CONVERGENCE OF OPTION REWARDS FOR MARKOV TYPE PRICE PROCESSES MODULATED BY STOCHASTIC INDICES

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Abstract. A general price process represented by a two-component Markov process is considered. Its first component is interpreted as a price process and the second one as an index process modulating the price component. American type options with pay-off functions, which admit power type upper bounds, are studied. Both the transition characteristics of the price processes and the pay-off functions are assumed to depend on a perturbation parameter $\delta \geq 0$ and to converge to the corresponding limit characteristics as $\delta \to 0$. In the first part of the paper, asymptotically uniform skeleton approximations connecting reward functionals for continuous and discrete time models are given. In the second part of the paper, these skeleton approximations are used for getting results about the convergence of reward functionals for American type options for perturbed price processes in discrete and continuous time. Examples related to modulated exponential price processes with independent increments are given.

1. Introduction

This paper is devoted to studies of conditions for convergence of reward functionals for American type options under Markov type price processes modulated by stochastic indices.

The idea behind these models is that the stochasticity of these models depends on the global market environment through some indicators or indices. One example would be a model where the price process depends on the level of a market index reflecting a bullish, bearish, or stable market behaviour. Another example is a model where the overall market volatility is indicating high, moderate, or low volatility environment.

The main objective of the present paper is to study the continuous time optimal stopping problem originating from American option pricing under these processes and to derive approximations of the reward functionals for the continuous time models by imbedded discrete time models and the convergence of these reward functionals.

Markov type price processes modulated by stochastic indices and option pricing for such processes have been studied in [1, 4, 5, 10, 11, 16, 21, 22, 23, 24, 25, 30, 33, 34, 35, 40, 42, 46, 56, 59, 60, 61].

Date: October 13, 2008.

2000 *Mathematics Subject Classification.* Primary 60J05, 60H10; Secondary 91B28, 91B70.

Key words and phrases. Reward, convergence, optimal stopping, American option, skeleton approximation, Markov process, price process, modulation, stochastic index.

Part of this research has been done during the time while H. Jönsson was an EU-Marie Curie Intra-European Fellow with funding from the European Community's Sixth Framework Programme (MEIF-CT-2006-041115).

We also would like to refer the books [42, 44, 46, 47, 48] for an account of various models of stochastic price processes and optimal stopping problems for options. The books [31, 50] contain descriptions of a variety of models of stochastic processes with semi-Markov modulation (switchings).

We consider the variant of price processes modulated by stochastic indices as was introduced in [33, 34, 35]. The object of our study is a two-component process $Z^{(\delta)}(t) = (Y^{(\delta)}(t), X^{(\delta)}(t))$, where the first component $Y^{(\delta)}(t)$ is a real-valued càdlàg process and the second component $X^{(\delta)}(t)$ is a measurable process with a general metric phase space. The first component is interpreted as a log-price process while the second component is interpreted as a stochastic index modulating the price process.

As was mentioned above, the process $X^{(\delta)}(t)$ can be a global price index "modulating" market prices, or a jump process representing some market regime index. The stochastic index can indicate, for example, growing, declining, or stable market situation, or high, moderate, or low level of volatility, or describe credit rating dynamics modulating the price process $Y^{(\delta)}(t)$.

The log-price process $Y^{(\delta)}(t)$ as well as the corresponding price process $S^{(\delta)}(t) = e^{Y^{(\delta)}(t)}$ are themselves not assumed to be Markov processes but the two-component process $Z^{(\delta)}(t)$ is assumed to be a continuous time inhomogeneous two-component Markov process. Thus, the component $X^{(\delta)}(t)$ represents information which in addition to the information represented by the log-price process $Y^{(\delta)}(t)$ makes the two-component process $(Y^{(\delta)}(t), X^{(\delta)}(t))$ a Markov process.

In the literature, the values of options in discrete time markets have been used to approximate the value of the corresponding option in continuous time. Convergence of European option values for the Binomial tree model to the Black-Scholes value for geometrical Brownian motion was shown in the seminal paper [8].

Further results on convergence of the values of European and American options can be found in [2, 3, 7, 9, 15, 27, 36, 39, 41, 43, 56, 59]. In particular, conditions for convergence of the values for American options in a discrete-time model to the value of the option in a continuous-time model, under the assumption that the sequence of processes describing the value of the underlying asset converge weakly to a diffusion is given in [2]. There are also results presented for the case when the limiting process is a diffusion with discrete jumps at fixed dates. Recent results on weak convergence in financial markets based on martingale methods, for both European and American type options, are presented in [43]. We would also like to mention the papers [12, 13, 14, 17, 18, 19, 37, 38], where convergence in optimal stopping problems are studied for general Markov processes.

It is well known that there does not exist explicit formulas for optimal rewards for American type options even for standard payoff functions and simple price processes. The methods used in this case are based on approximations of price processes by simpler ones, for example Binomial tree price processes. Models with complex non-standard payoff functions may also require to approximate these payoffs by simpler ones, for example by piece-wise linear payoff functions. Results concerning convergence of rewards for perturbed price processes play here a crucial role and serve as a substantiation for the corresponding approximation algorithms.

Our results differ from the results in the aforementioned papers by generality of models for price processes and non-standard pay-off functions as well as conditions of convergence.

We consider very general models of càdlàg Markov type price processes modulated by stochastic indices. So far, conditions of convergence for rewards were not investigated for such general models.

We consider so called triangular array models, in which the processes under consideration depend on a small perturbation parameter $\delta \geq 0$. It is assumed that the transition probabilities of the perturbed processes $Z^{(\delta)}(t)$ converge in some sense to the corresponding transition probabilities of the limiting process $Z^{(0)}(t)$ as $\delta \to 0$. That is, the processes $Z^{(\delta)}(t)$ can be considered to be a perturbed modification of the corresponding limit process $Z^{(0)}(t)$. An example is the Binomial tree model converging to the corresponding geometrical Brownian motion.

We do not involve directly the condition of finite-dimensional weak convergence for the corresponding processes, which is characteristic for general limit theorems for Markov type processes. Our conditions also do not use any assumptions about convergence of auxiliary processes in probability which is characteristic for martingale based methods. The latter type of conditions usually do involve some special imbedding constructions replacing perturbed and limiting processes on one probability space that may be difficult to realise for complex models of price processes.

Instead of the conditions mentioned above, we introduce new general conditions of local uniform convergence for the corresponding transition probabilities. These conditions do imply finite-dimensional weak convergence for the price processes and can be effectively used in applications. We also use effective conditions of exponential moment compactness for the increments of the log-price processes, which are natural for applications to Markov type processes.

We also consider American type options with non-standard payoff functions $g^{(\delta)}(t,s)$, which are assumed to be non-negative functions with not more than polynomial growth. The pay-off functions are also assumed to be perturbed and converge to the corresponding limit pay-off functions $g^{(0)}(t,s)$ as $\delta \to 0$. This is an useful assumption. For example, it has been shown in [33] how one can approximate reward functions for options with general convex payoff functions by reward functions for options with more simple piece-wise linear payoff functions.

As is well known, the optimal stopping moment for the exercise of an American option has the form of the first hitting time into the optimal price-time stopping domain. It is worth to note that, under the general assumptions on the payoff functions listed above, the structure of the reward functions and the corresponding optimal stopping domain can be rather complicated. For example, as shown in [26, 28, 29, 33, 34, 35] the optimal stopping domains can possess a multi-threshold structure.

Despite of this complexity, we can prove convergence of the reward functionals which represent the optimal expected rewards in the class of all Markov stopping moments.

Our approach is based on the use of skeleton approximations for price processes given in [34], where continuous time reward functionals have been approximated by their analogues for imbedded skeleton type discrete time models. In this paper, skeleton approximations were given in the form suitable for applications to continuous price processes. We improve these approximations to the form that let us apply them to càdlàg price processes and, moreover, give them in the form asymptotically uniform as the perturbation parameter $\delta \to 0$. Another important element of our approach is a recursive method for asymptotic analysis of reward functionals for discrete time models developed in [27]. Key examples of price processes modulated by semi-Markov indices and corresponding convergence results are also given in [56, 59].

The outline of the paper is as follows. In Section 2, we introduce Markov type price processes modulated by stochastic indices and American type options with general payoff functions. Section 3 contains results about asymptotically uniform skeleton approximations. These results have their own value and let one approximate reward functionals for continuous time price processes by similar functionals for simpler imbedded discrete time models. In Section 4, results concerning conditions for convergence of reward functionals in discrete time models are given. Section 5 presents general results on convergence of reward functionals for American type options. In Sections 6 and 7, we illustrate our general convergence results by applying them to exponential price processes with independent increments and exponential Lévy price processes modulated by semi-Markov stochastic indices, and some other models.

This paper is an improved and extended version of the report [54]. The main results are also presented in a short paper [55].

2. American type options under price processes modulated by stochastic indices

Let $Z^{(\delta)}(t) = (Y^{(\delta)}(t), X^{(\delta)}(t)), t \geq 0$ be, for every $\delta \geq 0$, a Markov process with the phase space space $\mathbb{Z} = \mathbb{R}_1 \times \mathbb{X}$, where $\mathbb{R}_1$ is the real line and $\mathbb{X}$ is a Polish space (a separable, complete metric space), transition probabilities $P^{(\delta)}(t, z, t+u, A)$ and an initial distribution $P^{(\delta)}(A)$.

It is useful to note that $\mathbb{Z}$ is also a Polish space with the metrics $d_{\mathbb{Z}}(z', z'') = (|y' - y''|^2 + d_{\mathbb{X}}(x', x'')^2)^{\frac{1}{2}}$, where $z' = (y', x'), z'' = (y'', x'')$, and $d_{\mathbb{X}}(x', x'')$ is the metrics in the space $\mathbb{X}$. The Borel σ–field $\mathcal{B}_{\mathbb{Z}} = \sigma(\mathcal{B}_1 \times \mathcal{B}_{\mathbb{X}})$, where $\mathcal{B}_1$ and $\mathcal{B}_{\mathbb{X}}$ are Borel σ–fields in $\mathbb{R}_1$ and $\mathbb{X}$, respectively, and the transition probabilities and the initial distribution are probability measures on $\mathcal{B}_{\mathbb{Z}}$.

The process $Z^{(\delta)}(t), t \geq 0$ is defined on a probability space $(\Omega^{(\delta)}, \mathcal{F}^{(\delta)}, \mathsf{P}^{(\delta)})$. Note that these spaces can be different for different δ, i.e., we consider a triangular array model.

We assume that the process $Z^{(\delta)}(t), t \geq 0$ is a measurable process, i.e., $Z^{(\delta)}(t, \omega)$ is a measurable function in $(t, \omega) \in [0, \infty) \times \Omega^{(\delta)}$. Also, we assume that the first component $Y^{(\delta)}(t), t \geq 0$ is a càdlàg process, i.e., a process that is almost surely continuous from the right and has limits from the left at all points $t \geq 0$.

We interpret the component $Y^{(\delta)}(t)$ as a log-price process and the component $X^{(\delta)}(t)$ as a stochastic index modulating the log-price process $Y^{(\delta)}(t)$.

Let us define the price process,

$$S^{(\delta)}(t) = \exp\{Y^{(\delta)}(t)\}, \ t \geq 0, \tag{1}$$

and consider the two-component process $V^{(\delta)}(t) = (S^{(\delta)}(t), X^{(\delta)}(t)), \ t \geq 0$. Due to the one-to-one mapping and continuity properties of exponential function, $V^{(\delta)}(t)$ is also a measurable Markov process, with the phase space $\mathbb{V} = (0, \infty) \times \mathbb{X}$ and its first component $S^{(\delta)}(t), t \geq 0$ is a càdlàg process. The process $V^{(\delta)}(t)$ has the transition probabilities $Q^{(\delta)}(t, v, t+u, A) = P^{(\delta)}(t, z, t+u, \ln A)$, and the initial distribution $Q^{(\delta)}(A) = P^{(\delta)}(\ln A)$, where $v = (s, x) \in \mathbb{V}, z = (\ln s, x) \in \mathbb{Z}$, and

$\ln A = \{z = (y, x) : y = \ln s, (s, x) \in A\}, A \in \mathcal{B}_{\mathbb{V}} = \sigma(\mathcal{B}_+ \times \mathcal{B}_{\mathbb{X}})$, where $\mathcal{B}_+$ is the Borel σ-algebra of subsets of $(0, \infty)$.

Let $g^{(\delta)}(t, s), (t, s) \in [0, \infty) \times (0, \infty)$ be, for every $\delta \geq 0$, a pay-off function. We assume that $g^{(\delta)}(t, s)$ is a nonnegative measurable (Borel) function.

The typical example of pay-off function is

$$g^{(\delta)}(t, s) = e^{-R_t^{(\delta)}} a_t^{(\delta)} [s - K_t^{(\delta)}]_+, \tag{2}$$

where $a_t^{(\delta)}, t \geq 0$ and $K_t^{(\delta)}, t \geq 0$ are two nonnegative measurable functions, and $R_t^{(\delta)}, t \geq 0$ is a nondecreasing function with $R_0^{(\delta)} = 0$.

Here, $R_t^{(\delta)}$ is accumulated continuously compounded riskless interest rate. Typically, $R_t^{(\delta)} = \int_0^t r^{(\delta)}(s) ds$, where $r^{(\delta)}(s) \geq 0$ is a nonnegative measurable function representing an instant riskless interest rate at moment s.

As far as functions $a_t^{(\delta)}, t \geq 0$ and $K_t^{(\delta)}, t \geq 0$ are concerned, these are parameters of an option contract. The case, where $a_t^{(\delta)} = a^{(\delta)}$ and $K_t^{(\delta)} = K^{(\delta)}$ do not depend on t, corresponds to the standard American call option.

Let $\mathcal{F}_t^{(\delta)}, t \geq 0$ be a natural filtration of σ-fields, associated with the process $Z^{(\delta)}(t), t \geq 0$. We shall consider Markov moments $\tau^{(\delta)}$ with respect to the filtration $\mathcal{F}_t^{(\delta)}, t \geq 0$. It means that $\tau^{(\delta)}$ is a random variable which takes values in $[0, \infty]$ and with the property $\{\omega : \tau^{(\delta)}(\omega) \leq t\} \in \mathcal{F}_t^{(\delta)}, t \geq 0$.

It is useful to note that $\mathcal{F}_t^{(\delta)}, t \geq 0$ is also a natural filtration of σ-fields, associated with process $V^{(\delta)}(t), t \geq 0$.

Let us denote $\mathcal{M}_{max,T}^{(\delta)}$, the class of all Markov moments $\tau^{(\delta)} \leq T$, where $T > 0$, and consider a class of Markov moments $\mathcal{M}_T^{(\delta)} \subseteq \mathcal{M}_{max,T}^{(\delta)}$.

Our goal is to maximize an expected pay-off for a given stopping moment over a class $\mathcal{M}_T^{(\delta)}$,

$$\Phi(\mathcal{M}_T^{(\delta)}) = \sup_{\tau^{(\delta)} \in \mathcal{M}_T^{(\delta)}} \mathsf{E} g^{(\delta)}(\tau^{(\delta)}, S^{(\delta)}(\tau^{(\delta)})). \tag{3}$$

The reward functional $\Phi(\mathcal{M}_T^{(\delta)})$ can take the value $+\infty$. However, we shall impose below conditions on price processes and pay-off functions which will guarantee that, for all δ small enough, $\Phi(\mathcal{M}_{max,T}^{(\delta)}) < \infty$.

Note that we do not impose on the pay-off functions $g^{(\delta)}(t, s)$ any monotonicity conditions. However, it is worth noting that the cases where the pay-off function $g^{(\delta)}(t, s)$ is non-decreasing or non-increasing in argument s correspond to call and put American type options, respectively.

The first condition assumes the absolute continuity of pay-off functions and imposes power type upper bounds on their partial derivatives:

$\mathbf{A_1}$: There exist $\delta_0 > 0$ such that for every $0 \leq \delta \leq \delta_0$: **(a)** function $g^{(\delta)}(t, s)$ is absolutely continuous in t with respect to the Lebesgue measure for every fixed $s \in (0, \infty)$ and in s with respect to the Lebesgue measure for every fixed $t \in [0, T]$; **(b)** for every $s \in (0, \infty)$, the partial derivative $|\frac{\partial g^{(\delta)}(t,s)}{\partial t}| \leq K_1 + K_2 s^{\gamma_1}$ for almost all $t \in [0, T]$ with respect to the Lebesgue measure, where $0 \leq K_1, K_2 < \infty$ and $\gamma_1 \geq 0$; **(c)** for every $t \in [0, T]$, the partial derivative $|\frac{\partial g^{(\delta)}(t,s)}{\partial s}| \leq K_3 + K_4 s^{\gamma_2}$ for almost all $s \in (0, \infty)$ with respect to the Lebesgue measure, where $0 \leq K_3, K_4 < \infty$ and $\gamma_2 \geq 0$; **(d)**

for every $t \in [0, T]$, the function $g^{(\delta)}(t, 0) = \overline{\lim}_{s \to 0} g^{(\delta)}(t, s) \leq K_5$, where $0 \leq K_5 < \infty$.

Note that condition $\mathbf{A_1}$ **(a)** admits the case where the corresponding partial derivatives exist in points from $[0, T]$ or $(0, \infty)$, respectively, except some subsets with zero Lebesgue measures, while conditions $\mathbf{A_1}$ **(b)** and **(c)** admit the case where the corresponding upper bounds hold in points from the sets where the corresponding derivatives exist except some subsets (of these sets) with zero Lebesgue measures.

It is useful to note that condition $\mathbf{A_1}$ implies that function $g^{(\delta)}(t, s)$ is jointly continuous in arguments $t \in [0, T]$ and $s \in (0, \infty)$.

For example, condition $\mathbf{A_1}$ holds for the pay-off function given in (2) if functions $R_t^{(\delta)}, a_t^{(\delta)}$ and $K_t^{(\delta)}$ have bounded first derivatives in the interval $[0, T]$. In this case $\gamma_1 = 1$ and $\gamma_2 = 0$.

Taking into account formula $S^{(\delta)}(t) = e^{Y^{(\delta)}(t)}$ connecting the price process $S^{(\delta)}(t)$ and the log-price process $Y^{(\delta)}(t)$, condition $\mathbf{A_1}$ can be re-written in the equivalent form in terms of function $g^{(\delta)}(t, e^y), (t, y) \in [0, T] \times \mathbb{R}_1$.

Let us denote $g_1^{(\delta)}(t, s) = \frac{\partial g^{(\delta)}(t,s)}{\partial t}$ and $g_2^{(\delta)}(t, s) = \frac{\partial g^{(\delta)}(t,s)}{\partial s}$. Then $\frac{\partial g^{(\delta)}(t,e^y)}{\partial t}$ $= g_1^{(\delta)}(t, e^y)$ and $\frac{\partial g^{(\delta)}(t,e^y)}{\partial y} = g_2^{(\delta)}(t, e^y)e^y$, and the equivalent variant of condition $\mathbf{A_1}$ takes the following form:

$\mathbf{A_1'}$: There exist $\delta_0 > 0$ such that for every $0 \leq \delta \leq \delta_0$: **(a)** function $g^{(\delta)}(t, e^y)$ is absolutely continuous upon t with respect to the Lebesgue measure for every fixed $y \in \mathbb{R}_1$ and in y with respect to the Lebesgue measure for every fixed $t \in [0, T]$; **(b)** for every $y \in \mathbb{R}_1$, the partial derivative $|\frac{\partial g^{(\delta)}(t,e^y)}{\partial t}| \leq K_1 + K_2 e^{\gamma_1 y}$ for almost all $t \in [0, T]$ with respect to the Lebesgue measure, where $0 \leq K_1, K_2 < \infty$ and $\gamma_1 \geq 0$; **(c)** for every $t \in [0, T]$, the partial derivative $|\frac{\partial g^{(\delta)}(t,e^y)}{\partial y}| \leq (K_3 + K_4 e^{\gamma_2 y})e^y$ for almost all $y \in \mathbb{R}_1$ with respect to the Lebesgue measure, where $0 \leq K_3, K_4 < \infty$ and $\gamma_2 \geq 0$; **(d)** for every $t \in [0, T]$, the function $g^{(\delta)}(t, -\infty) = \overline{\lim}_{y \to -\infty} g^{(\delta)}(t, e^y) \leq K_5$, where $0 \leq K_5 < \infty$.

As usual we use notations $\mathsf{E}_{z,t}$ and $\mathsf{P}_{z,t}$ for expectations and probabilities calculated under condition that $Z^{(\delta)}(t) = z$.

Let us define, for $\beta, c, T > 0$, an exponential moment modulus of compactness for the càdlàg process $Y^{(\delta)}(t), t \geq 0$,

$$\Delta_\beta(Y^{(\delta)}(\cdot), c, T) = \sup_{0 \leq t \leq t+u \leq t+c \leq T} \sup_{z \in \mathbb{Z}} \mathsf{E}_{z,t}(e^{\beta |Y^{(\delta)}(t+u) - Y^{(\delta)}(t)|} - 1).$$

We need also the following conditions of exponential moment compactness for log-price processes:

$\mathbf{C_1}$: $\lim_{c \to 0} \overline{\lim}_{\delta \to 0} \Delta_\beta(Y^{(\delta)}(\cdot), c, T) = 0$ for some $\beta > \gamma = \max(\gamma_1, \gamma_2 + 1)$, where γ_1 and γ_2 are the parameters introduced in condition $\mathbf{A_1}$,

and

$\mathbf{C_2}$: $\overline{\lim}_{\delta \to 0} \mathsf{E} e^{\beta |Y^{(\delta)}(0)|} < \infty$, where β is the parameter introduced in condition $\mathbf{C_1}$.

Let us get asymptotically uniform upper bounds for moments of the maximums of log-price and price processes. Explicit expressions for the constants are given in the proofs of the corresponding lemmas.

Lemma 1. *Let conditions* $\mathbf{C_1}$ *and* $\mathbf{C_2}$ *hold. Then, there exist* $0 < \delta_1 \leq \delta_0$ *and a constant* $L_1 < \infty$ *such that for every* $\delta \leq \delta_1$,

$$\mathsf{E} \exp\{\beta \sup_{0 \leq u \leq T} |Y^{(\delta)}(u)|\} \leq L_1. \tag{4}$$

Lemma 2. *Let conditions* $\mathbf{A_1}$, $\mathbf{C_1}$, *and* $\mathbf{C_2}$ *hold. Then, there exists a constant* $L_2 < \infty$ *such that for every* $\delta \leq \delta_1$,

$$\mathsf{E}(\sup_{0 \leq u \leq T} g^{(\delta)}(u, S^{(\delta)}(u)))^{\frac{\beta}{\gamma}} \leq L_2. \tag{5}$$

Proof of Lemma 1. Let us define the random variables

$$S^{(\delta)}_\beta(t) = \exp\{\beta \sup_{0 \leq u \leq t} |Y^{(\delta)}(u)|\}.$$

Note that

$$S^{(\delta)}_\beta(t) = \begin{cases} \exp\{\beta|Y^{(\delta)}(0)|\}, & \text{if} \quad t = 0, \\ \sup_{0 \leq u \leq t} \exp\{\beta|Y^{(\delta)}(u)|\}, & \text{if} \quad 0 < t \leq T. \end{cases} \tag{6}$$

Let us also introduce random variables

$$W^{(\delta)}_\beta[t', t''] = \sup_{t' \leq t \leq t''} \exp\{\beta|Y^{(\delta)}(t) - Y^{(\delta)}(t')|\}, \ 0 \leq t' \leq t'' \leq T.$$

Let us use a partition $\tilde{\Pi}_m = \{0 = v_{0,m} < \cdots < v_{m,m} = T\}$ of interval $[0, T]$ by points $v_{n,m} = nT/m, n = 0, \ldots, m$. Using equality (6) we can get the following inequalities $n = 1, \ldots m$,

$$\begin{aligned} S^{(\delta)}_\beta(v_{n,m}) &\leq S^{(\delta)}_\beta(v_{n-1,m}) + \sup_{v_{n-1,m} \leq u \leq v_{n,m}} \exp\{\beta|Y^{(\delta)}(u)|\} \\ &\leq S^{(\delta)}_\beta(v_{n-1,m}) + \exp\{\beta|Y^{(\delta)}(v_{n-1,m})|\} W^{(\delta)}_\beta[v_{n-1,m}, v_{n,m}] \\ &\leq S^{(\delta)}_\beta(v_{n-1,m})(W^{(\delta)}_\beta[v_{n-1,m}, v_{n,m}] + 1). \end{aligned} \tag{7}$$

Condition $\mathbf{C_1}$ implies that for any constant $e^{-\beta} < L_5 < 1$ one can choose $c = c(L_5) > 0$ and then $\delta_1 = \delta_1(c) \leq \delta_0$ such that for $\delta \leq \delta_1$,

$$\frac{\Delta_\beta(Y^{(\delta)}(\cdot), c, T) + 1}{e^\beta} \leq L_5. \tag{8}$$

Also condition $\mathbf{C_2}$ implies that δ_1 can be chosen in such a way that, for some constant $L_6 = L_6(\delta_1) < \infty$, the following inequality holds for $\delta \leq \delta_1$,

$$\mathsf{E} \exp\{\beta|Y^{(\delta)}(0)|\} \leq L_6. \tag{9}$$

The process $Y^{(\delta)}(t)$ is not a Markov process. Despite this, an analogue of the Kolmogorov inequality can be obtained by a slight modification of its standard proof for Markov processes (See, for example, [20]). Let us formulate it in the form of a lemma. Note that we do assume in this lemma that the two-component process $Z^{(\delta)}(t)$ is a Markov process.

Lemma 3. *Let* $a, b > 0$ *and for the process* $Y^{(\delta)}(t)$ *the following condition holds* $\sup_{z \in \mathbb{Z}} \mathsf{P}_{z,t}\{|Y^{(\delta)}(t'') - Y^{(\delta)}(t)| \geq a\} \leq L < 1$, $t' \leq t \leq t''$. *Then, for any point* $z_0 \in \mathbb{Z}$,

$$\mathsf{P}_{z_0,t'}\{ \sup_{t' \leq t \leq t''} |Y^{(\delta)}(t) - Y^{(\delta)}(t')| \geq a + b\} \leq \frac{1}{1 - L} \mathsf{P}_{z_0,t'}\{|Y^{(\delta)}(t'') - Y^{(\delta)}(t')| \geq b\}. \tag{10}$$

We refer to the report [49], where one can find the corresponding proof.

Let us use Lemma 3 to show that the following inequality holds for $\delta \leq \delta_1$,

$$\sup_{0 \leq t' \leq t'' \leq t'+c \leq T} \sup_{z \in \mathbb{Z}} \mathsf{E}_{z,t'} W_\beta^{(\delta)}[t', t''] \leq L_7, \tag{11}$$

where

$$L_7 = \frac{e^\beta (e^\beta - 1) L_5}{1 - L_5} < \infty. \tag{12}$$

Relation (8) implies that for every $\delta \leq \delta_1$,

$$\begin{aligned} &\sup_{0 \leq t' \leq t \leq t'' \leq t'+c \leq T} \sup_{z \in \mathbb{Z}} \mathsf{P}_{z,t}\{|Y^{(\delta)}(t'') - Y^{(\delta)}(t)| \geq 1\} \\ &\leq \sup_{0 \leq t' \leq t \leq t'' \leq t'+c \leq T} \sup_{z \in \mathbb{Z}} \frac{\mathsf{E}_{z,t} \exp\{\beta |Y^{(\delta)}(t'') - Y^{(\delta)}(t)|\}}{e^\beta} \\ &\leq \frac{\Delta_\beta(Y^{(\delta)}(\cdot), c, T) + 1}{e^\beta} \leq L_5 < 1. \end{aligned} \tag{13}$$

By applying Lemma 3, we get for every $\delta \leq \delta_1$, $0 \leq t' \leq t'' \leq t' + c \leq T$, $z \in \mathbb{Z}$, and $b > 0$,

$$\mathsf{P}_{z,t'}\{\sup_{t' \leq t \leq t''} |Y^{(\delta)}(t) - Y^{(\delta)}(t')| \geq 1 + b\} \leq \frac{1}{1 - L_5} \mathsf{P}_{z,t'}\{|Y^{(\delta)}(t'') - Y^{(\delta)}(t')| \geq b\}. \tag{14}$$

To shorten notations let us denote the random variable $W = |Y^{(\delta)}(t'') - Y^{(\delta)}(t')|$ and $W^+ = \sup_{t' \leq t \leq t''} |Y^{(\delta)}(t) - Y^{(\delta)}(t')|$. Note that $e^{\beta W^+} = W_\beta^{(\delta)}[t', t'']$.

Relations (8) and (14) imply that for every $\delta \leq \delta_1$, $0 \leq t' \leq t'' \leq t' + c \leq T$, $z \in \mathbb{Z}$,

$$\begin{aligned} \mathsf{E}_{z,t'} e^{\beta W^+} &= 1 + \beta \int_0^\infty e^{\beta b} \mathsf{P}_{z,t'}\{W^+ \geq b\} db \\ &\leq 1 + \beta \int_0^1 e^{\beta b} db + \beta \int_1^\infty e^{\beta b} \mathsf{P}_{z,t'}\{W^+ \geq b\} db \\ &= e^\beta + \beta \int_0^\infty e^{\beta(1+b)} \mathsf{P}_{z,t'}\{W^+ \geq 1 + b\} db \\ &\leq e^\beta + \frac{\beta e^\beta}{1 - L_5} \int_0^\infty e^{\beta b} \mathsf{P}_{z,t'}\{W \geq b\} db \\ &= e^\beta + \frac{\beta e^\beta}{1 - L_5} \frac{\mathsf{E}_{z,t'} e^{\beta W} - 1}{\beta} = \frac{e^\beta}{1 - L_5} (\mathsf{E}_{z,t'} e^{\beta W} - L_5) \\ &\leq \frac{e^\beta}{1 - L_5} (\Delta_\beta(Y^{(\delta)}(\cdot), c, T) + 1 - L_5) \leq \frac{e^\beta (e^\beta - 1) L_5}{1 - L_5} = L_7. \end{aligned} \tag{15}$$

Since inequality (15) holds for every $\delta \leq \delta_1$ and $0 \leq t' \leq t'' \leq t' + c \leq T$, $z \in \mathbb{Z}$, it imply relation (11).

Now we can complete the proof of Lemma 1. Using condition $\mathbf{C_2}$, relations (7), (9) – (12), and the Markov property of the process $Z^{(\delta)}(t)$ we get, for $\delta \leq \delta_1$ and $m = [T/c] + 1$, where $[x]$ denotes integer part of x (in this case $T/m \leq c$),

$n = 1, \ldots, m$,

$$\begin{aligned} \mathsf{E}S_\beta^{(\delta)}(v_{n,m}) &\leq \mathsf{E}\{S_\beta^{(\delta)}(v_{n-1,m})\mathsf{E}\{(W_\beta^{(\delta)}[v_{n-1,m}, v_{n,m}] + 1)/Z^{(\delta)}(v_{n-1,m})\}\} \\ &\leq \mathsf{E}S_\beta^{(\delta)}(v_{n-1,m})(L_7 + 1) \leq \cdots \leq \mathsf{E}S_\beta^{(\delta)}(0)(L_7 + 1)^n \leq L_6(L_7 + 1)^n. \end{aligned} \tag{16}$$

Finally, we get, for $\delta \leq \delta_1$,

$$\mathsf{E}\exp\{\beta \sup_{0 \leq u \leq T} |Y^{(\delta)}(u)|\} = \mathsf{E}S_\beta^{(\delta)}(v_{m,m}) \leq L_6(L_7 + 1)^m. \tag{17}$$

Relation (17) obviously implies that inequality (4) given in Lemma 1 holds, for $\delta \leq \delta_1$, with the constant,

$$L_1 = L_6(L_7 + 1)^m. \tag{18}$$

The proof of Lemma 1 is complete.

Proof of Lemma 2. According condition $\mathbf{A_1}$ **(c)** and **(d)** and since $\gamma_2 + 1 \leq \gamma$, the following inequality holds, for $\delta \leq \delta_0$,

$$\begin{aligned} g^{(\delta)}(u, S^{(\delta)}(u)) &\leq \int_0^{S^{(\delta)}(u)} |\frac{\partial g^{(\delta)}(u, s)}{\partial s}| ds + g^{(\delta)}(u, 0) \\ &\leq K_3 S^{(\delta)}(u) + \frac{K_4}{\gamma_2 + 1} S^{(\delta)}(u)^{\gamma_2 + 1} + K_5 \leq L_8 e^{\gamma |Y^{(\delta)}(u)|}, \end{aligned} \tag{19}$$

where

$$L_8 = K_3 + \frac{K_4}{\gamma_2 + 1} + K_5 < \infty. \tag{20}$$

Relation (6) and inequality (19) implies that

$$(\sup_{0 \leq u \leq T} g^{(\delta)}(u, S^{(\delta)}(u)))^{\frac{\beta}{\gamma}} \leq (L_8)^{\frac{\beta}{\gamma}} \exp\{\beta \sup_{0 \leq u \leq T} |Y^{(\delta)}(u))|\}. \tag{21}$$

Inequalities (4) and (21) obviously imply that inequality (5) holds, for $\delta \leq \delta_1$, with the constant,

$$L_2 = L_1(L_8)^{\frac{\beta}{\gamma}} < \infty. \tag{22}$$

The proof of Lemma 2 is complete. $\diamond$

Relation (5) given in Lemma 2 implies that for $\delta \leq \delta_1$,

$$\Phi(\mathcal{M}_{max,T}^{(\delta)}) \leq \mathsf{E} \sup_{0 \leq u \leq T} g^{(\delta)}(u, S^{(\delta)}(u)) \leq (L_2)^{\frac{\gamma}{\beta}} < \infty. \tag{23}$$

Therefore, functional $\Phi(\mathcal{M}_{max,T}^{(\delta)})$ is well defined for $\delta \leq \delta_1$. In what follows we take $\delta \leq \delta_1$.

3. Skeleton Approximations

In this section we derive skeleton approximations for the reward functional $\Phi(\mathcal{M}_{max,T}^{(\delta)})$ by a similar functional for an imbedded discrete time model.

Let $\Pi = \{0 = t_0 < t_1 < \ldots t_N = T\}$ be a partition of the interval $[0, T]$. We consider the class $\hat{\mathcal{M}}_{\Pi,T}^{(\delta)}$ of all Markov moments from $\mathcal{M}_{max,T}^{(\delta)}$, which only take the values $t_0, t_1, \ldots t_N$, and the class $\mathcal{M}_{\Pi,T}^{(\delta)}$ of all Markov moments $\tau^{(\delta)}$ from $\hat{\mathcal{M}}_{\Pi,T}^{(\delta)}$

such that event $\{\omega : \tau^{(\delta)}(\omega) = t_k\} \in \sigma[Z^{(\delta)}(t_0), \dots, Z^{(\delta)}(t_k)]$ for $k = 0, \dots N$. By definition,

$$\mathcal{M}^{(\delta)}_{\Pi,T} \subseteq \hat{\mathcal{M}}^{(\delta)}_{\Pi,T} \subseteq \mathcal{M}^{(\delta)}_{max,T}. \tag{24}$$

Relations (23) and (24) imply that, under conditions of Lemma 2,

$$\Phi(\mathcal{M}^{(\delta)}_{\Pi,T}) \leq \Phi(\hat{\mathcal{M}}^{(\delta)}_{\Pi,T}) \leq \Phi(\mathcal{M}^{(\delta)}_{max,T}) < \infty. \tag{25}$$

The reward functionals $\Phi(\mathcal{M}^{(\delta)}_{max,T})$, $\Phi(\hat{\mathcal{M}}^{(\delta)}_{\Pi,T})$, and $\Phi(\mathcal{M}^{(\delta)}_{\Pi,T})$ correspond to the models of American type option in continuous time, Bermudan type option in continuous time, and American type option in discrete time, respectively.

In the first two cases, the underlying price process is a continuous time Markov type price process modulated by a stochastic index while in the third case the corresponding price process is a discrete time Markov type process modulated by a stochastic index.

Indeed, the random variables $Z^{(\delta)}(t_0), Z^{(\delta)}(t_1), \dots, Z^{(\delta)}(t_N)$ are connected in a discrete time inhomogeneous Markov chain with the phase space $\mathbb{Z}$, the transition probabilities $P^{(\delta)}(t_n, z, t_{n+1}, A)$, and the initial distribution $P^{(\delta)}(A)$.

Note that we have slightly modified the standard definition for a discrete time Markov chain by counting moments $t_0, \dots, t_N$ as the moments of jumps for the Markov chain $Z^{(\delta)}(t_n)$ instead of the moments $0, \dots, N$. This is done in order to synchronize the discrete and continuous time models.

Thus, the optimisation problem (3) for the class $\mathcal{M}^{(\delta)}_{\Pi,T}$ is really a problem of optimal expected reward for American type options in discrete time.

Now we are ready to formulate the first main result of the paper concerning skeleton approximations of the reward functional in the continuous time model by the corresponding reward functional in the corresponding discrete time model. Note that skeleton approximations have asymptotically uniform with respect to perturbation parameter form. This is very essential for using these approximations in convergence results given in the second part of the paper.

We use the method developed in [34]. However, we essentially improve the skeleton approximation obtained in this paper, where the difference $\Phi(\mathcal{M}^{(\delta)}_{max,T}) - \Phi(\mathcal{M}^{(\delta)}_{\Pi,T})$ have been estimated from above via the modulus of compactness for the uniform topology for the price processes. This estimate could only be used for continuous price processes. In the present paper, we get alternative estimates based on the exponential moment modulus of compactness $\Delta_\beta(Y^{(\delta)}(\cdot), c, T)$. These estimates can be effectively used for càdlàg price processes.

The following theorem presents this result. The explicit expression for the constants in the corresponding estimate will be given in the proof of the theorem.

Theorem 1. *Let conditions* $\mathbf{A_1}$, $\mathbf{C_1}$, *and* $\mathbf{C_2}$ *hold, and let also* $\delta \leq \delta_1$ *and* $d(\Pi) \leq c$ *where* c *and* δ_1 *are defined in relations* (8) *and* (9). *Then there exist constants* $L_3, L_4 < \infty$ *such that the following skeleton approximation inequality holds,*

$$\Phi(\mathcal{M}^{(\delta)}_{max,T}) - \Phi(\mathcal{M}^{(\delta)}_{\Pi,T}) \leq L_3 d(\Pi) + L_4(\Delta_\beta(Y^{(\delta)}(\cdot), d(\Pi), T))^{\frac{\beta-\gamma}{\beta}}. \tag{26}$$

Proof of Theorem 1. Let us begin from the following important fact which plays an important role in the proof of Theorem 1.

Lemma 4. *For any partition* $\Pi = \{0 = t_0 < t_1 < \ldots < t_N = T\}$ *of interval* $[0, T]$,

$$\Phi(\mathcal{M}^{(\delta)}_{\Pi,T}) = \Phi(\hat{\mathcal{M}}^{(\delta)}_{\Pi,T}). \tag{27}$$

Proof of Lemma 4. A similar result was given in [33, 35] and we shortly present the modified version of the corresponding proof.

The optimisation problem (3) for the class $\hat{\mathcal{M}}^{(\delta)}_{\Pi,T}$ can be considered as a problem of optimal expected reward for American type options with discrete time. To see this let us add to the random variables Z_{t_n} additional components $\bar{Z}^{(\delta)}_n = \{Z^{(\delta)}(t), t_{n-1} < t \leq t_n\}$ with the corresponding phase space $\bar{\mathbb{Z}}$ endowed by the corresponding cylindrical σ-field. As $\bar{Z}^{(\delta)}_0$ we can take an arbitrary point in $\bar{\mathbb{Z}}$. Consider the extended Markov chain $\tilde{Z}^{(\delta)}_n = (Z^{(\delta)}(t_n), \bar{Z}^{(\delta)}_n)$ with the phase space $\tilde{\mathbb{Z}} = \mathbb{Z} \times \bar{\mathbb{Z}}$. As above, we slightly modify the standard definition and count moments $t_0, \ldots, t_N$ as moments of jumps for the this Markov chain instead of moments $0, \ldots, N$. This is done in order to synchronize the discrete and continuous time models.

Let us denote by $\tilde{\mathcal{M}}^{(\delta)}_{\Pi,T}$ the class of all Markov moments $\tau^{(\delta)} \leq t_N$ for discrete time Markov chain $\tilde{Z}^{(\delta)}_n$ and let us also consider the reward functional,

$$\Phi(\tilde{\mathcal{M}}^{(\delta)}_{\Pi,T}) = \sup_{\tau^{(\delta)} \in \tilde{\mathcal{M}}^{(\delta)}_{\Pi,T}} \mathsf{E} g^{(\delta)}(\tau^{(\delta)}, S^{(\delta)}(\tau^{(\delta)})). \tag{28}$$

It is readily seen that the optimisation problem (3) for the class $\hat{\mathcal{M}}^{(\delta)}_{\Pi,T}$ is equivalent to the optimisation problem (28), i.e.,

$$\Phi(\hat{\mathcal{M}}^{(\delta)}_{\Pi,T}) = \Phi(\tilde{\mathcal{M}}^{(\delta)}_{\Pi,T}). \tag{29}$$

As is known, (See, for example, [45]) the optimal stopping moment $\tau^{(\delta)}$ exists in any discrete time Markov model, and the optimal decision $\{\tau^{(\delta)} = t_n\}$ depends only on the value $\tilde{Z}^{(\delta)}_n$. Moreover the optimal Markov moment has the first hitting time structure, i.e., it has the form $\tau^{(\delta)} = \min(t_n : \tilde{Z}^{(\delta)}_n \in \tilde{\mathbb{D}}^{(\delta)}_n)$, where $\tilde{\mathbb{D}}^{(\delta)}_n, n = 0, \ldots, N$ are some measurable subsets of the phase space $\tilde{\mathbb{Z}}$. The optimal stopping domains are determined by the transition probabilities of the extended Markov chain $\tilde{Z}^{(\delta)}_n$.

However, the extended Markov chain $\tilde{Z}^{(\delta)}_n$ has transition probabilities depending only on values of the first component $Z^{(\delta)}(t_n)$. As was shown in [35], the optimal Markov moment has in this case the first hitting time structure of the form $\tau^{(\delta)} = \min(t_n : Z^{(\delta)}(t_n) \in \mathbb{D}^{(\delta)}_n)$, where $\mathbb{D}^{(\delta)}_n, n = 0, \ldots, N$ are some measurable subsets of the phase space of the first component $\mathbb{Z}$.

Therefore, for the optimal stopping moment $\tau^{(\delta)}$ the decision $\{\tau^{(\delta)} = t_n\}$ depends only on the value $Z^{(\delta)}(t_n)$, and $\tau^{(\delta)} \in \mathcal{M}^{(\delta)}_{\Pi,T}$. Hence,

$$\Phi(\mathcal{M}_{\Pi,T}) \geq \mathsf{E} g^{(\delta)}(\tau^{(\delta)}, S^{(\delta)}(\tau^{(\delta)})) = \Phi(\hat{\mathcal{M}}^{(\delta)}_{\Pi,T}). \tag{30}$$

Inequalities (25) and (30) imply the equality (27). $\diamond$

For any Markov moment $\tau^{(\delta)} \in \mathcal{M}^{(\delta)}_{max,T}$ and a partition $\Pi = \{0 = t_0 < t_1 < \ldots < t_N = T\}$ one can define the discretisation of this moment,

$$\tau^{(\delta)}[\Pi] = \begin{cases} 0, & \text{if } \ \tau^{(\delta)} = 0, \\ t_k, & \text{if } \ t_{k-1} < \tau^{(\delta)} \leq t_k, \ k = 1, \ldots N. \end{cases}$$

Let $\tau_\varepsilon^{(\delta)}$ be ε-optimal Markov moment in the class $\mathcal{M}^{(\delta)}_{max,T}$, i.e.,

$$\mathsf{E}g^{(\delta)}(\tau_\varepsilon^{(\delta)}, S^{(\delta)}(\tau_\varepsilon^{(\delta)})) \geq \Phi(\mathcal{M}^{(\delta)}_{max,T}) - \varepsilon. \tag{31}$$

Such ε-optimal Markov moment always exists for any $\varepsilon > 0$, by definition of the reward functional $\Phi(\mathcal{M}^{(\delta)}_{max,T})$.

By definition, the Markov moment $\tau_\varepsilon^{(\delta)}[\Pi] \in \hat{\mathcal{M}}^{(\delta)}_{\Pi,T}$. This fact and relation (27) given in Lemma 4 implies that

$$\mathsf{E}g^{(\delta)}(\tau_\varepsilon^{(\delta)}[\Pi], S^{(\delta)}(\tau_\varepsilon^{(\delta)}[\Pi])) \leq \Phi(\hat{\mathcal{M}}^{(\delta)}_{\Pi,T}) = \Phi(\mathcal{M}^{(\delta)}_{\Pi,T}) \leq \Phi(\mathcal{M}^{(\delta)}_{max,T}). \tag{32}$$

Let us denote

$$d(\Pi) = \max\{t_k - t_{k-1}, k = 1, \dots N\}.$$

Obviously,

$$\tau_\varepsilon^{(\delta)} \leq \tau_\varepsilon^{(\delta)}[\Pi] \leq \tau_\varepsilon^{(\delta)} + d(\Pi). \tag{33}$$

Now inequalities (31) and (32) imply the following skeleton approximation inequality,

$$\begin{aligned} 0 &\leq \Phi(\mathcal{M}^{(\delta)}_{max,T}) - \Phi(\mathcal{M}^{(\delta)}_{\Pi,T}) \\ &\leq \varepsilon + \mathsf{E}g^{(\delta)}(\tau_\varepsilon^{(\delta)}, S^{(\delta)}(\tau_\varepsilon^{(\delta)})) - \mathsf{E}g^{(\delta)}(\tau_\varepsilon^{(\delta)}[\Pi], S^{(\delta)}(\tau_\varepsilon^{(\delta)}[\Pi])) \\ &\leq \varepsilon + \mathsf{E}\left| g^{(\delta)}(\tau_\varepsilon^{(\delta)}, S^{(\delta)}(\tau_\varepsilon^{(\delta)})) - g^{(\delta)}(\tau_\varepsilon^{(\delta)}[\Pi], S^{(\delta)}(\tau_\varepsilon^{(\delta)}[\Pi]))\right|. \end{aligned} \tag{34}$$

To shorten notations let us denote, for the moment, the random variables $\tau' = \tau_\varepsilon^{(\delta)}$, $\tau'' = \tau_\varepsilon^{(\delta)}[\Pi]$, and $Y' = Y^{(\delta)}(\tau')$, $Y'' = Y^{(\delta)}(\tau'')$. Let also denote $Y^+ = Y' \vee Y''$, $Y^- = Y' \wedge Y''$. By the definition, $0 \leq \tau' \leq \tau'' \leq T$ and $Y^- \leq Y^+$.

Using these notations and condition $\mathbf{A_1'}$ we get the following inequalities,

$$\begin{aligned} &|g^{(\delta)}(\tau', e^{Y'}) - g^{(\delta)}(\tau'', e^{Y''})| \\ &\quad \leq |g^{(\delta)}(\tau', e^{Y'}) - g^{(\delta)}(\tau'', e^{Y'})| + |g^{(\delta)}(\tau'', e^{Y'}) - g^{(\delta)}(\tau'', e^{Y''})| \\ &\quad \leq \int_{\tau'}^{\tau''} |g_1^{(\delta)}(t, e^{Y'})| dt + \int_{Y^-}^{Y^+} |g_2^{(\delta)}(\tau'', e^y) e^y| dy \\ &\quad \leq \int_{\tau'}^{\tau''} (K_1 + K_2 e^{\gamma_1 Y'}) dt + \int_{Y^-}^{Y^+} (K_3 e^y + K_4 e^{(\gamma_2+1)y}) dy \\ &\quad \leq (K_1 + K_2 e^{\gamma_1 |Y'|})(\tau'' - \tau') + (K_3 e^{|Y^+|} + K_4 e^{(\gamma_2+1)|Y^+|})(Y^+ - Y^-) \\ &\quad \leq (K_1 + K_2) \exp\{\gamma_1 \sup_{0 \leq u \leq T} |Y^{(\delta)}(u)|\}(\tau'' - \tau') \\ &\qquad + (K_3 + K_4) \exp\{(\gamma_2 + 1) \sup_{0 \leq u \leq T} |Y^{(\delta)}(u)|\} |Y' - Y''|. \end{aligned} \tag{35}$$

Recall that $0 \leq \tau'' - \tau' \leq d(\Pi)$ and $\gamma_1 \vee (\gamma_2 + 1) = \gamma < \beta$. Now, applying Hölder's inequality (with parameters $p = \beta/\gamma$ and $q = \beta/(\beta - \gamma)$) to the corresponding products of random variables on the right hand side in (35), and using inequality (4) given in Lemma 1, we can write down the following estimate for the expectation on the right hand side in (34), for $\delta \leq \delta_1$,

$$
\begin{aligned}
& \mathsf{E}|g^{(\delta)}(\tau_\varepsilon^{(\delta)}, S^{(\delta)}(\tau_\varepsilon^{(\delta)})) - g^{(\delta)}(\tau_\varepsilon^{(\delta)}[\Pi], S^{(\delta)}(\tau_\varepsilon^{(\delta)}[\Pi]))| \\
& \quad = \mathsf{E}|g^{(\delta)}(\tau', e^{Y'}) - g^{(\delta)}(\tau'', e^{Y''})| \\
& \quad \leq (K_1 + K_2)\mathsf{E}\exp\{\gamma \sup_{0 \leq u \leq T} |Y^{(\delta)}(u)|\} d(\Pi) \\
& \qquad + (K_3 + K_4)\mathsf{E}\exp\{\gamma \sup_{0 \leq u \leq T} |Y^{(\delta)}(u)|\} |Y' - Y''| \\
& \quad \leq (K_1 + K_2)[L_1]^{\frac{\gamma}{\beta}} d(\Pi) + (K_3 + K_4)[L_1]^{\frac{\gamma}{\beta}} (\mathsf{E}|Y' - Y''|^{\frac{\beta}{\beta-\gamma}})^{\frac{\beta-\gamma}{\beta}}. \qquad (36)
\end{aligned}
$$

The next step in the proof is to show that, for $\delta \leq \delta_1$,

$$
\begin{aligned}
\mathsf{E}|Y' - Y''|^{\frac{\beta}{\beta-\gamma}} & = \mathsf{E}|Y^{(\delta)}(\tau_\varepsilon^{(\delta)})) - Y^{(\delta)}(\tau_\varepsilon^{(\delta)}[\Pi])|^{\frac{\beta}{\beta-\gamma}} \\
& \leq L_9 \Delta_\beta(Y^{(\delta)}(\cdot), d(\Pi), T), \qquad (37)
\end{aligned}
$$

where

$$
L_9 = \sup_{y \geq 0} \frac{y^{\frac{\beta}{\beta-\gamma}}}{e^{\beta y} - 1} < \infty. \qquad (38)
$$

In order get inequality (37), we employ the method for estimation of moments for increments of stochastic processes stopped at Markov type moments, from [47].

By the definition $\tau'' = \tau' + f_\Pi(\tau')$ where function $f_\Pi(t) = t - t_k$ for $t_k \leq t < t_{k+1}, k = 0, \ldots, N-1$ and 0 for $t = t_N$. Obviously function $f_\Pi(t)$ is continuous from the right on the interval $[0, T]$ and $0 \leq f_\Pi(t) \leq d(\Pi)$.

Let us now use again the partition $\tilde{\Pi}_m$ of interval $[0, T]$ by points $v_{n,m} = nT/m$, $n = 0, \ldots, m$. Consider random variables,

$$
\tau'[\tilde{\Pi}_m] = \begin{cases} 0, & \text{if} \quad \tau' = 0, \\ v_{k,m}, & \text{if} \quad v_{k-1,m} < \tau' \leq v_{k,m},\ k = 1, \ldots N. \end{cases}
$$

Obviously $\tau' \leq \tau'[\tilde{\Pi}_m] \leq \tau' + T/m$. Thus random variables $\tau'[\tilde{\Pi}_m] \stackrel{a.s.}{\longrightarrow} \tau'$ as $m \to \infty$ (a.s. is an abbreviation for almost surely). Since the $Y^{(\delta)}(t)$ is a càdlàg process, we get also the following relation,

$$
\begin{aligned}
Q_m^{(\delta)} & = |Y^{(\delta)}(\tau'[\tilde{\Pi}_m]) - Y^{(\delta)}(\tau'[\tilde{\Pi}_m] + f_\Pi(\tau'[\tilde{\Pi}_m]))|^{\frac{\beta}{\beta-\gamma}} \\
& \stackrel{a.s.}{\longrightarrow} Q^{(\delta)} = |Y^{(\delta)}(\tau') - Y^{(\delta)}(\tau' + f_\Pi(\tau'))|^{\frac{\beta}{\beta-\gamma}} \text{ as } m \to \infty. \qquad (39)
\end{aligned}
$$

Note also that $Q_m^{(\delta)}$ are non-negative random variables and the following estimate holds for any $m = 1, \ldots,$,

$$
\begin{aligned}
Q_m^{(\delta)} & \leq (|Y^{(\delta)}(\tau'[\tilde{\Pi}_m])| + |Y^{(\delta)}(\tau'[\tilde{\Pi}_m] + f_\Pi(\tau'[\tilde{\Pi}_m]))|)^{\frac{\beta}{\beta-\gamma}} \\
& \leq 2^{\frac{\beta}{\beta-\gamma}-1}(|Y^{(\delta)}(\tau'[\tilde{\Pi}_m])|^{\frac{\beta}{\beta-\gamma}} + |Y^{(\delta)}(\tau'[\tilde{\Pi}_m] + f_\Pi(\tau'[\tilde{\Pi}_m]))|^{\frac{\beta}{\beta-\gamma}}) \\
& \leq 2^{\frac{\beta}{\beta-\gamma}}(\sup_{0 \leq u \leq T} |Y^{(\delta)}(u)|)^{\frac{\beta}{\beta-\gamma}} \leq 2^{\frac{\beta}{\beta-\gamma}} L_9 \exp\{\beta \sup_{0 \leq u \leq T} |Y^{(\delta)}(u)|\}. \qquad (40)
\end{aligned}
$$

Taken into account inequality (4) given in Lemma 1, which implies that the random variable on the right hand side in (40) has a finite expectation, and relations (39) and (40), we get by Lebesgue theorem that, for $\delta \leq \delta_1$,

$$
\mathsf{E}Q_m^{(\delta)} \to \mathsf{E}Q^{(\delta)} \text{ as } m \to \infty. \qquad (41)
$$

Let us now estimate $\mathsf{E}Q_m^{(\delta)}$. To reduce notation let us denote for the moment $Y'_{n+1} = Y^{(\delta)}(v_{n+1,m})$ and $Y''_{n+1} = Y^{(\delta)}(v_{n+1,m} + f_\Pi(v_{n+1,m}))$. Recall that τ' is a Markov moment for the Markov process $Z^{(\delta)}(t)$. Thus, random variables $\chi(v_{n,m} < \tau' \leq v_{n+1,m})$ and $|Y'_{n+1} - Y''_{n+1}|^{\frac{\beta}{\beta-\gamma}}$ are conditionally independent with respect to random variable $Z^{(\delta)}(v_{n+1,m})$. Using this fact and inequality $f_\Pi(v_{n+1,m}) \leq d(\Pi)$, we get, for $\delta \leq \delta_1$,

$$\begin{aligned}
\mathsf{E}Q_m^{(\delta)} &= \mathsf{E}|Y^{(\delta)}(\tau'[\tilde{\Pi}_m]) - Y^{(\delta)}(\tau'[\tilde{\Pi}_m] + f_\Pi(\tau'[\tilde{\Pi}_m]))|^{\frac{\beta}{\beta-\gamma}} \\
&= \sum_{n=0}^{m-1} \mathsf{E}|Y'_{n+1} - Y''_{n+1}|^{\frac{\beta}{\beta-\gamma}} \chi(v_{n,m} < \tau' \leq v_{n+1,m}) \\
&= \sum_{n=0}^{m-1} \mathsf{E}\{\chi(v_{n,m} < \tau' \leq v_{n+1,m}) \mathsf{E}\{|Y'_{n+1,} - Y''_{n+1}|^{\frac{\beta}{\beta-\gamma}} / Z^{(\delta)}(v_{n+1,m})\}\} \\
&\leq \sum_{n=0}^{m-1} \sup_{z\in\mathbb{Z}} \mathsf{E}_{z,v_{n+1,m}} |Y'_{n+1,} - Y''_{n+1}|^{\frac{\beta}{\beta-\gamma}} \mathsf{P}\{v_{n,m} < \tau' \leq v_{n+1,m}\} \\
&\leq \sum_{n=0}^{m-1} L_9 \sup_{z\in\mathbb{Z}} \mathsf{E}_{z,v_{n+1,m}} \exp\{\beta|Y'_{n+1,} - Y''_{n+1}|\} \mathsf{P}\{v_{n,m} < \tau' \leq v_{n+1,m}\} \\
&\leq \sum_{n=0}^{m-1} L_9 \Delta_\beta(Y^{(\delta)}(\cdot), d(\Pi), T) \mathsf{P}\{v_{n,m} < \tau' \leq v_{n+1,m}\} \\
&\leq L_9 \Delta_\beta(Y^{(\delta)}(\cdot), d(\Pi), T). \qquad (42)
\end{aligned}$$

Relations (41) and (42) imply that, for $\delta \leq \delta_1$,

$$\begin{aligned}
\mathsf{E}Q^{(\delta)} &= \mathsf{E}|Y^{(\delta)}(\tau') - Y^{(\delta)}(\tau' + f_\Pi(\tau'))|^{\frac{\beta}{\beta-\gamma}} \\
&\leq L_9 \Delta_\beta(Y^{(\delta)}(\cdot), d(\Pi), T). \qquad (43)
\end{aligned}$$

This inequality is equivalent to inequality (37) since, by introduced notations, $|Y^{(\delta)}(\tau') - Y^{(\delta)}(\tau' + f_\Pi(\tau'))| = |Y^{(\delta)}(\tau_\varepsilon^{(\delta)}) - Y^{(\delta)}(\tau_\varepsilon^{(\delta)}[\Pi])|$.

If (37) is proved then the estimate (36) can be continued and transformed, for $\delta \leq \delta_1$, to the following form,

$$\begin{aligned}
&\mathsf{E}|g^{(\delta)}(\tau_\varepsilon^{(\delta)}, S^{(\delta)}(\tau_\varepsilon^{(\delta)})) - g^{(\delta)}(\tau_\varepsilon^{(\delta)}[\Pi], S^{(\delta)}(\tau_\varepsilon^{(\delta)}[\Pi]))| \\
&\quad \leq L_3 d(\Pi_N) + L_4(\Delta_\beta(Y^{(\delta)}(\cdot), d(\Pi), T))^{\frac{\beta-\gamma}{\beta}}, \qquad (44)
\end{aligned}$$

where

$$L_3 = (K_1 + K_2)[L_1]^{\frac{\gamma}{\beta}}, \ L_4 = (K_3 + K_4)(L_1)^{\frac{\gamma}{\beta}}(L_9)^{\frac{\beta-\gamma}{\beta}}. \qquad (45)$$

Note that the quantity on the right hand side in (44) does not depend on ε. Thus, we can substitute it in (34) and then to pass ε to zero in this relation that will result inequality (26) given in Theorem 1.

The proof of Theorem 1 is complete. $\diamond$

In conclusion, we would like to note that the skeleton approximations given in Theorem 1 have their own value beyond their use in convergence theorems that will presented in the second part of the present paper.

Indeed, one of the main approaches used to evaluate reward functional for American type options is based on the use of Monte Carlo algorithms, which obviously require that the corresponding continuous time price processes to be replaced by their more simple discrete time models usually constructed on the base of the corresponding skeleton approximations. Theorem 1 gives explicit estimates for the accuracy of the corresponding approximations of reward functionals for continuous time price processes by the corresponding reward functionals for skeleton type discrete time price processes.

4. Convergence of rewards for discrete time options

In this section we give conditions of convergence for discrete time reward functionals $\Phi(\mathcal{M}^{(\delta)}_{\Pi,T})$ for a given partition $\Pi = \{0 = t_0 < t_1 \cdots < t_N = T\}$ of interval $[0,T]$.

In this case, it is natural to use conditions based on the transition probabilities between the sequential moments of this partition and values of the pay-off functions at the moments of this partition.

In the continuous time case, the derivatives of the pay-off functions were involved in condition $\mathbf{A_1}$. The corresponding assumptions implied continuity of the pay-off functions. These assumptions played an essential role in the proof of Theorem 1, where skeleton approximations were obtained.

In the discrete time case, the derivatives of the pay-off functions are not involved. In this case, the pay-off functions can be discontinuous.

We replace condition $\mathbf{A_1}$ by a simpler condition:

$\mathbf{A_2}$: There exist $\delta_0 > 0$ such that, for every $0 \leq \delta \leq \delta_0$, function $g^{(\delta)}(t_n, s) \leq K_6 + K_7 s^\gamma$, for $n = 0, \ldots, N$ and $s \in (0, \infty)$ for some $\gamma \geq 1$ and constants $K_6, K_7 < \infty$.

We need also an assumption about convergence of payoff functions. We require locally uniform convergence for pay-off functions on some sets, which later will be assumed to have the value 1 for the corresponding limit transition probabilities and the limit initial distribution:

$\mathbf{A_3}$: There exists a measurable set $\mathbb{S}_{t_n} \subseteq (0,\infty)$ for every $n = 0, \ldots, N$, such that $g^{(\delta)}(t_n, s_\delta) \to g^{(0)}(t_n, s)$ as $\delta \to 0$ for any $s_\delta \to s \in \mathbb{S}_{t_n}$ and $n = 0, \ldots, N$.

Let us also denote as $\mathbb{V}_{t_n} = \mathbb{S}_{t_n} \times \mathbb{X}$.

Obviously, condition $\mathbf{A_3}$ can be re-written in terms of function $g^{(\delta)}(t, e^y)$, $(t, y) \in [0, \infty) \times \mathbb{R}_1$:

$\mathbf{A'_3}$: There exists a measurable set $\mathbb{Y}'_{t_n} \subseteq \mathbb{R}_1$ for every $n = 0, \ldots, N$, such that $g^{(\delta)}(t_n, e^{y_\delta}) \to g^{(0)}(t_n, e^y)$ as $\delta \to 0$ for any $y_\delta \to y \in \mathbb{Y}'_{t_n}$ and $n = 0, \ldots, N$.

It is obvious that the sets $\mathbb{S}_{t_n}$ and $\mathbb{Y}'_{t_n}$ are connected by the relations $\mathbb{Y}'_{t_n} = \ln \mathbb{S}_{t_n} = \{y = \ln s, s \in \mathbb{S}_{t_n}\}, n = 0, \ldots, N$.

Let us also denote $\mathbb{Z}'_{t_n} = \mathbb{Y}'_{t_n} \times \mathbb{X}$.

The typical examples are where the sets $\bar{\mathbb{Y}}'_{t_n} = \emptyset$ or where $\bar{\mathbb{Y}}'_{t_n}$ are finite or countable sets. For example, if pay-off functions $g^{(\delta)}(t, e^y)$ are monotonic functions in y, the point-wise convergence $g^{(\delta)}(t, e^y) \to g^{(0)}(t, e^y)$ as $\delta \to 0$, $y \in \mathbb{Y}^*_{t_n}$, for every $n = 0, \ldots, N$, where $\mathbb{Y}^*_{t_n}$ are some countable dense sets in $\mathbb{R}_1$, implies the locally uniform convergence required in condition $\mathbf{A'_3}$ for sets $\mathbb{Y}'_{t_n}$, which are the sets of continuity points for the limit functions $g^{(0)}(t_n, e^y)$, as functions in y, for every $n = 0, \ldots, N$. Due to monotonicity of these functions, $\bar{\mathbb{Y}}'_{t_n}$ are at most countable sets.

Symbol $\Rightarrow$ is used below to denote weak convergence of probability measures, i.e. convergence of their values for sets of continuity for the corresponding limit measure or to denote weak convergence for the corresponding random variables.

We need also conditions on convergence of transition probabilities of price processes between sequential moments of a time partition $\Pi = \{0 = t_0 < t_1 \cdots < t_N = T\}$:

$\mathbf{B_1}$: There exist measurable sets $\mathbb{Z}_{t_n} \subseteq \mathbb{Z}$, $n = 0, \ldots, N$ such that **(a)** $P^{(\delta)}(t_n, z_\delta, t_{n+1}, \cdot) \Rightarrow P^{(0)}(t_n, z, t_{n+1}, \cdot)$ as $\delta \to 0$, for any $z_\delta \to z \in \mathbb{Z}_{t_n}$ as $\delta \to 0$ and $n = 0, \ldots, N-1$; **(b)** $P^{(0)}(t_n, z, t_{n+1}, \mathbb{Z}'_{t_{n+1}} \cap \mathbb{Z}_{t_{n+1}}) = 1$ for every $z \in \mathbb{Z}_{t_n}$ and $n = 0, \ldots, N-1$, where $\mathbb{Z}'_{t_{n+1}}$ are the sets introduced in condition $\mathbf{A'_3}$.

The typical example is where the sets $\bar{\mathbb{Z}}'_{t_n} \cup \bar{\mathbb{Z}}_{t_n} = \emptyset$. In this case, condition $\mathbf{B_1}$ **(b)** automatically holds. Another typical example is where $\mathbb{Z}'_{t_n} = \mathbb{Y}'_{t_n} \times \mathbb{X}$ and $\mathbb{Z}_{t_n} = \mathbb{Y}_{t_n} \times \mathbb{X}$, where the sets $\bar{\mathbb{Y}}'_{t_n}$ and $\bar{\mathbb{Y}}_{t_n}$ are at most finite or countable sets. In this case, the assumption that the measures $P^{(0)}(t, z, t+u, A \times \mathbb{X}), A \in \mathcal{B}_1$ have no atoms implies that condition $\mathbf{B_1}$ **(b)** holds.

As far as condition of convergence for initial distributions is concerned, we shall require weak convergence for the initial distributions to some distribution that is assumed to be concentrated on the intersections of the sets of convergence for the corresponding transition probabilities and pay-off functions:

$\mathbf{B_2}$: **(a)** $P^{(\delta)}(\cdot) \Rightarrow P^{(0)}(\cdot)$ as $\delta \to 0$; **(b)** $P^{(0)}(\mathbb{Z}'_{t_0} \cap \mathbb{Z}_{t_0}) = 1$, where $\mathbb{Z}'_{t_0}$ and $\mathbb{Z}_{t_0}$ are the sets introduced in conditions $\mathbf{A'_2}$ and $\mathbf{B_1}$.

The typical example is where the sets $\bar{\mathbb{Z}}'_{t_0} \cup \bar{\mathbb{Z}}_{t_0} = \emptyset$. In this case, condition $\mathbf{B_2}$ **(b)** automatically holds. Another typical example is where $\mathbb{Z}'_{t_0} = \mathbb{Y}'_{t_0} \times \mathbb{X}$ and $\mathbb{Z}_{t_0} = \mathbb{Y}_{t_0} \times \mathbb{X}$, where the sets $\bar{\mathbb{Y}}'_{t_0}$ and $\bar{\mathbb{Y}}_{t_0}$ are at most finite or countable sets. In this case, the assumption that the measure $P^{(0)}(A \times \mathbb{X}), A \in \mathcal{B}_1$ has no atoms implies that condition $\mathbf{B_2}$ **(b)** holds.

Condition $\mathbf{B_2}$ holds, for example, if the initial distributions $P^{(\delta)}(A) = \chi_A(z_0)$ are concentrated in a point $z_0 \in \mathbb{Z}'_{t_0} \cap \mathbb{Z}_{t_0}$, for all $\delta \geq 0$. This condition also holds if the initial distributions $P^{(\delta)}(A) = \chi_A(z_\delta)$ for $\delta \geq 0$, where $z_\delta \to z_0$ as $\delta \to 0$ and $z_0 \in \mathbb{Z}'_{t_0} \cap \mathbb{Z}_{t_0}$.

We also weaken condition $\mathbf{C_1}$ by replacing it by a simpler condition:

$\mathbf{C_3}$: $\overline{\lim}_{\delta\to 0} \sup_{z\in\mathbb{Z}} \mathsf{E}_{z,t_n}(e^{\beta|Y^{(\delta)}(t_{n+1})-Y^{(\delta)}(t_n)|} - 1) < \infty$, $n = 0, \ldots, N-1$, for some $\beta > \gamma$, where γ is the parameter introduced in condition $\mathbf{A_2}$.

Condition $\mathbf{C_2}$ does not change and takes the following form:

$\mathbf{C_4}$: $\overline{\lim}_{\delta\to 0} \mathsf{E} e^{\beta|Y^{(\delta)}(t_0)|} < \infty$, where β is the parameter introduced in condition $\mathbf{C_3}$.

The following theorem is the second main result of the present paper.

Theorem 2. *Let conditions* $\mathbf{A_2}$, $\mathbf{A_3}$, $\mathbf{B_1}$, $\mathbf{B_2}$, $\mathbf{C_3}$, *and* $\mathbf{C_4}$ *hold. Then, the following asymptotic relation holds for the partition* $\Pi = \{0 = t_0 < t_1 \cdots < t_N = T\}$ *of interval* $[0, T]$,

$$\Phi(\mathcal{M}^{(\delta)}_{\Pi,T}) \to \Phi(\mathcal{M}^{(0)}_{\Pi,T}) \text{ as } \delta \to 0. \tag{46}$$

Proof. We improve the method based on recursive asymptotic analysis of reward functions used in [27].

The reward functions are defined by the following recursive relations,

$$w^{(\delta)}(t_N, z) = g^{(\delta)}(t_N, e^y),\ z = (y, x) \in \mathbb{Z},$$

and, for $n = 0, \dots, N-1$,

$$w^{(\delta)}(t_n, z) = \max(g^{(\delta)}(t_n, e^y),\ \mathsf{E}_{z,t_n} w^{(\delta)}(t_{n+1}, Z^{(\delta)}(t_{n+1}))),\ z = (y, x) \in \mathbb{Z}.$$

As follows from general results on optimal stopping for discrete time Markov processes ([6] and [45]), the reward functional,

$$\Phi(\mathcal{M}^{(\delta)}_{\Pi,T}) = \mathsf{E} w^{(\delta)}(t_0, Z^{(\delta)}(0)). \tag{47}$$

Note that, by definition, the reward functions $w^{(\delta)}(t_n, z) \geq 0, z \in \mathbb{Z}, n = 0, \dots, N$.

Condition $\mathbf{C_3}$ implies that there exists a constant $L_{10} < \infty$ and $\delta_2 \leq \delta_0$ such that for $n = 0, \dots, N-1$ and $\delta \leq \delta_2$,

$$\sup_{z \in \mathbb{Z}} \mathsf{E}_{z,t_n}(e^{\beta|Y^{(\delta)}(t_{n+1}) - Y^{(\delta)}(t_n)|} - 1) \leq L_{10}. \tag{48}$$

Also condition $\mathbf{C_4}$ implies that δ_2 can be chosen in such a way that, for some constant $L_{11} < \infty$, the following inequality holds for $\delta \leq \delta_2$,

$$\mathsf{E} e^{\beta|Y^{(\delta)}(0)|} \leq L_{11}. \tag{49}$$

Condition $\mathbf{A_2}$ directly implies that the following power upper bound for the reward function $w^{(\delta)}(t_N, z)$ holds, for $\delta \leq \delta_2$,

$$w^{(\delta)}(t_N, z) \leq L_{1,N} + L_{2,N} e^{\gamma|y|},\ z = (y, x) \in \mathbb{Z}, \tag{50}$$

where

$$L_{1,N} = K_6,\ L_{2,N} = K_7 < \infty. \tag{51}$$

Also, according to condition $\mathbf{A'_3}$, for an arbitrary $z_\delta \to z_0$ as $\delta \to 0$, where $z_0 \in \mathbb{Z}'_{t_N} \cap \mathbb{Z}_{t_N}$,

$$w^{(\delta)}(t_N, z_\delta) \to w^{(0)}(t_N, z_0) \text{ as } \delta \to 0. \tag{52}$$

Let us prove that relations similar with (50), (51), and (52) hold for the reward functions $w^{(\delta)}(t_{N-1}, z)$.

We get, using relation (50), for $z = (y, x) \in \mathbb{Z}$ and $\delta \leq \delta_2$,

$$\begin{aligned} \mathsf{E}_{z,t_{N-1}} g^{(\delta)}(t_N, e^{Y^{(\delta)}(t_N)}) &\leq L_{1,N} + L_{2,N} \mathsf{E}_{z,t_{N-1}} e^{\gamma|Y^{(\delta)}(t_N)|} \\ &\leq L_{1,N} + L_{2,N} \mathsf{E}_{z,t_{N-1}} e^{\gamma|y|} e^{\gamma|Y^{(\delta)}(t_N) - y|} \\ &\leq L_{1,N} + L_{2,N} e^{\gamma|y|} \mathsf{E}_{z,t_{N-1}} e^{\gamma|Y^{(\delta)}(t_N) - Y^{(\delta)}(t_{N-1})|} \\ &\leq L_{1,N} + L_{2,N}(L_{10} + 1) e^{\gamma|y|}. \end{aligned} \tag{53}$$

Relation (53) implies that, for $z = (y, x) \in \mathbb{Z}$ and $\delta \leq \delta_2$,

$$\begin{aligned} w^{(\delta)}(t_{N-1}, z) &= \max(g^{(\delta)}(t_{N-1}, e^y), \mathsf{E}_{z,t} w^{(\delta)}(t_N, Z^{(\delta)}(t_N)) \\ &\leq K_6 + K_7 e^{\gamma|y|} + L_{1,N} + L_{2,N}(L_{10} + 1) e^{\gamma|y|} \\ &\leq L_{1,N-1} + L_{2,N-1} e^{\gamma|y|}, \end{aligned} \tag{54}$$

where

$$L_{1,N-1} = K_6 + L_{1,N},\ L_{2,N-1} = K_7 + L_{2,N}(L_{10} + 1) < \infty. \tag{55}$$

Let us introduce, for every $n = 0, \dots, N-1$ and $z \in \mathbb{Z}$ random variables $Z_n^{(\delta)}(z) = (Y_n^{(\delta)}(z), X_n^{(\delta)}(z))$ such that $\mathsf{P}\{Z_n^{(\delta)}(z) \in A\} = P^{(\delta)}(t_n, z, t_{n+1}, A)$, $A \in \mathcal{B}_{\mathbb{Z}}$.

Let us prove that, for any $z_\delta \to z_0 \in \mathbb{Z}'_{t_{N-1}} \cap \mathbb{Z}_{t_{N-1}}$ as $\delta \to 0$, the following relation takes place,

$$w^{(\delta)}(t_N, Z_{N-1}^{(\delta)}(z_\delta)) \Rightarrow w^{(0)}(t_N, Z_{N-1}^{(0)}(z_0)) \text{ as } \delta \to 0. \tag{56}$$

Relation (56) follows from general results on weak convergence for compositions of random functions given in [51]. However, the external functions $w^{(\delta)}(t_N, \cdot)$ in the composition on the right hand side in (56) is non-random. This let us give a simpler proof of this relation.

Let us take an arbitrary sequence $\delta_k \to \delta_0 = 0$ as $k \to \infty$. According to condition $\mathbf{B_1}$, (a) the random variables $Z_{N-1}^{(\delta_k)}(z_{\delta_k}) \Rightarrow Z_{N-1}^{(\delta_0)}(z_{\delta_0})$ as $k \to \infty$, for an arbitrary $z_{\delta_k} \to z_{\delta_0} \in \mathbb{Z}'_{t_{N-1}} \cap \mathbb{Z}_{t_{N-1}}$ as $k \to \infty$, and (b) $\mathsf{P}\{Z_{N-1}^{(\delta_0)}(z_{\delta_0}) \in \mathbb{Z}'_{t_N} \cap \mathbb{Z}_{t_N}\} = 1$. Now, according the representation theorem by Skorokhod [57], one can construct random variables $\tilde{Z}_{N-1}^{(\delta_k)}(z_{\delta_k}), k = 0, 1, \dots$ on some probability space $(\Omega, \mathcal{F}, \mathsf{P})$ such that (c) $\mathsf{P}\{\tilde{Z}_{N-1}^{(\delta_k)}(z_{\delta_k}) \in A\} = \mathsf{P}\{Z_{N-1}^{(\delta_k)}(z_{\delta_k}) \in A\}, A \in \mathcal{B}_{\mathbb{Z}}$, for every $k = 0, 1, \dots$, and (d) $\tilde{Z}_{N-1}^{(\delta_k)}(z_{\delta_k}) \xrightarrow{a.s.} \tilde{Z}_{N-1}^{(\delta_0)}(z_{\delta_0})$ as $k \to \infty$. Let $A_{N-1} = \{\omega \in \Omega : \tilde{Z}_{N-1}^{(\delta_k)}(z_{\delta_k}, \omega) \to \tilde{Z}_{N-1}^{(\delta_0)}(z_{\delta_0}, \omega) \text{ as } k \to \infty\}$ and $B_{N-1} = \{\omega \in \Omega : \tilde{Z}_{N-1}^{(\delta_0)}(z_{\delta_0}, \omega) \in \mathbb{Z}'_{t_N} \cap \mathbb{Z}_{t_N}\}$. Relation (d) implies that $\mathsf{P}(A_{N-1}) = 1$. Relations (b) and (c) imply that $\mathsf{P}(B_{N-1}) = 1$. These two relations imply that $\mathsf{P}(A_{N-1} \cap B_{N-1}) = 1$. By relation (52) and the definition of the sets A_{N-1} and B_{N-1}, functions $w^{(\delta_k)}(t_N, \tilde{Z}_{N-1}^{(\delta_k)}(z_{\delta_k}, \omega)) \to w^{(\delta_0)}(t_N, \tilde{Z}_{N-1}^{(\delta_0)}(z_{\delta_0}, \omega))$ as $k \to \infty$, for $\omega \in A_{N-1} \cap B_{N-1}$. Thus, (e) the random variables $w^{(\delta_k)}(t_N, \tilde{Z}_{N-1}^{(\delta_k)}(z_{\delta_k})) \xrightarrow{a.s.} w^{(\delta_0)}(t_N, \tilde{Z}_{N-1}^{(\delta_0)}(z_{\delta_0}))$ as $k \to \infty$. Relation (c) implies that (f) $\mathsf{P}\{w^{(\delta_k)}(t_N, \tilde{Z}_{N-1}^{(\delta_k)}(z_{\delta_k})) \in A\} = \mathsf{P}\{w^{(\delta_k)}(t_N, Z_{N-1}^{(\delta_k)}(z_{\delta_k})) \in A\}, A \in \mathcal{B}_{\mathbb{Z}}$, for every $k = 0, 1, \dots$. Relations (e) and (f) imply that (g) the random variables $w^{(\delta_k)}(t_N, Z_{N-1}^{(\delta_k)}(z_{\delta_k})) \Rightarrow w^{(\delta_0)}(t_N, Z_{N-1}^{(\delta_0)}(z_{\delta_0}))$ as $k \to \infty$. Because of the arbitrary choice of the sequence $\delta_k \to \delta_0$, relation (g) implies relation (56).

Using inequality (54) and condition $\mathbf{C_3}$ we get for any sequence $z_\delta = (y_\delta, x_\delta) \to z_0 = (y_0, x_0) \in \mathbb{Z}'_{t_{N-1}} \cap \mathbb{Z}_{t_{N-1}}$ as $\delta \to 0$, and for $\delta \leq \delta_2$,

$$\begin{aligned}
\mathsf{E}(w^{(\delta)}(t_N, Z_{N-1}^{(\delta)}(z_\delta)))^{\frac{\beta}{\gamma}} &= \mathsf{E}_{z_\delta, t_{N-1}}(w^{(\delta)}(t_N, Z^{(\delta)}(t_N)))^{\frac{\beta}{\gamma}} \\
&\leq \mathsf{E}_{z_\delta, t_{N-1}}(L_{1,N} + L_{2,N} e^{\gamma |Y^{(\delta)}(t_N)|})^{\frac{\beta}{\gamma}} \\
&\leq 2^{\frac{\beta}{\gamma} - 1}([L_{1,N}]^{\frac{\beta}{\gamma}} + [L_{2,N}]^{\frac{\beta}{\gamma}} \mathsf{E}_{z_\delta, t_{N-1}} e^{\beta |y_\delta|} e^{\beta |Y^{(\delta)}(t_N) - y_\delta|}) \\
&\leq 2^{\frac{\beta}{\gamma} - 1}([L_{1,N}]^{\frac{\beta}{\gamma}} + [L_{2,N}]^{\frac{\beta}{\gamma}} (L_{10} + 1) e^{\beta |y_\delta|})
\end{aligned} \tag{57}$$

and, therefore,

$$\overline{\lim}_{\delta \to 0} \mathsf{E}(w^{(\delta)}(t_N, Z_{N-1}^{(\delta)}(z_\delta)))^{\frac{\beta}{\gamma}} < \infty. \tag{58}$$

Relations (56) and (58) imply that for any sequence $z_\delta \to z_0 \in \mathbb{Z}'_{t_{N-1}} \cap \mathbb{Z}_{t_{N-1}}$ as $\delta \to 0$,

$$\mathsf{E}_{z_\delta, t_{N-1}} w^{(\delta)}(t_N, Z^{(\delta)}(t_N)) \to \mathsf{E}_{z_0, t_{N-1}} w^{(0)}(t_N, Z^{(0)}(t_N)) \text{ as } \delta \to 0. \tag{59}$$

Relation (59) and condition $\mathbf{A_3'}$ imply that for any sequence $z_\delta = (y_\delta, x_\delta) \to z_0 = (y_0, x_0) \in \mathbb{Z}'_{t_{N-1}} \cap \mathbb{Z}_{t_{N-1}}$ as $\delta \to 0$,

$$\begin{aligned} w^{(\delta)}(t_{N-1}, z_\delta) &= g^{(\delta)}(t_{N-1}, e^{y_\delta}) \vee \mathsf{E}_{z_\delta, t_{N-1}} w^{(\delta)}(t_N, Z^{(\delta)}(t_N)) \\ &\to w^{(0)}(t_{N-1}, z_0) \\ &= g^{(0)}(t_{N-1}, e^{y_0}) \vee \mathsf{E}_{z_0, t_{N-1}} w^{(0)}(t_N, Z^{(0)}(t_N)) \text{ as } \delta \to 0. \end{aligned} \tag{60}$$

Relations (54), (55), and (60) are analogues of relations (50), (51), and (52). By repeating, the recursive procedure described above we finally get that for every $n = 0, 1, \ldots, N$, and for $\delta \leq \delta_2$,

$$w^{(\delta)}(t_n, z) \leq L_{1,n} + L_{2,n} e^{\gamma |y|}, \ z = (y, x) \in \mathbb{Z}, \tag{61}$$

for some constants,

$$L_{1,n}, L_{2,n} < \infty, \tag{62}$$

and that, for an arbitrary $z_{\delta,n} \to z_{0,n}$ as $\delta \to 0$, where $z_{0,n} \in \mathbb{Z}'_{t_n} \cap \mathbb{Z}_{t_n}$, and for every $n = 0, 1, \ldots, N$,

$$w^{(\delta)}(t_n, z_{\delta,n}) \to w^{(0)}(t_n, z_{0,n}) \text{ as } \delta \to 0. \tag{63}$$

Let us take an arbitrary sequence $\delta_k \to \delta_0 = 0$ as $k \to \infty$. According to condition $\mathbf{B_2}$, the random variables (h) $Z^{(\delta_k)}(0) \Rightarrow Z^{(\delta_0)}(0)$ as $k \to \infty$ and (i) $\mathsf{P}\{Z^{(\delta_0)}(0) \in \mathbb{Z}'_{t_0} \cap \mathbb{Z}_{t_0}\} = 1$. According to Skorokhod Representation Theorem, one can construct random variables $\tilde{Z}^{(\delta_k)}(0), k = 0, 1, \ldots$ on some probability space $(\Omega, \mathcal{F}, \mathsf{P})$ such that (j) $\mathsf{P}\{\tilde{Z}^{(\delta_k)}(0) \in A\} = \mathsf{P}\{Z^{(\delta_k)}(0) \in A\}, A \in \mathcal{B}_\mathbb{Z}$, for every $k = 0, 1, \ldots$, and (k) $\tilde{Z}^{(\delta_k)}(0) \xrightarrow{a.s.} \tilde{Z}^{(\delta_0)}(0)$ as $k \to \infty$. Let us denote $A = \{\omega \in \Omega : \tilde{Z}^{(\delta_k)}(0, \omega) \to \tilde{Z}^{(\delta_0)}(0, \omega) \text{ as } k \to \infty\}$ and $B = \{\omega \in \Omega : \tilde{Z}^{(\delta_0)}(0, \omega) \in \mathbb{Z}'_{t_0} \cap \mathbb{Z}_{t_0}\}$. Relation (k) implies that $\mathsf{P}(A) = 1$. Relations (i) and (j) imply that $\mathsf{P}(B) = 1$. These two relations imply that $\mathsf{P}(A \cap B) = 1$. By condition $\mathbf{B_2}$, relation (63), and the definition of sets A and B, functions $w^{(\delta_k)}(t_0, \tilde{Z}^{(\delta_k)}(0, \omega)) \to w^{(\delta_0)}(t_0, \tilde{Z}^{(\delta_0)}(0, \omega))$ as $k \to \infty$, for $\omega \in A \cap B$. Thus, (l) the random variables $w^{(\delta_k)}(t_0, \tilde{Z}^{(\delta_k)}(0)) \xrightarrow{a.s.} w^{(\delta_0)}(t_0, \tilde{Z}^{(\delta_0)}(0))$ as $k \to \infty$. Relation (j) implies that (m) $\mathsf{P}\{w^{(\delta_k)}(t_0, \tilde{Z}^{(\delta_k)}(0)) \in A\} = \mathsf{P}\{w^{(\delta_k)}(t_0, Z^{(\delta_k)}(0)) \in A\}, A \in \mathcal{B}_\mathbb{Z}$, for every $k = 0, 1, \ldots$. Relations (l) and (m) imply that (n) the random variables $w^{(\delta_k)}(t_N, Z^{(\delta_k)}(0)) \Rightarrow w^{(\delta_0)}(t_N, Z^{(\delta_0)}(0))$ as $k \to \infty$. Because the sequence $\delta_k \to \delta_0$ was arbitrary, relation (n) implies that,

$$w^{(\delta)}(t_0, Z^{(\delta)}(0)) \Rightarrow w^{(0)}(t_0, Z^{(0)}(0)) \text{ as } \delta \to 0. \tag{64}$$

Using inequality (61) and condition $\mathbf{C_4}$, we get for $\delta \leq \delta_3$,

$$\begin{aligned} \mathsf{E}(w^{(\delta)}(t_0, Z^{(\delta)}(0)))^{\frac{\beta}{\gamma}} &\leq \mathsf{E}(L_{1,0} + L_{2,0} e^{\gamma |Y^{(\delta)}(0)|})^{\frac{\beta}{\gamma}} \\ &\leq 2^{\frac{\beta}{\gamma} - 1}((L_{1,0})^{\frac{\beta}{\gamma}} + (L_{2,0})^{\frac{\beta}{\gamma}} \mathsf{E} e^{\beta |Y^{(\delta)}(0)|}) \\ &\leq 2^{\frac{\beta}{\gamma} - 1}((L_{1,0})^{\frac{\beta}{\gamma}} + (L_{2,0})^{\frac{\beta}{\gamma}} L_{11}), \end{aligned} \tag{65}$$

and, therefore,

$$\overline{\lim}_{\delta \to 0} \mathsf{E}(w^{(\delta)}(t_0, Z^{(\delta)}(0)))^{\frac{\beta}{\gamma}} < \infty. \tag{66}$$

Relations (64) and (66) imply that,

$$\mathsf{E} w^{(\delta)}(t_0, Z^{(\delta)}(0)) \to \mathsf{E} w^{(0)}(t_0, Z^{(0)}(0)) \text{ as } \delta \to 0. \tag{67}$$

Formula (47) and relation (67) imply relation (46) given in Theorem 2.

The proof of Theorem 2 is complete. $\diamond$

In order to provide convergence of the reward functionals $\Phi(\mathcal{M}^{(\delta)}_{\Pi_N,T})$ for any partition Π_N of the interval $[0,T]$, one can require the conditions of Theorem 3 to hold for any partition of this interval. Note that these conditions also would not involve the derivatives of the pay-off functions. In this case, the pre-limit and the limit pay-off functions can be discontinuous.

5. Convergence of rewards for continuous time price processes

As was mentioned above, in the discrete time case, the pay-off functions can be discontinuous. In the continuous time case, the derivatives of the pay-off functions are involved in condition $\mathbf{A_1}$. The corresponding assumptions imply continuity of the pay-off functions.

This give us possibility to weaken the assumption concerning the convergence of the pay-off functions and just to require their pointwise convergence:

$\mathbf{A_4}$: $g^{(\delta)}(t,s) \to g^{(0)}(t,s)$ as $\delta \to 0$, for every $(t,s) \in [0,T] \times (0,\infty)$.

Obviously, condition $\mathbf{A_4}$ can be re-written in terms of function $g^{(\delta)}(t,e^y)$, $(t,y) \in [0,\infty) \times \mathbb{R}_1$:

$\mathbf{A_4'}$: $g^{(\delta)}(t,e^y) \to g^{(0)}(t,e^y)$ as $\delta \to 0$, for every $(t,y) \in [0,T] \times \mathbb{R}_1$.

Let us now formulate conditions assumed for the transition probabilities and the initial distributions of process $Z^{(\delta)}(t)$.

The first condition assumes weak convergence of the transition probabilities that should be locally uniform with respect to initial states from some sets, and also that the corresponding limit measures are concentrated on these sets:

$\mathbf{B_3}$: There exist measurable sets $\mathbb{Z}_t \subseteq \mathbb{Z}$, $t \in [0,T]$ such that: **(a)** $P^{(\delta)}(t, z_\delta, t+u, \cdot) \Rightarrow P^{(0)}(t,z,t+u,\cdot)$ as $\delta \to 0$, for any $z_\delta \to z \in \mathbb{Z}_t$ as $\delta \to 0$ and $0 \le t < t+u \le T$; **(b)** $P^{(0)}(t,z,t+u,\mathbb{Z}_{t+u}) = 1$ for every $z \in \mathbb{Z}_t$ and $0 \le t < t+u \le T$.

The typical example is where the sets $\bar{\mathbb{Z}}_t = \emptyset$. In this case, condition $\mathbf{B_3}$ **(b)** automatically holds. Another typical example is where $\mathbb{Z}_t = \mathbb{Y}_t \times \mathbb{X}$, where the sets $\bar{\mathbb{Y}}_t$ are at most finite or countable sets. In this case, the assumption that the measures $P^{(0)}(t,z,t+u,A \times \mathbb{X}), A \in \mathcal{B}_1$ have no atoms implies that conditions $\mathbf{B_3}$ **(b)** holds.

The second condition assumes weak convergence of the initial distributions to some distribution that is assumed to be concentrated on the sets of convergence for the corresponding transition probabilities:

$\mathbf{B_4}$: **(a)** $P^{(\delta)}(\cdot) \Rightarrow P^{(0)}(\cdot)$ as $\delta \to 0$; **(b)** $P^{(0)}(\mathbb{Z}_0) = 1$, where $\mathbb{Z}_0$ is the set introduced in condition $\mathbf{B_3}$.

The typical example is again when the set $\bar{\mathbb{Z}}_0$ is empty. In this case condition $\mathbf{B_4}$ **(b)** holds automatically. Also in the case, where $\mathbb{Z}_0 = \mathbb{Y}_0 \times \mathbb{X}$ and $\bar{\mathbb{Y}}_0$ is at most a finite or countable set, the assumption that measure $P^{(0)}(A \times \mathbb{X}), A \in \mathcal{B}_1$ has no atoms implies that conditions $\mathbf{B_4}$ **(b)** holds.

Condition $\mathbf{B_4}$ holds, for example, if the initial distributions $P^{(\delta)}(A) = \chi_A(z_0)$ are concentrated in a point $z_0 \in \mathbb{Z}_0$, for all $\delta \ge 0$. This condition also holds, if the initial distributions $P^{(\delta)}(A) = \chi_A(z_\delta)$ for $\delta \ge 0$, where $z_\delta \to z_0$ as $\delta \to 0$ and $z_0 \in \mathbb{Z}_0$.

The following theorem, presenting conditions for convergence of reward functionals $\Phi(\mathcal{M}^{(\delta)}_{max,T})$, is the third main result of the present paper.

Theorem 3. *Let conditions* $\mathbf{A_1}$, $\mathbf{A_4}$, $\mathbf{B_3}$, $\mathbf{B_4}$, $\mathbf{C_1}$, *and* $\mathbf{C_2}$ *hold. Then,*

$$\Phi(\mathcal{M}^{(\delta)}_{max,T}) \to \Phi(\mathcal{M}^{(0)}_{max,T}) < \infty \text{ as } \delta \to 0. \tag{68}$$

Proof of Theorem 3. Let $\Pi_N = \{0 = t_{0,N} < t_{1,N} < \dots t_{N,N} = T\}$ be a sequence of partitions such that $d(\Pi_N) \to 0$ as $N \to \infty$.

Lemma 5. *Let conditions* $\mathbf{A_1}$, $\mathbf{C_1}$, *and* $\mathbf{C_2}$, *the following relation holds for any sequence of partitions* Π_N *such that* $d(\Pi_N) \to 0$ *as* $N \to \infty$,

$$\lim_{N\to\infty} \overline{\lim}_{\delta\to 0}(\Phi(\mathcal{M}^{(\delta)}_{max,T}) - \Phi(\mathcal{M}^{(\delta)}_{\Pi_N,T})) = 0. \tag{69}$$

Proof of Lemma 5. This lemma is a direct corollary of Theorem 1, which implies that, under conditions $\mathbf{A_1}$, $\mathbf{C_1}$, and $\mathbf{C_2}$, there exist constants $L_3, L_4 < \infty$ such that the following skeleton approximation inequality holds for $\delta \leq \delta_1$ and N such that $d(\Pi_N) \leq c$ (δ_1 and c were defined in Theorem 1),

$$\Phi(\mathcal{M}^{(\delta)}_{max,T}) - \Phi(\mathcal{M}^{(\delta)}_{\Pi_N,T}) \leq L_3 d(\Pi_N) + L_4(\Delta_\beta(Y^{(\delta)}(\cdot), d(\Pi_N), T))^{\frac{\beta-\gamma}{\beta}}. \tag{70}$$

This estimate directly implies relation (69). ⋄

The following lemma show that conditions of Theorem 3 do imply that conditions of Theorem 2 hold for any partition Π_N of the interval $[0, T]$.

Lemma 6. *Let conditions* $\mathbf{A_1}$, $\mathbf{A_4}$, $\mathbf{B_3}$, $\mathbf{B_4}$, $\mathbf{C_1}$, *and* $\mathbf{C_2}$ *hold. Then, conditions of Theorem* 2 *hold for any partition* $\Pi_N = \{0 = t_0 < t_1 \cdots < t_N = T\}$ *of interval* $[0, T]$, *and therefore the following asymptotic relation holds,*

$$\Phi(\mathcal{M}^{(\delta)}_{\Pi_N,T}) \to \Phi(\mathcal{M}^{(0)}_{\Pi_N,T}) \text{ as } \delta \to 0. \tag{71}$$

Proof. Conditions $\mathbf{A_1}$ **(c)** and **(d)** imply that (t) for any $t \in [0, T]$, $0 < s < \infty$, and $\delta \leq \delta_0$,

$$\begin{aligned} g^{(\delta)}(t, s) &\leq \int_0^s |\frac{\partial g^{(\delta)}(t, u)}{\partial u}| du + g^{(\delta)}(t, 0) \\ &\leq \int_0^s (K_3 + K_4 u^{\gamma_2}) du + K_5 \\ &= K_3 s + \frac{K_4}{\gamma_2 + 1} s^{\gamma_2+1} + K_5 \leq K_6 + K_7 s^\gamma, \end{aligned} \tag{72}$$

where $K_6 = K_3 + K_4 + K_5$ and $K_7 = (K_3 + K_4)$.

Thus condition $\mathbf{A_2}$ holds for any partition Π_N of the interval $[0, T]$.

Condition $\mathbf{A_1}$ **(c)** imply also that, for any $t \in [0, T]$, $0 < s' < s'' < \infty$, and $\delta \leq \delta_0$,

$$\begin{aligned} |g^{(\delta)}(t, s'') - g^{(\delta)}(t, s'')| &\leq \int_{s'}^{s''} |\frac{\partial g^{(\delta)}(t, u)}{\partial u}| du \\ &\leq \int_{s'}^{s''} (K_3 + K_4 u^{\gamma_2}) du \leq (K_3 + K_4 (s'')^\gamma)(s'' - s'). \end{aligned} \tag{73}$$

Thus, for any $t \in [0, T]$, $0 < s^- < s^+ < \infty$, and $\delta \leq \delta_0$, the following inequality holds for the modula of compactness in uniform topology for payoff functions,

$$\sup_{s^- \leq s' \leq s'' \leq s' + c \leq s^+} |g^{(\delta)}(t, s') - g^{(\delta)}(t, s'')| \leq (K_3 + K_4 (s^+)^\gamma) c. \tag{74}$$

Relation (74) and condition $\mathbf{A_4}$ imply that the conditions of the Ascoli-Arzelá Theorem holds for pay-off functions $g^{(\delta)}(t,s), s \in [s^-, s^+]$, for every $t \in [0,T]$ and $0 < s^- < s^+ < \infty$. Thus, for every $t \in [0,T]$, these functions converge uniformly, i.e.,

$$\sup_{s^- \leq s \leq s^+} |g^{(\delta)}(t,s) - g^{(0)}(t,s)| \to 0 \text{ as } \delta \to 0. \tag{75}$$

Relation (75) implies, in an obvious way, that the condition of locally uniform convergence $\mathbf{A_3}$ holds for any partition Π_N of the interval $[0,T]$ with the corresponding sets $\mathbb{S}_{t_n} = (0,\infty)$. Note that in this case sets $\mathbb{Y}_{t_n} = \ln \mathbb{S}_{t_n} = \mathbb{R}_1$ and therefore the sets $\mathbb{Z}'_{t_n} = \mathbb{Z}$.

If $\mathbb{Z}'_{t_n} = \mathbb{Z}$ for any $t_n \in [0,T]$ then condition $\mathbf{B_3}$ implies that condition $\mathbf{B_1}$ holds with the sets $\mathbb{Z}_{t_n}$ from condition $\mathbf{B_3}$ for any partition Π_N of the interval $[0,T]$, and condition $\mathbf{B_4}$ implies that condition $\mathbf{B_2}$ holds with the sets $\mathbb{Z}'_{t_n} = \mathbb{Z}$ for any partition Π_N of the interval $[0,T]$.

It remain to show that condition $\mathbf{C_1}$ implies that condition $\mathbf{C_3}$ holds. Condition $\mathbf{C_1}$ implies that for any constant $L_{12} < \infty$ one can choose $c = c(L_{12}) > 0$ and then $\delta_3 = \delta_3(c) < \delta_0$ such that for $\delta \leq \delta_3$,

$$\Delta_\beta(Y^{(\delta)}(\cdot), c, T) \leq L_{12}. \tag{76}$$

Take an arbitrary integer $0 \leq n < N$ and consider the uniform partition $t_n = u_{0,m} < \cdots < u_{m,m} = t_{n+1}$ of the interval $[t_n, t_{n+1}]$ by points $u_{k,m} = \frac{(t_{n+1}-t_n)k}{m}$. Using relation (76) and the Markov property of the process $Z^{(\delta)}(t)$ we get, for $\delta \leq \delta_3$, $m = [\frac{(t_{n+1}-t_n)}{c}] + 1$ (in this case $\frac{(t_{n+1}-t_n)}{m} \leq c$), $z \in \mathbb{Z}$, and $k = 1, \ldots, m$,

$$\begin{aligned}
&\mathsf{E}_{z,t_n}(e^{\beta|Y^{(\delta)}(u_{k,m})-Y^{(\delta)}(u_{0,m})|} - 1) \\
&\quad \leq \mathsf{E}_{z,t_n} e^{\beta|Y^{(\delta)}(u_{k-1,m})-Y^{(\delta)}(u_{0,m})|} e^{\beta|Y^{(\delta)}(u_{k,m})-Y^{(\delta)}(u_{k-1,m})|} - 1 \\
&\quad = \mathsf{E}_{z,t_n}((e^{\beta|Y^{(\delta)}(u_{k-1,m})-Y^{(\delta)}(u_{0,m})|} - 1) e^{\beta|Y^{(\delta)}(u_{k,m})-Y^{(\delta)}(u_{k-1,m})|} \\
&\qquad + e^{\beta|Y^{(\delta)}(u_{k,m})-Y^{(\delta)}(u_{k-1,m})|} - 1) \\
&\quad = \mathsf{E}_{z,t_n}\{(e^{\beta|Y^{(\delta)}(u_{k-1,m})-Y^{(\delta)}(u_{0,m})|} - 1) \\
&\qquad \times \mathsf{E}\{(e^{\beta|Y^{(\delta)}(u_{k,m})-Y^{(\delta)}(u_{k-1,m})|} - 1 + 1)/Z^{(\delta)}(u_{k-1,m})\}\} \\
&\qquad + \mathsf{E}_{z,t_n}\{\mathsf{E}\{(e^{\beta|Y^{(\delta)}(u_{k,m})-Y^{(\delta)}(u_{k-1,m})|} - 1)/Z^{(\delta)}(u_{k-1,m})\}\} \\
&\quad \leq \mathsf{E}_{z,t_n}(e^{\beta|Y^{(\delta)}(u_{k-1,m})-Y^{(\delta)}(u_{0,m})|} - 1)(L_{12} + 1) + L_{12}. \qquad (77)
\end{aligned}$$

Finally, we get, for $\delta \leq \delta_3$ and $z \in \mathbb{Z}$,

$$\begin{aligned}
&\mathsf{E}_{z,t_n}(e^{\beta|Y^{(\delta)}(t_{n+1})-Y^{(\delta)}(t_n)|} - 1) \\
&\quad = \mathsf{E}_{z,t_n}(e^{\beta|Y^{(\delta)}(u_{m,m})-Y^{(\delta)}(u_{0,m})|} - 1) \\
&\quad \leq (L_{12} + 1)^m + L_{12} \sum_{k=0}^{m-1} (L_{12} + 1)^k = 2(L_{12} + 1)^m - 1 < \infty. \qquad (78)
\end{aligned}$$

Thus condition $\mathbf{C_3}$ holds. Finally, condition $\mathbf{C_2}$ is equivalent to condition $\mathbf{C_4}$. The proof is complete. ◇

Lemmas 5 and 6 let us make the final fifth step in the proof of Theorem 3. We employ the following obvious inequality that can be written down for any partition

Π_N,

$$|\Phi(\mathcal{M}^{(\delta)}_{max,T}) - \Phi(\mathcal{M}^{(0)}_{max,T})| \leq |\Phi(\mathcal{M}^{(\delta)}_{max,T}) - \Phi(\mathcal{M}^{(\delta)}_{\Pi_N,T})| \\ + |\Phi(\mathcal{M}^{(0)}_{max,T}) - \Phi(\mathcal{M}^{(0)}_{\Pi_N,T})| + |\Phi(\mathcal{M}^{(\delta)}_{\Pi_N,T}) - \Phi(\mathcal{M}^{(0)}_{\Pi_N,T})|. \tag{79}$$

Using this inequality and relation (71) given in Lemma 6 we get for any partition Π_N,

$$\overline{\lim}_{\delta\to 0}|\Phi(\mathcal{M}^{(\delta)}_{max,T}) - \Phi(\mathcal{M}^{(0)}_{max,T})| \\ \leq \overline{\lim}_{\delta\to 0}|\Phi(\mathcal{M}^{(\delta)}_{max,T}) - \Phi(\mathcal{M}^{(\delta)}_{\Pi_N,T})| + |\Phi(\mathcal{M}^{(0)}_{max,T}) - \Phi(\mathcal{M}^{(0)}_{\Pi_N,T})|. \tag{80}$$

Finally, relation (69) given in Lemma 5 implies (note that relation $\delta \to 0$ admit also the case where $\delta = 0$) that the expression on the right hand side in (80) can be forced to take a value less then any $\varepsilon > 0$ by choosing the partition Π_N with the diameter $d(\Pi_N)$ small enough.

This proves the asymptotic relation (68) and completes the proof of Theorem 3. $\diamond$

6. Compactness conditions for log-price processes

Let us make several useful remarks concerning the evaluation of the exponential modulus of compactness $\Delta_\beta(Y^{(\delta)}(\cdot), c, T)$.

The following representation formula takes place for the exponential modulus of compactness $\Delta_\beta(Y^{(\delta)}(\cdot), c, T)$,

$$\Delta_\beta(Y^{(\delta)}(\cdot), c, T) = \sup_{0\leq t\leq t+u\leq t+c\leq T}\ \sup_{z\in\mathbb{Z}}(\mathsf{E}_{z,t}e^{\beta|Y^{(\delta)}(t+u)-Y^{(\delta)}(t)|} - 1) \\ = \sup_{0\leq t\leq t+u\leq t+c\leq T}\ \sup_{z\in\mathbb{Z}}\beta\int_0^\infty e^{\beta y}\mathsf{P}_{z,t}\{|Y^{(\delta)}(t+u) - Y^{(\delta)}(t)| > y\}dy.$$

This representation shows that the compactness condition $\mathbf{C_1}$ can be effectively used if the tail probabilities for increments $|Y^{(\delta)}(t+u) - Y^{(\delta)}(t)|$ are given explicitly or can be effectively estimated.

The condition of exponential moment compactness $\mathbf{C_1}$ can also be connected with the traditional condition of compactness in J-topology for Markov type càdlàg processes. Let us introduce the modulus of J-compactness,

$$\Delta(Y^{(\delta)}(\cdot), h, c, T) = \sup_{0\leq t\leq t+u\leq t+c\leq T}\ \sup_{z\in\mathbb{Z}}\mathsf{P}_{z,t}\{|Y^{(\delta)}(t+u) - Y^{(\delta)}(t)| \geq h\}.$$

The following condition of J-compactness plays the key role in functional limit theorems for Markov type càdlàg processes:

$\mathbf{C_5}$: $\lim_{c\to 0}\overline{\lim}_{\delta\to 0}\Delta(Y^{(\delta)}(\cdot), h, c, T) = 0,\ h > 0$.

Let also introduce the quantity, which represents the maximum of the moment generating functions for increments of the log-price process $Y^{(\delta)}(t)$:

$$\Xi_\beta(Y^{(\delta)}(\cdot), T) = \sup_{0\leq t\leq t+u\leq T}\ \sup_{z\in\mathbb{Z}}\mathsf{E}_{z,t}e^{\beta(Y^{(\delta)}(t+u) - Y^{(\delta)}(t))}.$$

The following condition, formulated in terms of these moment generating functions, can be effectively verified in many cases:

$\mathbf{C_6}$: $\overline{\lim}_{\delta\to 0}\Xi_{\pm\beta'}(Y^{(\delta)}(\cdot), T) < \infty$, for some $\beta' > \beta$, where β is the parameter in condition $\mathbf{C_1}$.

Lemma 7. *Conditions* $\mathbf{C_5}$ *and* $\mathbf{C_6}$ *imply condition* $\mathbf{C_1}$ *to hold.*

Proof of Lemma 7. Using Hölder's inequality we get the following estimates, for every $0 \leq t \leq t+u \leq T$ and $z \in \mathbb{Z}$,

$$\begin{aligned} & \mathsf{E}_{z,t} e^{\beta |Y^{(\delta)}(t+u) - Y^{(\delta)}(t)|} - 1 \\ & \quad \leq (e^{\beta h} - 1) + \mathsf{E}_{z,t} e^{\beta |Y^{(\delta)}(t+u) - Y^{(\delta)}(t)|} \chi(|Y^{(\delta)}(t+u) - Y^{(\delta)}(t)| \geq h) \\ & \quad \leq (e^{\beta h} - 1) + (\mathsf{E}_{z,t} e^{\beta' |Y^{(\delta)}(t+u) - Y^{(\delta)}(t)|})^{\frac{\beta}{\beta'}} \mathsf{P}_{z,t}\{|Y^{(\delta)}(t+u) - Y^{(\delta)}(t)| \geq h\}. \end{aligned} \tag{81}$$

The following inequality, which connect the exponential moment modulus of compactness with the modulus of J-compactness, follows readily from (81),

$$\begin{aligned} \Delta_\beta(Y^{(\delta)}(\cdot), c, T) & \leq (e^{\beta h} - 1) \\ & \quad + (\Delta_{\beta'}(Y^{(\delta)}(\cdot), c, T) + 1)^{\frac{\beta}{\beta'}} \Delta(Y^{(\delta)}(\cdot), h, c, T). \end{aligned} \tag{82}$$

Also, the following estimate takes place, for every $0 \leq t \leq t+u \leq T$ and $z \in \mathbb{Z}$,

$$\begin{aligned} \mathsf{E}_{z,t} e^{\beta' |Y^{(\delta)}(t+u) - Y^{(\delta)}(t)|} & = \mathsf{E}_{z,t} e^{\beta' (Y^{(\delta)}(t+u) - Y^{(\delta)}(t))} \chi(Y^{(\delta)}(t+u) \geq Y^{(\delta)}(t)) \\ & \quad + \mathsf{E}_{z,t} e^{-\beta' (Y^{(\delta)}(t+u) - Y^{(\delta)}(t))} \chi(Y^{(\delta)}(t+u) < Y^{(\delta)}(t)) \\ & \leq \mathsf{E}_{z,t} e^{\beta' (Y^{(\delta)}(t+u) - Y^{(\delta)}(t))} + \mathsf{E}_{z,t} e^{-\beta' (Y^{(\delta)}(t+u) - Y^{(\delta)}(t))}. \end{aligned} \tag{83}$$

This estimate implies the following inequality,

$$\Delta_{\beta'}(Y^{(\delta)}(\cdot), c, T) + 1 \leq \Xi_{\beta'}(Y^{(\delta)}(\cdot), T) + \Xi_{-\beta'}(Y^{(\delta)}(\cdot), T). \tag{84}$$

Relations (88) and (89) imply the statement of Lemma 7. $\diamond$

7. Examples

In this section, we illustrate the theoretical results given in Theorems 1–3 by several examples.

Let us consider the model without modulation that is where the phase space $\mathbb{X} = \{x_0\}$ degenerates to an one-point set while the log-price process $Y^{(\delta)}(t), t \geq 0$ is a càdlàg processes with independent increments.

We also assume for simplicity that the initial state of this process $y_\delta = Y^{(\delta)}(0)$ is a constant.

The process $Y^{(\delta)}(t)$ is a càdlàg Markov process with transition probabilities which are connected with the distributions of increments for this process $P^{(\delta)}(t, t+u, A)$ by the following relation,

$$\begin{aligned} P^{(\delta)}(t, y, t+u, A) & = P^{(\delta)}(t, t+u, A - y) \\ & = \mathsf{P}\{y + Y^{(\delta)}(t+u) - Y^{(\delta)}(t) \in A\}. \end{aligned} \tag{85}$$

Let us assume the following standard condition of weak convergence for distributions of increments for log-price processes:

$\mathbf{D_1}$: $P^{(\delta)}(t, t+u, \cdot) \Rightarrow P^{(0)}(t, t+u, \cdot)$ as $\delta \to 0$, $0 \leq t \leq t+u \leq T$.

Representation (85) implies in an obvious way that condition $\mathbf{B_3}$ holds with the sets $\mathbb{Y}_t = \mathbb{R}_1, t \in [0, T]$, i.e., distributions of increments for the processes $Y^{(\delta)}(t)$ locally uniformly weakly converge, if condition $\mathbf{D_1}$ holds. Thus, in the case of processes with independent increments, condition $\mathbf{B_3}$ with the sets $\mathbb{Y}_t = \mathbb{R}_1$ is, in fact, equivalent to the standard condition of weak convergence for such processes.

In this case the J-compactness modulus $\Delta(Y^{(\delta)}(\cdot), h, c, T)$ take the following form,

$$\Delta(Y^{(\delta)}(\cdot), h, c, T) = \sup_{0 \leq t \leq t+u \leq t+c \leq T} \mathsf{P}\{|Y^{(\delta)}(t+u) - Y^{(\delta)}(t)| \geq h\}.$$

Thus, condition $\mathbf{C_5}$ is reduced to the standard J-compactness condition for the log-price processes:

$\mathbf{D_2}$: $\overline{\lim}_{c \to 0} \lim_{\delta \to 0} \Delta(Y^{(\delta)}(t), h, c, T) = 0$, $h > 0$,

Note that conditions $\mathbf{D_1}$ and $\mathbf{D_2}$ imply J-convergence of processes $Y^{(\delta)}(t), t \in [0, T]$ to process $Y^{(0)}(t), t \in [0, T]$ as $\delta \to 0$ and stochastic continuity of the limit process.

Also, in this case, the quantity $\Xi_\beta(Y^{(\delta)}(\cdot), T)$ takes the simplified form,

$$\Xi_\beta(Y^{(\delta)}(\cdot), T) = \sup_{0 \leq t \leq t+u \leq T} \mathsf{E} e^{\beta(Y^{(\delta)}(t+u) - Y^{(\delta)}(t))}.$$

Therefore, condition $\mathbf{C_6}$ take the following form:

$\mathbf{D_3}$: $\overline{\lim}_{\delta \to 0} \Xi_{\pm\beta'}(Y^{(\delta)}(\cdot), T) < \infty$, for some $\beta' > \beta$, where β is the parameter penetrating condition $\mathbf{C_1}$.

According to Lemma 7, conditions $\mathbf{D_1}$ and $\mathbf{D_2}$ imply condition $\mathbf{C_1}$ to hold.

Condition $\mathbf{B_4}$ is reduced in this case to the following condition:

$\mathbf{D_4}$: $\lim_{\delta \to 0} y_\delta = y_0$.

Note that y_0 can be any real number since the set $\mathbb{Y}_0 = \mathbb{R}_1$.

Obviously, condition $\mathbf{D_4}$ implies also condition $\mathbf{C_4}$ to hold.

Summarising the remarks above, one can conclude that the conditions and, therefore, the statement of Theorem 3 hold for the exponential price processes with independent increments $S^{(\delta)}(t) = e^{Y^{(\delta)}(t)}$, if conditions $\mathbf{A_1}$, $\mathbf{A_4}$, and $\mathbf{D_1}$ – $\mathbf{D_4}$ hold.

The skeleton approximations $Y^{(\delta)}(t) = Y^{(0)}([t/\delta])$, $t \geq 0$ for a stochastically continuous càdlàg log-prise process $Y^{(0)}(t)$, $t \geq 0$ with independent increments give an example of the model introduced above.

In this case, conditions $\mathbf{D_1}$ and $\mathbf{D_2}$ automatically hold.

As far as condition $\mathbf{D_3}$ is concerned it is implied by the following condition:

$\mathbf{D_5}$: $\Xi_{\pm\beta'}(Y^{(0)}(\cdot), T) < \infty$, for some $\beta' > \beta$, where β is the parameter penetrating condition $\mathbf{C_1}$.

Thus, the statement of Theorem 3 hold for the exponential price processes $S^{(\delta)}(t) = e^{Y^{(0)}([t/\delta])}, t \in [0, T]$, if conditions $\mathbf{A_1}$, $\mathbf{A_4}$, and $\mathbf{D_5}$ hold.

It is not out of the picture to note that, in this case, Theorem 1, given in the first part of this paper, yields a stronger result in the form of explicit estimates for accuracy of skeleton approximations for reward functions.

Note also that the optimal expected rewards for the skeleton price processes $S^{(\delta)}(t) = e^{Y^{(0)}([t/\delta])}$ can be estimated with the use of Monte Carlo method. The corresponding algorithms are described, for example, in [27, 28, 29, 34, 35, 52, 53].

Combination of these Monte Carlo based estimates with the skeleton approximations described above yields the effective approximation methods for optimal expected rewards for American type options with non-standard payoffs.

Now, let us consider the model, where the log-price process $Y^{(\delta)}(t), t \geq 0$ is, for every $\delta \geq 0$, a càdlàg Lévy process, which Lévy–Khintchine representation has the following form,

$$
\begin{aligned}
\varphi_t^{(\delta)}(v) &= \mathsf{E}\exp\{\mathrm{i}v(Y^{(\delta)}(t) - y_\delta)\} \\
&= \exp\{\mathrm{i}ta_\delta v - \frac{1}{2}tb_\delta^2 v^2 + t\int_{\mathbb{R}_1}(e^{\mathrm{i}vs} - 1 - \frac{\mathrm{i}vs}{1+s^2})\Pi_\delta(ds)\},\ v \in \mathbb{R}_1,\ t \geq 0, \qquad (86)
\end{aligned}
$$

where (a) $a_\delta \in \mathbb{R}_1$; (b) $b_\delta \geq 0$; (c) $\Pi_\delta(A)$ is a measure on $\mathbb{R}_1$ satisfying the integral condition $\int_{|s|\leq 1} s^2\Pi_\delta(ds) < \infty$; (d) $y_\delta = Y^{(\delta)}(0)$ is an initial state of this process, assumed for simplicity to be a constant.

Let us assume the following standard condition that is necessary and sufficient condition for weak convergence of increments of the processes $Y^{(\delta)}(t)$:

E₁: **(a)** $\lim_{\delta\to 0} a_\delta = a_0$; **(b)** $\overline{\lim}_{\varepsilon\to 0}\lim_{\delta\to 0}|b_0^2 - b_\delta^2 - \int_{|s|\leq\varepsilon} s^2\Pi_\delta(ds)| = 0$; **(c)** $\lim_{\delta\to 0}\int_{\mathbb{R}_1} f(s)\Pi_\delta(ds) = \int_{\mathbb{R}_1} f(s)\Pi_0(ds)$ for any continuous bounded function $f(s)$ vanishing to 0 in some neighbourhood of zero.

As is well known (see, for example, [58]) condition $\mathbf{E_1}$ implies, in this case, both conditions $\mathbf{D_1}$ and $\mathbf{D_2}$ to hold.

Also, as known, the moment generating function $\mathsf{E}\exp\{wY^{(\delta)}(t)\} = \psi_t^{(\delta)}(w)$ exists for given $w \in \mathbb{R}_1$ if and only if $\int_{|s|>1} e^{ws}\Pi_\delta(ds) < \infty$. Moreover, it is connected with the corresponding characteristic function by the formula $\psi_t^{(\delta)}(w) = \varphi^{(\delta)}(w/\mathrm{i})$.

It readily follows from these facts that condition $\mathbf{D_3}$ is implied, under condition $\mathbf{D_5}$, by the following condition:

E₂: $\overline{\lim}_{\delta\to 0}\int_{|s|>1} e^{\beta'|s|}\Pi_\delta(ds) < \infty$, where β' is the parameter given in condition $\mathbf{C_6}$.

Summarising the remarks above, one can conclude that the conditions and, therefore, the statement of Theorem 3 hold for the exponential Lévy prise processes introduced above, if conditions $\mathbf{A_1}$, $\mathbf{A_4}$, $\mathbf{D_4}$, and $\mathbf{E_1} - \mathbf{E_2}$ hold.

As far as the case with skeleton approximations $S^{(\delta)}(t) = e^{Y^{(0)}([t/\delta])}$ is concerned, let just note that condition $\mathbf{D_5}$ is implied, in this case, by the following condition:

E₃: $\int_{|s|>1} e^{\beta'|s|}\Pi_0(ds) < \infty$, where β' is the parameter given in condition $\mathbf{C_6}$.

Thus, the statement of Theorem 3 hold for the exponential price processes $S^{(\delta)}(t) = e^{Y^{(0)}([t/\delta])}$, if conditions $\mathbf{A_1}$, $\mathbf{A_4}$, and $\mathbf{E_3}$ hold.

The result formulated above can be readily generalised to the model of non-homogeneous in time stochastically continuous càdlàg exponential prise processes with independent increments.

Let us now consider a price process $S^{(\delta)}(t) = e^{Y^{(\delta)}(t)}$ which can be referred as an exponential process with independent increments modulated by semi-Markov stochastic index process. In this model, the log-price process $Y^{(\delta)}(t)$ is given in the form,

$$
\begin{aligned}
Y^{(\delta)}(t) = y_\delta &+ \sum_{n=0}^{N^{(\delta)}(t)-1} Y^{(\delta)}_{n,X_n^{(\delta)}}(T_{n+1}^{(\delta)} - T_n^{(\delta)}) \\
&+ Y^{(\delta)}_{N^{(\delta)}(t),X^{(\delta)}(t)}(T^{(\delta)}(t)), t \geq 0, \qquad (87)
\end{aligned}
$$

where (e) $X^{(\delta)}(t), t \geq 0$ is a continuous from the right semi-Markov process with a finite set of states $\mathbb{X} = \{1, \ldots, m\}$ and transition probabilities $Q_{ij}^{(\delta)}(t), t \geq 0, i, j \in \mathbb{X}$;

(f) $N^{(\delta)}(t)$ is the number of jumps for the semi-Markov process $X^{(\delta)}(u)$ in interval $[0,t]$; (g) $X_n^{(\delta)}, n = 0, 1, \ldots$ are the states of the semi-Markov process $X^{(\delta)}(t)$ at sequential moments of jumps; (h) $T_n^{(\delta)}, n = 0, 1, \ldots$ are the sequential moments of jumps of the semi-Markov process $X^{(\delta)}(t)$, (t) $T^{(\delta)}(t) = t - T^{(\delta)}_{N^{(\delta)}(t)}$ is the time between moment t and the moment of the last jump of process $X^{(\delta)}(u)$ before t; (i) $Y_{n,i}^{(\delta)}(t), t \geq 0$ is, for every $n = 0, 1, \ldots$ and $i \in \mathbb{X}$, a real-valued càdlàg process with independent increments with an initial state $Y_{n,i}^{(\delta)}(0) = 0$ and distributions of increments $P_i^{(\delta)}(t, t+s, A) = \mathsf{P}\{Y_{n,i}^{(\delta)}(t+s) - Y_{n,i}^{(\delta)}(t) \in A\}$ which do not depend on $n = 0, 1, \ldots$; (j) the processes $Y_{n,i}^{(\delta)}(t), t \geq 0$ for $n = 0, 1, \ldots, i \in \mathbb{X}$ and the process $X^{(\delta)}(t), t \geq 0$ are mutually independent.

We assume for simplicity that (k) the initial state of the log-price process $Y^{(\delta)}(0) = y_0 \in \mathbb{R}_1$ and the semi-Markov index $X^{(\delta)}(0) = i_0 \in \mathbb{X}$ are a constants, and also that (l) the semi-Markov index has no instant transition, i.e., $Q_{ij}^{(\delta)}(0) = 0, i, j \in \mathbb{X}$.

In this case, the two-component process $(Y^{(\delta)}(t), X^{(\delta)}(t))$ is not a Markov process, but the process $\tilde{X}^{(\delta)}(t) = (X^{(\delta)}(t), T^{(\delta)}(t))$ is a homogeneous Markov process with the phase space $\tilde{\mathbb{X}} = \{1, \ldots, m\} \times [0, \infty)$. This process should be interpreted as a stochastic index modulating the log-price process $Y^{(\delta)}(t)$. The phase space of this process is a Polish space with the standard metric $\rho((i,s),(j,t)) = \sqrt{\chi(i \neq j) + |s-t|^2}$.

The process $\tilde{Z}^{(\delta)}(t) = (Y^{(\delta)}(t), \tilde{X}^{(\delta)}(t)), t \geq 0$ is a homogeneous Markov process. Note that it is so even if the processes $Y_{n,i}^{(\delta)}(t), t \geq 0$ are non-homogeneous in time processes with independent increments.

The results formulated above for exponential processes with independent increments can be generalised to the model introduced here.

The transition probabilities of the semi-Markov process $X^{(\delta)}(t)$ can always be represented in the form $Q_{ij}^{(\delta)}(t) = p_{ij}^{(\delta)} F_{ij}^{(\delta)}(t), t \geq 0, i, j \in \mathbb{X}$, where $p_{ij}^{(\delta)}$ are transition probabilities of the imbedded Markov chain $X_n^{(\delta)}$ while $F_{ij}^{(\delta)}(t)$ are the distributions of inter-jump times for this process conditioned on the states before and after jumps.

In this case, one should first assume the following condition of weak convergence for transition probabilities for the stochastic semi-Markov indices:

$\mathbf{F_1}$: **(a)** $p_{ij}^{(\delta)} \to p_{ij}^{(0)}$ as $\delta \to 0, i, j \in \mathbb{X}$; **(b)** $F_{ij}^{(\delta)}(\cdot) \Rightarrow F_{ij}^{(0)}(\cdot)$ as $\delta \to 0, i, j \in \mathbb{X}$, where the limit distribution functions are continuous for every $i, j \in \mathbb{X}$.

As far as log-price processes $Y_{n,i}^{(\delta)}(t)$ are concerned the following conditions should be imposed:

$\mathbf{F_2}$: Conditions $\mathbf{D_1} - \mathbf{D_3}$ hold for processes $Y_{0,i}^{(\delta)}(t), t \in [0,T]$ for every $i \in \mathbb{X}$.

Introduce the distribution of sojourn times $F_i^{(\delta)}(t) = \sum_{j \in \mathbb{X}} p_{ij}^{(\delta)} F_{ij}^{(\delta)}(t)$ and introduce the set $\tilde{\mathbb{Y}} = \{(i,t) : F_i^{(0)}(t) < 1, i \in \mathbb{X}\}$.

Conditions $\mathbf{F_1}$ and $\mathbf{F_2}$ imply that, for the processes $\tilde{Z}^{(\delta)}(t), t \in [0,T]$, condition $\mathbf{B_1}$ holds, with the sets $\mathbb{Z}_t = \mathbb{R}_1 \times \tilde{\mathbb{Y}}, 0 \leq t \leq T$, as well as that conditions $\mathbf{C_5} - \mathbf{C_6}$ hold.

We refer to the work in [59], where one can find the related proofs based on estimates for the corresponding quantities related to log-price processes under fixed trajectories of the semi-Markov index; integration of these estimates with respect to the measure generated by the semi-Markov index in the space of its trajectories; and estimates for the number of jumps of this index in the finite interval $[0, T]$.

For example, let us shortly present a sketch of the corresponding proof related to condition $\mathbf{D_3}$.

The assumption that condition $\mathbf{D_3}$ holds for every processes $Y_{0,i}^{(\delta)}(t), t \in [0, T]$ for every $i \in \mathbb{X}$ implies that there exists $\delta_4 > 0$ such that

$$\Xi_{\pm\beta'}(T) = \sup_{\delta \leq \delta_4} \max_{i \in \mathbb{X}} \Xi_{\pm\beta'}(Y_{0,i}^{(\delta)}(\cdot), T) < \infty. \tag{88}$$

Also, condition $\mathbf{F_1}$ implies that for any $\alpha > 0$ there exist $\delta_5 = \delta_5(\alpha) > 0$ such that

$$\Upsilon(\alpha, T) = \sup_{\delta \leq \delta_5} \mathsf{E} e^{\alpha N^{(\delta)}(T)} < \infty. \tag{89}$$

Due to assumptions (e) - (j), the log-price process $Y^{(\delta)}(t)$ is a process with independent increments conditionally independent with respect to the modulating stochastic semi-Markov index $X^{(\delta)}(t)$. This let us get, using (88) - (89), the following estimate, for $\delta \leq \delta_6 = \delta_4 \wedge \delta_5(\ln \Xi_{\pm\beta'}(T))$, and $0 \leq t \leq t + u \leq T$,

$$\begin{aligned}
&\mathsf{E} e^{\pm\beta'(Y^{(\delta)}(t+u) - Y^{(\delta)}(t))} \\
&\quad = \mathsf{E} \sum_{n=0}^{\infty} \sum_{i_1, \ldots, i_n \in \mathbb{X}} \int_{0 < t_1 < \cdots < t_n \leq T} \mathsf{E}\{e^{\pm\beta'(Y^{(\delta)}(t+u) - Y^{(\delta)}(t))} / X_k^{(\delta)} = i_k, \\
&\quad\quad T_k^{(\delta)} \in dt_k, k = 1, \ldots, n, N^{(\delta)}(T) = n\} \mathsf{P}\{X_k^{(\delta)} = i_k, \\
&\quad\quad T_k^{(\delta)} \in dt_k, k = 1, \ldots, n, N^{(\delta)}(T) = n\} \\
&\quad \leq \sum_{n=0}^{\infty} \Xi_{\pm\beta'}(T)^{n+1} \mathsf{P}\{N^{(\delta)}(T) = n\} \\
&\quad = \Xi_{\pm\beta'}(T) \Upsilon(\ln(\Xi_{\pm\beta'}(T)), T) < \infty.
\end{aligned} \tag{90}$$

Relation (90) obviously implies that condition $\mathbf{D_3}$ holds for processes $Y^{(\delta)}(t)$.

Note also that assumption (k) automatically implies that condition $\mathbf{B_4}$ and $\mathbf{C_4}$ holds for any initial states y_0 and i_0.

Summarising the remarks above, one can conclude that the conditions and, therefore, the statement of Theorem 3 hold for the exponential price processes with independent increments modulated by the semi-Markov stochastic indices introduced above, if conditions $\mathbf{A_1}$, $\mathbf{A_4}$, and $\mathbf{F_1} - \mathbf{F_2}$ hold.

In particular, the statement of Theorem 3 hold for the exponential Lévy price processes modulated by semi-Markov stochastic indices introduced above, if conditions $\mathbf{A_1}$, $\mathbf{A_4}$, $\mathbf{F_1}$, and also conditions $\mathbf{E_1} - \mathbf{E_2}$ hold for Lévy processes $Y_{0,i}^{(\delta)}(t), t \in [0, T]$ for every $i \in \mathbb{X}$.

In conclusion, let us consider the case where the log-price processes $Y_{n,i}^{(\delta)}(t)$ are of trinomial type, i.e., have, for $\delta > 0$ the following form,

$$Y_{n,i}^{(\delta)}(t) = \sum_{1 \leq k \leq [t/\delta]} Y_{n,i,k}^{(\delta)}, \ t \geq 0, \tag{91}$$

where $Y^{(\delta)}_{n,i,k}, k = 1, 2, \dots, i \in \mathbb{X}, n = 0, 1, \dots$ are independent random variables which take three values $u^{(\delta)}_i > 0$, 0, and $-u^{(\delta)}_i$ with probability $p^{(\delta)}_i$, $r^{(\delta)}_i$ and $q^{(\delta)}_i$, respectively, where $p^{(\delta)}_i + r^{(\delta)}_i + q^{(\delta)}_i = 1$.

As far as the limiting log-price processes $Y^{(0)}_{n,i}(t)$ are concerned, they have the following form,

$$Y^{(\delta)}_{n,i}(t) = \mu_i t + \sigma_i W_{n,i}(t),\ t \geq 0, \tag{92}$$

where $W_{n,i}(t)$, $t \geq 0$ are standard Wiener processes mutually independent for $i \in \mathbb{X}, n = 0, 1, \dots$.

In this case, $S^{(0)}(t) = \exp\{Y^{(0)}(t)\}$ is a geometric Brownian motion modulated by a semi-Markov stochastic index $X^{(0)}(t)$.

It is also naturally to assume that the stochastic index $X^{(\delta)}(t)$ can jump only at the moments $\delta, 2\delta, \dots$ that is provided by the assumption that (m) the transition probabilities $Q^{(\delta)}_{ij}(t), i, j \in \mathbb{X}$, as functions of t, may possess jumps only at moments $\delta, 2\delta, \dots$.

Let us choose $\sigma > \max_{i\in\mathbb{X}} \sigma_i$. In this case, $p^{(\delta)}_i$ and $q^{(\delta)}_i$ defined below in condition $\mathbf{G_1}$ are positive numbers and their sum is less than 1 for all $i \in \mathbb{X}$ and $0 < \delta < \delta_0 = \min_{i\in\mathbb{X}} \frac{(\sigma^2-\sigma_i^2)\wedge\sigma_i^2}{\mu_i^2+1}$. The condition mentioned above is:

$\mathbf{G_1}$: $u^{(\delta)}_i = \sigma\sqrt{\delta}$, $p^{(\delta)}_i = \frac{\sigma_i^2}{2\sigma^2} + \frac{\mu_i}{2\sigma}\sqrt{\delta} + \frac{\mu_i^2}{2\sigma^2}\delta$, $q^{(\delta)}_i = \frac{\sigma_i^2}{2\sigma^2} - \frac{\mu_i}{2\sigma}\sqrt{\delta} + \frac{\mu_i^2}{2\sigma^2}\delta$, $i \in \mathbb{X}$.

Condition $\mathbf{G_1}$ implies that $\mathsf{E}Y^{(\delta)}_{0,i,1} = \mu_i\delta$, $\mathsf{Var}Y^{(\delta)}_{0,i,1} = \sigma_i^2\delta$; provide J-convergence of the processes $Y^{(\delta)}_{0,i}(t), t \in [0, T]$ to the processes $Y^{(0)}_{0,i}(t), t \in [0, T]$ as $\delta \to 0$, for every $i \in \mathbb{X}$; and that conditions $\mathbf{D_1}$ and $\mathbf{D_2}$ hold, for every $i \in \mathbb{X}$.

Also, the moment generation function $\mathsf{E}\exp\{\beta Y^{(\delta)}_{0,i}(t)\}$ exist for any $\beta \in \mathbb{R}_1$ and has an explicit form, namely,

$$\mathsf{E}\exp\{\beta Y^{(\delta)}_{0,i}(t)\} = \begin{cases} (e^{\beta\sigma\sqrt{\delta}}p^{(\delta)}_i + r^{(\delta)}_i + e^{-\beta\sigma\sqrt{\delta}}q^{(\delta)}_i)^{[t/\delta]}, & \text{if } \delta > 0, \\ e^{\beta\mu_i t + \frac{\beta^2\sigma_i^2 t}{2}}, & \text{if } \delta = 0. \end{cases} \tag{93}$$

This makes it easy to check that condition $\mathbf{D_3}$ holds for these processes for any $\beta' > \beta$ and for every $i \in \mathbb{X}$. Thus, condition $\mathbf{F_2}$ holds.

Let assume that the semi-Markov stochastic index is a Markov chain and has transition probabilities that satisfy the relation $Q^{(\delta)}_{ij}(t) = p^{(\delta)}_{ij} Q^{(\delta)}_i(t)$, where $Q^{(\delta)}_i(t)$ are geometric distributions of random sojourn times which take values $n\delta$ with probabilities $Q^{(\delta)}_i(1 - Q^{(\delta)}_i)^{n-1}, n = 1, 2, \dots$.

In this case, condition $\mathbf{F_1}$ is implied by the following condition:

$\mathbf{F_3}$: **(a)** $p^{(\delta)}_{ij} \to p^{(0)}_{ij}$ as $\delta \to 0, i, j \in \mathbb{X}$; **(b)** $Q^{(\delta)}_i/\delta \to \lambda_i > 0$ as $\delta \to 0, i \in \mathbb{X}$.

The limiting transition probabilities takes the form,

$$Q^{(0)}_{ij}(t) = p^{(0)}_{ij}(1 - e^{-\lambda_i t}),\ t \geq 0,\ i, j \in \mathbb{X}. \tag{94}$$

Summarising the remarks above, one can conclude that conditions $\mathbf{A_1}$, $\mathbf{A_4}$, $\mathbf{F_1}$ and $\mathbf{G_1}$ imply that the conditions and, therefore, the statement of Theorem 3 hold, i.e., $\Phi(\mathcal{M}^{(\delta)}_{max,T}) \to \Phi(\mathcal{M}^{(0)}_{max,T})$ as $\delta \to 0$, for the exponential trinomial price processes $S^{(\delta)}(t) = e^{Y^{(\delta)}(t)}$ introduced above in (87) and (91) – (92).

Let assume for simplicity that $\delta = T/N$ and consider the partition $\Pi_\delta =< t_0 = 0 < t_1 = \delta < \cdots < t_{N-1} = (N-1)\delta < t_N = T >$ of interval $[0, T]$.

In this case, the Markov chain $(n, Y^{(\delta)}(n\delta), X^{(\delta)}(n\delta)), n = 0, 1, \dots$ is a trinomial tree model with the initial node $(0, y_0, i_0)$ and $(2n+1)m$ nodes of the form $(n, y_\delta \pm k\sigma\sqrt{\delta}, i), k = 0, 1, \dots, n,\ i = 1, \dots, m$ after $n \geq 1$ steps.

The standard backward procedure can be applied in order to find optimal expected reward at moment 0 for the for the discrete time exponential trinomial price process with Markov modulation $(S^{(\delta)}(n\delta), X^{(\delta)}(n\delta))$. This optimal expected reward coincides, in this case, with the reward functional $\Phi(\mathcal{M}^{(\delta)}_{\Pi_\delta,T})$ for the exponential trinomial price processes $S^{(\delta)}(t) = e^{Y^{(\delta)}(t)}$ introduced above in (87) and (91) – (92).

To estimate the deference $\Phi(\mathcal{M}^{(\delta)}_{max,T}) - \Phi(\mathcal{M}^{(\delta)}_{\Pi_\delta,T})$ we can use Theorem 1. In this case, $d(\Pi_\delta) = \delta$ and $\Delta_\beta(Y^{(\delta)}(\cdot), \delta, T) = \max_{i\in\mathbb{X}}(\mathsf{E}e^{\beta|Y^{(\delta)}_{1,i,1}|} - 1) \leq e^{\beta\sigma\sqrt{\delta}} - 1$.

Theorem 1 yields in this case the following estimate

$$\Phi(\mathcal{M}^{(\delta)}_{max,T}) - \Phi(\mathcal{M}^{(\delta)}_{\Pi_\delta,T}) \leq L_3\delta + L_4(e^{\beta\sigma\sqrt{\delta}} - 1)^{\frac{\beta-\gamma}{\beta}} \to 0 \text{ as } \delta \to 0. \tag{95}$$

Thus, Theorem 3 guarantees that the optimal expected reward $\Phi(\mathcal{M}^{(\delta)}_{\Pi_\delta,T})$ converge, under conditions $\mathbf{A_1}$, $\mathbf{A_4}$, $\mathbf{F_3}$, and $\mathbf{G_1}$, to the reward functional $\Phi(\mathcal{M}^{(0)}_{max,T})$ for the geometric Brownian motion defined in (87) and (92) modulated by the continuous time homogeneous Markov chain with transition probabilities given in (94).

Note that we use the trinomial model instead of the standard binomial one in order to be able to fit the corresponding values for expectations and variances for price jumps $Y^{(\delta)}_{0,i,1}$ by choosing (n) the specified values of probabilities $p^{(\delta)}_i$ and $q^{(\delta)}_i$, for every $i \in \mathbb{X}$, and, at the same time, (o) the same values of jumps $u^{(\delta)}_i = \sigma\sqrt{\delta}$, for all $i \in \mathbb{X}$. The latter property (o) provides automatically necessary recombining properties for the corresponding trinomial tree model, which makes it possible to construct the tree model for the modulated log-price processes with the number of nodes which has a linear rate of growth as a function of the number of steps.

In the case of approximation of the continuous type option with maturity T by the corresponding discrete time model with time step $\delta = T/N$ the corresponding tree has N steps, and therefore $(2N+1)m$ nodes after the last N-th step, $(2(N-1)+1)m$ after $(N-1)$-th step, etc.

The above backward algorithm can be also generalised to the general case of trinomial tree model with semi-Markov index $X^{(\delta)}(t)$ making jumps only at the moments $\delta, 2\delta, \dots$. In this case, one should consider the Markov chain

$$(n, Y^{(\delta)}(n\delta), X^{(\delta)}(n\delta), T^{(\delta)}(n\delta)), \quad n = 0, 1, \dots$$

as a tree model with the initial node $(0, y_0, i_0, 0)$ and $(2n+1)mn$ nodes of the form $(n, y_0 \pm k\sigma\sqrt{\delta}, i, l), k = 0, 1, \dots, n,\ i = 1, \dots, m, l = 0, 1, \dots, n$ after $n \geq 1$ steps.

In the case, the corresponding tree with number of notes which has a quadratic rate of growth as a function of the number of steps. In the case of approximation of the continuous type option with maturity T by the corresponding discrete time model with time step $\delta = 1/N$ the corresponding tree has N steps, and therefore $(2N+1)mN$ notes after the last N-th step, $(2(N-1)+1)m(N-1)$ after $(N-1)$-th step, etc.

Theorem 3 guarantees that these optimal expected reward at time 0 for the described above trinomial tree model with semi-Markov modulation converge, under conditions $\mathbf{A_1}$, $\mathbf{A_4}$, $\mathbf{F_1}$, and $\mathbf{G_1}$, to the corresponding optimal expected reward functional $\Phi(\mathcal{M}^{(0)}_{max,T})$ for the geometrical Brownian motion (defined in (87) and (92)) modulated by the semi-Markov index with the transition probabilities $Q^{(0)}_{ij}(t) = p^{(0)}_{ij} F^{(0)}_{ij}(t)$ given in condition $\mathbf{F_1}$.

Numerical algorithms for approximation of optimal reward values for exponential price processes modulated by stochastic market indices shortly described above do require a more detailed consideration. The corresponding results will be presented in a separate paper.

In conclusion, we would like also to note that similar examples for multivariate exponential price processes with independent increments are also considered in [32].

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