound on the denominator that is polynomial in n .

For the circular chromatic number it is known that we can write $\chi_c(G) = a/b$ with a equal to the length of some cycle and b equal to the cardinality of some stable set of G – of course provided G has at least one cycle (see [16] for a neat proof). A more ambitious direction for further work

would thus be to see if a similar description of $\text{cch}(G)$ in terms of other characteristics of G can be derived.

Another very natural question that presents itself is the following:

Question 13. Is there a graph G with $\text{cch}(G) = q$ for every rational number $q \geq 2$?

The answer to the corresponding question for the circular chromatic number is yes. For natural numbers $a \geq 2b$, the *circular clique* $K_{a/b}$ is defined by setting $V(K_{a/b}) = \{0, \dots, a-1\}$ and putting $ij \in E(K_{a/b})$ iff $|i - j|_a \geq b$. It can be shown that $\chi_c(K_{a/b}) = a/b$. Zhu asked in [17] whether it is also true that $\text{cch}(K_{a/b}) = a/b$ for all $a \geq 2b$, but this was observed to be false in [2].

In his thesis [13] the second author introduced and studied the *choosability ratio*, a graph parameter that is closely related to the circular choosability. The choosability ratio $\sigma(G)$ is essentially a “non-circular” version of $\text{cch}(G)$. It is defined analogously to $\text{cch}(G)$ with the important difference that the lists are now subsets of \mathbb{R} instead of a circle $S(r)$. In fact some of the results, proofs and conjectures in [13] are strikingly similar to results, proofs and conjectures in [2] and [17] (which were found independently).

Theorem 5.12 in [13] is the choosability ratio analogue of Proposition 8 and the proof in [13] inspired the proof of Proposition 8. It should however be mentioned that the proof of Proposition 8 given here is by no means a straightforward adaptation of the proof of Theorem 5.12 in [13] – the “circularity” adds substantial technical difficulty. On the other hand it is straightforward to adapt the proof of Theorem 1 to show that $\sigma(G) \in \mathbb{Q}$ for all finite G . We have chosen to omit this here.

In his thesis [13] and in [14] the second author introduced and studied the *consecutive choosability ratio* $\tau(G)$ which is defined similarly to $\sigma(G)$ with the difference that all lists are intervals. He showed that $\tau(G)$ can be written as $\tau(G) = a/b$ with $b \leq n$. A very similar concept is the *circular consecutive choosability*, introduced by Lin et al. [3] and studied further by Norin et al. [6] and Pan and Zhu [9]. The circular consecutive choosability is almost the same as our $\text{cch}_1(G)$; the lists live on a circle $S(r)$ and consist of a single interval, but the difference is that in addition it is required that $r \geq \chi_c(G)$. Again it is clear that a straightforward adaptation of the proof of Theorem 1 above will show that the circular consecutive choosability is always a rational number. (And again we have chosen to omit this here.)

Acknowledgements

We would like to thank Frédéric Havet, Ross Kang, Sergey Norin and Jean-Sébastien Sereni for helpful discussions.

References

- [1] D. L. Cohn. *Measure theory*. Birkhäuser Boston, Mass., 1980.
- [2] F. Havet, R. J. Kang, T. Müller, and J.-S. Sereni. Circular choosability. preprint.
- [3] W. Lin, D. Yang, C. Yang, and X. Zhu. Circular consecutive choosability of graphs. *Taiwanese Journal of Mathematics*, to appear.
- [4] B. Mohar. Choosability for the circular chromatic number. Problem of the month, January 2002, <http://www.fmf.uni-lj.si/~mohar/>.
- [5] S. Norin. On two questions concerning circular choosability. *J. Graph Theory*, to appear.
- [6] S. Norin, Z. Pan, and X. Zhu. Bounds on circular consecutive choosability. preprint.
- [7] S. Norin, T.-L. Wong, and X. Zhu. Circular choosability via combinatorial nullstellensatz. preprint.
- [8] S. Norin and X. Zhu. Circular degree choosability. preprint.

- [9] Z. Pan and X. Zhu. Every 2-choosable graph is circular consecutive 2-choosable. preprint.
- [10] A. Raspaud and X. Zhu. List circular coloring of trees and cycles. *J. Graph Theory*, 55(3):249–265, 2007.
- [11] A. Schrijver. *Theory of linear and integer programming*. Wiley-Interscience Series in Discrete Mathematics. John Wiley & Sons Ltd., Chichester, 1986. A Wiley-Interscience Publication.
- [12] A. Vince. Star chromatic number. *J. Graph Theory*, 12:551–559, 1988.
- [13] R. J. Waters. Graph colouring and frequency assignment. Doctoral thesis, University of Nottingham, 2005.
- [14] R. J. Waters. Consecutive list colouring and a new graph invariant. *J. London Math. Soc. (2)*, 73(3):565–585, 2006.
- [15] J.-G. Yu, G.-H. Wang, and G.-Z. Liu. Girth and circular choosability of series-parallel graphs. *J. Math. Res. Exposition*, 26(3):495–498, 2006.
- [16] X. Zhu. Circular chromatic number: a survey. *Discrete Math.*, 229(1-3):371–410, 2001.
- [17] X. Zhu. Circular choosability of graphs. *J. Graph Theory*, 48(3):210–218, 2005.