The spectrum of the random environment and localization of noise

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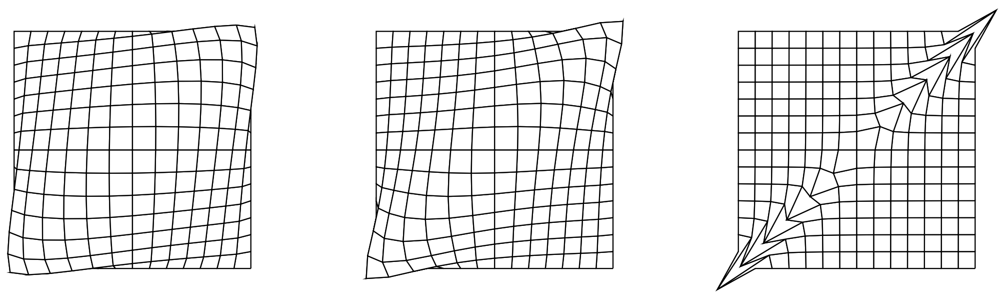
Abstract

We consider random walk on a mildly random environment on finite transitive d-regular graphs of increasing girth. After scaling and centering, the analytic spectrum of the transition matrix converges in distribution to a Gaussian noise. An interesting phenomenon occurs at $d = 2$: as the limit graph changes from a regular tree to the integers, the noise becomes localized.



1 Introduction

Localization phenomena for eigenvalues of random media have received considerable attention lately. Among the several results, we point out a specific one. If one perturbs the Laplacian of the nearest-neighbor graph on the integers by a very small i.i.d. potential, its spectrum becomes pure point immediately. In contrast, the same change on a higher degree regular tree preserves the continuous spectrum, see Klein (1998).

The results of this paper point to an analogous, but distinct phenomenon. While the above localization phenomena are due to large-scale behavior of the eigenvectors, we find a similar dichotomy for the local behavior.

For this, we consider sequences of vertex-transitive finite graphs G_n with degree $d \geq 2$ and increasing girth (the length of the shortest simple cycle); these converge locally to either the integers ($d = 2$) or a regular tree of higher degree ($d \geq 3$). On these graphs, we consider a perturbation of the Laplacian, given by the well-studied random walk on random environment model with small noise (see, for example Zeitouni (2004)). The noise in the Laplacian creates a Gaussian noise in the spectrum. Our main discovery is that in the limit this noise is local for the integers and has long-range correlations for higher-degree trees.

We first consider more general graphs G. We will have the following standing assumptions.

Assumptions. G is a vertex-transitive graph of finite degree. Let M denote the transition probability matrix for nearest-neighbor simple random walk on G. For the random environment on G one randomly modifies transition probabilities along edges. Its transition matrix is defined as $M + \varepsilon B$, where B is a random matrix satisfying the following.

The random variables $B_{u,v}$ have mean zero and variance 1. Further, the $B_{u,v}$'s corresponding to different u are independent, and for some constant c_1 and all vertices $u, v \in G$ we have

$$|B_{u,v}| \leq c_1 M_{u,v}, \tag{1}$$

$$\sum_{g \in \mathrm{Stab}(u)} B_{u,gv} = 0. \tag{2}$$

Here $\mathrm{Stab}(u)$ is the stabilizer of u in the automorphism group of G. Condition (2) means that the sum of the bias of the random environment over symmetric directions is zero, which is a bit stronger than just requiring $M + \varepsilon B$ to be stochastic. Condition (1) implies that $B_{u,v}$ vanishes when uv is not an edge and that for small ε the entries of $B + \varepsilon M$ are in $[0, 1]$. These assumptions will be in effect for the rest of the paper.

For a finite G, let μ_ε denote the empirical probability measure of the eigenvalues of $M + \varepsilon B$. Then as $\varepsilon \to 0$ we will show an expansion

$$\mu_\varepsilon = \mu_0 + \frac{1}{2}\mu''\varepsilon^2 + o(\varepsilon^2),$$

where μ'' is a random functionals. We define the second difference quotient

$$m_\varepsilon = |G|^{1/2}\frac{\mu_\varepsilon - \mu_0}{\varepsilon^2},$$

where the extra factor upfront makes the scaling consistent as G changes. This is the centered and scaled empirical eigenvalue measure of $M + \varepsilon B$. Let $\mathcal{X}$ denote the space of complex functions analytic in a neighborhood of the closed unit disk. We call f real if it maps $\mathbb{R}$ to $\mathbb{R}$. In Proposition 4, we will show that for $f \in \mathcal{X}$ the limit

$$T_G(f) = \lim_{\varepsilon \to 0} \int f \, dm_\varepsilon \tag{3}$$

exists. Our first theorem identifies the covariance structure of $T_G(f)$ as f ranges over real functions in $\mathcal{X}$.

For a possibly infinite vertex-transitive graph G and complex $|\lambda| < 1$, let

$$p_\lambda(x) = 1/(1 - \lambda x), \qquad \mathcal{G}_\lambda = p_\lambda(M) = (I - \lambda M)^{-1},$$

the Green's function corresponding to G. Let o be a marked vertex of G and define

$$H_G(p_\lambda, p_\mu) := \frac{\lambda\mu}{2} \partial^2_{\lambda,\mu} \left(\lambda^2 \mu^2 \sum_{v \neq o} \mathbf{E} \left[(B\mathcal{G}_\lambda)_{ov} (B\mathcal{G}_\mu)_{ov} \right] \mathbf{E} \left[(B\mathcal{G}_\lambda)_{vo} (B\mathcal{G}_\mu)_{vo} \right] \right). \tag{4}$$

We will show in Lemma 7 and Proposition 8 that $H_G(p_\lambda, p_\mu)$ is well-defined and extends uniquely to a bilinear form H_G on $\mathcal{X}$.

Theorem 1. *For any finite vertex-transitive graph G and $f, g \in \mathcal{X}$ we have*

$$\mathbf{E} T_G(f) = 0, \qquad \mathbf{E}[T_G(f) T_G(g)] = H_G(f, g).$$

We now consider sequences of transitive graphs $G_n \to G$ locally, which means that for every r the r-neighborhood of a fixed vertex of G_n eventually agrees with that in G.

Theorem 2 (Normality)**.** *Let $G_n \to G$, an infinite graph, and let G_n, G satisfy the assumptions above with uniform constant c_1. Then as $n \to \infty$, jointly for all real $f \in \mathcal{X}$ the random variables $T_{G_n}(f)$ converge weakly to mean zero normal random variables $T(f)$. Moreover, for all $f, g \in \mathcal{X}$ we have*

$$\mathbf{E} \left[T_{G_n}(f) T_{G_n}(g) \right] \to \mathbf{E} \left[T(f) T(g) \right] = H_G(f, g).$$

Finally, we consider the case when $G_n \to \mathbb{T}_d$, the d-regular tree. Since $\mathbb{T}_d$ is bipartite, we expect that the random functionals m_ε have a symmetric limit. Therefore it is natural to consider T_n for even functions of the form $f \circ s$ where $s(x) = x^2$ and $f \in \mathcal{X}$. Note that for any finite graph G

$$T_G(f \circ s) = \tilde{T}_G(f),$$

where $\tilde{T}_G(f)$ is defined analogously to $T_G(f)$ but in terms of $\tilde{\mu}_\varepsilon$, the empirical eigenvalue measure of $(M+\varepsilon B)^2$.

It is known (see Linial and Widgerson, Theorem 13.4) that $\tilde{\mu}_n$ corresponding to G_n converges weakly to the measure with density

$$a_d(x) := \frac{2d^2}{\pi} \frac{\sqrt{x(\rho - x)}}{1-x} \mathbf{1}_{0<x<\rho}, \tag{5}$$

where $\rho = \rho_d := 4(d-1)/d^2$, the squared spectral radius of the walk on the infinite d-regular tree $\mathbb{T}_d$. Our next theorem gives an explicit expression for the limiting covariance structure.

Theorem 3 (Tree limits)**.** *Assume that $G_n \to \mathbb{T}_d$, the d-regular tree. Then jointly for all real $f \in \mathcal{X}$ we have the convergence in distribution $\tilde{T}_{G_n}(f) \to \tilde{T}(f)$, where the $\tilde{T}(f)$ are jointly normal and have mean 0. Moreover,*

$$\mathbf{E}[\tilde{T}(f)\tilde{T}(g)] = \int_0^\rho \int_0^\rho f'(x)g'(y)\,\beta_d(x,y)\,dxdy, \tag{6}$$

where for $d>2$ the kernel β_d is given by

$$\beta_d(x,y) = \frac{2d^4}{\pi^2} \frac{(d-2)\,\kappa(x)\kappa(y)}{16(2d-3)(x-y)^2 + (d-2)^2 A(d,x,y)}, \tag{7}$$

with the semicircle function

$$\begin{aligned} \kappa(x) &:= 2d\sqrt{x(\rho-x)} \qquad \text{and} \\ A(d,x,y) &:= \rho\kappa\Big(\frac{x+y}{2}\Big)^2 + 4(d+3)(x-y)^2 + \rho^2(d-2)^2. \end{aligned}$$

For $d=2$, the covariance is given by

$$\frac{32}{\pi} \int_0^1 f'(x)g'(x)\sqrt{x(1-x)}dx,$$

which corresponds to $\beta_2(x,y) = \frac{32}{\pi}\sqrt{x(1-x)}\ \delta_x(y)$.

The kernel function β_d becomes singular as $d \downarrow 2$. This can be seen in the figure on the first page showing graphs $\beta_d : [0,\rho]^2 \to \mathbb{R}$ for $d = 4, 3, 2.1$ from above. In words, in the $d=2$ case the centered and scaled empirical measure converges to the so-called H_1-noise with a certain density. In contrast with the $H_{1/2}$-noise that arises in the limit of the Gaussian Orthogonal Ensemble and Haar unitary random matrix models (among others; see, for example Diaconis and Evans (2001), Anderson and Zeitouni (2006) and the references therein), this noise is local, as the δ-function in the covariance formula shows. H_1 noise appears typically for complex eigenvalues, see Rider and Virág (2007). Formula (6) is written in a form to show

how the kernel becomes singular (localized) as $d \downarrow 2$. Recall that $\pi^{-1}\varepsilon/(x^2+\varepsilon^2)$ converges as $\varepsilon \to 0$ to the delta function at $x = 0$.

The presence of f', g' in the formula for the covariance is a priori expected. When the matrix is randomly perturbed, the eigenvalues move in random directions. For every eigenvalue λ, the function $f(\lambda)$ changes by a small amount proportional to f' times the change in λ. However, the picture here is more complicated because the perturbation is of order ε^2. The kernel β reflects the correlation between the random change in eigenvalues as well as their density.

Theorem 3 motivates the following question:

Question 1. *Which natural sequences of graphs give rise to localized noise?*

Our strategy is as follows. To prove normality, we use a central limit theorem for dependent variables based on Stein's method, rather than the usual method of moment computations. For computing the covariance, we consider traces, and so we will have to count certain paths. We use the Green function to jointly treat paths of different lengths and to avoid complicated computations with orthogonal polynomials that normally arise in this context. Theorems 1, 2 and 3 are proved in Sections 4, 5 and 7 respectively.

The finite graphs that we study arise naturally. The simplest way to pick an interesting transitive graph is to consider a Cayley graph of a sufficiently complicated finite group. As it is discussed in Gamburd et al., such Cayley graphs will typically have girth tending to infinity; for example, random Cayley graphs of simple groups of increasing order will have this property. Of course, for $d = 2$, the only examples are graphs consisting of cycles.

Random perturbations of eigenvalues have been extensively studied in the literature. Reference vom Scheidt and Purkert (1983) studies properties of the eigenvalues of a random perturbation of a fixed matrix without the restriction for stochasticity or positivity of the matrices involved (see Sections 1.3 and 2.2). Another possible approach to our problem is via perturbation expansions for eigenvalues (see for example Deif (1991) Sections 6.3, 6.4), but this requires more control of the eigenvectors.

2 The $\varepsilon \to 0$ limit

We first show that for finite graphs G the limit (3) defining $T_G(f)$ exists and identify it. We will rely on the Assumptions about the perturbation matrix B, in particular, (2). In fact, throughout this paper, we only need a simple consequence of (2). Namely, for all $k \geq 1$ and

$v \in G$, we have

$$(BM^k)_{v,v} = 0. \tag{8}$$

Indeed, for any $g \in \mathrm{Stab}(v)$ this equals

$$\sum_{w \in G} B_{v,w} M^k_{w,v} = \sum_{w \in G} B_{gv,gw} M^k_{gw,gv} = \sum_{w \in G} B_{v,gw} M^k_{w,v}.$$

Averaging each term over all $g \in \mathrm{Stab}(v)$ and using (2) we get (8).

Proposition 4 (The limit as $\varepsilon \to 0$)**.** *For any $f = \sum a_j z^j \in \mathcal{X}$, the limit $T_G(f)$ exists, and*

$$T_G(f) = \frac{1}{2} \operatorname{Tr} \left(\partial_{\varepsilon\varepsilon} f(M + \varepsilon B)|_{\varepsilon=0} \right) = \sum_{j=0}^{\infty} a_j T_G(z^j). \tag{9}$$

Moreover, for integers $j \geq 0$ we have

$$T_G(z^j) = |G|^{-1/2} \, \frac{j}{2} \sum_{k_1+k_2+2=j} \operatorname{Tr}(BM^{k_1} BM^{k_2}).$$

Proof. We use the fact (see, for example, Theorem 6.2.8 in Horn and Johnson (1991)) that for f an analytic function with radius of convergence $r > 0$, and for A in the set $\mathcal{M}_r$ of matrices of some fixed dimension and with spectral radius less than r, the power series $f(A)$ is absolutely convergent and is analytic as a function of (the entries of) A. Thus $\operatorname{Tr} f(M+\varepsilon B)$ is an analytic function of all of its (at most) $2|G|^2 + 1$ variables near M, B, and $\varepsilon = 0$. We are free to rearrange its absolutely convergent multiple power series expansion in any way we like. So the first derivative with respect to ε at 0 is the coefficient of the ε terms; this is a multiple of

$$\sum_{k_1+k_2=j-1} \operatorname{Tr}(M^{k_1} BM^{k_2}) = \sum_{k_1+k_2=j-1} \operatorname{Tr}(BM^{k_1+k_2})$$

and each of the summands is zero by (8). We have

$$|G|^{1/2} \int f \, dm_\varepsilon = \varepsilon^{-2} (\operatorname{Tr} f(M + \varepsilon B) - \operatorname{Tr} f(M)) \to \frac{1}{2} \partial_{\varepsilon\varepsilon} \operatorname{Tr} f(M + \varepsilon B)|_{\varepsilon=0}$$

when $\varepsilon \to 0$, since the first derivative vanishes at $\varepsilon = 0$. Now the right hand side equals the coefficient of ε^2 in the expansion. This gives (9).

Since the trace of a product does not change when we cyclically permute the factors, we see that

$$\partial_\varepsilon \operatorname{Tr}((M + \varepsilon B)^j) = \operatorname{Tr}(\partial_\varepsilon (M + \varepsilon B)^j) = j \operatorname{Tr}(B(M + \varepsilon B)^{j-1}).$$

Taking another derivative, we get the second claim of the proposition. ■

3 Properties of the bilinear form H_G

The goal of this section is to define the bilinear form H_G introduced in (4) and establish its continuity properties.

Recall that $\mathcal{X}$ denotes the set of power series centered at zero, and with radius of convergence more than 1. For a power series f, let $[x^k]f(x)$ denote the coefficient of x^k in the expansion of f. Given a finite or infinite graph G (with B and M) as in the introduction, we fix a vertex o of G, and re-define the bilinear form on $\mathcal{X}$ as

$$\begin{aligned} H_G(f,g) :&= \frac{1}{2}\sum_{ij} ij\alpha_{ij}\,[x^i]f(x)[x^j]g(x) \\ &= \frac{1}{2}\sum_{ij} \alpha_{ij}\,[x^{i-1}]f'(x)[x^{j-1}]g'(x), \end{aligned} \tag{10}$$

where

$$\alpha_{ij} := \mathbf{E}\left[\sum_{v\in G\setminus\{o\}} Y_i(G,o,v)Y_j(G,o,v)\right],$$

and for any vertices v, w of G, we define

$$Y_j(G,v,w) := \sum_{j_1+j_2+2=j} (BM^{j_1})_{v,w}(BM^{j_2})_{w,v}. \tag{11}$$

The last sum is always finite and symmetric in v, w. In Proposition 8 we will show that this definition agrees with (4) for the functions p_λ. Note that the definition of H_G does not depend on o if G is vertex-transitive. For any graph G, the bilinear form $H_G(f,g)$ clearly makes sense for polynomials f, g. To go beyond polynomials, we need to show that the infinite sum in (10) is well-defined. For $f = \sum a_k z^k \in \mathcal{X}$, set

$$\|f\|_* = \sum_{k=1}^{+\infty} |a_k|k^2,$$

which defines a norm on $\mathcal{X}$. (To be precise, it is a norm on $\mathcal{X}$ modulo the constants; however, we note that for the purposes of this paper the constant terms for functions in $\mathcal{X}$ do not matter, so we may thing of $\mathcal{X}$ itself as the space of functions modulo the constants.) Note that on $\mathcal{X}$ this norm is finite.

Remark 5. On the subspace of $\mathcal{X}$ consisting of power series with radius of convergence strictly larger that a fixed $r > 1$, the norm $\|\cdot\|_*$ is dominated by the supremum norm on $D_r := \{z : |z| \le r\}$. Indeed, for $f(z) = \sum_{k=0} a_k z^k$ in that subspace, Cauchy's inequalities give $|a_k| \le r^{-k}\|f\|_{D_r}^\infty$. So that

$$\|f\|_* \le \frac{r(1+r)}{(r-1)^3}\|f\|_{D_r}^\infty.$$

As a direct consequence, we get

Lemma 6. *Polynomials are $\|\cdot\|_*$ - dense in $\mathcal{X}$.*

We are now ready to show that H_G is well-defined and continuous.

Lemma 7. *For $f, g \in \mathcal{X}$ and any G, the sum giving $H_G(f,g)$ is absolutely convergent, and*

$$|H_G(f,g)| \le c_1^4 \|f\|_* \|g\|_*.$$

Proof. Note that since $|B_{vw}| \le c_1 M_{vw}$, we have

$$\sum_{w\in G} |Y_j(G,v,w)| \le c_1^2 \sum_{w\in G} \sum_{j_1+j_2+2=j} M_{v,w}^{j_1+1} M_{w,v}^{j_2+1} = c_1^2 (j-1)(M^j)_{v,v} \le c_1^2 j (M^j)_{v,v}. \tag{12}$$

In particular, the same bound holds for each term $|Y_j(G,v,w)|$. Therefore

$$\frac{1}{c_1^4 ij} \sum_{v\in G\setminus\{o\}} Y_i(G,o,v) Y_j(G,o,v) \le \sum_{v\in G\setminus\{o\}} M_{v,v}^i M_{v,v}^j \le \sum_{v\in G\setminus\{o\}, w\in G} M_{v,w}^i M_{w,v}^j = M_{v,v}^{i+j} \le 1.$$

The claim now follows by summing over all i, j. ■

Proposition 8. *H_G satisfies* (4)*. Moreover,* (4) *uniquely defines H_G as a $\|\cdot\|_*$-continuous bilinear form.*

Proof. The absolute convergence of the series defining H_G, implies

$$H_G(p_\lambda, p_\mu) = \frac{1}{2} \sum_{i,j\ge 1} ij\, \alpha_{ij} \lambda^i \mu^j = \frac{1}{2} \lambda\mu\, \partial_\lambda \partial_\mu \sum_{i,j\ge 1} \alpha_{ij} \lambda^i \mu^j.$$

Using the definition of α_{ij}, we write the sum above as

$$\begin{aligned}
&\sum_{\substack{v\in G\setminus\{o\}\\ k,\ell,k',\ell'\ge 0}} \lambda^{k+\ell+2} \mu^{k'+\ell'+2} \mathbf{E} (BM^k)_{ov} (BM^\ell)_{vo} (BM^{k'})_{ov} (BM^{\ell'})_{vo} \\
&= \lambda^2 \mu^2 \sum_{v\in G\setminus\{o\}} \mathbf{E} (B\mathcal{G}_\lambda)_{ov} (B\mathcal{G}_\lambda)_{vo} (B\mathcal{G}_\mu)_{ov} (B\mathcal{G}_\mu)_{vo} \\
&= \lambda^2 \mu^2 \sum_{v\in G\setminus\{o\}} \mathbf{E} \left[(B\mathcal{G}_\lambda)_{ov} (B\mathcal{G}_\mu)_{ov} \right] \mathbf{E} \left[(B\mathcal{G}_\lambda)_{vo} (B\mathcal{G}_\mu)_{vo} \right].
\end{aligned}$$

This proves the first part. For the second, it suffices to show that the linear span of $\{p_\lambda : |\lambda| < 1\}$ is $\|\cdot\|_*$-dense in $\mathcal{X}$; this will be done in the next lemma. ■

Lemma 9. *The vector space generated by $\{p_\lambda : |\lambda| < 1\}$ is $\|\cdot\|_*$ - dense in $\mathcal{X}$.*

Proof. Because of the density of polynomials in $\mathcal{X}$ and Remark 5, it suffices to prove that any polynomial P can be approximated in the supremum norm on a disk D_r with $r > 1$ by elements of the linear span $\{p_\lambda : |\lambda| < 1\}$.

Pick $1 < r < r_1$, and let $C_{r_1} : \{z : |z| = r_1\}$. Then

$$P(z) = \frac{1}{2\pi i} \int_{C_{r_1}} \frac{P(\zeta)}{\zeta - z} d\zeta = \frac{1}{2\pi i} \int_{C_{r_1}} \frac{P(\zeta)}{\zeta} p_{1/\zeta}(z) d\zeta.$$

Call $R_n(z)$ the Riemann sum corresponding to an equipartition of the circle with n pieces. The sequence of functions R_n is equicontinuous on $\{z : |z| \le r\}$ and it converges pointwise to $P(z)$, thus the convergence is uniform on that set. So that for a given $\delta > 0$, there is a finite linear combination

$$A_\delta(z) := c_1 p_{1/\zeta_1}(z) + c_2 p_{1/\zeta_2}(z) + \ldots + c_k p_{1/\zeta_k}(z)$$

such that

$$|f(z) - A_\delta(z)| \le \delta \tag{13}$$

for all $z \in \mathbb{C}$ with $|z| \le r$. This completes the proof of the lemma. ■

Finally, we check that $H_G(f, g)$ is continuous in G as well. Recall the definition of local convergence of graphs from the introduction. For the following lemma, we use, as usual, the assumptions from the introduction for each G_n and G.

Lemma 10. *If $G_n \to G$ locally, then for $f, g \in \mathcal{X}$ we have $H_{G_n}(f, g) \to H_G(f, g)$.*

Proof. Note that as $n \to \infty$, the neighborhood of radius $i + j$ of o in G_n stabilizes to look like the same neighborhood in the limit graph G.

For polynomials f, g, and for all large n, we have $H_{G_n}(f, g) = H_G(f, g)$. This is because $H_{G_n}(f, g)$ only depends on a neighborhood of the root o of radius given by the maximal degree of f and g.

Now we have that the sequence of functions $H_{G_n}(f, g)$ is equicontinuous on $\mathcal{X}^2$ (by Lemma 7) and converges on a dense set. Thus, by Lemma (15) they converge on the entire set to a continuous limit. $H_G(\cdot, \cdot)$ is continuous (Lemma 7), and this finishes the proof. ■

4 The covariance structure

After establishing some properties of the bilinear form H_G, we show that for finite graphs G it gives the covariance structure of $T_G(f)$.

Lemma 11. *For any finite vertex-transitive graph G and complex polynomials f, g, we have*

$$\mathbf{E}(T_G(f)) = 0,$$
$$\mathbf{E}(T_G(f)T_G(g)) = H_G(f, g).$$

Proof. We will show this for monomials. The extension to polynomials is straightforward from bilinearity. First, we have

$$\begin{aligned} T_G(z^j) &= \frac{1}{\sqrt{|G|}}\frac{j}{2} \operatorname{Tr} \sum_{j_1+j_2+2=j} BM^{j_1}BM^{j_2} \\ &= \frac{1}{\sqrt{|G|}}\frac{j}{2} \sum_{\substack{v,w\in G \\ v\neq w}} Y_j(G, v, w). \end{aligned} \tag{14}$$

Note that the $v = w$ terms vanish by (8). Each $Y_j(G, v, w)$ with $v \neq w$ has zero mean because different rows of B are independent, with entries having zero mean. Thus, $T(z^j)$ has also zero mean. Again because of independence of rows of B, when we compute second moments, the terms in the sum below with $\{v, w\} \neq \{v', w'\}$ vanish. That is

$$\begin{aligned} \mathbf{E}(T_G(z^i)T_G(z^j)) &= \frac{ij}{4|G|} \sum_{\substack{v,w,v',w'\in G \\ v\neq w, v'\neq w'}} \mathbf{E}Y_i(G, v, w)Y_j(G, v', w') \\ &= \frac{ij}{4|G|} \sum_{v,w\in G: v\neq w} \mathbf{E}Y_i(G, v, w)\left[Y_j(G, v, w) + Y_j(G, w, v)\right] \\ &= \frac{ij}{2} \sum_{v\in G\setminus\{o\}} \mathbf{E}Y_i(G, o, v)Y_j(G, o, v) \end{aligned}$$

For the last equality we fixed a vertex o of G and used the vertex-transitivity of the graph and the symmetry of Y in its last two parameters. ■

Lemma 12. *For G finite, $|T_G(f)| \leq c_1^2|G|^{1/2}\|f\|_*$.*

Proof. This follows directly from (9), (14), and (12). ■

We are now ready to prove Theorem 1.

Proof of Theorem 1. Lemma 7 and Lemma 12 show that $\mathbf{E}[T_G(\cdot)T_G(\cdot)]$ and $H_G(\cdot, \cdot)$ are $\|\cdot\|_*$-continuous in f, g. We conclude the proof by approximating f and g with polynomials and using Lemma 6 and Lemma 11. ■

5 Asymptotic normality

For this section we consider a sequence of transitive graphs G_n converging to locally to limit G. The assumptions from the introduction hold for all G_n and the limit G with the same constant c_1. We may also assume that all graphs have degree d.

We will prove convergence of certain sequences to normal under a topology which we introduce now.

Let $\mathcal{Y}$ denote the space of probability measures on $\mathbb{R}$ with finite second moment, equipped with the 2-Wasserstein distance, call it d_2 (see Villani, Chapter 6). For two measures μ, ν in the space, this is the minimal L^2-distance over all possible couplings of them, i.e.,

$$d_2(\mu, \nu) := \inf_k \Big(\int |x-y|^2 \, dk(x,y) \Big)^{1/2},$$

where the infimum is over the set of probability measures on $\mathbb{R}^2$ with first marginal μ, and second ν.

The space $(\mathcal{Y}, d_2)$ is complete, and its topology is stronger than weak convergence. It is easy to show that for a sequence X_n of random variables, $X_n \to X$ in this topology (i.e., the corresponding laws converge) if and only if $X_n \to X$ weakly and $\mathbf{E}X_n^2 \to \mathbf{E}X^2$. In particular, in this topology the function variance, $\mathrm{Var} : \mathcal{Y} \to \mathbb{R}$, is continuous.

We will need the following case of Lemma 2.4 of Chen and Shao (2004). It is a normal approximation theorem for dependent variables. It is proved with the use of Stein's method.

Lemma 13 (Chen and Shao (2004)). *Let $\mathcal{I}$ be an index set, $\{X_i : i \in \mathcal{I}\}$ a family of random variables, and for $A \subset \mathcal{I}$, let $X_A := \{X_i : i \in A\}$. Assume that*

(1) For each $i \in \mathcal{I}$, there exist $A_i \subset B_i \subset C_i \subset \mathcal{I}$ such that X_i is independent of $X_{\mathcal{I} \setminus A_i}$, X_{A_i} is independent of $X_{\mathcal{I} \setminus B_i}$, and X_{B_i} is independent of $X_{\mathcal{I} \setminus C_i}$.

(2) There exists a constant γ so that for all $i \in \mathcal{I}$ we have

$$\max(|N(C_i)|, |\{j : i \in C_j\}|) \leq \gamma,$$

where

$$N(C_i) := \{j \in \mathcal{I} : C_i \cap B_j \neq \emptyset\}.$$

(3) Each X_i has zero mean and finite variance, and $W := \sum_{i \in \mathcal{I}} X_i$ satisfies $\mathrm{Var}(W) = 1$.

Then for $2 < p \leq 3$, we have

$$\sup_{z\in\mathbb{R}} |F(z)-\Phi(z)| \le 75\gamma^{p-1}\sum_{i\in\mathcal{I}} \mathbf{E}|X_i|^p,$$

where F, Φ are the distribution functions of W and of the standard normal $N(0,1)$.

An immediate consequence is the following convergence result.

Lemma 14. *For any real polynomial f, the sequence $T_{G_n}(f)$ converges as $n \to \infty$ in the 2-Wasserstein distance to a normal random variable with zero mean and variance $H_G(f,f)$.*

Proof. We will apply Lemma 13. Let $f(x) = \sum_{j=0}^k a_j z^j$, and $\mathcal{I}$ the set of vertices of G_n. In the following, we will omit the subscripts for the matrices M, B associated with the graph G_n. We have

$$T_{G_n}(f) = \frac{1}{\sqrt{|G_n|}} \operatorname{Tr}\left(\sum_{j=0}^k \frac{j}{2} a_j \sum_{k_1+k_2+2=j} BM^{k_1}BM^{k_2}\right) = \sum_{v\in G_n} Y_{n,v},$$

where

$$Y_{n,v} := \frac{1}{\sqrt{|G_n|}} \sum_{j=0}^k \frac{j}{2} a_j \sum_{k_1+k_2+2=j} (BM^{k_1}BM^{k_2})_{v,v}.$$

It holds $\lim_{n\to+\infty} \operatorname{Var}\ T_{G_n}(f) = H_G(f,f)$ by Lemma 10. If this limit is zero, then the sequence $T_{G_n}(f)$ will converge to δ_0 in the 2-Wasserstein topology, and the result is proved. We may therefore assume that the limit is positive, and $\operatorname{Var}\ T_{G_n}(f) > 0$ for all n.

For $v \in \mathcal{I}$, define

$$X_v := \frac{Y_{n,v}}{\operatorname{Var}\ T_{G_n}(f)}, \qquad W := \sum_{v\in G_n} X_v = \frac{T_{G_n}(f)}{\operatorname{Var}\ T_{G_n}(f)},$$

and the sets

$$\begin{aligned} A_v &:= \{w : \operatorname{dist}(v,w) \le k\}, \\ B_v &:= \{w : \operatorname{dist}(v,w) \le 2k\}, \\ C_v &:= \{w : \operatorname{dist}(v,w) \le 3k\}. \end{aligned}$$

These sets satisfy the conditions of Lemma 13 (because the X_v's corresponding to vertices that are distance at least $k+1$ apart are independent random variables), and for all v, we have $|N(C_v)| \le |\{w : \operatorname{dist}(v,w) \le 5k\}| \le d(d-1)^{5k-1}$, and $|\{w : v \in C_w\}| = |C_v| \le d(d-1)^{3k-1}$. [Here we used the regularity of the graphs, but of course a uniform bound on the degree of the vertices of all graphs would work in the same way.] Also, the X_v's have zero mean and finite variance by Lemma 11. Pick any $p \in (2,3]$. Lemma 13 gives that the distribution functions F_W, F_Z of W and of the standard $N(0,1)$ satisfy

$$\sup_{x\in\mathbb{R}} |F_W(x) - F_Z(x)| \leq C|G_n|^{1-\frac{p}{2}}\mathbf{E}|A|^p \tag{15}$$

with $C := 75(d(d-1)^{5k-1})^{p-1}$ and

$$A := \sum_{j=0}^{k} \frac{j}{2} a_j \sum_{k_1+k_2+2=j} (BM^{k_1}BM^{k_2})_{o,o},$$

which is independent of n for large n because the sequence converges locally to G. Thus relation (15), $|G_n| \to \infty$, and $p > 2$ imply that the sequence

$$\frac{T_{G_n}(f)}{\text{Var }\ T_{G_n}(f)}$$

converges to a standard normal random variable. By Theorem 1 and Lemma 10 we have $\text{Var }\ T_{G_n}(f) \to H_G(f,f)$, and the result follows. ∎

Our next goal is to strengthen Lemma 14 to get Theorem 2 by showing that its conclusion is true also for all functions f in $\mathcal{X}$. The proof is simply by approximation. We will use the fact that the set of polynomials is $\|\cdot\|_*-$ dense in $\mathcal{X}$ (Lemma 6) and the following simple lemma (Lemma 38, Chapter 7 of Royden (1988)).

Lemma 15. *Let X be a metric space, and Y a complete metric space. Assume that $f_n : X \to Y$ is an equicontinuous sequence of functions that converge pointwise on a dense subset of X. Then the sequence f_n converges pointwise on the entire X to a continuous limit.*

We are ready to prove Theorem 2.

Proof of Theorem 2. Let $\hat{T}_n(f)$ denote the distribution of $T_{G_n}(f)$. The sequence of functions $\hat{T}_n : \mathcal{X} \to \mathcal{Y}$ is uniformly equicontinuous because, by Lemma 7 and Theorem 1 we have

$$\mathbf{E}|T_{G_n}(f) - T_{G_n}(g)|^2 \leq c_1^4 \|f-g\|_*^2.$$

By Lemma 14, they converge pointwise at polynomials, which form a dense subset of $\mathcal{X}$ by Lemma 6. By Lemma 15, the limit $\hat{T}(f)$ of $\hat{T}_n(f)$ exists for all functions $f \in \mathcal{X}$ and is continuous. Also, for the 2-Wasserstein distance between $\hat{T}(f)$ and $\hat{T}(g)$, we have

$$d_2(\hat{T}(f), \hat{T}(g)) \leq c_1^2 \|f-g\|_*. \tag{16}$$

Since the limit is normal on a dense set of points, and limits of normal random variables are normal, it follows that all limits $\hat{T}(f)$ are normal. Also the functionals

$$f \mapsto \operatorname{Var} \hat{T}(f), \;\; f \mapsto H_G(f,f)$$

are $\|\cdot\|_*$- continuous (the first because of (16) and a property of the 2-Wasserstein distance, the second by Lemma 7) and they agree on a $\|\cdot\|_*$-dense set by Lemma 14. Thus, $\operatorname{Var} \hat{T}(f) = H_G(f,f)$ for all $f \in \mathcal{X}$. ∎

6 A formula for the covariance

The goal of this section is to reduce the problem of computing the covariance kernel β of Theorem 3 to inverting a Stieltjes transform.

When the limit graph G is bipartite (see the discussion before Theorem 3), we would like to write the limiting covariance in the form

$$H_G(f \circ s, g \circ s) = \int f'(x) g'(y) d\beta(x,y), \tag{17}$$

where $s(x) = x^2$. It is sufficient to check the identity (17) for the functions $f = p_{\lambda^2}$ and $g = p_{\mu^2}$, since their linear span is a dense subset in $\mathcal{X}$ with respect to the norm $\|\cdot\|_*$ (Lemma 9), and the integral on the right is clearly continuous with respect to the product topology based on this norm. For this choice of f, g the right hand side of (17) equals

$$\int \frac{\lambda^2}{(1-\lambda^2 x)^2} \frac{\mu^2}{(1-\mu^2 y)^2} \, d\beta(x,y) = \frac{1}{4} \lambda\mu \, \partial^2_{\lambda,\mu} \int \frac{1}{(x-\lambda^{-2})} \frac{1}{(y-\mu^{-2})} \, d\beta(x,y).$$

Note that $p_\lambda - p_{\lambda^2} \circ s$ is an odd function. For bipartite graphs G it is clear from the expressions (10) and (11) that H_G vanishes if one its arguments is odd. So we have

$$H_G(p_{\lambda^2} \circ s, p_{\mu^2} \circ s) = H_G(p_\lambda, p_\mu).$$

In light of the expression (4) we arrive at the following:

Proposition 16. *If the Stieltjes transform relation*

$$\lambda^2 \mu^2 \sum_{v \neq o} 2\mathbf{E}\Big[(B\mathcal{G}_\lambda)_{o,v} (B\mathcal{G}_\mu)_{o,v}\Big] \mathbf{E}\Big[(B\mathcal{G}_\lambda)_{v,o} (B\mathcal{G}_\mu)_{v,o}\Big] = \int \frac{1}{(x-\lambda^{-2})} \frac{1}{(y-\mu^{-2})} \, d\beta(x,y) \tag{18}$$

holds for all $|\lambda|, |\mu| < 1$, *then* (17) *holds for all* $f, g \in \mathcal{X}$.

7 Explicit formulas for trees

In this section, we look at the case where the limiting graph G is the infinite d-regular tree $\mathbb{T}_d$, we compute the measure $d\beta$ introduced in (17), and thus complete the proof of Theorem 3.

From now on we define square roots for complex numbers with a branch cut at the negative real axis. More precisely, for $z = |z|e^{i\theta}$, with $\theta \in (-\pi, \pi]$, we set $\sqrt{z} = \sqrt{|z|}e^{i\theta/2}$.

Proof of the covariance formula in Theorem 3. We start by computing the left hand side of (18) explicitly. The Green's function for the infinite d-regular tree, $\mathcal{G}_\lambda(v,w) = \mathcal{G}_\lambda(\mathrm{dist}(v,w))$, is given by

$$\mathcal{G}_\lambda(r) = b_\lambda a_\lambda^r,$$

where

$$a_\lambda := d\,\frac{1-\sqrt{1-\rho\lambda^2}}{2(d-1)\lambda}, \quad b_\lambda := (1-\lambda a_\lambda)^{-1},$$

and $\rho = 4(d-1)/d^2$ as in the introduction. See Woess (2000), Lemma 1.24.

First, an observation. For two vertices i, j, let ℓ denote the unique neighbor of i closest to j, r the distance between i and j, and define $B_{ij}^* = B_{i\ell}$. For λ with modulus less than 1, by symmetry and since $\sum_k B_{ik} = 0$, we get

$$(B\mathcal{G}_\lambda)_{ij} = B_{i\ell}\mathcal{G}_\lambda(r-1) + \sum_{\substack{k\sim i\\ k\neq \ell}} B_{ik}\mathcal{G}_\lambda(r+1) = B_{ij}^*\mathcal{G}_\lambda^*(r),$$

where

$$\mathcal{G}_\lambda^*(r) = \mathcal{G}_\lambda(r-1) - \mathcal{G}_\lambda(r+1) = b_\lambda(1-a_\lambda^2)a_\lambda^{r-1}.$$

Thus for any vertex $w \neq o$, with $r = \mathrm{dist}(o,w)$, we have

$$\mathbf{E}(B\mathcal{G}_\lambda)_{o,w}(B\mathcal{G}_\mu)_{o,w} = \mathbf{E}B_{ow}^{*2}\mathcal{G}_\lambda^*(r)\mathcal{G}_\mu^*(r) = \mathcal{G}_\lambda^*(r)\mathcal{G}_\mu^*(r),$$

because $\mathbf{E}{B_{o,w}^*}^2 = 1$, and therefore the sum in the left hand side of (18) equals

$$\sum_{w\in\mathbb{T}^d\setminus\{o\}} 2\mathcal{G}_\lambda^*(r)^2\mathcal{G}_\mu^*(r)^2 = 2d\sum_{r=1}^{\infty}(d-1)^{r-1}\,\mathcal{G}_\lambda^*(r)^2\mathcal{G}_\mu^*(r)^2 = 2\frac{(b_\lambda b_\mu)^2(1-a_\lambda^2)^2(1-a_\mu^2)^2 d}{1-(d-1)(a_\lambda a_\mu)^2}. \quad (19)$$

Now this can be expressed in terms of $s = \sqrt{1-\rho\lambda^2}$ and $t = \sqrt{1-\rho\mu^2}$. Indeed, in these variables, we have the simpler expressions

$$a_\lambda^2 = \frac{1-s}{(d-1)(1+s)}, \qquad b_\lambda = \frac{2(d-1)}{(d-2)+sd},$$

and (19) becomes

$$\frac{32\,(d-1)d}{(1+s)(1+t)\,((d-2)(1+s)(1+t)+2(s+t))}. \tag{20}$$

Introduce new variables u, v by $u = \lambda^{-2}, v = \mu^{-2}$, and let $\hat{\beta}(u,v)$ denote expression (20) as a function of them. In terms of these variables, relation (18) becomes

$$\frac{\hat{\beta}(u,v)}{uv} = \int \frac{1}{(x-u)}\frac{1}{(y-v)}\, d\beta(x,y).$$

Due to our convention for square roots (see beginning of this section), the quantity s is an analytic function of u in $\mathbb{C}\setminus[0,\rho]$, and the same holds for t as a function of v.

Assume that $d > 2$. Then, the denominator in (20) does not vanish because s, t have nonnegative real parts (if we set the last factor in the denominator equal to zero and solve for t, we get a quantity with negative real part). So $\hat{\beta}$ defined by (20) is a holomorphic function of (u, v) on $(\mathbb{C}\setminus[0,\rho])^2$. In fact, even the limits of the denominator when u or v approaches $[0,\rho)$ are not zero.

Since the function $h(u,v) := (uv)^{-1}\hat{\beta}(u,v)$ is holomorphic in $(\mathbb{C}\setminus[0,\rho])^2$, and it decays as $(uv)^{-1}$ near infinity, Cauchy's formula will express its values in terms of double contour integrals around the segment $[0,\rho]$. Shrinking the contour around $[0,\rho]$, we get a line integral, and we take into account the different limits of h as one of its arguments approaches the segment from the upper or the lower half plane. That is, when u approaches $x \in [0,\rho)$ from the upper half plane, we have $s(u) \to s(x)$. While when u approaches x from the lower half plane, we have $s(u) \to -s(x)$. The difference comes from the branch cut discontinuity of square root in the definition of s. Thus we have

$$\frac{\hat{\beta}(x,y)}{xy} = -\frac{1}{4\pi^2}\int_0^\rho\int_0^\rho \frac{1}{(u-x)(v-y)}\frac{\tilde{\beta}(u,v)\,du\,dv}{uv},$$

where for $u, v \in [0,\rho]$ we have

$$\begin{aligned} \tilde{\beta}(u,v) &= \sum_{\sigma,\tau=\pm1} \sigma\tau\hat{\beta}[\sigma s, \tau t] \\ &= \frac{512(d-1)^2 d^2 (d-2)\, st}{(d^2(s^2+t^2)-(d-2)^2(1+s^2t^2))^2-(8(d-1)st)^2}, \end{aligned}$$

and $\hat{\beta}[\sigma s, \tau t]$ refers to the expression (20) with (s,t) replaced by $(\sigma s, \tau t)$.

Consequently, the density of the measure $d\beta$ is

$$\beta_d(u,v) = -\frac{1}{4\pi^2}\frac{\tilde{\beta}(u,v)}{uv}.$$

Substituting the expressions of s, t in terms of u, v, we find

$$\beta_d(u,v) = \frac{128}{\pi^2} d^2(d-1)^2(d-2)\frac{\sqrt{u(\rho-u)}\sqrt{v(\rho-v)}}{A},$$

with

$$\begin{aligned} A :&= u^2v^2\Big((d^2(s^2+t^2) - (d-2)^2(1+s^2t^2))^2 - (8(d-1)st)^2 \Big) \\ &= \rho^2 \left[16(2d-3)(u-v)^2 + (d-2)^2 \Big(\rho\kappa\big(\frac{u+v}{2}\big)^2 + 4(d+3)(u-v)^2 + \rho^2(d-2)^2 \Big) \right], \end{aligned}$$

where $\kappa(u) = 2d\sqrt{u(\rho-u)}$. Then

$$\beta_d(u,v) = \frac{2\,\kappa(u)\kappa(v)\,(d-2)\,d^4\,\pi^{-2}}{16(2d-3)(u-v)^2 + (d-2)^2\big(\rho k\big(\frac{u+v}{2}\big)^2 + 4(d+3)(u-v)^2 + \rho^2(d-2)^2\big)}.$$

This proves the $d > 2$ case.

The case $d = 2$ can be easily shown by using continuity in the formulas as $d \downarrow 2$. We get

$$\beta_2(x,y) = \frac{8}{\pi}\,\kappa(x)\delta_x(y) = \frac{32}{\pi}\sqrt{x(1-x)}\;\delta_x(y).$$

The qualitative difference here is that for $d = 2$ the denominator of (20) does vanish along a line. ■

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References

G. W. Anderson and O. Zeitouni (2006). A CLT for a band matrix model. *Probab. Theory Related Fields*, 134(2):283–338.

A. S. Deif. *Advanced matrix theory for scientists and engineers.* Abacus Press, New York, second edition, 1991.

P. Diaconis and S. N. Evans (2001). Linear functionals of eigenvalues of random matrices. *Trans. Amer. Math. Soc.*, 353(7):2615–2633.

A. Gamburd, S. Hoory, M. Shahshahani, A. Shalev, and B. Virág. On the girth of random cayley graphs. To appear in Random Structures Algorithms. arxiv:0707.1833.

R. A. Horn and C. R. Johnson. *Topics in matrix analysis.* Cambridge University Press, Cambridge, 1991.

A. Klein (1998). Extended states in the Anderson model on the Bethe lattice. *Adv. Math.*, 133(1):163–184.

N. Linial and A. Widgerson. Expander graphs and their applications. Lecture notes available at http://www.cs.princeton.edu/∼boaz/ExpanderCourse/.

B. Rider and B. Virág (2007). The noise in the circular law and the Gaussian free field. *Int. Math. Res. Not. IMRN*, (2):Art. ID rnm006, 33.

H. L. Royden. *Real analysis.* Macmillan Publishing Company, New York, third edition, 1988.

C. Villani. Optimal transport, old and new. Available at http://www.umpa.ens-lyon.fr/∼cvillani/surveys.html.

J. vom Scheidt and W. Purkert. *Random eigenvalue problems*, volume 63. Akademie-Verlag, Berlin, 1983.

W. Woess. *Random walks on infinite graphs and groups*, volume 138 of *Cambridge Tracts in Mathematics.* Cambridge University Press, Cambridge, 2000.

O. Zeitouni. Lecture notes on random walks in random environment. In *Lectures on probability theory and statistics*, volume 1837 of *Lecture Notes in Mathematics.* Springer-Verlag, Berlin, 2004. Lectures from the 31st Summer School on Probability Theory held in Saint-Flour, July 8–25, 2001, Edited by Jean Picard.

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