An infinite-server queue influenced by a semi-Markovian environment

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26th June 2008

Abstract

We consider an infinite-server queue, where the arrival and service rates are both governed by a semi-Markov process that's independent of all other aspects of the queue. In particular, we derive a system of equations that are satisfied by various "parts" of the generating function of the steady-state queue-length, while assuming that all arrivals bring an amount of work to the system that's either Erlang or hyperexponentially distributed. These equations are then used to show how to derive all moments of the steady-state queue-length. We then conclude by showing how these results can be slightly extended, and used, along with a transient version of Little's law, to generate rigorous approximations of the steady-state queue length in the case that the amount of work brought by a given arrival is of an arbitrary distribution.

1 Introduction

Queues with a randomly-varying arrival and service rate have recently received quite a bit of attention in the queueing literature. The typical setting is as follows: there is an external stochastic processes, known as the environment, that takes values in some state space $\mathbb{E}$, and these values tend to control various aspects of the queueing system, such as the arrival rate of customers, and the speed at which their work is processed.

Examples of such models in the single-server setting include the recent work of Nain and Nunez-Queija [12], along with the work of Takine [16]. In [12], they consider an $M/M/1$ queue that is influenced by a semi-Markovian environment which takes values in the state space $\{0, 1\}$. The amount of time spent in state 0 has a distribution that possesses a rational Laplace-Stieltjes transform (LST), while the amount of time spent in state 1 is generally distributed, or in particular heavy-tailed, in the sense that it has an infinite moment-generating function. There, the main goal is to compute the z-transform of the steady-state queue length distribution, by using techniques from complex analysis. However, things are looked at from a different perspective in the work of [16]: in this case the environment process is a continuous-time Markov chain, but multiple customer classes are considered, and both the arrival and service rates change according to the environment. Furthermore, in this model each customer brings an amount of work to the system that's generally distributed, with the distribution depending on the class of the customer. This paper focuses more on properties of the busy period, and the waiting time distribution of a customer that arrives to the system during steady-state.

Work has been very recently published in the infinite-server setting as well. In [3], they consider an $M/M/\infty$ queue, where only the service rates are governed by an external environment process, which in this case is a continuous-time Markov chain (CTMC) that again takes values in two states. Their main results include showing that the steady-state queue-length can be written as the sum of two independent random variables, where one of these variables is Poisson, and it can also be

interpreted as the number of customers present in a standard $M/M/\infty$ queue. In the infinite-server setting, we interpret the term "queue-length" as the number of customers presently in the system. Later, D'Auria wrote a series of papers [4, 5, 6] on this topic as well. The main point of [5] is to show that by using basic properties of Poisson processes on $\mathbb{R}^2$, similar stochastic decomposition results can be obtained for the steady-state queue-length, without necessarily assuming that the environment process is a CTMC. In particular, his result allows for virtually any asymptotically stationary environment process, so long as that in steady state, the environment process is ergodic. He then obtains the results of [3] by performing a scheme that involves looking at areas of random sets, where these sets are constructed based on the behavior of the environment process. He then uses this same type of technique in the very recent paper [6] to analyze an $M/M/\infty$ queue with a semi-Markovian environment.

Very recently, Falin [7] has also shown that for an $M/M/\infty$ queue, where both the arrival rates and the service rates are influenced by an external environment that's semi-Markovian, it is possible to apply what is known as a supplementary variable technique to compute the mean steady-state queue length. This technique involves looking at a process that consists of three states: the number of customers in the system, the state of the environment, and the time until the next environment transition (this last state descriptor makes the process Markovian). This allows him to get an expression for the generating function of the number of customers present in the system, and this can be used to compute the steady-state mean queue length. Older references focusing on the infinite-server case include O'Cinneide and Purdue [13], and Keilson and Servi [11].

The goal of this paper is to show how these results can be extended to the case when all arrivals bring Erlang, or hyperexponential amounts of work to the system. Once these results are established, it is easy to see that the same technique can be used for the case when arrivals bring to the system an amount of work that consists of a mixture of Erlang distributions. However, it is well-known (see, for instance, Asmussen [1]) that these random variables are dense in the space of nonnegative random variables under the Prohorov metric (a weak convergence metric), so we conclude by using this fact, along with a transient version of Little's law, to show that some type of analysis is still possible when the service times have a general distribution.

The main technique we will use in this paper involves looking at the queue-length process at the transition epochs of the environment process. Once we obtain the steady-state behavior of the process at these epochs, we will use the inversion formula (see [2]) to relate this to the steady-state behavior of the process at an arbitrary time. It would also be possible to apply a semi-regenerative argument (see, for example Chapter 7, Section 5 of [1]) but in this case we would also have to assume the presence of some sort of nonlattice condition, because these arguments lead to a limiting result. The reader should keep in mind that the Palm approach only says something about the stationary process: however, if the process has a limiting distribution, it will coincide with the marginal distribution of our stationary process.

2   Model description

Our paper will focus on an infinite-server queueing system, where the arrivals and service rates are governed by a stationary semi-Markov process $C := \{C(t); t \in \mathbb{R}\}$, on a finite state space $\mathbb{E}$. The transition times of C will be denoted by $\{T_n\}_{n \in \mathscr{Z}}$, where $\mathscr{Z}$ represents the set of all integers. It shall be assumed that these random variables, along with all other random elements found in this paper, exist on a probability space $(\Omega, \mathcal{F}, P)$, where Ω is an arbitrary space, $\mathcal{F}$ is an appropriate σ-field of subsets of Ω, and P is a probability measure that is defined on $\mathcal{F}$.

To completely describe how C evolves through time, it will suffice to give a pathwise description of both how long it spends in each state, and how it makes transitions from one state to the next. Hence, let $A_{i,j}(x) = P(T_{n+1} - T_n \le x, C(T_{n+1}) = j | C(T_n) = i)$. This represents the probability that, given at time T_n the process C has just made a transition to state i, it will make another transition before time $T_n + x$, and that transition will consist of a jump from state i to state j. Moreover, clearly $r_{i,j} := A_{i,j}(\infty)$ represents the probability that the environment jumps directly to state j from state i. We will let $\alpha_{i,j}(s) = \int_0^\infty e^{-sx} dA_{i,j}(x)$ denote the LST of $A_{i,j}$, and

$\alpha_i(s) = \sum_{j \in \mathbb{E}} \alpha_{i,j}(s)$. Finally, let $\{\pi_i\}_{i \in \mathbb{E}}$ denote the stationary distribution of C at its transition times, and let $v_i = E[T_{n+1} - T_n | C(T_n) = i]$, and $v = \sum_{i \in \mathbb{E}} v_i \pi_i$. Our assumption of a finite $\mathbb{E}$ ensures that the process C is regular, in that the expected number of its transitions in any compact set is finite. Such conditions will not explicitly be needed throughout our analysis, but they are needed in order to ensure that our semi-Markov process does not actually terminate at some point.

We also consider an infinite-server queueing system, which is influenced by the environment in the following way: while the environment is in state i, the system serves the remaining amount of work that's possessed by each customer currently in the system at a rate μ_i, and new arrivals show up in accordance with a Poisson process with rate λ_i, where the nth arrival to the system brings a random amount of work B_n, with distribution function $B(t)$. Here we follow the convention that, if $\{X_n\}$ denotes the arrival times of customers to the system, then $X_0 \leq 0 < X_1$. Throughout the paper, we will assume that $\mu_i > 0$ for all $i \in \mathbb{E}$, but this assumption is not essential for the analysis found within the next three sections; it is only used to make the paper slightly more readable. Finally, we also assume that the service requirements of all customers are independent of one another.

Thus, one could think of the arrival process in this case as being a "semi-Markov-Modulated Poisson process", but of course this arrival process is not independent of how the customers are served in the system. To specify exactly what sort of dependence structure is being assumed here among all of the stochastic elements present in this model, it suffices to say that conditional on a given sample path of the environment process, the arrivals arrive according to a nonhomogeneous Poisson process, and if someone arrives at time t and brings an amount of work W, it will be in the system for an amount of time $T(t)$, where

$$W = \int_t^{t+T(t)} \mu_{C(s)} ds.$$

In both [5] and [6], heavy use is made of the following fact from point process theory.

Theorem 2.1 *Suppose N is a nonhomogeneous Poisson process on $\mathbb{R}$ with points $\{X_n\}_{n \in \mathcal{Z}}$, where for each Borel set A,*

$$N(A) = \sum_{n \in \mathcal{Z}} \delta_{X_n}(A)$$

where $\delta_x(A) = 1$ if $x \in A$, and $\delta_x(A) = 0$ otherwise. If we add to each point X_n a random variable Y_n that is independent of all other locations of N and their marks, then the new point process

$$M(A \times B) = \sum_{n \in \mathcal{Z}} \delta_{(X_n, Y_n)}(A \times B)$$

is a nonhomogeneous Poisson process on $\mathbb{R}^2$.

When we say that M is a nonhomogeneous Poisson process on $\mathbb{R}^2$, we mean that the number of points in any Borel set $A \in \mathbb{R}^2$ is Poisson distributed, and for any disjoint collection of Borel sets $A_1, \ldots, A_n \subseteq \mathbb{R}^2$, the random variables $M(A_1), \ldots, M(A_n)$ are independent.

Keeping our model in mind, we see that if we add to each customer arrival time X_n its service requirement B_n, and we condition on the sample path of the environment, the resulting point process on $\mathbb{R}^2$ is Poisson. This implies that while conditioning on the environment, the number of customers in the system at time zero is Poisson as well. Proofs of these well-known facts can be found, for instance, in Serfozo [14].

This interesting property of Poisson processes was used in [5] to obtain a stochastic decomposition of the queue-length into two independent quantities, where one of those quantities represents the stationary distribution of a standard $M/G/\infty$ queue. Indeed, a stochastic decomposition that's similar in type to that discussed in [5] can also be easily generalized to our setting.

Theorem 2.2 *Consider an $M/G/\infty$ system with a semi-Markov modulated arrival and service rate, where $\lambda := \inf_{i \in \mathbb{E}} \lambda_i > 0$, and $\mu := \sup_{i \in \mathbb{E}} \mu_i > 0$. Then the steady-state number of customers in the system can be decomposed into the sum of two independent random variables:*

$$Q = Q_{\lambda,\mu}^{M/G/\infty} + Q_C$$

where $Q_{\lambda,\mu}^{M/G/\infty}$ represents the steady-state number of customers in the system in a standard $M/G/\infty$ queue that processes work at a rate μ, and Q_C is a randomized Poisson random variable, with a parameter that depends on the environment process C.

The proof of this result is essentially the same as the proof of the decomposition result found in [5], with only a slight modification: a semi-Markov modulated Poisson process can be represented as an independent sum of a homogeneous Poisson process with rate λ, along with another semi-Markov modulated Poisson process that at times has a rate of zero. Furthermore, as is done in [5], we will follow with a practical description of the result, by saying that the steady-state population can be decomposed into two classes: those that would have still been in the system, regardless of the increase of arrivals or the decrease in service speed, and the rest.

This result could very well be useful towards determining the steady-state queue length, but our approach will not involve its use. Rather, we will first derive a system of equations containing functions that can be interpreted as "parts" of the generating function of the steady-state queue-length, in the special case that all customer bring an amount of work that is either Erlang or hyperexponentially distributed. These results are then used to show how one can generate a system of equations that consist of unknowns, which once solved for can be used to compute the mean queue-length in steady-state. We will then conclude by providing a rigorous approximation of the mean queue-length in steady-state for the case when the services are allowed to have any type of distribution, by using Little's law, along with the fact that mixtures of Erlang distributions can be used to approximate any type of distribution with nonnegative support, with respect to the weak convergence metric.

Throughout the paper, we will use results from stationary point process theory to relate the stationary distribution of our process at the transition epochs of the environment to the stationary distribution of the process at an arbitrary time. In particular, this will involve the use of the Palm measure P_0 that's induced by the point process consisting of the transition times of the environment. We will not go into great mathematical details to explain exactly how P_0 is defined: indeed, the interested reader has many references to choose from regarding this topic, with recent ones including [2], Chapter 7, Section 6 of [1], and Chapter 6 of [15]. The reader merely needs to be aware of the following facts: (i) for an event A, $P_0(A)$ can be interpreted as the probability of A, given that the environment process changes state at time zero, and (ii) under the measure P_0, the joint distribution of $(Q(T_n), C(T_n))$ is the same for all $n \in \mathcal{Z}$.

3   Exponential services

We will first consider the case when each customer brings an amount of service that is exponentially distributed with rate ν, i.e., for each $t \geq 0$,

$$B(t) = 1 - e^{-\nu t}.$$

The approach we will use to compute all of the factorial moments of the queue-length will basically involve coming up with a system of equations that the functions $m_j(z)$ satisfy, where $m_j(z) = E_0[z^{Q(0)}\mathbf{1}(C(0) = j)]$.

Our first result provides an expression for the generating function of the steady-state queue-length. This will be used to compute the factorial moments $E[(Q(0))_n]$ for $n \geq 1$, where for $x \in \mathbb{R}$, $(x)_n = x(x-1)\cdots(x-n+1)$.

Theorem 3.1 *The functions m_j, $j \in \mathbb{E}$ satisfy the following system of equations: for each $j \in \mathbb{E}$,*

$$m_j(z) = \sum_{i \in \mathbb{E}} \int_0^\infty e^{-\frac{\lambda_i(1-e^{-\mu_i \nu t})(1-z)}{\mu_i \nu}} m_i(1 - e^{-\mu_i \nu t}(1-z)) dA_{i,j}(t). \tag{1}$$

Proof It was mentioned in the previous section that $m_i(z)$ is equal to $E_0[z^{Q(T_1)}\mathbf{1}(C(T_1) = i)]$, so to compute this we will first condition on $Q(0)$ and $C(0)$.

Suppose $Q(0) = m$ and $C(0) = i$, where $m \geq 0$ and $i \in \mathbb{E}$. After conditioning on the event that $T_1 = t$, we see that each of these m customers will be in the system at time t with probability $e^{-\mu_i \nu t}$. Furthermore, there could also be customers present in the system at time t that originally were not there at time zero. For a given customer that arrives to the system after time 0 but before time t, we see that the probability he or she is still in the system at time t is given by

$$\int_0^t e^{-\mu_i \nu (t-s)} \frac{1}{t} ds = \frac{1 - e^{-\mu_i \nu t}}{\mu_i \nu t}.$$

Since the number of customers that arrive in $(0, t]$ is Poisson with rate $\lambda_i t$, we see that

$$
\begin{aligned}
&E_0[z^{Q(T_1)}\mathbf{1}(C(T_1) = j)|Q(0) = m, C(0) = i] \\
=\ & \int_0^\infty \sum_{n=0}^\infty \frac{(\lambda_i t)^n e^{-\lambda_i t}}{n!} \left(1 - e^{-\mu_i \nu t}(1-z)\right)^m \left(1 - \frac{1 - e^{-\mu_i \nu t}}{\mu_i \nu t}(1-z)\right)^n dA_{i,j}(t) \\
=\ & \int_0^\infty e^{-\frac{\lambda_i (1 - e^{-\mu_i \nu t})(1-z)}{\mu_i \nu}} \left(1 - e^{-\mu_i \nu t}(1-z)\right)^m dA_{i,j}(t).
\end{aligned}
$$

After unconditioning, we conclude that

$$m_j(z) = \sum_{i \in \mathbb{E}} \int_0^\infty e^{-\frac{\lambda_i (1 - e^{-\mu_i \nu t})(1-z)}{\mu_i \nu}} m_i(1 - e^{-\mu_i \nu t}(1-z)) dA_{i,j}(t)$$

which completes the proof. $\Diamond$

The following corollary then immediately follows from this result.

Corollary 3.1 *The moments* $z_{n,j} = E_0[(Q(0))_n \mathbf{1}(C(0) = j)]$ *satisfy the following system of equations: for* $n \geq 1$ *and* $j \in \mathbb{E}$,

$$z_{n,j} = \sum_{i \in \mathbb{E}} \sum_{k=0}^n \binom{n}{k} \left(\frac{\lambda_i}{\mu_i \nu}\right)^{n-k} \left[\sum_{l=0}^{n-k} (-1)^l \binom{n-k}{l} \alpha_{i,j}((k+l)\mu_i \nu)\right] z_{k,i}. \tag{2}$$

Remark Notice that (2) can be used to compute all embedded factorial moments, in a recursive fashion: indeed, notice that the first factorial moments can be computed by solving a system of linear equations, where the number of equations and unknowns are equal to the cardinality of $\mathbb{E}$. Furthermore, once these are known, they can be plugged into a second system, and we end up with another system that's of the same size as the previous system. This procedure can clearly be repeated to produce higher moments as well.

Proof The proof of this result merely involves differentiating (1). From Leibniz rule, we see that for each $n \geq 1$,

$$m_j^{(n)}(z) = \sum_{i \in \mathbb{E}} \int_0^\infty e^{-\frac{\lambda_i (1 - e^{-\mu_i \nu t})(1-z)}{\mu_i \nu}} \sum_{k=0}^n \binom{n}{k} e^{-k\mu_i \nu t} \left(\frac{\lambda_i}{\mu_i \nu}(1 - e^{-\mu_i \nu t})\right)^{n-k} m_i^{(k)}(1 - e^{-\mu_i \nu t}(1-z)) dA_{i,j}(t). \tag{3}$$

By letting $z = 1$ in (3), we observe that

$$
\begin{aligned}
E_0[(Q)_n \mathbf{1}(C = j)] &= \sum_{i \in \mathbb{E}} \int_0^\infty \sum_{k=0}^n \binom{n}{k} e^{-k\mu_i \nu t} \left(\frac{\lambda_i}{\mu_i \nu}\right)^{n-k} \sum_{l=0}^{n-k} (-1)^l \binom{n-k}{l} e^{-l\mu_i \nu t} E_0[(Q)_k \mathbf{1}(C = i)] dA_{i,j}(t) \\
&= \sum_{i \in \mathbb{E}} \sum_{k=0}^n \binom{n}{k} \left(\frac{\lambda_i}{\mu_i \nu}\right)^{n-k} \left[\sum_{l=0}^{n-k} (-1)^l \binom{n-k}{l} \alpha_{i,j}((k+l)\mu_i \nu)\right] E_0[(Q)_k \mathbf{1}(C = i)].
\end{aligned}
$$

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$\diamondsuit$

The following formula can be used to relate the moments at a transition epoch to the moments at an arbitrary epoch.

Theorem 3.2 *For each $j \in \mathbb{E}$,*

$$E[z^{Q(0)}\mathbf{1}(C(0) = j)] \quad = \quad \frac{1}{\upsilon}\int_0^\infty e^{-\frac{\lambda_j(1-e^{-\mu_j\nu t})(1-z)}{\mu_j\nu}}m_j(1-e^{\mu_j\nu t}(1-z))\overline{A}_j(t)\pi_j dt, \qquad (4)$$

where $\overline{A}_j(t) = P(T_{n+1} - T_n \leq t|C(T_n) = j)$.

Proof From the inversion formula (see [2]), we also find that

$$E[z^{Q(0)}\mathbf{1}(C(0) = j)] \quad = \quad \frac{1}{\upsilon}E_0\left[\int_0^{T_1} z^{Q(t)}\mathbf{1}(C(t) = j)dt\right]$$

$$= \quad \frac{1}{\upsilon}\int_0^\infty E_0[z^{Q(t)}|T_1 > t, C(t) = j]\overline{A}_j(t)\pi_j dt.$$

However, under the measure P_0, $T_0 = 0$ almost surely, and furthermore,

$$E_0[z^{Q(t)}|T_1 > t, C(t) = j]\pi_j = \pi_j e^{-\frac{\lambda_j}{\mu_j\nu}(1-e^{-\mu_j\nu t})(1-z)}m_j(1 - e^{-\mu_j\nu t}(1-z)).$$

Therefore,

$$E[z^{Q(0)}\mathbf{1}(C(0) = j)] \quad = \quad \frac{1}{\upsilon}\int_0^\infty e^{-\frac{\lambda_j(1-e^{-\mu_j\nu t})(1-z)}{\mu_j\nu}}m_j(1-e^{\mu_j\nu t}(1-z))\overline{A}_j(t)\pi_j dt.$$

This proves the claim. $\diamondsuit$

Corollary 3.2 *The nth factorial moment of the steady-state queue-length distribution is*

$$E[(Q(0))_n] = \quad \sum_{i\in\mathbb{E}}\frac{\pi_i}{\upsilon}\sum_{k=0}^n\binom{n}{k}\left(\frac{\lambda_i}{\mu_i\nu}\right)^{n-k}\sum_{l=0}^{n-k}(-1)^l\binom{n-k}{l}$$

$$\times \quad \left[\frac{1}{(k+l)\mu_i\nu}\left(1 - \alpha_i((k+l)\mu_i\nu)\right)\mathbf{1}(k+l > 0) + \upsilon_i\mathbf{1}(k+l=0)\right]z_{k,i}.$$

Proof After applying Leibniz rule to (4), we also see that

$$E[(Q(0))_n z^{Q(0)-n}\mathbf{1}(C(0) = j)]$$

$$= \quad \frac{1}{\upsilon}\int_0^\infty e^{-\frac{\lambda_j(1-e^{-\mu_j\nu t})(1-z)}{\mu_j\nu}}\sum_{k=0}^n\binom{n}{k}e^{-k\mu_j\nu t}\left(\frac{\lambda_j}{\mu_j\nu}(1-e^{-\mu_j\nu t})\right)^{n-k}m_j^{(k)}(1-e^{-\mu_j\nu t}(1-z))\overline{A}_j(t)\pi_j dt$$

and so, putting $z = 1$ gives

$$E[(Q(0))_n\mathbf{1}(C(0) = j)]$$

$$= \quad \frac{1}{\upsilon}\int_0^\infty \sum_{k=0}^n\binom{n}{k}e^{-k\mu_j\nu t}\left(\frac{\lambda_j}{\mu_j\nu}\right)^{n-k}\sum_{l=0}^{n-k}(-1)^l\binom{n-k}{l}e^{-l\mu_j\nu t}E_0[(Q)_k\mathbf{1}(C = j)]\overline{A}_j(t)\pi_j dt$$

$$= \quad \frac{\pi_j}{\upsilon}\sum_{k=0}^n\binom{n}{k}\left(\frac{\lambda_j}{\mu_j\nu}\right)^{n-k}\sum_{l=0}^{n-k}(-1)^l\binom{n-k}{l}\left[\frac{(1 - \alpha_j((k+l)\mu_j\nu))}{(k+l)\mu_j\nu}\mathbf{1}(k+l > 0) + \upsilon_j\mathbf{1}(k+l=0)\right]$$

$$\times \quad E_0[(Q)_k\mathbf{1}(C = j)].$$

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This concludes the proof. $\diamondsuit$

4 Hyperexponential services

We continue by considering the case where the amount of service each customer brings to the system is hyperexponentially distributed, i.e., for each $t \geq 0$,

$$B(t) = \sum_{k=1}^{N} p_i(1 - e^{-\nu_i t})$$

where N is an integer, p_i, $1 \leq i \leq N$ are probabilities summing to 1 and $\nu_i > 0$, for $1 \leq i \leq N$. In our model, we can say that a given customer that arrives to the system is a type-i customer with probability p_i, and type-i customers bring an exponentially distributed (with rate ν_i) amount of work to the system. For ease of exposition, we will focus on the case when $N = 2$, but it will become clear that the same type of reasoning can be used for arbitrary N.

Again, our first step will involve deriving a system of equations, which are satisfied by the functions $m_j(z_1, z_2) = E_0[z_1^{Q_1(0)} z_2^{Q_2(0)} \mathbf{1}(C(0) = j)]$.

Theorem 4.1 *The functions m_j, $j \in \mathbb{E}$ satisfy the following system of equations: for each $j \in \mathbb{E}$,*

$$m_j(z_1, z_2) = \sum_{i \in \mathbb{E}} \int_0^\infty e^{-p_1 \frac{\lambda_i}{\mu_i \nu_1}(1 - e^{-\mu_i \nu_1 t})(1 - z_1)} e^{-p_2 \frac{\lambda_i}{\mu_i \nu_2}(1 - e^{-\mu_i \nu_2 t})(1 - z_2)}$$

$$\times \quad m_i(1 - e^{-\mu_i \nu_1 t}(1 - z_1), 1 - e^{-\mu_i \nu_2 t}(1 - z_2)) dA_{i,j}(t). \tag{5}$$

Proof We recall that $(Q_1(0), Q_2(0), C(0)) \stackrel{d}{=} (Q_1(T_1), Q_2(T_1), C(T_1))$, so to compute each m_j it will again be helpful to condition on $Q_1(0)$, $Q_2(0)$, and $C(0)$. Once we have conditioned on these variables, it will be useful to also condition on the number of new arrivals in the interval $(0, T_1]$, which we also condition on by assuming that it is of length t. Therefore, assuming $C(0) = i$, there are four different populations to consider: (1) type-1 customers currently in the system at 0, that leave before time T_1 with probability $1 - \eta_i$, (2) type-2 customers currently in the system at 0, that leave before T_1 with probability $1 - \delta_i$, (3) new type-1 customers that arrive in $(0, T_1]$, which leave before T_1 with probability $1 - \beta_i$, and finally (4) new type-2 customers that arrive in $(0, T_1]$, which leave before T_1 with probability $1 - \gamma_i$. Thus, $\eta_i = e^{-\mu_i \nu_1 t}$, $\delta_i = e^{-\mu_i \nu_2 t}$,

$$\beta_i = \int_0^t e^{-\mu_i \nu_1(t-s)} \frac{1}{t} ds = \frac{1 - e^{-\mu_i \nu_1 t}}{\mu_i \nu_1}$$

and

$$\gamma_i = \int_0^t e^{-\mu_i \nu_2(t-s)} \frac{1}{t} ds = \frac{1 - e^{-\mu_i \nu_2 t}}{\mu_i \nu_2}.$$

Now we are ready to begin our computations. Here

$$E_0[z_1^{Q_1(T_1)} z_2^{Q_2(T_1)} \mathbf{1}(C(T_1) = j) | Q_1(0) = i_1, Q_2(0) = i_2, C(0) = i]$$

$$= \int_0^\infty \sum_{n_1=0}^\infty \sum_{n_2=0}^\infty \frac{(p_1 \lambda_i t)^{n_1} e^{-p_1 \lambda_i t}}{n_1.} \frac{(p_2 \lambda_i t)^{n_2} e^{-p_2 \lambda_i t}}{n_2.} (1 - \eta_i + \eta_i z_1)^{n_1} (1 - \delta_i + \delta_i z_2)^{n_2}$$

$$\times (1 - \beta_i + \beta_i z_1)^{i_1} (1 - \gamma_i + \gamma_i z_2)^{i_2} dA_{i,j}(t)$$

$$= \int_0^\infty e^{-p_1 \lambda_i t \eta_i (1 - z_1)} e^{-p_2 \lambda_i t \delta_i (1 - z_2)} (1 - \beta + \beta z_1)^{i_1} (1 - \gamma + \gamma z_2)^{i_2} dA_{i,j}(t)$$

$$= \int_0^\infty e^{-p_1 \frac{\lambda_i}{\mu_i \nu_1}(1 - e^{-\mu_i \nu_1 t})(1 - z_1)} e^{-p_2 \frac{\lambda_i}{\mu_i \nu_2}(1 - e^{-\mu_i \nu_2 t})(1 - z_2)} (1 - e^{-\mu_i \nu_1 t}(1 - z_1))^{i_1} (1 - e^{-\mu_i \nu_2 t}(1 - z_2))^{i_2} dA_{i,j}(t)$$

After unconditioning, we then see that

$$m_j(z_1, z_2) = \sum_{i \in \mathbb{E}} \int_0^\infty e^{-p_1 \frac{\lambda_i}{\mu_i \nu_1}(1 - e^{-\mu_i \nu_1 t})(1 - z_1)} e^{-p_2 \frac{\lambda_i}{\mu_i \nu_2}(1 - e^{-\mu_i \nu_2 t})(1 - z_2)}$$

$$\times \quad m_i(1 - e^{-\mu_i \nu_1 t}(1 - z_1), 1 - e^{-\mu_i \nu_2 t}(1 - z_2))dA_{i,j}(t)$$

and so (5) holds. $\diamond$

This immediately gives the following corollary.

Corollary 4.1 *The first moments* $z_{i,j} = E_0[Q_i(0)\mathbf{1}(C(0) = j)]$ *satisfy the following system of equations: for each* $j \in \mathbb{E}$,

$$z_{1,j} = \sum_{i \in \mathbb{E}} \left[\pi_i p_1 \frac{\lambda_i}{\mu_i \nu_1}(r_{i,j} - \alpha_{i,j}(\mu_i \nu_1)) + z_{1,i} \alpha_{i,j}(\mu_i \nu_1) \right]$$

and

$$z_{2,j} = \sum_{i \in \mathbb{E}} \left[\pi_i p_2 \frac{\lambda_i}{\mu_i \nu_2}(r_{i,j} - \alpha_{i,j}(\mu_i \nu_2)) + z_{2,i} \alpha_{i,j}(\mu_i \nu_2) \right].$$

Proof After taking partial derivatives in (5) and setting $(z_1, z_2) = (1,1)$, we end up with the following system of equations:

$$E_0[Q_1(0)\mathbf{1}(C(0) = j)] = \sum_{i \in \mathbb{E}} \left[\pi_i p_1 \frac{\lambda_i}{\mu_i \nu_1}(r_{i,j} - \alpha_{i,j}(\mu_i \nu_1)) + E_0[Q_1(0)\mathbf{1}(C(0) = i)]\alpha_{i,j}(\mu_i \nu_1) \right]$$

and

$$E_0[Q_2(0)\mathbf{1}(C(0) = j)] = \sum_{i \in \mathbb{E}} \left[\pi_i p_2 \frac{\lambda_i}{\mu_i \nu_2}(r_{i,j} - \alpha_{i,j}(\mu_i \nu_2)) + E_0[Q_2(0)\mathbf{1}(C(0) = i)]\alpha_{i,j}(\mu_i \nu_2) \right],$$

which is the same as the equations given in the second part of the corollary. Notice that these equations are no more difficult to solve than the ones found in [7], which are the same as system (2); as a matter of fact, they can be split into two sets, with each set being of the same form as the system of equations in [7]. $\diamond$

Again, we can relate these moments to the moments at an arbitrary time.

Theorem 4.2 *For each* $j \in \mathbb{E}$,

$$E[z_1^{Q_1(0)} z_2^{Q_2(0)} \mathbf{1}(C(0) = j)] = \frac{1}{\upsilon} \int_0^\infty e^{-p_1 \frac{\lambda_j}{\mu_j \nu_1}(1 - e^{-\mu_j \nu_1 t})(1 - z_1)} e^{-p_2 \frac{\lambda_j}{\mu_j \nu_2}(1 - e^{-\mu_j \nu_2 t})(1 - z_2)}$$

$$\times \quad m_j(1 - e^{-\mu_j \nu_1 t}(1 - z_1), 1 - e^{-\mu_j \nu_2 t}(1 - z_2))\overline{A}_j(t)dt.$$

Proof To compute the LST of the steady-state queue-length at an arbitrary instant, we can again use the Inversion formula to conclude that

$$E[z_1^{Q_1(0)} z_2^{Q_2(0)} \mathbf{1}(C(0) = j)] = \frac{1}{\upsilon} E_0 \left[\int_0^\infty z_1^{Q_1(t)} z_2^{Q_2(t)} \mathbf{1}(C(t) = j)\mathbf{1}(T_1 > t)dt \right]$$

$$= \frac{1}{\upsilon} \int_0^\infty E_0[z_1^{Q_1(t)} z_2^{Q_2(t)} | T_1 > t, C(t) = j]\overline{A}_j(t)\pi_j dt.$$

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But $T_0 = 0$ almost surely under P_0, and furthermore,

$$E_0[z_1^{Q_1(t)} z_2^{Q_2(t)} | T_1 > t, C(t) = j]\pi_j = \pi_j \ e^{-p_1 \frac{\lambda_j}{\mu_j \nu_1}(1 - e^{-\mu_j \nu_1 t})(1 - z_1)} e^{-p_2 \frac{\lambda_j}{\mu_j \nu_2}(1 - e^{-\mu_j \nu_2 t})(1 - z_2)}$$
$$\times \ m_j(1 - e^{-\mu_i \nu_1 t}(1 - z_1), 1 - e^{-\mu_i \nu_2 t}(1 - z_2)).$$

Therefore,

$$E[z_1^{Q_1(0)} z_2^{Q_2(0)} \mathbf{1}(C(0) = j)] = \frac{1}{\upsilon} \int_0^\infty e^{-p_1 \frac{\lambda_j}{\mu_j \nu_1}(1 - e^{-\mu_j \nu_1 t})(1 - z_1)} e^{-p_2 \frac{\lambda_j}{\mu_j \nu_2}(1 - e^{-\mu_j \nu_2 t})(1 - z_2)}$$
$$\times \ m_j(1 - e^{-\mu_j \nu_1 t}(1 - z_1), 1 - e^{-\mu_j \nu_2 t}(1 - z_2))\overline{A}_j(t)dt.$$

$\diamondsuit$

We can then get an expression for any moment we would like of the queue-length at an arbitrary time, by taking the appropriate derivatives.

Corollary 4.2 *The mean steady-state queue-length is given by*

$$E[Q(0)] = \sum_{j \in \mathbb{E}} \frac{1}{\upsilon} \left[\pi_j \frac{\lambda_j p_1}{\mu_j \nu_1} \upsilon_j + \left[z_{1,j} - \frac{\lambda_j p_1}{\mu_j \nu_1} \upsilon_j \right] \frac{1 - \alpha_j(\mu_j \nu_1)}{\mu_j \nu_1} \right]$$
$$+ \sum_{j \in \mathbb{E}} \frac{1}{\upsilon} \left[\pi_j \frac{\lambda_j p_2}{\mu_j \nu_2} \upsilon_j + \left[z_{2,j} - \frac{\lambda_j p_2}{\mu_j \nu_2} \upsilon_j \right] \frac{1 - \alpha_j(\mu_j \nu_2)}{\mu_j \nu_2} \right] \quad (6)$$

Proof After taking derivatives, we see that

$$E[Q_1(0)\mathbf{1}(C(0) = j)] = \frac{1}{\upsilon} \left[\pi_j \frac{\lambda_j p_1}{\mu_j \nu_1} \upsilon_j + \left[E_0[Q_1(0)\mathbf{1}(C(0) = j)] - \frac{\lambda_j p_1}{\mu_j \nu_1} \upsilon_j \right] \frac{1 - \alpha_j(\mu_j \nu_1)}{\mu_j \nu_1} \right]$$

and

$$E[Q_2(0)\mathbf{1}(C(0) = j)] = \frac{1}{\upsilon} \left[\pi_j \frac{\lambda_j p_2}{\mu_j \nu_2} \upsilon_j + \left[E_0[Q_2(0)\mathbf{1}(C(0) = j)] - \frac{\lambda_j p_2}{\mu_j \nu_2} \upsilon_j \right] \frac{1 - \alpha_j(\mu_j \nu_2)}{\mu_j \nu_2} \right].$$

These expressions can then be used to derive (6). $\diamondsuit$

Remark As was mentioned in the proof, higher moments of the queue-length can also be computed as well. For instance, the system of equations that can be used to compute $E[Q_1(0)Q_2(0)]$ is as follows:

$$E_0[Q_1(0)Q_2(0)\mathbf{1}(C(0) = j)] = \sum_{i \in \mathbb{E}} \frac{p_1}{\nu_1} \frac{\lambda_i}{\mu_i} \frac{p_2}{\nu_2} \frac{\lambda_i}{\mu_i} \pi_i \left(r_{i,j} - \alpha_{i,j}(\mu_i \nu_1) - \alpha_{i,j}(\mu_i \nu_2) + \alpha_{i,j}(\mu_i(\nu_1 + \nu_2)) \right)$$
$$+ \sum_{i \in \mathbb{E}} \frac{p_2}{\nu_2} \frac{\lambda_i}{\mu_i} \left(\alpha_{i,j}(\mu_i \nu_1) - \alpha_{i,j}(\mu_i(\nu_1 + \nu_2)) \right) E_0[Q_1(0)\mathbf{1}(C(0) = i)]$$
$$+ \sum_{i \in \mathbb{E}} \frac{p_1}{\nu_1} \frac{\lambda_i}{\mu_i} \left(\alpha_{i,j}(\mu_i \nu_2) - \alpha_{i,j}(\mu_i(\nu_1 + \nu_2)) \right) E_0[Q_2(0)\mathbf{1}(C(0) = i)]$$
$$+ \sum_{i \in \mathbb{E}} \alpha_{i,j}(\mu_i(\nu_1 + \nu_2)) E_0[Q_1(0)Q_2(0)\mathbf{1}(C(0) = i)].$$

5   Erlang services

We now consider the case where the services are Erlang distributed, with $N \geq 1$ phases. In this case,

$$B(t) = 1 - \sum_{k=0}^{N-1} \frac{(\nu t)^k e^{-\nu t}}{k!}.$$

Again, we will assume that $N = 2$, but it will be clear as to how to proceed for large N.

Let $Q_i(t)$ denote the number of queueing customers present in the system that are in phase i, for $i = 1, 2$. Here a phase refers to a characteristic of the customer's service. In particular, each customer's service amount is Erlang distributed, and this can be broken up into two exponential random variables. A given customer is said to be in phase 1 during the times when the server is processing work from this first exponential amount, and he or she is in phase two during the processing of the second amount. Throughout the rest of the paper, customers that are in phase i will be referred to as type i customers.

Again, we can derive the same type of equations for the functions $m_j(z_1, z_2) = E_0[z_1^{Q_1(0)} z_2^{Q_2(0)} \mathbf{1}(C(0) = j)]$.

Theorem 5.1 *The functions m_j satisfy the following system of equations: for each $j \in \mathbb{E}$,*

$$m_j(z_1, z_2) = \sum_{i \in \mathbb{E}} \int_0^\infty e^{-\lambda_i t \eta_{i,1}(1-z_1)} e^{-\lambda_i t \eta_{i,2}(1-z_2)} m_i(1 - \beta_{i,1} - \beta_{i,2} + \beta_{i,1} z_1 + \beta_{i,2} z_2, 1 - \gamma_{i,2} + \gamma_{i,2} z_2) dA_{i,j}(t),$$

where $\eta_{i,n} = \frac{1}{\mu_i \nu t}\left[1 - \sum_{k=0}^{n-1} \frac{(\mu_i \nu t)^k e^{-\mu_i \nu t}}{k!}\right]$, $\beta_{i,n} = \frac{(\mu_i \nu t)^{n-1} e^{-\mu_i \nu t}}{(n-1)!}$, and $\gamma_{i,n} = \frac{(\mu_i \nu t)^{n-2} e^{-\mu_i \nu t}}{(n-2)!}$.

Proof Clearly, we see that

$$E_0[z_1^{Q_1(T_1)} z_2^{Q_2(T_1)} \mathbf{1}(C(T_1) = j) | Q_1(0) = i_1, Q_2(0) = i_2, C(0) = i]$$

$$= \int_0^\infty E_0[z_1^{Q_1(t)} z_2^{Q_2(t)} | Q_1(0) = i_1, Q_2(0) = i_2, C(0) = i, C(t) = j] dA_{i,j}(t)$$

$$= \int_0^\infty \sum_{n=0}^\infty E[z_1^{Q_1(t)} z_2^{Q_2(t)} | Q_1(0) = i_1, Q_2(0) = i_2, C(0) = i, C(t) = j, N(0,t] = n] \frac{(\lambda_i t)^n e^{-\lambda_i(t)}}{n.} dA_{i,j}(t).$$

At this point we have to consider three different types of populations: type 1 customers that were present in the system at time 0, type 2 customers present at 0, and new arrivals in $(0, t]$. If, at the time of a transition the environment is in state i, each type 1 customer will, at time t, either be a type 1 customer, a type 2 customer, or it will leave the system, with probabilities $\beta_{i,1}$, $\beta_{i,2}$, and $1 - \beta_{i,1} - \beta_{i,2}$, respectively. A type 2 customer will stay as a type 2 with probability $\gamma_{i,2}$, and a new arrival will be type 1, type 2, or will leave with probabilities $\eta_{i,1}$, $\eta_{i,2}$, or $1 - \eta_{i,1} - \eta_{i,2}$, respectively. The behavior of each customer is independent of all other customers present in the system, and because of this we are able to just use multinomial transforms to compute the above expression. Therefore,

$$E_0[z_1^{Q_1(T_1)} z_2^{Q_2(T_1)} \mathbf{1}(C(T_1) = j) | Q_1(0) = i_1, Q_2(0) = i_2, C(0) = i]$$

$$= \int_0^\infty \sum_{n=0}^\infty \frac{(\lambda_i t)^n e^{-\lambda_i t}}{n!} (1 - \eta_{i,1} - \eta_{i,2} + \eta_{i,1} z_1 + \eta_{i,2} z_2)^n (1 - \beta_{i,1} - \beta_{i,2} + \beta_{i,1} z_1 + \beta_{i,2} z_2)^{i_1}$$

$$\times (1 - \gamma_{i,2} + \gamma_{i,2} z_2)^{i_2} dA_{i,j}(t)$$

$$= \int_0^\infty e^{-\lambda_i t(\eta_{i,1}(1-z_1) + \eta_{i,2}(1-z_2))} (1 - \beta_{i,1} - \beta_{i,2} + \beta_{i,1} z_1 + \beta_{i,2} z_2)^{i_1} (1 - \gamma_{i,2} + \gamma_{i,2} z_2)^{i_2} dA_{i,j}(t).$$

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Again, after unconditioning we find that

$$m_j(z_1, z_2) = \sum_{i \in \mathbb{E}} \int_0^\infty e^{-\lambda_i t \eta_{i,1}(1-z_1)} e^{-\lambda_i t \eta_{i,2}(1-z_2)} m_i(1 - \beta_{i,1} - \beta_{i,2} + \beta_{i,1} z_1 + \beta_{i,2} z_2, 1 - \gamma_{i,2} + \gamma_{i,2} z_2) dA_{i,j}(t).$$

All we need to do now is compute the η's, β's and γ's. When computing each $\eta_{i,n}$, we can apply the Campbell-Mecke formula (see [2]) to quickly deduce that for each n,

$$
\begin{aligned}
\eta_{i,n} &= \int_0^t P(N_{\mu_i \nu}(t - s) = n - 1) \frac{1}{t} ds \\
&= \frac{1}{\mu_i \nu t} \left[1 - \sum_{k=0}^{n-1} \frac{(\mu_i \nu t)^k e^{-\mu_i \nu t}}{k!} \right].
\end{aligned}
$$

Here N_μ is used to represent a homogeneous Poisson process with rate μ. Furthermore,

$$\beta_{i,n} = P(N(t) = n - 1) = \frac{(\mu_i \nu t)^{n-1} e^{-\mu_i \nu t}}{(n-1)!}$$

and

$$\gamma_{i,n} = P(N(t) = n - 2) = \frac{(\mu_i \nu t)^{n-2} e^{-\mu_i \nu t}}{(n-2)!}.$$

This concludes the proof. $\diamond$

Corollary 5.1 *The first moments $z_{i,j} = E[Q_i(0)\mathbf{1}(C(0) = j)]$ can be found by solving the following systems of equations: for $j \in \mathbb{E}$,*

$$z_{1,j} = \sum_{i \in \mathbb{E}} \left[\pi_i \frac{\lambda_i}{\mu_i \nu} (r_{i,j} - \alpha_{i,j}(\mu_i \nu)) + \alpha_{i,j}(\mu_i \nu) z_{1,i} \right]$$

and

$$z_{2,j} = \sum_{i \in \mathbb{E}} \left[\pi_i \frac{\lambda_i}{\mu_i \nu} (r_{i,j} - \alpha_{i,j}(\mu_i \nu) + \mu_i \nu \alpha'_{i,j}(\mu_i \nu)) - \mu_i \nu \alpha'_{i,j}(\mu_i \nu) z_{1,i} + \alpha_{i,j}(\mu_i \nu) z_{2,i} \right].$$

Proof After taking derivatives, setting $z = 1$ and plugging in the correct expressions for η_n, β_n and γ_n, we conclude that

$$E_0[Q_1(0)\mathbf{1}(C(0) = j)] = \sum_{i \in \mathbb{E}} \left[\pi_i \frac{\lambda_i}{\mu_i \nu} (r_{i,j} - \alpha_{i,j}(\mu_i \nu)) + \alpha_{i,j}(\mu_i \nu) E_0[Q_1(0)\mathbf{1}(C(0) = i)] \right]$$

and

$$
\begin{aligned}
E_0[Q_2(0)\mathbf{1}(C(0) = j)] = &\sum_{i \in \mathbb{E}} \left[\pi_i \frac{\lambda_i}{\mu_i \nu} (r_{i,j} - \alpha_{i,j}(\mu_i \nu) + \mu_i \nu \alpha'_{i,j}(\mu_i \nu)) - \mu_i \nu \alpha'_{i,j}(\mu_i \nu) E_0[Q_1(0)\mathbf{1}(C(0) = i)] \right] \\
&+ \sum_{i \in \mathbb{E}} \alpha_{i,j}(\mu_i \nu) E_0[Q_2(0)\mathbf{1}(C(0) = i)].
\end{aligned}
$$

◇

The moments at an arbitrary time can also be computed through the use of exactly the same methods as was used before.

Theorem 5.2 *For each $j \in \mathbb{E}$, we see that*

$$E[z_1^{Q_1(0)} z_2^{Q_2(0)} \mathbf{1}(C(0) = j)] = \quad \frac{1}{\upsilon} \int_0^\infty e^{-\lambda_j t \eta_{j,1}(1-z_1)} e^{-\lambda_j t \eta_{j,2}(1-z_2)}$$

$$\times \quad m_j(1 - \beta_{j,1} - \beta_{j,2} + \beta_{j,1} z_1 + \beta_{j,2} z_2, 1 - \gamma_{j,2} + \gamma_{j,2} z_2) \overline{A}_j(t) dt.$$

Proof In order to convert from the embedded steady-state distribution to the steady-state distribution at an arbitrary time, we'll again use the Inversion formula:

$$E[z_1^{Q_1(0)} z_2^{Q_2(0)} \mathbf{1}(C(0) = j)] = \quad \frac{1}{\upsilon} E_0 \left[\int_0^\infty z_1^{Q_1(t)} z_2^{Q_2(t)} \mathbf{1}(T > t, C(t) = j) dt \right]$$

$$= \quad \frac{1}{\upsilon} \int_0^\infty e^{-\lambda_j t \eta_{j,1}(1-z_1)} e^{-\lambda_j t \eta_{j,2}(1-z_2)}$$

$$\times \quad E_0((1 - \beta_{j,1} - \beta_{j,2} + \beta_{j,1} z_1 + \beta_{j,2} z_2)^{Q_1(0)} (1 - \gamma_{j,2} + \gamma_{j,2} z_2)^{Q_2(0)} \mathbf{1}(C(0) = j))$$

$$\times \quad \overline{A}_j(t) dt$$

$$= \quad \frac{1}{\upsilon} \int_0^\infty e^{-\lambda_j t \eta_{j,1}(1-z_1)} e^{-\lambda_j t \eta_{j,2}(1-z_2)}$$

$$\times \quad m_j(1 - \beta_{j,1} - \beta_{j,2} + \beta_{j,1} z_1 + \beta_{j,2} z_2, 1 - \gamma_{j,2} + \gamma_{j,2} z_2) \overline{A}_j(t) dt.$$

◇

Again, computing any steady-state moments of interest will involve plugging in the appropriate values for the $\eta_{j,k}$, $\beta_{j,k}$ and $\gamma_{j,k}$ probabilities, and then differentiating. Thus,

Corollary 5.2 *The mean number of customers in the system during steady-state is just*

$$E[Q(0)] = \sum_{j \in \mathbb{E}} E[Q_1(0) \mathbf{1}(C(0) = j)] + E[Q_2(0) \mathbf{1}(C(0) = j)]$$

where

$$E[Q(0) \mathbf{1}(C(0) = j)] = \frac{1}{\upsilon} \left[\frac{\lambda_j \pi_j}{\mu_j \nu} \left[\upsilon_i - \frac{1 - \alpha_j(\mu_j \nu)}{\mu_j \nu} \right] + \frac{1 - \alpha_j(\mu_j \nu)}{\mu_j \nu} z_{1,j} \right]$$

and

$$E[Q_2(0) \mathbf{1}(C(0) = j)] = \quad \frac{1}{\upsilon} \left[\frac{\lambda_j \pi_j}{\mu_j \nu} \left[\upsilon_j - \frac{1 - \alpha_j(\mu_j \nu)}{\mu_j \nu} - \frac{1 - \alpha_j(\mu_j \nu)}{\mu_j \nu} + \alpha_j'(\mu_j \nu) \right] \right]$$

$$+ \frac{1}{\upsilon} \left[\left(\frac{1 - \alpha_j(\mu_j \nu)}{\mu_j \nu} - \alpha_j'(\mu_j \nu) \right) z_{1,j} \right]$$

$$+ \frac{1}{\upsilon} \frac{1 - \alpha_j(\mu_j \nu)}{\mu_j \nu} z_{2,j}.$$

Proof After taking derivatives and letting $z = 1$, we get

$$E[Q_1(0)\mathbf{1}(C(0) = j)] = \frac{1}{v}\left[\frac{\lambda_j \pi_j}{\mu_j \nu}\left[v_i - \frac{1 - \alpha_j(\mu_j\nu)}{\mu_j \nu}\right] + \frac{1 - \alpha_j(\mu_j\nu)}{\mu_j \nu}E_0[Q_1(0)\mathbf{1}(C(0) = j)]\right]$$

and

$$
\begin{aligned}
E[Q_2(0)\mathbf{1}(C(0) = j)] &= \frac{1}{v}\left[\frac{\lambda_j \pi_j}{\mu_j \nu}\left[v_j - \frac{1 - \alpha_j(\mu_j\nu)}{\mu_j \nu} - \frac{1 - \alpha_j(\mu_j\nu)}{\mu_j \nu} + \alpha_j'(\mu_j\nu)\right]\right] \\
&+ \frac{1}{v}\left[\left(\frac{1 - \alpha_j(\mu_j\nu)}{\mu_j \nu} - \alpha_j'(\mu_j\nu)\right)E_0[Q_1(0)\mathbf{1}(C(0) = j)]\right] \\
&+ \frac{1}{v}\frac{1 - \alpha_j(\mu_j\nu)}{\mu_j \nu}E_0[Q_2(0)\mathbf{1}(C(0) = j)].
\end{aligned}
$$

This concludes the proof. $\diamondsuit$

6 General services

At this point it is also clear that expressions for the steady-state moments of the queue-length can be computed in the case when $B(t)$ is a mixture of Erlang distributions, and from a result in Asmussen [1] we also know that these distributions are dense in the space of distributions with nonnegative support. Our goal now is to prove a continuity theorem, which will allow us to approximate all steady-state moments of the queue-length by approximating arbitrary services with Erlang mixtures. In this section it will be necessary to assume that $\mu_i > 0$, for $i \in \mathbb{E}$; we assumed it as well in previous sections, but there the assumption wasn't strictly needed.

Theorem 6.1 *Let $\{Q_m(t); t \in \mathbb{R}\}$ denote a stationary version of the queue-length process described above, where each customer brings an amount of work B_n^m, $n \in \mathcal{Z}$. If, for each n, B_n^m converges weakly to B_n as $m \to \infty$, then for each integer k,*

$$\lim_{m\to\infty} E[Q_m(0)^k] = E[Q(0)^k].$$

Proof To prove this result, we will first show that, as $m \to \infty$, $E[Q_m(0)|C] \to E[Q(0)|C]$, where $C = \{C(t); t \in \mathbb{R}\}$ represents the entire path of our semi-Markovian environment. Proving this will immediately imply that $E[Q_m(0)^k|C] \to E[Q(0)^k|C]$ for all $k \geq 2$ as well, since conditional on C, the random variables $Q_m(0)$ and $Q(0)$ are all Poisson distributed (from Theorem 2.1), and the kth factorial moment of a Poisson random variable is just the mean raised to the kth power.

Unfortunately, conditioning on the sample-path of C no longer allows us to use any stationary properties of our system, so we will have to introduce a small amount of notation. Let $\{W(s); s \in \mathbb{R}\}$ denote a stochastic process, where $W(s)$ denotes the amount of work brought to the system by the last customer to arrive at or before time s. Then, from a transient version of Little's law (see Fralix et al. [8]), we see that

$$E[Q_m(0)|C] = \int_{-\infty}^0 \mathcal{P}_s\left(W_m(s) > \int_s^0 \mu_{C(x)}dx\right)\lambda_{C(s)}ds.$$

Here $\mathcal{P} : \mathbb{R} \times \mathcal{F} \to [0, 1]$ is the Palm probability kernel induced by the non-homogeneous Poisson arrival process (we're now conditioning on C). We will not go into great detail to explain how these probabilities are derived: rather, we will simply state that for a given event $A \in \mathcal{F}$, $\mathcal{P}_s(A)$ can be interpreted as the probability of A, given that there is an arrival at time s. Details behind the construction of these measures can be found in, for instance, Chapter 10 of [10].

Clearly, for each s, $\mathcal{P}_s(W_m(s) > \int_s^0 \mu_{C(x)}dx)$ represents the probability that the person arriving at time s is still in the system at time zero. It is a simple exercise to verify that the distribution of $W(s)$ under $\mathcal{P}_s$ is indeed the distribution of an arbitrary service time, so under $\mathcal{P}_s$, $W_m(s) \Rightarrow W(s)$ as $m \to \infty$, where $\Rightarrow$ is used to denote weak convergence. Moreover, since $\mathcal{P}_s(W_m(s) > \int_s^0 \mu_{C(x)}dx)\lambda_{C(s)} \le \mathcal{P}_s(W_m(s) > -\mu s)\lambda$, the dominated convergence theorem allows us to conclude that the conditional first moments converge.

To show that the unconditional moments converge, notice that for each k, $E[Q_m(0)^k|C] \le E[Q_{m,\lambda^*,\mu^*}(0)^k]$, which represents the kth moment of an $M/G/\infty$ queue with arrival rate $\lambda^* = \sup_{i\in\mathbb{E}}\lambda_i$ and service speed $\mu^* = \inf_{i\in\mathbb{E}}\mu_i$, that observes the same work sequence as the mth queue. Furthermore, from a slight extension of the dominated convergence theorem (see Kallenberg [9]) we can conclude that all moments converge as well. $\diamondsuit$

7 Conclusions

We have now shown that an embedded process approach can be used to both provide yet another method of computing the moments and the generating function of the steady-state queue length of a semi-Markov-modulated $M/M/\infty$ queue. Moreover, the approach can also be used when mixtures of Erlang services are considered, and this immediately gives an approximation of all moments of the steady-state queue-length in such a system that has generally distributed service times.

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