December 5, 2008

FIXED CYCLE SINGLE-ITEM PRODUCTION SYSTEMS

J. BRUIN* AND J. VAN DER WAL◇

*EURANDOM
◇Eindhoven University of Technology

Abstract. This paper considers a cyclic production system. The cycle consists of a production period with a fixed number of production times that can be used for production or idling, followed by a vacation. The duration of the vacation is independent of the production period. Demand arrives according to a (compound) Poisson process and is satisfied from stock or backlogged. The embedded process is modeled in discrete time and analyzed using generating functions. The optimal base stock level is derived from a news vendor type relation. The model is also extended to one with time slot dependent base stock levels.

1. Introduction

We consider a cyclic production system. In the cycle one distinguishes two parts: a production period and a vacation period. The production period consists of a fixed number, g, of production times. Each production time can be used for production or for idling. So the cycle consists of $g+1$ (possibly random) intervals, that are assumed to be independent. Demand arrives according to a (compound) Poisson process and is satisfied from stock or backlogged. A production interval is used for production if the stock level, just before the beginning of the interval, is below a certain base stock level. The goal is to find the base stock level that minimizes the average costs (a combination of holding and backlogging costs). By embedding the system on the time points the intervals start we obtain a discrete time periodic Markov chain. The state of the Markov chain is the number of items on stock or the number backlogged and the position within the cycle.

Erkip et al. [5] study this model in a multi-item production setting and present three algorithms to find the optimal lengths of the production periods of the different items. Furthermore, the optimal decision level is derived for a single period model and a matrix analytic method is used to find the optimal decision level for the infinite horizon model.

A more general queueing model with vacations is studied in Fuhrmann and Cooper [6]. They give a decomposition result for the distribution of the number of customers present in a queueing model with vacations which holds under certain conditions. These conditions include: Customers arrive according to a Poisson process, the customers are served in an order that is independent of their service times and service is nonpreemptive. The decomposition consists of the number of customers present in a standard $M|G|1$ queue and

Key words and phrases. queue, periodic, production, vacation, base stock level
AMS 2000 Subject Classification. Analysis of Manufacturing Systems.

the number of customers who arrive during a residual vacation. Unfortunately, our model does not satisfy all necessary conditions, because if the base stock level is reached during a production period, the system idles during one production time. This idling time is also called a vacation and therefore, the number of customers that arrive during a vacation is not independent of the number of customers present in the system when the vacation began. However, the limiting distribution of the stock out (which can be seen as the queue length distribution of items that need to be produced) can be found in a more direct way, without using the decomposition result from [6].

In the present study, a news vendor type expression for the optimal decision level is derived. This expression contains the limiting probabilities of the number of items short to the base stock level. A generating function approach is used to determine this limiting distribution. For heavily loaded systems, an approximation of the optimal decision level is obtained and numerical results are given to show how accurate this approximation is. The model is then slightly adjusted allowing the base stock levels to be slot dependent.

This paper is structured as follows. In Section 2 we explain the periodic production model in more detail. We derive an expression for the generating function of the number of products short compared to the base stock level in Section 3 and obtain the corresponding limiting distribution via the balance equations in Section 4. Next the optimal base stock level is derived from a news vendor type equation. For the limiting distribution, a tail approximation is given to cope with numerical problems and an approximation for the optimal decision level is derived from this tail approximation. Some numerical results are presented to give an idea about the quality of the approximation. In Section 8 we consider the case of time slot dependent base stock levels. Again a news vendor type equation is given and the equilibrium distribution of the number of items on stock or backlogged is found in Section 9. Numerical results on this extension are given in Section 11. Section 12 presents some conclusions and suggestions for further research.

2. Cyclic production

So, we consider a cyclic production system where each cycle starts with g production times and is concluded by a vacation. Production times and vacation are possibly random but independent. Demand arrives according to a (compound) Poisson process. The system is embedded on the instances corresponding to the start of a production time or the start of a vacation. The time intervals in this chain will be called *slots*, where we distinguish between production slots and vacation slots. Each cycle consists of $g+1$ slots. The decision to use a production slot for production or for idling is read from a base stock level S. If at the start of a production slot the inventory is less than S, then the slot is used to produce exactly one item. Given the assumptions about the demand process and the production rule the embedded system becomes a periodic (cyclic) embedded Markov chain.

The state of this chain is described as the number of products short to the level S at the beginning of a slot and the slot number within the cycle. Formulated in this way, the (limiting) behaviour of the Markov chain is independent of the value of S. Linear cost functions are considered for the number of items on stock and the backlog. Hence, as we will show, if the distribution of the number of products short to the base stock level S is known, an expression for the optimal value of S^* can be derived from a news vendor type equation.

The generating function for the demand in a production slot is denoted by $\mathcal{A}_P(z) = \sum_{k=0}^{\infty} a_P(k)z^k$, with $a_P(k)$ the probability that the demand in a production slot is equal to k. The average length of a production slot is denoted by T_P. Similarly, we define $\mathcal{A}_V(z)$, $a_V(k)$ and T_V for the vacation slot. Further, λ denotes the mean demand per time unit. We define X as the number of products short to the level S and look at the value of X at slot boundaries. In order to find the limiting distribution of X, a generating function approach will be used.

3. Generating function

Define $X_{n,t}$ as the value of X at slot boundary n in cycle t for $n = 1, \ldots, g+1$. Let us look at the limiting random variable

$$X_n = \lim_{t\to\infty} X_{n,t}, \quad n = 1, \ldots, g+1.$$

This limiting distribution is denoted by

$$p(k,n) = \mathbb{P}(X_n = k), \quad k \geq 0,\ n = 1, \ldots, g+1$$

and the generating function of X_n is defined as

$$\mathcal{G}_n(z) = \sum_{k=0}^{\infty} p(k,n)z^k, \quad n = 1, \ldots, g+1.$$

The distribution of X_n exists if the Markov chain $\{X_{n,t}, t = 1, 2, \ldots\}$ is aperiodic and irreducible (which is immediate from the (compound) Poisson demand assumption) and provided the system is stable, i.e. if $\lambda\,(gT_P + T_V) < g$. We assume this is the case.

We have

$$\begin{aligned} X_1 &= X_{g+1} + A_{g+1}, \\ X_n &= X_{n-1} + A_{n-1} - I_{\{X_{n-1}>0\}}, \quad n = 2, \ldots, g+1, \end{aligned}$$

where A_n denotes the demand that occurs in time slot n. From these equations we get

$$\begin{aligned} \mathcal{G}_1(z) &= \mathcal{G}_{g+1}(z)\mathcal{A}_V(z), \\ \mathcal{G}_n(z) &= \frac{1}{z}\mathcal{A}_P(z)\left[\mathcal{G}_{n-1}(z) + p(0,n-1)(z-1)\right], \quad n = 2, \ldots, g+1. \end{aligned}$$

As one easily verifies, this leads by iteration to

$$\mathcal{G}_1(z) = \frac{\sum_{m=1}^{g} \mathcal{A}_P^{g+1-m}(z)\mathcal{A}_V(z)(z^m - z^{m-1})p(0,m)}{z^g - \mathcal{A}_P^g(z)\mathcal{A}_V(z)}, \tag{3.1}$$

$$\mathcal{G}_n(z) = \left(\frac{\mathcal{A}_P(z)}{z}\right)^{n-m}\mathcal{G}_1(z) + \sum_{m=1}^{n-1} p(0,m)(z-1)\left(\frac{\mathcal{A}_P(z)}{z}\right)^{n-m}, \quad n = 2, \ldots, g+1. \tag{3.2}$$

The generating function of X_1 is of indeterminate form, but the g boundary probabilities $p(0,n),\ n = 0, \ldots, g$, can be determined by considering the zeros of the denominator in

(3.1) that lie on or within the unit circle. The following Rouché type lemma is taken from [1] and is specialized to our case.

Lemma 1. *If* $\rho := \frac{\lambda(gT_P+T_V)}{g} < 1$ *and* $\mathcal{A}_V(0)\mathcal{A}_P^g(0) \neq 0$, *then* $z^g = \mathcal{A}_P^g(z)\mathcal{A}_V(z)$ *has* g *roots on or within the unit circle.*

Denote the g roots of $z^g = \mathcal{A}^{g+1}(z)$ in $|z| \leq 1$ by $z_0 = 1, z_1, \ldots, z_{g-1}$. Since the function $\mathcal{G}_1(z)$ is finite on and inside the unit circle, the numerator of the right-hand side of (3.1) needs to be zero for each of the g roots, i.e., the numerator should vanish at the exact points where the denominator of the right-hand side of (3.1) vanishes. Lemma 1 and (3.1) together lead to g equations in terms of the g boundary probabilities, from which the latter can be determined. The roots can be determined using methods from [7]. It is assumed that these g roots are all different. If there is multiplicity of the roots greater than two, the derivatives (up to the number of multiplicity) of the numerator of (3.1) can be set to zero to obtain sufficiently many equations.

In principle, the probabilities $p(k, n)$ can be found by numerically inverting $\mathcal{G}_n(z)$. However, in this case, we can also derive the probabilities from the g boundary probabilities directly.

4. The limiting distribution

In order to find all limiting probabilities $p(k, n)$ from the probabilities $p(0, n),\ n = 1, \ldots, g$ (obtained via Lemma 1), we use the balance equations:

$$p(k,1) = \sum_{j=0}^{k} p(j, g+1)a_v(k-j), \tag{4.1}$$

$$p(k,n) = \sum_{j=1}^{k+1} p(j, n-1)a_p(k+1-j) + p(0, n-1)a_p(k),\ n = 2, \ldots, g+1, \tag{4.2}$$

Using (4.1) with $k = 0$ the probability $p(0, g + 1)$ can be obtained from:

$$p(0, g+1) = \frac{1}{a_v(0)} p(0, 1).$$

Next let us rewrite equation (4.2) for $n = 2, \ldots, g + 1$ as

$$p(k+1, n-1) = \frac{1}{a_p(0)} \left(p(k,n) - \sum_{j=1}^{k} p(j, n-1)a_p(k+1-j) - p(0, n-1)a_p(k) \right). \tag{4.3}$$

Then, starting with $k = 0$ we first find the probabilities $p(1, n),\ n = 2, \ldots, g + 1$. The probability $p(1, 1)$ can then be obtained from equation (4.1). Continuing in this way, one recursively gets the probabilities $p(k, n),\ k \geq 2$.

5. The optimal base-stock level

Now consider the following linear cost function, with weights based on the average slot duration:

$$C(S) = \sum_{n=1}^{g} \frac{T_P}{gT_P + T_V} \left(c_I \mathbb{E}(I_n) + c_B \mathbb{E}(B_n)\right)) + \frac{T_V}{gT_P + TV} \left(c_I \mathbb{E}(I_{g+1}) + c_B \mathbb{E}(B_{g+1})\right), \quad (5.1)$$

where I_n is the number of items on stock at slot boundary n and B_n the backlog at slot boundary n. Because the cost function is a weighted sum of costs at different time slots, we also look at the following weighted limiting distribution:

$$p(k) = \sum_{n=1}^{g} \frac{T_P}{gT_P + T_V} p(k, n) + \frac{T_V}{gT_P + T_V} p(k, g+1) \ .$$

The optimal base-stock level S^* for this 'news vendor problem' is now readily obtained as:

$$S^* = \min \left\{ S \,\middle|\, \sum_{k=0}^{S} p(k) > \frac{c_B}{c_I + c_B} \right\}. \quad (5.2)$$

6. A geometric tail approximation

The probabilities $p(k)$ in (5.2) can be found using the recursive method from Section 4. However, we have experienced numerical problems with this procedure for large values of k. For these cases, we suggest to use the following approximative method to determine S^*.

In [4] Van Eenige uses an approximation from Tijms and van de Coevering [9] for the tail probabilities that is based on the following asymptotic behavior

$$\lim_{k \to \infty} \frac{p(k)}{p(k+1)} = \gamma,$$

with γ the unique root of $z^g - \mathcal{A}_P^g(z)\mathcal{A}_V(z)$ in $(1, \infty)$.

A similar approximation method is employed in Van Mieghem [8], who gives a dominant pole approximation for $\mathbb{P}(X > k)$:

$$\mathbb{P}(X > k) \approx \frac{c}{\gamma - 1} \gamma^{-k}, \quad (6.1)$$

with c a normalization constant. The root γ can be determined by bisection.

Upon substituting (6.1) into (5.2), one gets the following approximative value for S^*:

$$S^* \approx \left\lceil \frac{-\ln(c_I) + \ln(c_I + c_B) + \ln(c) - \ln(\gamma - 1)}{\ln(\gamma)} \right\rceil.$$

If the production times and the vacation period are both deterministic and demand is Poisson distributed, then the total demand in a cycle is Poisson distributed with mean $\lambda\,(gT_P + T_V)$ and it can be seen from a Taylor expansion that, $\gamma \approx 1 + 2(1 - \rho)$ as $\rho \uparrow 1$. Then, for high values of ρ, we have $\ln(\gamma) \approx \ln(1 + 2(1 - \rho)) \approx 2(1 - \rho)$. Furthermore, if the approximation in (6.1) is close enough to the exact value for small values of k, one easily sees that c should be close to $\rho(\gamma - 1)$ (choose $k = 0$). Substituting these results into the equation above, we obtain the approximation $\tilde{S}$ defined as

$$\tilde{S} = \left\lceil \frac{-\ln(c_I) + \ln(c_I + c_B) + \ln(\rho)}{2(1 - \rho)} \right\rceil. \quad (6.2)$$

In order to see whether (6.2) results in a good approximation, we give some numerical results comparing the approximation with the exact method.

7. Numerical results

For various parameters settings, for which we can determine the exact value of S^* numerically, the results for S^* and $\tilde{S}$ are presented in Table 1. These results are based on a fixed cycle in which all time slots are deterministic and the demand process is Poisson. The values of $\tilde{S}$ from (6.2) are given without taking the 'ceiling' to show the real difference with the value of S^*.

$c_I = 1, c_B = 10, g = 5, T_P = 1, T_V = 5$

ρ	$\mathbb{E}I$	$\mathbb{E}B$	S^*	costs	$\tilde{S}$	costs
0.50	1.26	0.12	2	2.50	1.70	2.50
0.60	1.92	0.11	3	2.98	2.36	2.98
0.70	2.48	0.14	4	3.89	3.40	3.89
0.80	3.65	0.21	6	5.78	5.44	5.78
0.90	7.32	0.44	12	11.68	11.46	11.68
0.95	14.74	0.89	24	23.62	23.47	23.62

$c_I = 1, c_B = 10, g = 10, T_P = 1, T_V = 10$

ρ	$\mathbb{E}I$	$\mathbb{E}B$	S^*	costs	$\tilde{S}$	costs
0.50	1.89	0.14	3	3.30	1.70	4.25
0.60	2.48	0.13	4	3.81	2.36	4.41
0.70	2.96	0.17	5	4.62	3.40	5.18
0.80	4.04	0.22	7	6.27	5.44	6.59
0.90	7.62	0.43	13	11.91	11.46	12.00
0.95	15.00	0.88	25	23.71	23.47	23.79

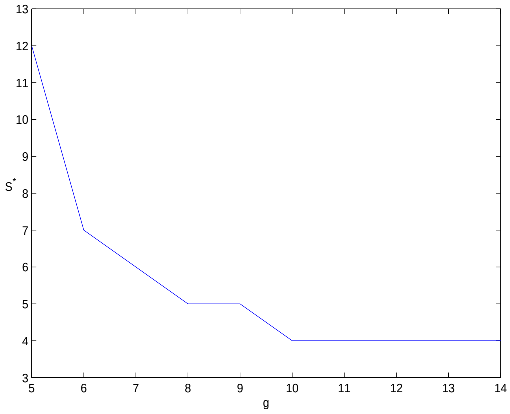
$c_I = 1, c_B = 10, g = 3, T_P = 1, T_V = 9$

ρ	$\mathbb{E}I$	$\mathbb{E}B$	S^*	costs	$\tilde{S}$	costs
0.50	1.31	0.10	2	2.29	1.70	2.29
0.60	2.00	0.09	3	2.85	2.36	2.85
0.70	2.58	0.12	4	3.79	3.40	3.79
0.80	3.78	0.19	6	5.71	5.44	5.71
0.90	7.46	0.42	12	11.66	11.46	11.66
0.95	14.89	0.87	24	23.57	23.47	23.57

Table 1. The values of $\mathbb{E}I$, $\mathbb{E}B$, S^* and $\tilde{S}$ for various model settings.

One sees that the approximation $\tilde{S}$ is correct for many parameter settings in Table 1. However, for higher values of g and T_V, the approximations gets worse. Fortunately, for higher values of ρ, the cost difference gets smaller. For these values of ρ, the difference between $\tilde{S}$ and S^* is mainly caused by the approximation of $\frac{c}{\gamma-1}$, which does not depend on the values of g and T_V anymore. The approximation of S^* is less accurate if S^* is low, because it is based on (6.1), which is only an approximation for the tail probabilities.

FIGURE 1



$T_P = 1, T_V = 4, \lambda = 0.5, c_I = 1, c_B = 10.$

Figure 1 gives the effect of g on the value of S^* and shows that S^* increases if g decreases. This is explained by the fact that decreasing g increases the effective utilization. Another point is that if g is somewhat longer, one needs the safety stock for the vacation period only at the end of the production period. This suggests that a cost reduction can be obtained with a base-stock level that is lower at the beginning of a production period and increases towards the end of the production period.

8. Time slot dependent base-stock levels

We now consider the system in which the base-stock level is time slot dependent. The different levels are denoted by $S_1, \ldots, S_g$, with S_n the base-stock level for time slot n, see Figure 2. Two assumptions are now stated and discussed below.

Assumption 1.

$$S_{n+1} \quad \leq \quad S_n + 1,\ n = 1, \ldots, g.$$

Assumption 2.

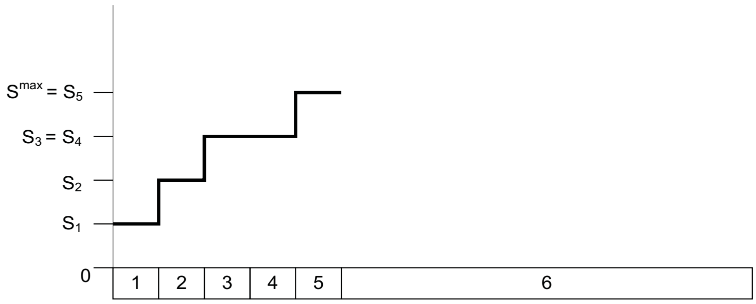
$$S_n \quad \leq \quad S_{n+1},\ n = 1, \ldots, g.$$

It seems reasonable to expect that the values of S_n are non-decreasing and that $S_n \leq S_{n-1} + 1$ for all n. However, it probably is possible to construct counterexamples for this based on the following intuition.

If the production times are highly variable and the non-production slots are not, then one might need a higher safety stock at the beginning of the production period than near the end of it.

If there is little variability in the slot durations, one wants to have the highest base stock level at the end of the production period. In the slot just before it, the safety stock is needed only because of the capacity limitation. Take the one but last slot. For a certain level, the inventory costs for the next slot might be just the reason not to produce. However, if (due to a somewhat lower than average demand in the previous inactive period) in the last production slot one already has a stock level that is one higher than the base stock level of the previous slot, one might still be willing to produce one more item.

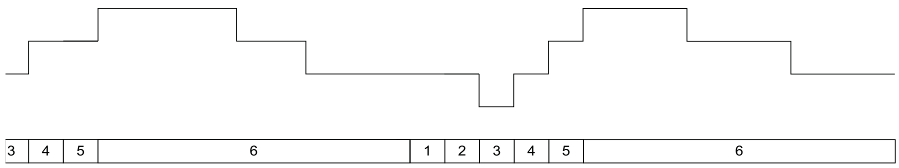
FIGURE 2



The base stock levels of 5 *production slots.*

In order to find the optimal values of $S_1, \ldots, S_g$, we slightly adapt the model description from Section 2. With different base-stock levels (and Poisson demand) in every slot the stock level can reach the maximum of $S_1, \ldots, S_g$, so in some slot(s) n the difference with the base-stock level S_n can be negative. This is shown in Figure 3, where the stock level just before the first production slot in the second cycle is higher than S_1. Therefore, we

FIGURE 3



The inventory level during two cycles.

define $S^{\max} = \max\{S_1, \ldots, S_g\}$ and $\tilde{X}_n$ as the number of products short compared to $S^{\max}$ at slot boundary n.

As before, if $\rho < 1$ the limiting distribution of $\tilde{X}_n$ exists and it is denoted by

$$\tilde{p}(k,n) = \lim_{t\to\infty} \mathbb{P}(\tilde{X}_{n,t} = k), \quad n = 1, \ldots, g+1, \quad k \geq 0,$$

with generating function

$$\tilde{\mathcal{G}}_n(z) = \sum_{k=0}^{\infty} \tilde{p}(k,n) z^k, \quad n = 1, \ldots, g+1.$$

Define

$$\delta(S_n) := S^{\max} - S_n, \quad n = 1, \ldots, g.$$

In the same way as in Section 3, we get

$$\tilde{\mathcal{G}}_1(z) = \frac{\sum_{m=1}^{g}\sum_{k=0}^{\delta(S_m)} \tilde{p}(k,m)(z^{k+m} - z^{k+m-1})\mathcal{A}_P^{g+1-m}(z)\mathcal{A}_V(z)}{z^g - \mathcal{A}_P^g(z)\mathcal{A}_V(z)}, \tag{8.1}$$

$$\tilde{\mathcal{G}}_n(z) = \tilde{\mathcal{G}}_1(z)\left(\frac{\mathcal{A}_P(z)}{z}\right)^{n-j} + \sum_{m=1}^{n-1}\sum_{k=0}^{\delta(S_m)} \tilde{p}(k,m)(z^k - z^{k-1})\left(\frac{\mathcal{A}_P(z)}{z}\right)^{n-m},$$
$$n = 2,\ldots,g+1,$$

The expressions for $\tilde{\mathcal{G}}_1(z),\ldots,\tilde{\mathcal{G}}_{g+1}(z)$, however, still contain the unknown boundary probabilities $\tilde{p}(k,n),\ k = 0,\ldots,\delta(S_n),\ n = 1,\ldots,g$. Lemma 1 gives g equations. Since there are more than g unknowns, we will have to construct a larger set of balance equations for these boundary probabilities. Because a similar problem is discussed in [3], the same approach is used to find the boundary probabilities in the paragraph below.

9. The boundary probabilities

The boundary probabilities we are looking for only concern probabilities from the production period. For ease of notation we combine the last production slot and the vacation period into one production slot.

With $a_g^*(k),\ k \geq 0$, denoting the distribution of the total demand in time slots g and $g+1$, the set of balance equations becomes:

$$\begin{aligned}
\tilde{p}(k,n) &= \sum_{m=\delta(S_{n-1})+1}^{k+1} \tilde{p}(m,n-1)a_p(k+1-m) &\quad (9.1)\\
&+ \sum_{m=0}^{\delta(S_{n-1})} \tilde{p}(m,n-1)a_p(k-m), \quad 2 \leq n \leq g, \quad k \geq \delta(S_{n-1}), &\\
\tilde{p}(k,n) &= \sum_{m=0}^{k} \tilde{p}(m,n-1)a_p(k-m), \quad 2 \leq n \leq g, \quad 0 \leq k < \delta(S_{n-1}), &\quad (9.2)\\
\tilde{p}(k,1) &= \sum_{m=\delta(S_g)+1}^{k+1} \tilde{p}(m,g)a_g^*(k+1-m) &\quad (9.3)\\
&+ \sum_{m=0}^{\delta(S_g)} \tilde{p}(m,g)a_g^*(k-m), \quad k \geq \delta(S_g), &\\
\tilde{p}(k,1) &= \sum_{m=0}^{k} \tilde{p}(m,g)a_g^*(k-m), \quad 0 \leq k < \delta(S_g). &\quad (9.4)
\end{aligned}$$

The algorithm below uses a set of unknown probabilities (variables), U, and a set of equations, E, from which the unknowns have to be obtained. Initially we define $U = \{\tilde{p}(k,n); k = 0,\ldots,\delta(S_n),\ n = 1,\ldots,g\}$ and we let E contain the g equations from Lemma 1 plus the equations described by (9.2) and (9.4). The balance equation with left-hand side

$\tilde{p}(k,n)$ will be labeled with (k,n). So U now contains the variables $\tilde{p}(k,n), k=0,\dots,\delta(S_n)$, and E the equations $(k,n),\ k=0,\dots,\delta(S_{n-1})-1$. Then the number of equations in E (including the ones from Lemma 1) and the number of unknowns in U are both equal to $\sum_n \delta(S_n)+g$.

Note that by construction for each equation in E the variables on the right-hand side belong to U. Under the assumption $S_n \le S_{n-1}+1$ for all n (see Assumption 1) the variable on the left-hand side will always be in U as well.

However, if the assumption $S_n \le S_{n-1}+1$ for all n does not hold, then for one or more of the equations in E the left-hand side probability is not in U. Then the sets U and E will (have to) be enlarged, slot by slot, in the following way.

Start with the slot just after the one with the lowest decision level, thus largest $\delta(S_n)$. Let n be the current production slot. For each $(k,n)\in E$ for which $\tilde{p}(k,n)$ is not yet in U the variable $\tilde{p}(k,n)$ is added to U. Next, for each of these variables an extra balance equation is added to E, namely equation $(k-1,n+1)$ (where $g+1$ is to be read as 1). All probabilities appearing at the right-hand side of this new equation are already in U and at most one extra unknown probability appears at the left-hand side. Then go to next slot, $n+1$. Again each variable that appears in E but is not in U is added to U, and in the same way as before, for a new variable $\tilde{p}(k,n)$ equation $(k-1,n+1)$ is added to E. Continue until all slots have been considered. In the last step (step g), the (a) slot with the highest value of $\delta(S_n)$ is reached. Therefore, the probabilities on the left-hand side of all equations added in the previous step are already in U, because k can not exceed $\max\{\delta(S_1),\dots,\delta(S_g)\}$.) This means that the construction ends with $|E|=|U|$ and the variables in U being the only ones appearing in E.

Assuming that all equations from the roots, l'Hôpital's rule, and this algorithm are linearly independent, the unknowns in U can be found.

10. Optimal maximum decision level

Denote the number of products short compared to $S^{\max}$ at weighted random slot boundaries by $\tilde{X}$. The generating function of $\tilde{X}$ is defined as

$$\tilde{\mathcal{G}}(z)=\sum_{k=0}^{\infty}\tilde{p}(k)z^k=\sum_{n=1}^{g}\frac{T_P}{gT_P+T_V}\mathcal{G}_n(z)+\frac{T_V}{gT_P+T_V}\mathcal{G}_{g+1}(z),$$

with

$$\tilde{p}(k)=\sum_{n=1}^{g}\frac{T_P}{gT_P+T_V}p(k,n)+\frac{T_V}{gT_P+T_V}p(k,g+1),k\quad\ge 0.$$

The limiting distributions of $\tilde{X},\tilde{X}_1,\dots,\tilde{X}_g$ can be found by inverting $\tilde{G}(z)$ and $\tilde{\mathcal{G}}_n(z),\ n=1,\dots,g$. The distribution of $\tilde{X}$ depends on $\delta(S_1),\dots,\delta(S_g)$, but not on $S^{\max}$. For a given vector $(\delta(S_1),\dots,\delta(S_g))$ the optimal value of $S^{\max}$ is given by

$$S^{\max *}=\min\left\{S^{\max}\,\middle|\,\sum_{k=0}^{S^{\max}}\tilde{p}(k)>\frac{c_B}{c_I+c_B}\right\}. \tag{10.1}$$

We emphasize that the distribution of $\tilde{X}$ depends on the whole vector $(\delta(S_1),\dots,\delta(S_g))$ and thus $S^{\max *}$ does as well. Furthermore, there is no expression for the optimal value of

every individual $S_n,\ n = 1, \ldots, g$. In the next Section we present a search for a presumably optimal vector $(\delta(S_1), \ldots, \delta(S_g))$ based on the two Assumptions 1 and 2.
In order to also find the expected costs $C(S_1, \ldots, S_g)$ for a given vector $\{S_1, \ldots, S_g\}$, we write

$$C(S_1, \ldots, S_g) = c_I \mathbb{E} I + c_B \mathbb{E} B = (c_I + c_B) \sum_{k=0}^{S^{\max}} \tilde{p}(k)(S^{\max} - k) + c_B(\mathbb{E}\tilde{X} - S^{\max}),$$

where the weights $\frac{T_P}{gT_P+T_V}$ and $\frac{G_v}{gT_P+T_V}$ are now hidden in $\tilde{p}(k)$ and $\mathbb{E}\tilde{X}$.

The finite sum $\sum_{k=0}^{S^{\max}} \tilde{p}(k)(S^{\max} - k)$ is obtained from the equilibrium probabilities. For $\mathbb{E}\tilde{X}$ we use $\mathbb{E}(\tilde{X}_1), \ldots, \mathbb{E}(\tilde{X}_{g+1})$. For the derivation of $\mathbb{E}(\tilde{X}_1)$, we refer to appendix A. The expressions for $\mathbb{E}(\tilde{X}_n),\ n = 2, \ldots, g+1$ are then obtained with

$$\tilde{X}_n \quad = \quad \tilde{X}_{n-1} + A_{n-1} - I_{\tilde{X}_{n-1} > \delta(S_{n-1})}, \qquad n = 2, \ldots, g+1,$$

with A_{n-1} the number of arrivals in time slot $n-1$ and $I_{\tilde{X}_{n-1} > \delta(S_{n-1})}$ the production indicator of time slot $n-1$. The result is:

$$\begin{aligned}
\mathbb{E}(\tilde{X}_1) \quad &= \quad \frac{1}{g - \lambda\,(gT_P + T_V)} \left(\sum_{m=1}^{g} \sum_{k=0}^{\delta(S_m)} \tilde{p}(k,m) \left[(gT_P + T_V)\,\lambda + k + m - 1 \right] \right. \\
&\quad - \left. \frac{1}{2} \left[g(g-1) - (gT_P + T_V)^2 \lambda^2 \right] \right), \\
\mathbb{E}(\tilde{X}_n) \quad &= \quad \mathbb{E}(\tilde{X}_1) + \lambda\,(gT_P + T_V) - \sum_{m=1}^{n-1} \sum_{k=0}^{\delta(S_m)} \tilde{p}(k,m), \qquad n = 2, \ldots, g+1.
\end{aligned}$$

11. Numerical Results

Next we present some numerical results comparing the expected costs per time unit for fixed cycles with one general decision level with the costs for fixed cycles with time slot dependent decision levels.

Recall that given the vector $(\delta(S_1), \ldots, \delta(S_g))$ the distribution of $\tilde{X}$ does not depend on $S^{\max}$ and that the corresponding optimal $S^{\max}$ is given by (10.1).

In order to limit the number of possible vectors with distances we use Assumptions 1 and 2 : $S_n \geq S_{n-1}, \quad n = 2, \ldots, g$ and $S_n \leq S_{n-1} + 1, \quad n = 2, \ldots, g$. Then the number of possible vectors with distances equals 2^{g-1}. In the numerical results below, the presented optimal values $\{S_1, \ldots, S_g\}$ are the optimal ones given these two restrictions.

Tables 2 and 3 show numerical results for $T_V = 5$, while in Tables 4 and 5, $T_V = 25$. Tables 2 and 4 show numerical results for $c_I = 1, c_B = 10$, while in Tables 3 and 5 $c_B = 20$. For almost every value of $\rho < 0.95$, Tables 4 and 5 show a larger cost reduction than Tables 2 and 3. Apparently, the length of the vacation period has a positive effect on the attainable cost reduction, while the value of ρ and the fraction $\frac{c_I}{c_I+c_B}$ have a negative effect on the attainable cost reduction. The first observation can be explained by the fact that the average arrivals per time slot decrease if T_V increases, so there is relatively more time to get a high stock level at the end of the production period. Therefore, the decision levels

TABLE 2

$c_I = 1, c_B = 10, g = 5, T_P = 1, T_V = 5$

ρ	S^*	costs	$[S_1, S_2, S_3, S_4, S_5]$	costs	cost reduction in %
0.75	5	4.642	[4 4 5 5 5]	4.608	0.716
0.8	6	5.782	[5 5 6 6 6]	5.762	0.334
0.85	8	7.731	[7 7 8 8 8]	7.716	0.195
0.9	12	11.682	[11 11 12 12 12]	11.672	0.089
0.95	24	23.630	[23 23 24 24 24]	23.625	0.023

TABLE 3

$c_I = 1, c_B = 20, g = 5, T_P = 1, T_V = 5$

ρ	S^*	costs	$[S_1, S_2, S_3, S_4, S_5]$	costs	cost reduction in %
0.75	6	5.807	[4 5 6 6 6]	5.699	1.869
0.8	8	7.262	[7 7 8 8 8]	7.239	0.324
0.85	10	9.794	[9 9 10 10 10]	9.769	0.248
0.9	15	14.749	[15 15 15 15 15]	14.749	0.000
0.95	30	30.000	[29 30 30 30 30]	29.963	0.110

TABLE 4

$c_I = 1, c_B = 10, g = 5, T_P = 1, T_V = 25$

ρ	S^*	costs	$[S_1, S_2, S_3, S_4, S_5]$	costs	cost reduction in %
0.75	5	4.984	[3 4 5 5 6]	4.933	1.024
0.8	7	6.039	[4 5 6 7 7]	6.006	0.551
0.85	9	7.931	[6 7 8 8 9]	7.902	0.365
0.9	12	11.816	[10 11 11 12 13]	11.798	0.150
0.95	24	23.692	[22 23 23 24 24]	23.688	0.015

TABLE 5

$c_I = 1, c_B = 20, g = 5, T_P = 1, T_V = 25$

ρ	S^*	costs	$[S_1, S_2, S_3, S_4, S_5]$	costs	cost reduction in %
0.75	7	6.118	[4 5 6 6 7]	5.976	2.329
0.8	8	7.448	[6 6 7 7 8]	7.425	0.309
0.85	11	9.961	[9 9 10 11 11]	9.894	0.670
0.9	16	14.946	[14 14 15 16 16]	14.856	0.604
0.95	31	30.029	[29 30 30 31 31]	29.987	0.140

at the beginning of the production period can be lower. The second observation is easy to explain: The values of ρ increases the load on the system, so the system should use its full capacity to get $S^{\max *}$ products on stock. The last observation is explained by the fact that if the backlogging costs are relatively high, one wants to prevent the system to create backlog. Therefore, the decision levels are all close to $S^{\max *}$.

12. Conclusion

In this paper, we analyzed a fixed cycle scheme in a single-item production setting. Besides the analysis for a fixed cycle with one base stock level for all decisions slots we also treat the situation with time slot dependent decision levels. An expression for the generating function for the stock out at random slot boundaries is found and the boundary probabilities that appear in the numerator of this expression are determined via the balance equations. The optimal values of S and $S^{\max}$ are found via news vendor type equations and in the case with a general decision level, an approximation based on the geometric tail behavior of the stock out distribution is given for S^*. Numerical results show that this approximation works well for high values of ρ. Time slot dependent decision levels seem to be useful for low values of ρ and $\frac{c_I}{c_I+c_B}$ and when the production period is long.

In a multi-item production setting, different fixed cycle schemes can be compared in order to find the optimal fixed cycle. If the lengths of the different production periods are determined, the resulting fixed cycle can be used as a basis for a one step improvement approach, see [2] for results on this approach in the multi-item production setting with lost sales.

Acknowledgement. We thank Johan van Leeuwaarden for the helpful discussions on the determination of the boundary probabilities in the expressions of $\mathcal{G}(z)$ and $\tilde{\mathcal{G}}(z)$.

Appendix A. Expectation in the first slot

We obtain the mean value of $\tilde{X}_1$ by taking the first derivative of $\tilde{\mathcal{G}}_1(z)$. We rewrite this generating function as $\frac{\mathcal{N}(z)}{\mathcal{D}(z)}$, with

$$\mathcal{N}(z) = \sum_{m=1}^{g} \sum_{k=0}^{\delta(S_m)} \tilde{p}(k,m)(z^{k+m} - z^{k+m-1})\mathcal{A}_P^{g+1-m}(z)\mathcal{A}_V(z)$$

and

$$\mathcal{D}(z) = z^g - \mathcal{A}_P^g(z)\mathcal{A}_V(z)$$

to keep the notation simple.

$\tilde{\mathcal{G}}_1'(1)$ can be rewritten as

$$\frac{\mathcal{N}'(z)\mathcal{D}(z) - \mathcal{D}'(z)\mathcal{N}(z)|_{z=1}}{\mathcal{D}^2(z)|_{z=1}} = \frac{\mathcal{N}'(z) - \mathcal{D}'(z)\tilde{\mathcal{G}}_1(z)|_{z=1}}{\mathcal{D}(z)|_{z=1}}.$$

Since $\tilde{\mathcal{G}}_1(1) = \frac{\mathcal{N}'(1)}{\mathcal{D}'(1)}$ by l'Hôpital and $\mathcal{D}(1) = 0$, we can use l'Hôpital again:

$$\tilde{\mathcal{G}}_1'(1) = \frac{\mathcal{N}''(z) - \mathcal{D}''(z)\tilde{\mathcal{G}}_1(z) - \mathcal{D}'(z)\tilde{\mathcal{G}}_1'(z)|_{z=1}}{\mathcal{D}'(z)|_{z=1}}$$

Using $\tilde{\mathcal{G}}_1(1) = 1$ and rearranging terms gives us:

$$\tilde{\mathcal{G}}_1'(1) = \frac{\mathcal{N}''(1) - \mathcal{D}''(1)}{2\mathcal{D}'(1)},$$

with

$$\begin{aligned}
\mathcal{D}'(1) &= g - (gT_P + T_V), \\
\mathcal{N}''(1) &= 2\sum_{m=1}^{g}\sum_{k=0}^{\delta(S_m)} \tilde{p}(k,m)\left[(gT_P + T_V)\lambda + k + m - 1\right], \\
\mathcal{D}''(1) &= \left[g(g-1) - (gT_P + T_V)^2\lambda^2\right].
\end{aligned}$$

References

[1] Adan, I.J.B.F., J.S.H. van Leeuwaarden and E.M.M. Winands (2006). On the application of Rouché's theorem in queueing theory. *Operations Research Letters* **34**: 355-360.

[2] Bruin, J. and J. van der Wal. A MDP decomposition approach on multi-item production systems *(in preparation)*.

[3] Denteneer, D., J.S.H. van Leeuwaarden and I.J.B.F. Adan (2007). The acquisition queue, *Queueing Systems* **56**: 229–240.

[4] van Eenige, M.J.A. (1996). Queueing Systems with Periodic Service, *Ph.D. thesis, Technical University Eindhoven, Eindhoven.*

[5] Erkip, N., R. Güllü and A. Kocabiyikoglu (2000). A quasi-birth-and-death model to evaluate fixed cycle time policies for stochastic multi-item production/inventory problem, *Proceedings of MSOM conference, Ann Harbor.*

[6] Fuhrmann, S.W. and R.B. Cooper (1985). Stochastic decomposition in an M/G/l queue with generalized vacations, *Operatoins Research* **33**: 11-17.

[7] Janssen, A.J.E.M. and J.S.H. van Leeuwaarden (2004). Analytic computation schemes for the discrete-time bulk service queue, *Queueing Systems* **50**: 141-163.

[8] van Mieghem, P. (1996). The asymptotic behavior of queueing systems, *Queueing Systems*, **23**: 27-55.

[9] Tijms, H.C. and M.C.T. van de Coevering (1991). A simple numerical approach for infinite-state Markov chains, *Probability in the Engineering and Informational Sciences* **5**: 85-295.

EURANDOM, 5612 AZ Eindhoven, The Netherlands
E-mail address: `bruin@eurandom.tue.nl`

Eindhoven Univerity of Technology, 5612 AZ Eindhoven, The Netherlands
E-mail address: `jan.v.d.wal@tue.nl`