Long-range self-avoiding walk converges to α-stable processes

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Abstract: We consider a long-range version of self-avoiding walk in dimension $d > 2(\alpha \wedge 2)$, where d denotes dimension and α the power-law decay exponent of the coupling function. Under appropriate scaling we prove convergence to Brownian motion for $\alpha \geq 2$, and to α-stable Lévy motion for $\alpha < 2$. This complements results by Slade (1988), who proves convergence to Brownian motion for nearest-neighbor self-avoiding walk in high dimension.

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1 Introduction and results

1.1 The model

We study self-avoiding walk on the hypercubic lattice $\mathbb{Z}^d$. We consider $\mathbb{Z}^d$ as a complete graph, i.e., the graph with vertex set $\mathbb{Z}^d$ and corresponding edge set $\mathbb{Z}^d \times \mathbb{Z}^d$. We assign each (undirected) bond $\{x, y\}$ a weight $D(x-y)$, where D is a probability distribution specified in Section 1.1 below. If $D(x-y) = 0$, then we can omit the bond $\{x, y\}$.

Two-point function. For every lattice site $x \in \mathbb{Z}^d$, we denote by

$$\mathcal{W}_n(x) = \{(w_0, \dots, w_n) \,|\, w_0 = 0,\, w_n = x,\, w_i \in \mathbb{Z}^d, 1 \leq i \leq n-1\} \tag{1.1}$$

the set of n-step walks from the origin 0 to x. We call such a walk $w \in \mathcal{W}_n(x)$ *self-avoiding* if $w_i \neq w_j$ for $i \neq j$ with $i, j \in \{0, \dots, n\}$. We define $c_0(x) = \delta_{0,x}$ and, for $n \geq 1$,

$$c_n(x) := \sum_{w \in \mathcal{W}_n(x)} \prod_{i=1}^{n} D(w_i - w_{i-1}) \, \mathbb{1}_{\{w \text{ is self-avoiding}\}}. \tag{1.2}$$

where D is specified below. We refer to D as the *step* distribution, having in mind a random walker taking steps that are distributed according to D. Without loss of generality we can assume here that $D(0) = 0$.

The self-avoiding walk measure is the measure $\mathbb{Q}_n$ on the set of n-step paths $\mathcal{W}_n = \bigcup_{x\in\mathbb{Z}^d} \mathcal{W}_n(x) = \{0\} \times \mathbb{Z}^{dn}$ defined by

$$\mathbb{Q}_n(w) := \frac{1}{c_n} \prod_{i=1}^{n} D(w_i - w_{i-1})\, \mathbb{1}_{\{w \text{ is self-avoiding}\}}, \tag{1.3}$$

where $c_n = \sum_{x\in\mathbb{Z}^d} c_n(x)$.

We consider the the Green's function $G_z(x)$, $x \in \mathbb{Z}^d$, defined by

$$G_z(x) = \sum_{n=0}^{\infty} c_n(x)\, z^n. \tag{1.4}$$

We further introduce the *susceptibility* as

$$\chi(z) := \sum_{x\in\mathbb{Z}^d} G_z(x) \tag{1.5}$$

and define z_c, the critical value of z, as the convergence radius of the power series (1.4), i.e.

$$z_c := \sup \{z \,|\, \chi(z) < \infty\}. \tag{1.6}$$

The main part of our analysis is based on Fourier space analysis. Unless specified otherwise, k will always denote an arbitrary element from the Fourier dual of the discrete lattice, which is the torus $[-\pi, \pi)^d$. The Fourier transform of a function $f\colon \mathbb{Z}^d \to \mathbb{C}$ is defined by $\hat{f}(k) = \sum_{x\in\mathbb{Z}^d} f(x)\, \mathrm{e}^{ik\cdot x}$.

The step distribution D. Let h be a non-negative bounded function on $\mathbb{R}^d$ which is almost everywhere continuous, and symmetric under the lattice symmetries of reflection in coordinate hyperplanes and rotations by ninety degrees. Assume that there is an integrable function H on $\mathbb{R}^d$ with $H(te)$ non-increasing in $t \geq 0$ for every unit vector $e \in \mathbb{R}^d$, such that $h(x) \leq H(x)$ for all $x \in \mathbb{R}^d$. Furthermore we require h to decay as $|x|^{-d-\alpha}$ as $|x| \to \infty$, where $\alpha > 0$ is a parameter of the model. In particular, there exists a positive constants c_h such that

$$h(x) \sim c_h |x|^{-d-\alpha} \qquad \text{whenever } |x| \to \infty, \tag{1.7}$$

where $\sim$ denotes asymptotic equivalence, i.e., $f(x) \sim g(x)$ if $f(x)/g(x) \to 1$. For $\alpha \leq 2$ we assume further that $h(x)$ can be extended to a function on $\mathbb{R}^d$ that is rotation invariant. The monotonicity and integrability hypothesis on H imply that $\sum_x h(x/L) < \infty$ for all L, with $x/L = (x_1/L, \ldots, x_d/L)$.

We then consider D of the form

$$D(x) = \frac{h(x/L)}{\sum_{y\in\mathbb{Z}^d} h(y/L)}, \qquad x \in \mathbb{Z}^d, \tag{1.8}$$

where L is a spread-out parameter (to be chosen large later on). We note that the κth moment $\sum_{x\in\mathbb{Z}^d} |x|^\kappa D(x)$ does not exist if $\kappa \geq \alpha$, but exists and equals $O(L^\kappa)$ if $\kappa < \alpha$.

Lemma 1.1 (Properties of D)**.** *The step distribution D satisfies the following properties:*

(i) there is a constant C such that, for all $L \geq 1$,

$$\|D\|_\infty \leq CL^{-d}; \tag{1.9}$$

(ii) there is a constants $c > 0$ such that

$$1 - \hat{D}(k) \;>\; c \qquad \textit{if } \|k\|_\infty \geq L^{-1}, \tag{1.10}$$

$$1 - \hat{D}(k) \;<\; 2 - c, \qquad k \in [-\pi, \pi)^d; \tag{1.11}$$

(iii) there is a constant $v_\alpha > 0$ such that, as $|k| \to 0$,

$$1 - \hat{D}(k) \sim \begin{cases} v_\alpha |k|^{\alpha\wedge 2} & \textit{if } \alpha \neq 2, \\ v_2 |k|^2 \log(1/|k|) & \textit{if } \alpha = 2. \end{cases} \tag{1.12}$$

Chen and Sakai [3, Prop. 1.1] show that D satisfies conditions (1.9)–(1.11). We prove in Appendix A that also (1.12) holds. It follows from [3, (1.7)] that $v_\alpha \leq O(L^{\alpha\wedge 2})$.

An example of h satisfying all of the above is

$$h(x) = (|x| \vee 1)^{-d-\alpha}, \tag{1.13}$$

in which case D has the form

$$D(x) = \frac{(|x/L| \vee 1)^{-d-\alpha}}{\sum_{y\in\mathbb{Z}^d} (|y/L| \vee 1)^{-d-\alpha}}, \qquad x \in \mathbb{Z}^d. \tag{1.14}$$

1.2 Weak convergence of the end-to-end displacement.

For $\alpha \in (0, \infty)$, we write

$$k_n := \begin{cases} k\,(v_\alpha n)^{-1/\alpha\wedge 2}, & \text{if } \alpha \neq 2 \\ k\,(v_2 n \log \sqrt{n})^{-1/2}, & \text{if } \alpha = 2 \end{cases} \tag{1.15}$$

so that

$$\lim_{n\to\infty} n\,[1 - \hat{D}(k_n)] = |k|^{\alpha\wedge 2}. \tag{1.16}$$

Theorem 1.2 (Weak convergence of end-to-end displacement)**.** *Assume that D is of the form (1.8), where the spread-out parameter L is sufficiently large. Then self-avoiding walk in dimension $d > d_c = 2(\alpha \wedge 2)$ satisfies*

$$\frac{\hat{c}_n(k_n)}{\hat{c}_n(0)} \to \exp\{-K_\alpha\, |k|^{\alpha\wedge 2}\} \qquad \textit{as } n \to \infty, \tag{1.17}$$

where

$$K_\alpha = \Big(1 + \sum_{x\in\mathbb{Z}^d} \sum_{n=2}^{\infty} n\, \pi_n(x)\, {z_c}^{n-1}\Big)^{-1} \begin{cases} 1, & \textit{if } \alpha \leq 2; \\ 1 + (2d\, v_\alpha)^{-1} \displaystyle\sum_{x\in\mathbb{Z}^d} \sum_{n=0}^{\infty} |x|^2\, \pi_n(x)\, {z_c}^{n}, & \textit{if } \alpha > 2. \end{cases} \tag{1.18}$$

The quantities $\pi_n(x)$ appearing in (1.18) are known as lace expansion coefficients. We do not perform the lace expansion in this paper. References to the derivation of the lace expansion and various bounds on these lace expansion coefficients are given later on. Under the conditions of Theorem 1.2, (2.21) and (2.58) below imply that K_α is a finite constant.

1.3 Mean-r displacement.

The *mean-r displacement* is defined as

$$\xi^{(r)}(n) := \left(\frac{\sum_{x\in\mathbb{Z}^d} |x|^r c_n(x)}{c_n} \right)^{1/r}, \tag{1.19}$$

where we recall $c_n = \sum_{x\in\mathbb{Z}^d} c_n(x) = \hat{c}_n(0)$. For $r = 2$ this is the mean-square displacement, and already well understood. For example, van der Hofstad and Slade [10] prove the following rather general version:

Theorem 1.3 (Mean-square displacement [10])**.** *Consider self-avoiding walk with step distribution D given in Section 1.1 with $\alpha > 2$. Then there is a constant $C > 0$ such that, as $n \to \infty$,*

$$\frac{1}{c_n} \sum_{x\in\mathbb{Z}^d} |x|^2 c_n(x) = C\, n\, (1 + o(1)). \tag{1.20}$$

The proof of Theorem 1.3 is also based on lace expansion. In the sequel we prove a complementary result for $r < 2$.

Theorem 1.4 (Mean-r displacement of order r)**.** *Under the assumptions of Theorem 1.2, for any $r < \alpha \wedge 2$,*

$$\xi^{(r)}(n) \asymp \begin{cases} n^{1/(\alpha\wedge 2)}, & \text{if } \alpha \neq 2, \\ (n \log n)^{1/2}, & \text{if } \alpha = 2, \end{cases} \tag{1.21}$$

as $n \to \infty$.

In view of (1.20) we conjecture that (1.21) actually holds for *all* positive values of r, even though our proof applies only to $r < \alpha \wedge 2$.

1.4 Convergence to Brownian motion and α-stable processes.

In order to deal with the cases $\alpha = 2$ and $\alpha \neq 2$ simultaneously, we write

$$f_\alpha(n) = \begin{cases} (v_\alpha n)^{-1/(\alpha\wedge 2)} & \text{if } \alpha \neq 2, \\ (v_2 n \log \sqrt{n})^{-1/2} & \text{if } \alpha = 2, \end{cases} \tag{1.22}$$

such that, for example, $k_n = f_\alpha(n)\, k$, cf. (1.15). Given an n-step self-avoiding walk w, define

$$X_n(t) = (2dK_\alpha)^{-\frac{1}{\alpha\wedge 2}}\ f_\alpha(n)\, w(\lfloor nt \rfloor), \qquad t \in [0,1]. \tag{1.23}$$

We aim to identify the scaling limit of X_n, and the appropriate space to study the limit is the space of $\mathbb{R}^d$-valued càdlàg-functions $D([0,1], \mathbb{R}^d)$ equipped with the Skorokhod topology.

For $\alpha \in (0, 2]$, $W^{(\alpha)}$ denotes the standard α-stable Lévy measure, normalized such that

$$\int e^{ik\cdot B^{(\alpha)}(t)}\ \mathrm{d}W^{(\alpha)} = e^{-|k|^\alpha t/(2d)}, \tag{1.24}$$

where $B^{(\alpha)}$ is a (càdlàg version of) standard symmetric α-stable Lévy motion (in the sense of [14, Definition 3.1.3]). Note that $W^{(2)}$ is the Wiener measure, and $B^{(2)}$ is Brownian motion. By $\langle \cdot \rangle_n$ we denote expectation with respect to the self-avoiding walk measure $\mathbb{Q}_n$ in (1.3).

Theorem 1.5 (Weak convergence to α-stable processes and Brownian motion)**.** *Under the assumptions in Theorem 1.2,*

$$\lim_{n\to\infty} \langle f(X_n)\rangle_n = \int f \, \mathrm{d}W^{(\alpha\wedge 2)}, \tag{1.25}$$

for every bounded continuous function $f\colon D([0,1],\mathbb{R}^d) \to \mathbb{R}$*. That is to say,* X_n *converges in distribution to an* α*-stable Lévy motion for* $\alpha < 2$*, and to Brownian motion for* $\alpha \geq 2$*. Equivalently,* $\mathbb{Q}_n$ *converges weakly to* $W^{(\alpha\wedge 2)}$*.*

In order to prove convergence in distribution, we need two properties: (i) the convergence of finite-dimensional distributions, and (ii) tightness of the family $\{X_n\}$. We shall now consider the former.

Convergence of finite-dimensional distributions means for every $N = 1, 2, 3, \dots$, any $0 < t_1 < \cdots < t_N \leq 1$, and any bounded continuous function $g\colon \mathbb{R}^{dN} \to \mathbb{R}$,

$$\lim_{n\to\infty} \big\langle g\big(X_n(t_1), \dots, X_n(t_N)\big)\big\rangle_n = \int g\big(B^{(\alpha\wedge 2)}(t_1), \dots, B^{(\alpha\wedge 2)}(t_N)\big) \, \mathrm{d}W^{(\alpha\wedge 2)}. \tag{1.26}$$

The distribution of a random variable is determined by its characteristic function, hence it suffices to consider functions g of the form

$$g(x_1, \dots, x_N) = \exp\{i\, \mathbf{k} \cdot (x_1, \dots, x_N)\}, \tag{1.27}$$

where $\mathbf{k} = \big(k^{(1)}, \dots, k^{(N)}\big) \in (-\pi, \pi]^{dN}$ and $x_i \in \mathbb{R}^d$, $i = 1, \dots, N$. We rather use the equivalent form

$$g(x_1, \dots, x_N) = \exp\{i\, \mathbf{k} \cdot (x_1, x_2 - x_1, \dots, x_N - x_{N-1})\}, \tag{1.28}$$

which better fits in our setting.

For $\mathbf{n} = (n^{(1)}, \dots, n^{(N)}) \in \mathbb{N}^N$, with $n^{(1)} < \cdots < n^{(N)}$, we define

$$\begin{aligned} \hat{\mathbf{c}}^{(N)}_{\mathbf{n}}(\mathbf{k}) := \sum_{x_1, x_2, \dots, x_{n^{(N)}}} & \exp\left\{ i \sum_{j=1}^{N} k^{(j)} \cdot \left(x_{n^{(j)}} - x_{n^{(j-1)}}\right) \right\} \\ & \times \prod_{i=1}^{n^{(N)}} D(x_i - x_{i-1}) \, \mathbb{1}_{\{(0, x_1, x_2, \dots, x_{n^{(N)}}) \text{ is self-avoiding}\}} \end{aligned} \tag{1.29}$$

as the N-dimensional version of (1.2), with $n^{(0)} = 0$. An alternative representation is

$$\hat{\mathbf{c}}^{(N)}_{\mathbf{n}}(\mathbf{k}) = \sum_{w \in \mathcal{W}_{n^{(N)}}} \mathrm{e}^{i\mathbf{k}\cdot\Delta w(\mathbf{n})} \, W(w) \, \mathbb{1}_{\{w \text{ is self-avoiding}\}}, \tag{1.30}$$

where $W(w) = \prod_{i=1}^{|w|} D(w_i - w_{i-1})$ is the *weight* of the walk w ($|w|$ denotes the length) and

$$\mathbf{k} \cdot \Delta w(\mathbf{n}) = \sum_{j=1}^{N} k^{(j)} \cdot \left(w_{n^{(j)}} - w_{n^{(j-1)}}\right).$$

We fix a sequence b_n converging to infinity slowly enough such that

$$f_\alpha(n)^{\alpha\wedge 1} \, b_n = o(1), \tag{1.31}$$

for example $b_n = \log n$.

Theorem 1.6 (Finite-dimensional distributions)**.** *Let N be a positive integer, $k^{(1)},\dots,k^{(N)}\in(-\pi,\pi]^d$, $0=t^{(0)}<t^{(1)}<\dots<t^{(N)}\in\mathbb{R}$, and $g=(g_n)$ a sequence of real numbers satisfying $0\le g_n\le b_n$. Denote*

$$\mathbf{k}_n=\left(k_n^{(1)},\dots,k_n^{(N)}\right)=f_\alpha(n)\left(k^{(1)},\dots,k^{(N)}\right),$$
$$n\mathbf{T}=\left(\lfloor nt^{(1)}\rfloor,\dots,\lfloor nt^{(N-1)}\rfloor,\lfloor nT\rfloor\right)$$

with $T=t^{(N)}(1-g_n)$. Under the conditions of Theorem 1.2,

$$\lim_{n\to\infty}\frac{\hat{\mathbf{c}}^{(N)}_{n\mathbf{T}}(\mathbf{k}_n)}{\hat{c}_{nT}(0)}=\exp\left\{-K_\alpha\sum_{j=1}^N|k^{(j)}|^{\alpha\wedge 2}\left(t^{(j)}-t^{(j-1)}\right)\right\}\tag{1.32}$$

holds uniformly in g.

Let us emphasize that (1.32) has indeed the required form. Let $g_n\equiv 0$ in Theorem 1.6, so that $n\mathbf{T}=\left(\lfloor nt^{(1)}\rfloor,\dots,\lfloor nt^{(N)}\rfloor\right)$. Then

$$\begin{aligned}\Big\langle\exp\left\{i\,\mathbf{k}\cdot\Delta X_n(n\mathbf{T})\right\}\Big\rangle_n &= \Big\langle\exp\left\{i\,(2dK_\alpha)^{-\frac{1}{\alpha\wedge 2}}\,\mathbf{k}_n\cdot\Delta\bullet(n\mathbf{T})\right\}\Big\rangle_n\\ &=\frac{\hat{\mathbf{c}}^{(N)}_{n\mathbf{T}}\left((2dK_\alpha)^{-\frac{1}{\alpha\wedge 2}}\,\mathbf{k}_n\right)}{\hat{c}_{nT}(0)},\end{aligned}$$

and this converges to

$$\exp\left\{-\frac{1}{2d}\sum_{j=1}^N|k^{(j)}|^{\alpha\wedge 2}\left(t^{(j)}-t^{(j-1)}\right)\right\}$$

as $n\to\infty$, as we aim to show for (1.26). Thus the finite dimensional distributions of (long-range) self-avoiding walk converge to those of an α-stable Lévy motion, which proves that this is the only possible scaling limit.

1.5 Discussion and related work

Long-range self-avoiding walk has rarely been studied. Klein and Yang [18] show that the endpoint of a weakly self-avoiding walk jumping m lattice sites *along the coordinate axes* with probability proportional to $1/m^2$, is Cauchy distributed. A similar result for strictly self-avoiding walk is obtained by Cheng [5].

In a previous paper [8] it is shown that long-range self-avoiding walk exhibits mean-field behavior above dimension $d_c=2(\alpha\wedge 2)$. More specifically, it is shown that under the conditions of Theorem 1.2, the Fourier transform of the critical two-point function satisfies $\hat{G}_{z_c}(k)=(1+O(\beta))/(1-\hat{D}(k))$, where $\beta=O(L^{-d})$ is an arbitrarily small quantity. Hence, on the level of Fourier transforms, the critical two-point functions of long-range self-avoiding walk and long-range *simple* random walk are very close. Indeed, the results in [8] suggest that the two models behave similar for $d>d_c$, and we prove this belief in a rather strong form by showing that both objects have the same scaling limit.

Chen and Sakai [4] prove an analogue of Theorem 1.2 for oriented percolation, and in fact our method of proving Theorem 1.2 is very much inspired by the method in [4]. The bounds on the diagrams are different for the two different models, but the general strategy works equally well with either model. In particular, the *spatial* fractional derivatives as in (2.30) are used for the first time in [4].

Slade [15, 16] proves convergence of the *nearest-neighbor* self-avoiding walk to Brownian motion in sufficiently high dimension, using a finite-memory cut-off. Hara and Slade [7] provide an alternative argument by using fractional derivative estimates. An account of the latter approach is contained in the monograph [13, Sect. 6.6]. All of these proofs use the lace expansion, which was introduced by Brydges and Spencer [2] to study weakly self-avoiding walk.

2 The scaling limit of the endpoint: Proof of Theorem 1.2

2.1 Overview of proof

The lace expansion obtains an expansion of the form

$$c_{n+1}(x) = (D * c_n)(x) + \sum_{m=2}^{n+1} (\pi_m * c_{n+1-m})(x) \tag{2.1}$$

for suitable coefficients $\pi_m(x)$, see e.g. [9, Sect. 2.2.1] or [17, Sect. 3] for a derivation of the lace expansion. We multiply (2.1) by z^{n+1} and sum over $n \geq 0$. By letting

$$\Pi_z(x) = \sum_{m=2}^{\infty} \pi_m(x) z^m \tag{2.2}$$

for $z \leq z_c$, and recalling $G_z(x) = \sum_{n=0}^{\infty} c_n(x) z^n$, this yields

$$G_z(x) = \delta_{0,x} + z(D * G_z)(x) + (G_z * \Pi_z)(x). \tag{2.3}$$

We proceed by proving Theorem 1.2 subject to certain bounds on the lace expansion coefficients $\pi_n(x)$ to be formulated below. A Fourier transformation of (2.3) yields

$$\hat{G}_z(k) = 1 + z\,\hat{D}(k)\,\hat{G}_z(k) + \hat{G}_z(k)\,\hat{\Pi}_z(k), \qquad k \in [-\pi, \pi)^d, \tag{2.4}$$

and this can be solved for $\hat{G}_z(k)$ as

$$\hat{G}_z(k)^{-1} = 1 - z\,\hat{D}(k) - \hat{\Pi}_z(k), \qquad k \in [-\pi, \pi)^d. \tag{2.5}$$

Since z_c is characterized by $\hat{G}_{z_c}(0)^{-1} = 0$, one has $\hat{\Pi}_{z_c}(0) = 1 - z_c$, and hence

$$\hat{G}_z(k)^{-1} = (z_c - z)\,\hat{D}(k) + \left(\hat{\Pi}_{z_c}(k) - \hat{\Pi}_z(k)\right) + z_c(1 - \hat{D}(k)) + \left(\hat{\Pi}_{z_c}(0) - \hat{\Pi}_{z_c}(k)\right). \tag{2.6}$$

If we let

$$A(k) := \hat{D}(k) + \partial_z \hat{\Pi}_z(k)\big|_{z=z_c}, \tag{2.7}$$

$$B(k) := 1 - \hat{D}(k) + \frac{1}{z_c}\left(\hat{\Pi}_{z_c}(0) - \hat{\Pi}_{z_c}(k)\right), \tag{2.8}$$

$$E_z(k) := \frac{\hat{\Pi}_{z_c}(k) - \hat{\Pi}_z(k)}{z_c - z} - \partial_z \hat{\Pi}_z(k)\big|_{z=z_c}, \tag{2.9}$$

then

$$\begin{aligned} z_c\,\hat{G}_z(k) &= \frac{1}{[1 - z/z_c]\,(A(k) + E_z(k)) + B(k)} \\ &= \frac{1}{[1 - z/z_c]\,A(k) + B(k)} - \Theta_z(k), \end{aligned} \tag{2.10}$$

where

$$\Theta_z(k) = \frac{[1 - z/z_c]\,E_z(k)}{\big([1 - z/z_c]\,(A(k) + E_z(k)) + B(k)\big)\,\big([1 - z/z_c]\,A(k) + B(k)\big)}. \tag{2.11}$$

If $\hat{G}_z(k)^{-1}$ is understood as a function of z, then $A(k)$ denotes the linear contribution, $E_z(k)$ denotes the higher order contribution (which will turn out to be asymptotically negligible), and $B(k)$ denotes the constant term.

For the first term in (2.10) we write

$$\frac{1}{[1-z/z_c]\,A(k)+B(k)}=\frac{1}{A(k)+B(k)}\sum_{n=0}^{\infty}\left(\frac{z}{z_c}\right)^n\left(\frac{A(k)}{A(k)+B(k)}\right)^n. \tag{2.12}$$

For $z<z_c$, we can write $\Theta_z(k)$ as a power series,

$$\Theta_z(k)=\sum_{n=0}^{\infty}\theta_n(k)\,z^n. \tag{2.13}$$

Since $\hat{G}_z(k)=\sum_{n=0}^{\infty}\hat{c}_n(k)z^n$ and $B(0)=0$, we thus obtained

$$\hat{c}_n(k)=\frac{1}{z_c}\left(\frac{z_c^{-n}}{A(k)+B(k)}\left(\frac{A(k)}{A(k)+B(k)}\right)^n+\theta_n(k)\right),\quad \hat{c}_n(0)=\frac{1}{z_c}\left(\frac{z_c^{-n}}{A(0)}+\theta_n(0)\right). \tag{2.14}$$

In Section 2.3 we prove the following bound on the error term θ_n:

Lemma 2.1. *Under the conditions of Theorem 1.2, $|\theta_n(k)|\le O(z_c^{-n}\,n^{-\varepsilon})$ for all $\varepsilon\in\big(0,\big(\frac{d}{\alpha\wedge 2}-2\big)\wedge 1\big)$ uniformly in $k\in[-\pi,\pi)^d$.*

Equation (2.14) and Lemma 2.1 imply the following corollary:

Corollary 2.2. *Under the conditions of Theorem 1.2,*

$$\hat{c}_n(0)=\Xi\,z_c^{-n}\left(1+O(n^{-\varepsilon})\right), \tag{2.15}$$

where $\varepsilon\in\big(0,\big(d/(\alpha\wedge 2)-2\big)\wedge 1\big)$ and

$$\Xi=[z_c\,A(0)]^{-1}=\left[z_c+\sum_{x\in\mathbb{Z}^d}\sum_{m=2}^{\infty}m\,\pi_m(x)\,z_c^m\right]^{-1}\in(0,\infty). \tag{2.16}$$

By (2.14) and Lemma 2.1, for $\varepsilon\in\big(0,\big(\frac{d}{\alpha\wedge 2}-2\big)\wedge 1\big)$,

$$\begin{aligned}\frac{\hat{c}_n(k_n)}{\hat{c}_n(0)} &= \big(1+O(n^{-\varepsilon})\big)\frac{A(0)}{A(k_n)+B(k_n)}\left(\frac{A(k_n)}{A(k_n)+B(k_n)}\right)^n+O(n^{-\varepsilon})\\ &= \big(1+O(n^{-\varepsilon})\big)\frac{A(0)}{A(k_n)+B(k_n)}\\ &\quad\times\left(1+\frac{-n(1-\hat{D}(k_n))\,A(k_n)^{-1}\,B(k_n)[1-\hat{D}(k_n)]^{-1}}{n}\right)^n+O(n^{-\varepsilon}).\end{aligned} \tag{2.17}$$

As $n\to\infty$, we have that $n(1-\hat{D}(k_n))\to|k|^{\alpha\wedge 2}$ by (1.16),

$$A(k_n)\to A(0)=1+\sum_{x\in\mathbb{Z}^d}\sum_{m=2}^{\infty}m\,\pi_m(x)\,z_c^{m-1}.$$

The convergence

$$\lim_{n\to\infty}\frac{B(k_n)}{1-\hat{D}(k_n)}=\begin{cases}1, & \text{if } \alpha\le 2;\\ 1+(2d\,v_\alpha)^{-1}\sum_{x\in\mathbb{Z}^d}|x|^2\,\Pi_{z_c}(x), & \text{if } \alpha>2.\end{cases} \tag{2.18}$$

follows directly from the following proposition:

Proposition 2.3. *Under the conditions of Theorem 1.2,*

$$\lim_{|k|\to 0} \frac{\hat{\Pi}_{z_c}(0)-\hat{\Pi}_{z_c}(k)}{1-\hat{D}(k)} = \begin{cases} 0, & \text{if } \alpha \le 2; \\ (2d\, v_\alpha)^{-1} \sum_{x\in\mathbb{Z}^d} |x|^2\, \Pi_{z_c}(x), & \text{if } \alpha > 2. \end{cases} \tag{2.19}$$

If a sequence h_n converges to a limit h, then $(1+h_n/n)^n$ converges to e^h. The above estimates imply

$$\lim_{n\to\infty} -n(1-\hat{D}(k_n))\, A(k_n)^{-1}\, B(k_n)[1-\hat{D}(k_n)]^{-1} = -K_\alpha\, |k|^{\alpha\wedge 2}$$

and

$$\lim_{n\to\infty} \frac{A(0)}{A(k_n)+B(k_n)} = 1.$$

We thus have proved Theorem 1.2 subject to Lemma 2.1 and Proposition 2.3. We want to emphasize that the bounds on the lace expansion coefficients $\pi_n(x)$ enter the calculation only through (2.19) and the error bound in Lemma 2.1.

2.2 Bounding the lace expansion coefficients

In this section we prove an estimate on moments of the lace expansion coefficients $\pi_n(x)$. This estimate is used to prove Proposition 2.3. Let us begin by stating the moment estimate.

Lemma 2.4 (Finite moments of the lace expansion coefficients)**.** *For $\alpha > 0$, $d > 2(\alpha\wedge 2)$ and L sufficiently large, we let*

$$\delta \begin{cases} \in \big(0\,,\, (\alpha\wedge 2)\wedge(2-2(\alpha\wedge 2))\big) & \text{if } \alpha \neq 2, \\ = 0 & \text{if } \alpha = 2. \end{cases} \tag{2.20}$$

Then, for any $z \le z_c$,

$$\sum_{x\in\mathbb{Z}^d} \sum_{n=0}^{\infty} |x|^{\alpha\wedge 2+\delta}\, |\pi_n(x)|\, z^n < \infty. \tag{2.21}$$

The fact that the $(\alpha\wedge 2+\delta)$th moment of $\Pi_{z_c}(x)$ exists is the key to the proof of (2.19). Interestingly, there is a crossover between the phases $\alpha < 2$ and $\alpha > 2$, with $\alpha = 2$ playing a special role. A version of Lemma 2.4 in the setting of oriented percolation is contained in [4, Proposition 3.1].

Before we start with the proof of Lemma 2.4, we shall review some basic facts about structure and convergence of quantities related to $\pi_n(x)$ introduced in (2.1)–(2.2). Our main reference for that is the monograph by Slade [17], who gives a detailed account of the lace expansion for percolation. Other references are [9, 13]. We shall also need results from [8], where a long-range version of the step distribution is considered. For $n \ge 2$, $N \ge 1$, $x \in \mathbb{Z}^d$, there exist quantities $\pi_n^{(N)}(x) \ge 0$ such that

$$\pi_n(x) = \sum_{N=1}^{\infty} (-1)^N \pi_n^{(N)}(x). \tag{2.22}$$

A combination of Theorem 4.1 with Lemma 5.10 (both references to Slade [17]), together with $\beta = O(L^{-d})$ [8, Prop. 2.2] shows

$$\sum_{x\in\mathbb{Z}^d} \sum_{n=2}^{\infty} \pi_n^{(N)}(x)\, z_c^n < O(L^{-d})^N, \tag{2.23}$$

where the constant in the O-term is uniform for all N. Consequently, (2.23) is summable in $N \geq 1$ provided that L is sufficiently large, and hence

$$\hat{\Pi}_{z_c}(k) \leq \sum_{x\in\mathbb{Z}^d}\sum_{n=2}^{\infty} |\pi_n(x)|\, z_c^n < \infty. \tag{2.24}$$

Lemma 2.4 implies Proposition 2.3, as we will show now.

Proof of Proposition 2.3 subject to Lemma 2.4. We first prove the assertion for $\alpha \leq 2$, and afterwards consider $\alpha > 2$.

For $\alpha \leq 2$, we choose $\delta \geq 0$ as in (2.20), hence $\alpha+\delta \leq 2$. Then we use $0 \leq 1-\cos(k\cdot x) \leq O(|k\cdot x|^{\alpha+\delta})$ to estimate

$$\begin{aligned} \left|\hat{\Pi}_{z_c}(0) - \hat{\Pi}_{z_c}(k)\right| &\leq \sum_{x\in\mathbb{Z}^d}\sum_{n=2}^{\infty}[1-\cos(k\cdot x)]\,|\pi_n(x)|\, z_c^n \\ &\leq \sum_{x\in\mathbb{Z}^d}\sum_{n=2}^{\infty} O(|k\cdot x|^{\alpha+\delta})\,|\pi_n(x)|\, z_c^n \\ &\leq O(1)\,|k|^{\alpha}\,|k|^{\delta}\sum_{x\in\mathbb{Z}^d}\sum_{n=2}^{\infty}|x|^{\alpha+\delta}\,|\pi_n(x)|\, z_c^n. \end{aligned} \tag{2.25}$$

We use (1.12) and Lemma 2.4 to bound further

$$\frac{|\hat{\Pi}_{z_c}(0) - \hat{\Pi}_{z_c}(k)|}{1-\hat{D}(k)} = \begin{cases} O(|k|^{\delta}) & \text{if } \alpha < 2, \\ O(1/\log(1/|k|)) & \text{if } \alpha = 2, \end{cases} \tag{2.26}$$

which proves (2.19) for $\alpha \leq 2$.

For $\alpha > 2$, we fix $\delta \in (0, 2\wedge(d-4))$. We apply the Taylor expansion

$$1-\cos(k\cdot x) = \frac{1}{2}(k\cdot x)^2 + O(|k\cdot x|^{2+\delta}), \tag{2.27}$$

together with spatial symmetry of the model and Lemma 2.4 to obtain

$$\hat{\Pi}_{z_c}(0) - \hat{\Pi}_{z_c}(k) = \sum_{x\in\mathbb{Z}^d}\sum_{n=2}^{\infty}[1-\cos(k\cdot x)]\,\pi_n(x)\, z_c^n = \frac{|k|^2}{2d}\sum_{x\in\mathbb{Z}^d}\sum_{n=2}^{\infty}|x|^2\,\pi_n(x)\, z_c^n + O(|k|^{2+\delta}). \tag{2.28}$$

Eq. (2.19) for $\alpha > 2$ now follows from (2.28) and (1.12). □

In the remainder of the section we prove Lemma 2.4. A key point in the proof is the use of a new form of (spatial) fractional derivative, first applied by Chen and Sakai [4] in the context of oriented percolation.

Proof of Lemma 2.4. For $t > 0$, $\zeta \in (0,2)$, we let

$$K'_\zeta := \int_0^\infty \frac{1-\cos(v)}{v^{1+\zeta}}\,\mathrm{d}v \in (0,\infty), \tag{2.29}$$

yielding

$$t^\zeta = \frac{1}{K'_\zeta}\int_0^\infty \frac{1-\cos(ut)}{u^{1+\zeta}}\,\mathrm{d}u. \tag{2.30}$$

For $\alpha > 0$ and $d > 2(\alpha \wedge 2)$, we choose δ as in (2.20). For $x \in \mathbb{Z}^d$ we write $x = (x_1, \ldots, x_d)$. Then by reflection and rotation symmetry of $\pi_n(x)$,

$$\sum_{x\in\mathbb{Z}^d}\sum_{n=0}^{\infty} |x|^{\alpha\wedge 2+\delta}\,|\pi_n(x)|\,z^n \le d^{\,(\alpha\wedge 2+\delta)/2+1}\sum_{x\in\mathbb{Z}^d}\sum_{n=0}^{\infty}|x_1|^{\alpha\wedge 2+\delta}\sum_{N=2}^{\infty}\pi_n^{(N)}(x)\,z_c^n, \tag{2.31}$$

cf. [4, Lemma 4.1]. We now apply (2.30) with $\zeta = \delta_1, \delta_2$, given by

$$\delta_1 \in \big(\delta\,,\,(\alpha\wedge 2)\wedge(2-2(\alpha\wedge 2))\big), \tag{2.32}$$

$$\delta_2 = \alpha\wedge 2+\delta-\delta_1. \tag{2.33}$$

This yields

$$O(1)\int_0^{\infty}\frac{\mathrm{d}u}{u^{1+\delta_1}}\int_0^{\infty}\frac{\mathrm{d}v}{v^{1+\delta_2}}\sum_{x\in\mathbb{Z}^d}\sum_{n=0}^{\infty}\sum_{N=2}^{\infty}[1-\cos(u\,x_1)]\,[1-\cos(v\,x_1)]\,\pi_n^{(N)}(x)\,z_c^n \tag{2.34}$$

as an upper bound of (2.31). We write the double integral appearing in (2.34) as the sum of four terms, $I_1 + I_2 + I_3 + I_4$, where

$$I_1 = \sum_{N=2}^{\infty}\int_0^1\frac{\mathrm{d}u}{u^{1+\delta_1}}\int_0^1\frac{\mathrm{d}v}{v^{1+\delta_2}}\sum_{x\in\mathbb{Z}^d}\sum_{n=0}^{\infty}[1-\cos(\vec{u}\cdot x)]\,[1-\cos(\vec{v}\cdot x)]\,\pi_n^{(N)}(x)\,z_c^n \tag{2.35}$$

with

$$\vec{u} = (u, 0, \ldots, 0) \in \mathbb{R}^d, \qquad \vec{v} = (v, 0, \ldots, 0) \in \mathbb{R}^d, \tag{2.36}$$

and I_2, I_3, I_4 are defined similarly:

$$I_2 = \int_0^1 \mathrm{d}u\int_1^{\infty}\mathrm{d}v\cdots, \quad I_3 = \int_0^1 \mathrm{d}u\int_1^{\infty}\mathrm{d}v\cdots, \quad I_4 = \int_1^{\infty}\mathrm{d}u\int_1^{\infty}\mathrm{d}v\cdots. \tag{2.37}$$

We now show that $I_1, \ldots, I_4$ are all finite, which implies (2.21). The bound $I_4 < \infty$ simply follows from $1 - \cos t \le 2$ and (2.24). In order to prove the bounds $I_1, I_2, I_3 < \infty$ we need the particular structure of the $\pi_n^{(N)}(x)$-terms.

To this end, we define

$$\tilde{G}_z(x) = z(D * G_z)(x), \qquad x \in \mathbb{Z}^d, \tag{2.38}$$

and

$$\tilde{B}(z) = \sup_{x\in\mathbb{Z}^d}(G_z * \tilde{G}_z)(x). \tag{2.39}$$

In [17, Theorem 4.1] it is shown that for $z \ge 0$, $N \ge 1$,

$$\sum_{x\in\mathbb{Z}^d}[1-\cos(k\cdot x)]\,\Pi_z^{(1)}(x) = 0 \tag{2.40}$$

and

$$\sum_{x\in\mathbb{Z}^d}[1-\cos(k\cdot x)]\,\Pi_z^{(N)}(x) \le \frac{N}{2}(N+1)\left(\sup_x\,[1-\cos(k\cdot x)]\,G_z(x)\right)\tilde{B}(z)^{N-1}, \quad N \ge 2. \tag{2.41}$$

These bounds are called *diagrammatic estimates*, because the lace expansion coefficients $\pi_z^{(N)}(x)$ are expressed in terms of diagrams, whose structure is heavily used in the derivation of the above bounds. The composition of the diagrams and their decomposition into two-point functions as in (2.40)–(2.41) is described in detail in [17, Sections 3 and 4]. It is clear that a slight modification of this procedure proves the bound

$$\begin{aligned}\sum_{x\in\mathbb{Z}^d}\sum_{n=0}^{\infty}&[1-\cos(\vec{v}\cdot x)]\,[1-\cos(\vec{u}\cdot x)]\,\pi_n^{(N)}(x)\,z^n\\ &\leq O(N^4)\,\tilde{B}(z)^{N-2}\left(\sup_x\,[1-\cos(\vec{v}\cdot x)]\,G_z(x)\right)\\ &\quad\times\left(\sup_y\sum_{x\in\mathbb{Z}^d}[1-\cos(\vec{u}\cdot x)]\,G_z(x)\,G_z(y-x)\right).\end{aligned}\tag{2.42}$$

Given (2.42), it remains to show the following three bounds:

$$\tilde{B}(z_c)=\sup_{x\in\mathbb{Z}^d}(G_{z_c}*\tilde{G}_{z_c})(x)\quad\leq\quad O\big(L^{-d}\big);\tag{2.43}$$

$$\sup_x\,[1-\cos(\vec{v}\cdot x)]\,G_{z_c}(x)\quad\leq\quad O\big(v^{\alpha\wedge 2}\big)\,;\tag{2.44}$$

$$\sup_y\sum_{x\in\mathbb{Z}^d}[1-\cos(\vec{u}\cdot x)]\,G_{z_c}(x)\,G_{z_c}(y-x)\quad\leq\quad O\Big(u^{(d-2(\alpha\wedge2))\wedge(\alpha\wedge2)}\Big)\,.\tag{2.45}$$

Suppose (2.43)–(2.45) were true, then

$$\begin{aligned}\sum_{x\in\mathbb{Z}^d}\sum_{n=0}^{\infty}&[1-\cos(\vec{u}\cdot x)]\,[1-\cos(\vec{v}\cdot x)]\,\pi_n^{(N)}(x)\,{z_c}^n\\ &\leq O\big(N^4\big)\,O\big(L^{-d}\big)^{N-2}\,O\big(v^{\alpha\wedge2}\big)\,O\Big(u^{(d-2(\alpha\wedge2))\wedge(\alpha\wedge2)}\Big)\,.\end{aligned}\tag{2.46}$$

Since $\delta_1<(\alpha\wedge2)\wedge(d-2(\alpha\wedge2))$ and $\delta_2<\alpha\wedge2$, we obtain that I_1 is finite for L sufficiently large, as desired. Similarly, it follows that I_2 and I_3 are finite. It remains to prove (2.43)–(2.45), and we use results from [8] to prove it.

We introduce the quantity

$$\lambda_z:=1-\frac{1}{\hat{G}_z(0)}=1-\frac{1}{\chi(z)}\in[0,1].\tag{2.47}$$

Then λ_z satisfies the equality

$$\hat{G}_z(0)=\hat{C}_{\lambda_z}(0),\tag{2.48}$$

where $\hat{C}_{\lambda_z}(k)=[1-\lambda_z\hat{D}(k)]^{-1}$ is the Fourier transform of the simple random walk Green's function. This definition is motivated by the intuition that $\hat{G}_z(k)$ and $\hat{C}_{\lambda_z}(k)$ are comparable in size and, moreover, the discretized second derivative

$$\Delta_k\hat{G}_z(l):=\hat{G}_z(l-k)+\hat{G}_z(l+k)-2\hat{G}(l)\tag{2.49}$$

is bounded by

$$U_{\lambda_z}(k,l):=200\,\hat{C}_{\lambda_z}(k)^{-1}\left\{\hat{C}_{\lambda_z}(l-k)\hat{C}_{\lambda_z}(l)+\hat{C}_{\lambda_z}(l)\hat{C}_{\lambda_z}(l+k)+\hat{C}_{\lambda_z}(l-k)\hat{C}_{\lambda_z}(l+k)\right\}.\tag{2.50}$$

To make this more precise, we consider the function $f\colon[0,z_c]\to\mathbb{R}$, defined by

$$f:=f_1\vee f_2\vee f_3\tag{2.51}$$

with

$$f_1(z) := z, \qquad f_2(z) := \sup_{k\in[-\pi,\pi)^d} \frac{\hat{G}_z(k)}{\hat{C}_{\lambda_z}(k)}, \tag{2.52}$$

and

$$f_3(z) := \sup_{k,l\in[-\pi,\pi)^d} \frac{|\Delta_k \hat{G}_z(l)|}{U_{\lambda_z}(k,l)}, \tag{2.53}$$

It is an important result in [8] that, under the conditions of Theorem 1.2, the function f is uniformly bounded on $[0, z_c)$, cf. [8, Prop. 2.5 and 2.6]. In fact, it is shown that $f(z) \le 1 + O(L^{-d})$, but for our need it suffices to have f uniformly bounded. Since the bound is uniform, we can conclude that even $f(z_c) < \infty$.

Indeed, (2.43) follows by standard methods from [8, Proposition 2.2], see e.g. [17, (5.28) in conjunction with Lemma 5.10]. Furthermore, (2.44) is proven in [8, Lemma B.5] in the context of the Ising model, but applies verbatim to self-avoiding walk. It remains to prove (2.45). Since

$$\begin{aligned} &\sup_y \sum_{x\in\mathbb{Z}^d} [1-\cos(\vec{u}\cdot x)]\, G_{z_c}(x)\, G_{z_c}(y-x) \\ =\;& \sup_y \int_{[-\pi,\pi)^d} \mathrm{e}^{-il\cdot y} \Big(\hat{G}_{z_c}(l) - \frac{1}{2}\Big(\hat{G}_{z_c}(l-\vec{u}) + \hat{G}_{z_c}(l+\vec{u})\Big)\Big)\, \hat{G}_{z_c}(l)\, \frac{\mathrm{d}l}{(2\pi)^d} \\ \le\;& \int_{[-\pi,\pi)^d} \Big|\frac{1}{2}\,\Delta_{\vec{u}}\, \hat{G}_{z_c}(l)\Big|\, \hat{G}_{z_c}(l)\, \frac{\mathrm{d}l}{(2\pi)^d}\,, \end{aligned} \tag{2.54}$$

our bounds $f_2(z_c) \le K$ and $f_3(z_c) \le K$, together with $\lambda_{z_c} = 1$, imply that

$$\begin{aligned} &\sup_y \sum_{x\in\mathbb{Z}^d} [1-\cos(\vec{u}\cdot x)]\, G_{z_c}(x)\, G_{z_c}(y-x) \\ &\quad\le 100K^2\, \hat{C}_1(\vec{u})^{-1} \int_{[-\pi,\pi)^d} \Big(\hat{C}_1(l-\vec{u})\,\hat{C}_1(l+\vec{u}) + \hat{C}_1(l-\vec{u})\,\hat{C}_1(l) \\ &\qquad\qquad + \hat{C}_1(l)\,\hat{C}_1(l+\vec{u})\Big)\, \hat{C}(l)\, \frac{\mathrm{d}l}{(2\pi)^d} \\ &\quad= O(1)\,[1-\hat{D}(\vec{u})] \int_{[-\pi,\pi)^d} \Bigg(\frac{1}{[1-\hat{D}(l-\vec{u})]\,[1-\hat{D}(l+\vec{u})]\,[1-\hat{D}(l)]} \\ &\qquad\qquad + \frac{1}{[1-\hat{D}(l-\vec{u})]\,[1-\hat{D}(l)]^2} + \frac{1}{[1-\hat{D}(l+\vec{u})]\,[1-\hat{D}(l)]^2}\Bigg)\, \frac{\mathrm{d}l}{(2\pi)^d}. \end{aligned} \tag{2.55}$$

Chen and Sakai show that the integral term on the right hand side of (2.55) is bounded above by $O\big(u^{(d-3(\alpha\wedge 2))\wedge 0}\big)$, cf. [4, (4.30)–(4.33)]. Furthermore, $1-\hat{D}(\vec{u}) \le O\big(u^{\alpha\wedge 2}\big)$ by (1.12). The combination of the above inequalities implies (2.45), and hence the claim follows. □

2.3 Error bounds

The proof of Lemma 2.1 is the final piece in the proof of Theorem 1.2. Our proof of Lemma 2.1 makes use of the following lemma:

Lemma 2.5. *Consider a function g given by the power series $g(z) = \sum_{n=0}^{\infty} a_n z^n$, with z_c as radius of convergence.*

(i) If $|g(z)| \leq O(|z_c - z|^{-b})$ *for some* $b \geq 1$*, then* $|a_n| \leq O(z_c^{-n} \log(n))$ *if* $b = 1$*, or* $|a_n| \leq O(z_c^{-n} n^{b-1})$ *if* $b > 1$.

(ii) If $|g'(z)| \leq O(|z_c - z|^{-b})$ *for some* $b > 1$*, then* $|a_n| \leq O(z_c^{-n} n^{b-2})$.

The proof of assertion (i) is contained in [6, Lemma 3.2], and (ii) is a direct consequence of (i) since (i) implies that $|n\, a_n| \leq O(z_c^{-n} n^{b-1})$. Lemma 2.5 is the key to the proof of Lemma 2.1.

Proof of Lemma 2.1. We recall

$$\Theta_z(k) = \sum_{n=0}^{\infty} \theta_n(k)\, z^n, \tag{2.56}$$

where

$$\Theta_z(k) = \frac{[1 - z/z_c]\, E_z(k)}{\big([1 - z/z_c]\,(A(k) + E_z(k)) + B(k)\big)\,\big([1 - z/z_c]\, A(k) + B(k)\big)}. \tag{2.57}$$

We fix $\varepsilon \in (0, (d(\alpha \wedge 2)^{-1} - 2) \wedge 1)$ and aim to prove $|\theta_n(k)| \leq O(z_c^{-n} n^{-\varepsilon})$, where the constant in the O-term is uniform for $k \in [-\pi, \pi)^d$. By Lemma 2.5 it is sufficient to show $|\partial_z \Theta_z(k)| \leq O\big(|z_c - z|^{-(2-\varepsilon)}\big)$, and we prove this now.

Before bounding $\partial_z \Theta_z(k)$, we consider derivatives of $\hat{\Pi}_z(k)$ (the Fourier transform of $\Pi_z(x)$ introduced in (2.2)). The first derivative of $\partial_z \hat{\Pi}_z(k)$ is converging absolutely for $z \leq z_c$, i.e.,

$$\sum_{x \in \mathbb{Z}^d} \sum_{n=2}^{\infty} n\, |\pi_n(x)|\, z_c^{n-1} < \infty, \tag{2.58}$$

cf. [13, Theorem 6.2.9] for a proof in the finite-range setting, and again [8] for the extension to long-range systems. Moreover, we claim that

$$\sum_{x \in \mathbb{Z}^d} \sum_{n=2}^{\infty} n(n-1)^{\varepsilon}\, |\pi_n(x)|\, z_c^{n-1} < \infty; \tag{2.59}$$

for $\varepsilon \in (0, (d(\alpha \wedge 2)^{-1} - 2) \wedge 1)$. The bound (2.59) can be proved by considering *temporal* fractional derivatives, as introduced in [13, Section 6.3]. In particular, the proof of [13, Theorem 6.4.2] shows

$$\sup_{x \in \mathbb{Z}^d} \sum_{n=2}^{\infty} n(n-1)^{\varepsilon}\, c_n(x)\, z_c^{n-1} \leq O(1) \int_{[-\pi,\pi)^d} \sum_{n \geq 2} n(n-1)^{\varepsilon} \hat{D}(k)^{n-2} \frac{\mathrm{d}k}{(2\pi)^d}, \tag{2.60}$$

(see the first displayed identity in [13, p. 196]). On the one hand, $\hat{D}(k) = 1 - (1 - \hat{D}(k)) \leq \mathrm{e}^{-(1-\hat{D}(k))} \leq \mathrm{e}^{-c_1 |k|^{\alpha \wedge 2}}$ for some constant $c_1 > 0$, by (1.12). On the other hand, $-\hat{D}(k) \leq 1 - c_2$ for a positive constant c_2, by (1.10). Together these bounds yield

$$\begin{aligned} \int_{[-\pi,\pi)^d} \hat{D}(k)^{n-2} \frac{\mathrm{d}k}{(2\pi)^d} \;&\leq\; \int_{\substack{k \in [-\pi,\pi)^d: \\ \hat{D}(k) \geq 0}} \mathrm{e}^{-c_1 (n-2) |k|^{\alpha \wedge 2}} \frac{\mathrm{d}k}{(2\pi)^d} \\ &\quad + \int_{\substack{k \in [-\pi,\pi)^d: \\ \hat{D}(k) < 0}} (1 - c_2)^{n-2} \frac{\mathrm{d}k}{(2\pi)^d} \\ &\leq\; O(n^{-d/(\alpha \wedge 2)}) + (1 - c_2)^{n-2} \leq O(n^{-d/(\alpha \wedge 2)}). \end{aligned} \tag{2.61}$$

Hence the right hand side of (2.60) is less than or equal to

$$\sum_{n\geq 2} n(n-1)^{\varepsilon}\, O(n^{-d/(\alpha\wedge 2)}), \tag{2.62}$$

and this is finite if $1+\varepsilon-d/(\alpha\wedge 2)<-1$. Furthermore, the proof of [13, Corollary 6.4.3] shows that

$$\sum_{x\in\mathbb{Z}^d}\sum_{n=2}^{\infty} n(n-1)^{\varepsilon}\,|\pi_n(x)|\, z_c^{n-1} \leq O(1)\left(\sup_{x\in\mathbb{Z}^d}\sum_{n=2}^{\infty} n(n-1)^{\varepsilon}\, c_n(x)\, z_c^{n-1}\right) \tag{2.63}$$

under the conditions of Theorem 1.2. This proves (2.59).

We first prove

$$E_z(k) \leq O(|z_c-z|^{\varepsilon}) \tag{2.64}$$

by considering the power series representation of $\hat{\Pi}_z(k)$ in (2.9):

$$E_z(k) = \frac{1}{z_c-z}\sum_x\sum_{n\geq 2} \mathrm{e}^{ik\cdot x}\,\pi_n(x)\,(z_c^n-z^n) - \sum_x\sum_{n\geq 2}\mathrm{e}^{ik\cdot x}\,\pi_n(x)\, n\, z_c^{n-1}. \tag{2.65}$$

Since

$$\frac{z_c^n-z^n}{z_c-z} = \sum_{i=0}^{n-1} z^i\, z_c^{(n-1)-i}, \tag{2.66}$$

one has

$$E_z(k) = \sum_x\sum_{n\geq 2}\mathrm{e}^{ik\cdot x}\,\pi_n(x)\sum_{i=1}^{n-1}\left(z^i-z_c^i\right) z_c^{(n-1)-i}. \tag{2.67}$$

For every $\zeta,\varepsilon\in(0,1)$ and $n\geq 2$,

$$\begin{aligned} \left|1-\zeta^{n-1}\right| &= \left|(1-\zeta^{n-1})^{1-\varepsilon}\left(\frac{1-\zeta^{n-1}}{1-\zeta}\right)^{\varepsilon}(1-\zeta)^{\varepsilon}\right| \\ &\leq \left|\sum_{l=0}^{n-2}\zeta^l\right|^{\varepsilon}(1-\zeta)^{\varepsilon} \leq (n-1)^{\varepsilon}\,(1-\zeta)^{\varepsilon}. \end{aligned} \tag{2.68}$$

Applying this for $\zeta=z/z_c$, we obtain for $z<z_c$ and $0<i<n$,

$$\begin{aligned} \left|z^i-z_c^i\right| z_c^{(n-1)-i} &= \left|1-\left(\frac{z}{z_c}\right)^i\right| z_c^{n-1} \leq \left|1-\left(\frac{z}{z_c}\right)^{n-1}\right| z_c^{n-1} \\ &\leq \left|1-\frac{z}{z_c}\right|^{\varepsilon}(n-1)^{\varepsilon}\, z_c^{n-1}. \end{aligned} \tag{2.69}$$

Insertion into (2.67) yields

$$|E_z(k)| \leq (z_c-z)^{\varepsilon}\sum_x\sum_{n\geq 2} n(n-1)^{\varepsilon}\,|\pi_n(x)|\, z_c^{n-1} \leq O(|z_c-z|^{\varepsilon}), \tag{2.70}$$

where the last bound uses (2.59). We further differentiate (2.9) to get

$$\begin{aligned} \partial_z E_z(k) &= \frac{(z_c-z)\,\partial_z\big(\hat{\Pi}_{z_c}(k)-\hat{\Pi}_z(k)\big)+\big(\hat{\Pi}_{z_c}(k)-\hat{\Pi}_z(k)\big)}{(z_c-z)^2} \\ &= \frac{1}{z_c-z}\left(\frac{\hat{\Pi}_{z_c}(k)-\hat{\Pi}_z(k)}{z_c-z}-\partial_z\hat{\Pi}_z(k)\right). \end{aligned} \tag{2.71}$$

A calculation similar to (2.65)–(2.70) shows

$$|\partial_z E_z(k)| \le \left|\frac{E_z(k)}{z_c - z}\right| + \frac{1}{z_c - z}\left|\sum_x \sum_{n\ge 2} \mathrm{e}^{ik\cdot x}\,\pi_n(x)\,n\,\big(z_c^{n-1} - z^{n-1}\big)\right| \le O(|z_c - z|^{\varepsilon - 1}). \tag{2.72}$$

We write D_1 and D_2 for the two factors in the denominator in (2.57). Then

$$\begin{aligned} z_c^2\,\partial_z\Theta_z(k) = & \frac{z_c}{D_1\,D_2}\,\big((z_c - z)\,\partial_z E_z(k) - E_z(k)\big) \\ & - \frac{z_c - z}{(D_1\,D_2)^2}\,E_z(k)\Big(\big(- A(k) - E_z(k) + (z_c - z)\,\partial_z E_z(k)\big)\,D_2 - D_1\,A(k)\Big). \end{aligned} \tag{2.73}$$

After further cancelation of D_1-, D_2-terms we are left with D_1 and D_2 in the denominator only, hence a lower bound on them suffices. Indeed, there is a constant $c > 0$ such that

$$|D_1| = \left|z_c\,\hat{G}_z(k)\right|^{-1} \ge z_c^{-1}\,\chi(z) \ge c\,(z_c - z)\,, \tag{2.74}$$

where the last bound follows from [8, (1.24) and Theorem 1.3]. Furthermore,

$$|D_2| \ge c\,(z_c - z) \tag{2.75}$$

because D_2 is a linear function in $(z_c - z)$. The lower bounds on D_1 and D_2, together with the bounds on $E_z(k)$ and $\partial_z E_z(k)$ in (2.64) and (2.72), prove that (2.73) is uniformly bounded for all $z \le z_c$, and in particular

$$\partial_z\Theta_z(k) \le O(|z_c - z|^{-(2-\varepsilon)}). \tag{2.76}$$

Finally, assertion (ii) in Lemma 2.5 implies

$$\theta_n(k) \le O(z_c^{-n}\,n^{-\varepsilon}) \tag{2.77}$$

for all $\varepsilon \in (0, (d(\alpha\wedge 2)^{-1} - 2) \wedge 1)$, uniformly in k. □

3 The mean-r displacement: proof of Theorem 1.4

Proof of Theorem 1.4. Our proof uses methods similar to those developed in Section 2.2, and again a key ingredient is the equality in (2.30). Recalling (1.22) we note that (1.21) can be rewritten as $\xi^{(r)}(n) \asymp f_\alpha(n)^{-1}$. Also, we write x_1 for the first component of the vector $x \in \mathbb{Z}^d$, and denote by $\vec{u}$ the vector $\vec{u} = (u, 0, \ldots, 0) \in \mathbb{R}^d$, see also (2.36). We use reflection and rotation symmetry of c_n in the first line, and (2.30) in the second line to obtain

$$\begin{aligned} \frac{1}{c_n}\sum_{x\in\mathbb{Z}^d} |x|^r c_n(x) &\asymp \sum_{x\in\mathbb{Z}^d} |x_1|^r \frac{c_n(x)}{c_n} \\ &\asymp \sum_{x\in\mathbb{Z}^d} \int_0^\infty \frac{\mathrm{d}u}{u^{1+r}}\,[1 - \cos(\vec{u}\cdot x)]\,\frac{c_n(x)}{c_n} \\ &= \int_{f_\alpha(n)}^\infty \frac{\mathrm{d}u}{u^{1+r}} \sum_{x\in\mathbb{Z}^d} [1 - \cos(\vec{u}\cdot x)]\,\frac{c_n(x)}{c_n} \\ &\quad + \int_0^{f_\alpha(n)} \frac{\mathrm{d}u}{u^{1+r}} \left(1 - \frac{\hat{c}_n(\vec{u})}{\hat{c}_n(0)}\right). \end{aligned} \tag{3.1}$$

For the first integral on the right hand side of (3.1) we use $0 \le [1-\cos(\vec{u}\cdot x)] \le 2$ yielding

$$0 \le \int_{f_\alpha(n)}^{\infty} \frac{\mathrm{d}u}{u^{1+r}} \sum_{x\in\mathbb{Z}^d} [1-\cos(\vec{u}\cdot x)]\, \frac{c_n(x)}{c_n} \le \int_{f_\alpha(n)}^{\infty} \frac{\mathrm{d}u}{u^{1+r}} = O\big(f_\alpha(n)^{-r}\big). \tag{3.2}$$

For the second integral, we substitute u by $f_\alpha(n)\,u$ to obtain

$$\int_0^{f_\alpha(n)} \frac{\mathrm{d}u}{u^{1+r}} \left(1-\frac{\hat c_n(\vec{u})}{\hat c_n(0)}\right) = f_\alpha(n)^{-r} \int_0^1 \frac{\mathrm{d}u}{u^{1+r}} \left(1-\frac{\hat c_n(\vec{u}_n)}{\hat c_n(0)}\right), \tag{3.3}$$

where $\vec{u}_n = f_\alpha(n)\vec{u}$ (compare with k_n in (1.15)). Suppose we know

$$\int_0^1 \frac{\mathrm{d}u}{u^{1+r}} \left(1-\frac{\hat c_n(\vec{u}_n)}{\hat c_n(0)}\right) \asymp 1, \tag{3.4}$$

then it would follow that $c_n^{-1}\sum_x |x|^r c_n(x) \asymp f_\alpha(n)^{-r}$, as desired.

It remains to show (3.4) is indeed true. The idea is the following. If the ratio $\hat c_n(\vec{u}_n)/\hat c_n(0)$ is replaced by its limit $\exp\{-K_\alpha u^{\alpha\wedge 2}\}$ (cf. Theorem 1.2), then Taylor expansion shows

$$1-\exp\{-K_\alpha u^{\alpha\wedge 2}\} = K_\alpha u^{\alpha\wedge 2} + O\big(u^{2(\alpha\wedge 2)}\big),$$

and since $\alpha\wedge 2-(1+r) > -1$, the integral in (3.4) converges. However, a careful consideration of error terms makes the argument look slightly more complicated.

We write

$$h_n = -n(1-\hat D(\vec{u}_n))\, A(\vec{u}_n)^{-1}\, B(\vec{u}_n)[1-\hat D(\vec{u}_n)]^{-1}. \tag{3.5}$$

By (2.17),

$$\left(1-\frac{\hat c_n(\vec{u}_n)}{\hat c_n(0)}\right) = \big(1+O(n^{-\varepsilon})\big)\left[1-\frac{A(0)}{A(\vec{u}_n)+B(\vec{u}_n)}\left(1+\frac{h_n}{n}\right)^n\right]. \tag{3.6}$$

Taylor expansion shows

$$n\log\left(1+\frac{h_n}{n}\right) = h_n + O\Big(\frac{h_n^2}{n}\Big)$$

and

$$\left(1+\frac{h_n}{n}\right)^n = \mathrm{e}^{n\log(1+h_n/n)} = \mathrm{e}^{h_n}\left(1+O\Big(\frac{h_n^2}{n}\Big)\right).$$

Insertion into (3.6) obtains

$$\begin{aligned}\left(1-\frac{\hat c_n(\vec{u}_n)}{\hat c_n(0)}\right) = \big(1+O(n^{-\varepsilon})\big)\,\frac{A(0)}{A(\vec{u}_n)+B(\vec{u}_n)}\Bigg[&\frac{A(\vec{u}_n)+B(\vec{u}_n)}{A(0)}-1+\Big(1-\mathrm{e}^{h_n}\Big)\\ &-O\Big(\frac{h_n^2}{n}\Big)\,\mathrm{e}^{h_n}\Bigg].\end{aligned} \tag{3.7}$$

We remark that the limit in (1.16) is uniform in $u\in(0,1]$, and the bound (2.26) implies that $B(\vec{u}_n) \asymp [1-\hat D(\vec{u}_n)]$ uniformly in $u\in(0,1]$. We show below that the limit $A(\vec{u}_n)\to A(0)$ is also uniform. Consequently, also $\lim_{n\to\infty} h_n = -Ku^{\alpha\wedge 2}$ is a uniform limit, and this is important since we are integrating u over the interval $(0,1]$.

We finally show that

$$\frac{A(\vec{u}_n)+B(\vec{u}_n)}{A(0)}-1=\frac{A(\vec{u}_n)-A(0)+B(\vec{u}_n)}{A(0)}\leq u^{\alpha\wedge 2}o(1) \tag{3.8}$$

as $n\to\infty$, uniformly in u. By (2.18), $B(\vec{u}_n)\asymp[1-\hat{D}(\vec{u}_n)]=O(1/n)\,u^{\alpha\wedge 2}$. We choose δ as in (2.20), so that in particular $0\leq(\alpha\wedge 2)+\delta\leq 2$. Consequently,

$$\begin{aligned}\Big|A(\vec{u}_n)-A(0)\Big| &\leq [1-\hat{D}(\vec{u}_n)]+\sum_{x\in\mathbb{Z}^d}\sum_{n=2}^{\infty}[1-\cos(\vec{u}_n\cdot x)]\,n\,|\pi_n(x)|\,z_c^{n-1}\\ &= u^{\alpha\wedge 2}\,O(1/n)+\sum_{x\in\mathbb{Z}^d}\sum_{n=2}^{\infty}O\big(|\vec{u}_n|^{(\alpha\wedge 2)+\delta}|x|^{(\alpha\wedge 2)+\delta}\big)\,n\,|\pi_n(x)|\,z_c^{n-1}\end{aligned}$$

Since $|\vec{u}_n|^{(\alpha\wedge 2)+\delta}\asymp u^{(\alpha\wedge 2)+\delta}/n^{1+\delta/(\alpha\wedge 2)}$ for $\alpha\neq 2$, and $|\vec{u}_n|^{(\alpha\wedge 2)+\delta}\asymp u^2/(n\log\sqrt{n})$ for $\alpha=2$, we bound further

$$\begin{aligned}&\sum_{x\in\mathbb{Z}^d}\sum_{n=2}^{\infty}O\big(|\vec{u}_n|^{(\alpha\wedge 2)+\delta}|x|^{(\alpha\wedge 2)+\delta}\big)\,n\,|\pi_n(x)|\,z_c^{n-1}\\ &\leq O\big(u^{(\alpha\wedge 2)+\delta}\big)\sum_{x\in\mathbb{Z}^d}\sum_{n=2}^{\infty}|x|^{(\alpha\wedge 2)+\delta}\,|\pi_n(x)|\,z_c^{n-1}\times\begin{cases}n^{-\delta/(\alpha\wedge 2)}, & \text{if } \alpha\neq 2,\\ (\log n)^{-1}, & \text{if } \alpha=2,\end{cases}\end{aligned} \tag{3.9}$$

and this is bounded above by $u^{\alpha\wedge 2}o(1)$ by appeal to Lemma 2.4. In particular, this implies that $A(\vec{u}_n)\to A(0)$ uniformly in u.

We have proven that the only non-vanishing contribution towards (3.7) comes from the term $1-\mathrm{e}^{h_n}$. Since the sequence h_n converges uniformly to the negative limit $-K_\alpha\,u^{\alpha\wedge 2}$, there is an n_0 such that for all $n\geq n_0$, $-2K_\alpha\,u^{\alpha\wedge 2}\leq h_n\leq -K_\alpha\,u^{\alpha\wedge 2}/2$. Consequently, $1-\mathrm{e}^{h_n}$ is positive for $n\geq n_0$, and $1-\mathrm{e}^{h_n}\leq O\big(u^{\alpha\wedge 2}\big)$ by Taylor expansion. Therefore,

$$0\leq\left(1-\frac{\hat{c}_n(\vec{u}_n)}{\hat{c}_n(0)}\right)\leq O\big(u^{\alpha\wedge 2}\big) \tag{3.10}$$

as $n\to\infty$, where the bounds on the error terms do not depend on n. Hence for $r<\alpha\wedge 2$, the integral

$$\int_0^1\frac{\mathrm{d}u}{u^{1+r}}\left(1-\frac{\hat{c}_n(\vec{u}_n)}{\hat{c}_n(0)}\right) \tag{3.11}$$

converges, and is positive for sufficiently large n. The combination of (3.2), (3.3) and (3.10) implies the claim. □

4 Convergence of finite dimensional distributions: proof of Theorem 1.6

Proof of Theorem 1.6. The proof is via induction over N, and is very much inspired by the proof of [13, Theorem 6.6.2], where finite-range models were considered. The flexibility in the last argument of $n\mathbf{T}$ is needed to perform the induction step. We shall further write $nt^{(j)}$ and nT instead of $\lfloor nt^{(j)}\rfloor$ and $\lfloor nT\rfloor$ for brevity.

To initialize the induction we consider the case $N = 1$. Since $\hat{\mathbf{c}}^{(1)}_{n\mathbf{T}}(\mathbf{k}_n) = \hat{c}_{nT}(k_n^{(1)})$, the assertion for $N = 1$ is a minor generalization of Theorem 1.2. In fact, if we replace n by nT, then instead of (1.16) we have

$$nT\,[1 - \hat{D}(k_n)] = nt^{(1)}(1-g_n)\left[1 - \hat{D}\big(f_\alpha(t^{(1)}n)\,k\,(t^{(1)})^{1/(\alpha\wedge 2)}\big)\right] \to |k|^{\alpha\wedge 2}\,t^{(1)} \quad \text{as } n\to\infty. \tag{4.1}$$

With an appropriate change in (2.17) we obtain (1.32) for $N = 1$ from Theorem 1.2.

To advance the induction we prove (1.32) assuming that it holds when N is replaced by $N-1$. For a path $w \in \mathcal{W}_n$ and $0 \le a \le b \le n$ it will be convenient to write

$$K_{[a,b]}(w) := \mathbb{1}_{\{(w_a,\ldots,w_b)\text{ is self-avoiding}\}}. \tag{4.2}$$

We further consider the quantity $J_{[a,b]}(w)$ that arises in the algebraic derivation of the lace expansion as in [17, Sect. 3.2]. For our needs it suffices to know that

$$\sum_{w\in\mathcal{W}_n(x)} W(w) J_{[0,n]}(w) = \pi_n(x) \tag{4.3}$$

and, for any integers $0 \le m \le n$ and paths $w \in \mathcal{W}_n$,

$$K_{[0,n]}(w) = \sum_{I\ni m} K_{[0,I_1]}(w)\, J_{[I_1,I_2]}(w)\, K_{[I_2,n]}(w), \tag{4.4}$$

where the sum is over all intervals $I = [I_1, I_2]$ of integers with either $0 \le I_1 < m < I_2 \le n$ or $I_1 = m = I_2$. We refer to [17, (3.13)] for (4.3), and to [13, Lemma 5.2.5] for (4.4). By (1.30) and (4.4),

$$\hat{\mathbf{c}}^{(N)}_{n\mathbf{T}}(\mathbf{k}_n) = \sum_{I\ni nt^{(N-1)}}\ \sum_{w\in\mathcal{W}_{nT}} \mathrm{e}^{i\mathbf{k}_n\cdot\Delta w(n\mathbf{T})}\, W(w)\; K_{[0,I_1]}(w)\, J_{[I_1,I_2]}(w)\, K_{[I_2,nT]}(w). \tag{4.5}$$

Let $\overset{\le}{\mathbf{c}}{}^{(N)}$ and $\overset{>}{\mathbf{c}}{}^{(N)}$ denote the contributions towards (4.5) corresponding to intervals I with length $|I| = I_2 - I_1 \le b_n$ and $|I| > b_n$, respectively. It will turn out that the latter contribution is negligible. We take n sufficiently large so that $(nt^{(N-1)} - nt^{(N-2)}) \vee (nt^{(N)} - nt^{(N-1)}) \ge b_n$ and

$$\begin{aligned}\overset{\le}{\hat{\mathbf{c}}}{}^{(N)}_{n\mathbf{T}}(\mathbf{k}_n) = &\sum_{\substack{I\ni\, nt^{(N-1)}\\ |I|\le b_n}} \hat{\mathbf{c}}^{(N-1)}_{(nt^{(1)},\ldots,nt^{(N-2)},I_1)}\big(k_n^{(1)},\ldots,k_n^{(N-1)}\big)\ \times\ \hat{c}_{nT-I_2}(k_n^{(N)})\\ &\times\sum_{w\in\mathcal{W}_{|I|}} \exp\big\{ik_n^{(N-1)}\cdot\, w_{nt^{(N-1)}-I_1} + ik_n^{(N)}\cdot\,(w_{I_2-I_1} - w_{nt^{(N-1)}-I_1})\big\}\; W(w)\, J_{[0,|I|]}(w).\end{aligned} \tag{4.6}$$

We use $\mathrm{e}^y = 1 + O(|y|^{\alpha\wedge 1})$ and (4.3) to see that the second line in (4.6) is equal to

$$\sum_x \big(1 + O(|f_\alpha(n)\,x|^{\alpha\wedge 1})\big)\,\pi_{|I|}(x). \tag{4.7}$$

By the induction hypothesis,

$$\begin{aligned}&\hat{\mathbf{c}}^{(N-1)}_{(nt^{(1)},\ldots,nt^{(N-2)},I_1)}\big(k_n^{(1)},\ldots,k_n^{(N-1)}\big)\\ &\qquad= \hat{c}_{I_1}(0)\,\exp\left\{-K_\alpha\sum_{j=1}^{N-1}|k^{(j)}|^{\alpha\wedge 2}\,(t^{(j)} - t^{(j-1)})\right\} + o(1)\end{aligned} \tag{4.8}$$

and

$$\hat{c}_{nT-I_2}(k_n^{(N)}) = \hat{c}_{nT-I_2}(0)\,\exp\big\{-K_\alpha\,|k^{(N)}|^{\alpha\wedge 2}\,(t^{(N)} - t^{(N-1)})\big\} + o(1), \tag{4.9}$$

where the error terms are uniform in $|I| \le b_n$.

Substituting (4.7)–(4.9) into (4.6) yields

$$\overset{\le}{\hat{\mathbf{c}}}{}^{(N)}_{n\mathbf{T}}(\mathbf{k}_n) = \exp\left\{-K_\alpha \sum_{j=1}^{N} |k^{(j)}|^{\alpha\wedge 2}\,(t^{(j)} - t^{(j-1)})\right\} \overset{\le}{\hat{\mathbf{c}}}{}^{(N)}_{n\mathbf{T}}(\mathbf{0}) + \Theta + o(1) \tag{4.10}$$

where

$$|\Theta| \le \sum_{\substack{I \ni nt^{(N-1)} \\ |I| \le b_n}} \hat{c}_{I_1}(0)\,\hat{c}_{nT-I_2}(0) \sum_x O\big(|f_\alpha(n)\,x|^{\alpha\wedge 1}\big)\,\pi_{|I|}(x). \tag{4.11}$$

In (4.11) there are precisely $m-1$ ways to choose the interval $I \ni nt^{(N-1)}$ of length $|I| = m$. We further bound

$$\begin{aligned}\frac{|\Theta|}{\hat{c}_{nT}(0)} &\le \sum_{m=1}^{b_n} m \sum_x O\big(|f_\alpha(n)\,x|^{\alpha\wedge 1}\big)\,\pi_m(x)\,z_c^m \\ &\le O(|f_\alpha(n)|^{\alpha\wedge 1}\,b_n) \sum_{m=1}^{\infty} \sum_x |x|^{\alpha\wedge 2}\,|\pi_m(x)|\,z_c^m = o(1), \end{aligned} \tag{4.12}$$

where Corollary 2.2 is used in the first inequality, $m \le b_n$ in the second, and the last estimate uses (1.31) and Lemma 2.4. Recalling $\hat{\mathbf{c}}^{(N)}_{n\mathbf{T}}(\mathbf{k}) = \overset{\le}{\hat{\mathbf{c}}}{}^{(N)}_{n\mathbf{T}}(\mathbf{k}) + \overset{>}{\hat{\mathbf{c}}}{}^{(N)}_{n\mathbf{T}}(\mathbf{k})$,

$$\frac{\overset{\le}{\hat{\mathbf{c}}}{}^{(N)}_{n\mathbf{T}}(\mathbf{k}_n)}{\hat{c}_{nT}(0)} = \exp\left\{-K_\alpha \sum_{j=1}^{N} |k^{(j)}|^{\alpha\wedge 2}\,(t^{(j)} - t^{(j-1)})\right\}\left(1 - \frac{\overset{>}{\hat{\mathbf{c}}}{}^{(N)}_{n\mathbf{T}}(\mathbf{0})}{\hat{c}_{nT}(0)}\right) + \frac{|\Theta|}{\hat{c}_{nT}(0)} + \frac{\overset{>}{\hat{\mathbf{c}}}{}^{(N)}_{n\mathbf{T}}(\mathbf{k}_n)}{\hat{c}_{nT}(0)}, \tag{4.13}$$

and it suffices to show $\overset{>}{\hat{\mathbf{c}}}{}^{(N)}_{n\mathbf{T}}(\mathbf{k}_n)/\hat{c}_{nT}(0) = o(1)$ as $n \to \infty$. By bounding $|\,\mathrm{e}^{i\mathbf{k}_n\cdot\Delta w(n\mathbf{T})}\,| \le 1$ in (4.5), and using again (4.3) and Corollary 2.2,

$$\frac{\overset{>}{\hat{\mathbf{c}}}{}^{(N)}_{n\mathbf{T}}(\mathbf{k}_n)}{\hat{c}_{nT}(0)} \le O(1) \sum_{m=b_n+1}^{\infty} m \sum_x |\pi_m(x)|\,z_c^m, \tag{4.14}$$

which vanishes as $n \to \infty$ by (2.58) and the fact that $b_n \to \infty$ as $n \to \infty$. We have completed the advancement of the induction, and all error terms occurring are uniform in sequences $g = (g_n)$ that satisfy $0 \le g_n \le b_n$. This proves (1.32) for all $N \ge 1$. □

5 Tightness

In this section we prove tightness of the sequence X_n, the missing piece for the proof of Theorem 1.5. Indeed, tightness is implied by Theorem 1.4 and the following tightness criterion.

Proposition 5.1 (Tightness criterion [1])**.** *The sequence $\{X_n\}$ is tight in $D([0,1],\mathbb{R}^d)$ if the limiting process X has a.s. no discontinuity at $t = 1$ and there exist constants $C > 0$, $r > 0$ and $a > 1$ such that for $0 \le t_1 < t_2 < t_3 \le 1$ and for all n,*

$$\langle |X_n(t_2) - X_n(t_1)|^r\,|X_n(t_3) - X_n(t_2)|^r \rangle_n \le C|t_3 - t_1|^a. \tag{5.1}$$

This proposition is a slight modification of Billingsley [1, Theorem 15.6], where (15.21) is replaced by the stronger moment condition on the bottom of page 128 (both references to Billingsley [1]).

Corollary 5.2 (Tightness)**.** *The sequence* $\{X_n\}$ *in* (1.23) *is tight in* $D([0,1],\mathbb{R}^d)$*.*

Proof. We first remark that α-stable Lèvy motion indeed has a version without jumps at fixed times, and hence no discontinuity at $t = 1$ occurs, see e.g. [11, Theorem 13.1]. Fix $r = 3/4\,(\alpha \wedge 2)$ (in fact, any choice $r \in ((\alpha \wedge 2)/2, \alpha \wedge 2)$ is possible). Again we write nt for $\lfloor nt \rfloor$, for brevity. The left hand side of (5.1) can be written as

$$\frac{f_\alpha(n)^{2r}}{c_n\,(2dK_\alpha)^{2r/(\alpha\wedge 2)}} \sum_{w\in\mathcal{W}_n} |w(nt_2)-w(nt_1)|^r\,|w(nt_3)-w(nt_2)|^r\,W(w)\,K_{[0,n]}(w), \tag{5.2}$$

where $K_{[0,n]}(w)$ was defined in (4.2). Since

$$K_{[0,n]}(w) \le K_{[0,nt_1]}(w)\,K_{[nt_1,nt_2]}(w)\,K_{[nt_2,nt_3]}(w)\,K_{[nt_3,n]}(w) \tag{5.3}$$

and, by Corollary 2.2,

$$c_n^{-1} \le O(1)\;c_{nt_1}^{-1}\,c_{nt_2-nt_1}^{-1}\,c_{nt_3-nt_2}^{-1}\,c_{n-nt_3}^{-1}, \tag{5.4}$$

we can bound (5.2) from above by

$$\begin{aligned} &\langle |X_n(t_2)-X_n(t_1)|^r\,|X_n(t_3)-X_n(t_2)|^r\rangle_n \\ &\quad\le\quad O(1)\,f_\alpha(n)^{2r}\,\frac{1}{c_{nt_2-nt_1}} \sum_{w\in\mathcal{W}_{nt_2-nt_1}} |w(nt_2-nt_1)|^r \\ &\qquad\qquad \times\,\frac{1}{c_{nt_3-nt_2}} \sum_{w\in\mathcal{W}_{nt_3-nt_2}} |w(nt_3-nt_2)|^r \\ &\quad=\quad O(1)\,f_\alpha(n)^{2r}\,\Big(\xi^{(r)}(nt_2-nt_1)\Big)^r\,\Big(\xi^{(r)}(nt_3-nt_2)\Big)^r. \end{aligned} \tag{5.5}$$

By Theorem 1.4 and (1.22),

$$\Big(\xi^{(r)}(nt^*-nt_*)\Big)^r \le O(1)\,f_\alpha(n)^{-r}\,(t^*-t_*)^{r/(\alpha\wedge 2)} \tag{5.6}$$

for any $0 \le t_* < t^* \le 1$, so that

$$\langle |X_n(t_2)-X_n(t_1)|^r\,|X_n(t_3)-X_n(t_2)|^r\rangle_n \le O(1)\,(t_3-t_1)^{2r/(\alpha\wedge 2)} = O(1)\,(t_3-t_1)^{3/2}. \tag{5.7}$$

This proves tightness of the sequence $\{X_n\}$. □

Proof of Theorem 1.5. The convergence in distribution in Theorem 1.5 is implied by convergence of finite dimensional distributions and tightness of the sequence X_n, see e.g. [1, Theorem 15.1]. Hence, Theorem 1.6 and Corollary 5.2 imply Theorem 1.5. □

A Aymptotics of the step distribution

Proof of (1.12)*.* We consider separately the cases $\alpha > 2$ and $\alpha \le 2$.

Case $\alpha > 2$. Since $\cos(t) = 1 - t^2/2 + o(t^2)$ as $t \to 0$, we have

$$\begin{aligned}\hat{D}(k) &= \sum_{x\in\mathbb{Z}^d} \mathrm{e}^{ik\cdot x}\, D(x) = \sum_{x\in\mathbb{Z}^d} \cos(k\cdot x)\, D(x)\\ &= \sum_{x\in\mathbb{Z}^d} D(x) - \sum_{x\in\mathbb{Z}^d} \left(\frac{1}{2}\sum_{j=1}^{d}(k_j\, x_j)^2 + o\big(|k\cdot x|^2\big)\right) D(x)\\ &= 1 - \frac{1}{2}\sum_{x\in\mathbb{Z}^d}\left(\sum_{j=1}^{d} k_j^2\, x_j^2 + 2\sum_{1\le j\le n\le d} k_j\, k_n\, x_j\, x_n\right) D(x) + o\big(|k|^2\big).\end{aligned} \tag{A.1}$$

By reflection symmetry,

$$\sum_{x\in\mathbb{Z}^d}\sum_{1\le j\le n\le d} k_j\, k_n\, x_j\, x_n\, D(x) = 0.$$

Furthermore, as D is symmetric under rotations by ninety degree,

$$\sum_{x\in\mathbb{Z}^d} x_1^2\, D(x) = \sum_{x\in\mathbb{Z}^d} x_2^2\, D(x) = \cdots = \frac{1}{d}\sum_{x\in\mathbb{Z}^d} |x|^2\, D(x),$$

so that

$$\hat{D}(k) = 1 - \frac{|k|^2}{2d}\sum_{x\in\mathbb{Z}^d} |x|^2\, D(x) + o\big(|k|^2\big). \tag{A.2}$$

Setting $\sum_{x\in\mathbb{Z}^d} |x|^2\, D(x) = 2d\, v_\alpha$ proves the claim.

Case $\alpha \le 2$. The case $\alpha \le 2$ requires a more elaborate calculation. This part of the proof is adapted from Koralov and Sinai [12, Lemma 10.18], who consider the one-dimensional continuous case. We can write $D(x)$ as

$$D(x) = c\,\frac{1+g(x)}{|x|^{d+\alpha}}, \tag{A.3}$$

where c is a positive constant and g is a bounded function on $\mathbb{R}^d$ obeying $g(x) \to 0$ as $|x| \to 0$. By our assumption, g is rotation invariant for $|x| > M$. We might limit ourselves to the case $|k| \le 1/M$ and split the sum defining $\hat{D}(k)$ as

$$\hat{D}(k) = \sum_{|x|\le M} \mathrm{e}^{ik\cdot x}\, D(x) + \sum_{M<|x|\le 1/|k|} \mathrm{e}^{ik\cdot x}\, D(x) + \sum_{1/|k|<|x|} \mathrm{e}^{ik\cdot x}\, D(x). \tag{A.4}$$

Denote by S_1, S_2 and S_3 the three sums on the right hand side of (A.4). A calculation similar to (A.2) shows

$$S_1 = \sum_{|x|\le M} D(x) + O\big(|k|^2\big) = \sum_{|x|\le M} D(x) + \begin{cases} o\big(|k|^\alpha\big) & \text{if } \alpha < 2,\\ o\big(|k|^2 \log \frac{1}{|k|}\big) & \text{if } \alpha = 2.\end{cases} \tag{A.5}$$

For S_3 we substitue x by $y/|k|$ yielding

$$S_3 = |k|^{d+\alpha} \sum_{\substack{y\in|k|\mathbb{Z}^d\\ |y|>1}} c\,\frac{1+g(y/|k|)}{|y|^{d+\alpha}}\, \mathrm{e}^{ie_k\cdot y}, \tag{A.6}$$

where $e_k = k/|k|$ is the unit vector in direction k. By translation invariance of g and Riemann sum approximation we obtain

$$S_3 = |k|^\alpha \left(\int_{|y|\geq 1} c \, \frac{1+g(y/|k|)}{|y|^{d+\alpha}} \, \mathrm{e}^{iy_1} \, \mathrm{d}y + o(1) \right), \tag{A.7}$$

with y_1 being the first coordinate of the vector y. Finally, the dominated convergence theorem obtains

$$S_3 = |k|^\alpha c \int_{|y|\geq 1} \frac{\mathrm{e}^{iy_1}}{|y|^{d+\alpha}} \, \mathrm{d}y + o\big(|k|^\alpha\big), \tag{A.8}$$

where the integral contributes towards v_α.

Since D is symmetric, the sum defining S_2 can be split as

$$S_2 = \sum_{M<|x|\leq 1/|k|} \Big(\mathrm{e}^{ik\cdot x} - 1 - ik\cdot x\Big) \, D(x) + \sum_{M<|x|} D(x) - \sum_{1/|k|<|x|} D(x). \tag{A.9}$$

Consider first the last sum. As before, we substitute x by $y/|k|$, use Riemann sum approximation and finally dominated convergence to obtain

$$\sum_{1/|k|<|x|} D(x) = |k|^{\alpha+d} \sum_{\substack{y\in|k|\mathbb{Z}^d \\ |y|>1}} c \, \frac{1+g(y/|k|)}{|y|^{d+\alpha}} = |k|^\alpha c \int_{|y|\geq 1} \frac{\mathrm{e}^{iy_1}}{|y|^{d+\alpha}} \, \mathrm{d}y + o\big(|k|^\alpha\big). \tag{A.10}$$

It remains to understand the first sum on the right hand side of (A.9). We treat this term with the same recipe as above yielding

$$\begin{aligned} &\sum_{M<|x|\leq 1/|k|} \Big(\mathrm{e}^{ik\cdot x} - 1 - ik\cdot x\Big) \, D(x) \\ &= |k|^\alpha c \int_{|k|\, M\leq|y|\leq 1} \frac{1+g(y/|k|)}{|y|^{d+\alpha}} \, \big(y_1^2 + O\big(|y_1|^{2+\varepsilon}\big)\big) \, \mathrm{d}y + o\big(|k|^\alpha\big). \end{aligned} \tag{A.11}$$

For $\alpha < 2$ the integral is uniformly bounded in k, and hence the dominated convergence theorem can be used one more time to obtain the desired asymptotics. However, if $\alpha = 2$ then the dominating contribution towards (A.11) is

$$|k|^2 \int_{|k|\, M\leq|y|\leq 1} \frac{y_1^2}{|y|^{d+\alpha}} \, \mathrm{d}y = \frac{|k|^2}{d} \int_{|k|\, M\leq|y|\leq 1} \frac{1}{|y|^d} \, \mathrm{d}y = \mathrm{const} \, |k|^2 \left(\log\frac{1}{|k|} + \log\frac{1}{M} \right). \tag{A.12}$$

Summarizing our calculations, we obtain

$$\hat{D}(k) = \sum_{x\in\mathbb{Z}^d} D(x) - v_\alpha |k|^\alpha + o\big(|k|^\alpha\big) = 1 - v_\alpha |k|^\alpha + o\big(|k|^\alpha\big) \tag{A.13}$$

for $\alpha < 2$, and

$$\hat{D}(k) = 1 - v_\alpha |k|^2 \log\frac{1}{|k|} + o\left(|k|^2 \log\frac{1}{|k|} \right) \tag{A.14}$$

for $\alpha = 2$, where v_α is composed of the various integrals arising during the proof. □

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