Numerical Accuracy of Real Inversion Formulas for the Laplace Transform

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December 2008

Abstract: In this paper we investigate and compare a number of real inversion formulas for the Laplace transform. The focus is on the accuracy and applicability of the formulas for numerical inversion. In this contribution, we study the performance of the formulas for measures concentrated on a positive half-line to continue with measures on an arbitrary half-line.

Keywords: Laplace transform, numerical inversion, Post-Widder inversion, Widder inversion, Shohat-Tamarkin formula, Gaver-Stehfest inversion.

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1 Introduction

In this paper we investigate and compare a number of real inversion formulas for the Laplace transform. The focus is on the accuracy and applicability of the formulas for numerical inversion. In this contribution, we study the performance of the formulas for measures concentrated on a positive half-line to continue with measures on an arbitrary half-line.

The remaining of the paper is organised as follows. In Section 2 we formulate the inversion formulas used to recover a measure defined on the positive half-line. Section 3 describes the potential of these formulas for numerical inversion of a probability measure. An extension of these formulas is given in Section 4. We compare the numerical results obtained by the original formulas on the one hand, and their extensions on the other. Section 5 is dedicated to another extension of the original formulas. In particular, we study the performance of the formulas adapted for the case where the measure is concentrated on an arbitrary half-line.

2 Preliminaries

In this section we describe the general framework and give an overview of the original inversion formulas.

Assume that μ is a bounded measure on the positive half-line. We define its *Laplace transform* by

$$\hat{\mu}(u) := \int_0^\infty e^{-ux}\, d\mu(x),$$

where $u \geq 0$. We are interested in recovering $\mu(.)$ from its Laplace transform $\hat{\mu}(.)$. For $0 \leq y_1 < y_2$, denote by $\mu\{y_1; y_2\}$ the inversion of the measure μ on $[y_1, y_2]$ such that

$$\mu\{y_1; y_2\} = \frac{1}{2}\mu\{y_1\} + \mu(y_1, y_2) + \frac{1}{2}\mu\{y_2\}\ .$$

Here $\mu\{y\}$ stands for the weight or measure at the point y while $\mu(a, b)$ is shorthand for the measure of the open interval (a, b).

For probabilistic proofs of a number of the inversion formulas used below, we refer to [8]. For ease of reference, we briefly go through these formulas.

2.1 Post-Widder Formula [2, 10]

This is one of the classical inversion formulas and it can be found in [10]. From [8] we have for $0 \leq y_1 < y_2$

$$\mu\{y_1; y_2\} = \lim_{n\to\infty} \int_{y_1}^{y_2} \frac{(-n)^n}{\Gamma(n)} \hat{\mu}^{(n)}(\frac{n}{t})\, \frac{dt}{t^{n+1}}\ . \tag{1}$$

Note that in order to recover the measure μ by the Post-Widder formula, one has to calculate all derivatives of the Laplace transform on the entire positive half-line.

2.2 Widder Formula [10]

From [8] we know that for $0 \le y_1 < y_2$

$$\mu\{y_1; y_2\} = \lim_{n\to\infty} \sum_{m=[ny_1]+1}^{[ny_2]} \frac{(-n)^m}{m!} \hat{\mu}^{(m)}(n) \ . \tag{2}$$

It follows from (2) that, in order to invert $\hat{\mu}$, one has to find all derivatives of $\hat{\mu}$ in the variable point n, $n \to \infty$.

2.3 Shohat-Tamarkin Formula [6]

For $0 \le y_1 < y_2$ we have

$$\mu\{y_1; y_2\} = \int_{y_1}^{y_2} \sum_{k=0}^{\infty} \sum_{k=0}^{n} \frac{n!}{k!^2 \ (n-k)!} \hat{\mu}^{(k)}(1) L_n(u) \, du \ , \tag{3}$$

where

$$L_n(u) = \sum_{r=0}^{n} \binom{n}{n-r} \frac{(-u)^r}{r!} \tag{4}$$

are the classical Laguerre polynomials. In order to recover the measure μ by the Shohat-Tamarkin formula, one has to calculate the Laplace transform $\hat{\mu}$ and all its derivatives in the single point 1. In [8] the formula (3) has been generalized in such a way that it requires the Laplace transform and its derivatives at an arbitrary point on the positive half-line.

2.4 Gaver-Stehfest Formula [4, 5]

From [8] we have for $0 \le y_1 < y_2$

$$\mu\{y_1; y_2\} = \lim_{n\to\infty} \sum_{k=0}^{n} b_{n,k} \left[\hat{\mu}\left(\frac{n+k}{y_2}\log 2\right) - \hat{\mu}\left(\frac{n+k}{y_1}\log 2\right) \right] \ , \tag{5}$$

where

$$b_{n,k} = \frac{(-1)^k n}{n+k} \binom{2n}{n} \binom{n}{k}. \tag{6}$$

It follows from (5) that the inversion requires the Laplace transform $\hat{\mu}$ in two real points but none of its derivatives.

3 Inversion Formulas on the Positive Half-line

As our trial distribution for μ on the positive half-line, we take Gamma(α, β), $\alpha > 0$, $\beta > 0$, distribution function $F_G(x)$, $x \ge 0$. The density is given by

$$f(x) = \frac{\beta^\alpha}{\gamma(\alpha)} x^{\alpha-1} e^{-\beta x}, \ x > 0.$$

We will test the inversion formulas listed above to recover the Gamma cumulative distribution function from its Laplace transform given by

$$\hat{\mu}(u) = \left(1 + \frac{u}{\beta}\right)^{-\alpha}, \ u > 0. \tag{7}$$

The Gamma family provides a rich family of distributions allowing a variety of desirable tail behavior that can be obtained by varying the shape parameter α. If we take $\alpha = 1$ then we obtain the exponential distribution that plays a benchmark role. By varying the shape parameter one obtains either heavier tails for $\alpha < 1$ or lighter tails $\alpha > 1$ than the tail of the exponential distribution. We choose the scale parameter β equal to $\sqrt{\alpha}$. In this way we reduce the number of parameters to one. In what follows we will refer to the Gamma$(\alpha, \sqrt{\alpha})$ distribution as to Gamma(α).

Consider the four inversion formulas: Post-Widder (P-W, 1), Widder (W, 2), Shohat-Tamarkin (S-T, 3), and Gaver-Stehfest (G-S, 5). We test these formulas for different values of α and on a number of intervals $[y_1, y_2]$, $0 \le y_1 < y_2 \le y^{up}$, where y^{up} is chosen such that $F_G(y^{up}) \approx 1$. More specifically, we calculate the relative errors of the numerical inversion by means of these formulas as these errors are used as the measure of the inversion accuracy. The relative error is calculated as

$$\frac{|(F_G(y_2) - F_G(y_1)) - \mu\{y_1; y_2\}|}{F_G(y_2) - F_G(y_1)} \cdot 100\%.$$

In all the considered inversion formulas we either need to take the limit when n goes to infinity or to calculate an unlimited sum. For the implementation we are forced to take some finite number, say $n = N$. The following range for N is recommended:

- for the Post-Widder formula (1), $70 \le N \le 130$;
- for the Widder formula (2) the choice depends strongly on y_2 and will be discussed below;
- in the Shohat-Tamarkin formula (3) the upper limit of the first summation is in the range $35 \le N \le 65$, the larger the α the larger the N.
- for the Graver-Stehfest formula (5), $10 \le N \le 25$.

3.1 The light tail case $\alpha \ge 1$

For this range of the parameter α, the Shohat-Tamarkin inversion formula (3) is almost always the most accurate. In Figures 1–4 we plotted the relative errors of numerical inversion by means of all four formulas on the intervals $[0.01, y_2]$ and $[5, y_2]$ for a number of values of y_2 and $\alpha = 0.1 \cdot j$, $j = 1, 2, \ldots, 100$. One can see in Figure 3 (left) that the relative error by the Shohat-Tamarkin formula (3) does not exceed 0.16% when the probability distribution has exponential or lighter than exponential tail ($1 \le \alpha \le 10$) and y_1 is very close to origin. In this case, the other formulas give lower precision, since the Post-Widder (1), Widder (2), and Gaver-Stehfest (5) formulas show relative errors up to 2.5% ($y_2 = 1$; $\alpha = 10$), 8% ($y_2 = 1$; $6 < \alpha < 9$), and even over 100% ($y_2 = 1$; $\alpha = 10$) respectively.

The inversion error increases as we increase the value of y_1 regardless which of the formulas we take. Nevertheless, the relative accuracy of each formula with respect to the others remains approximately the same. If we take for example $y_1 = 5$ and compare the relative errors of the four formulas then the error by the Shohat-Tamarkin (3) goes up to 25 % ($y_2 = 12$; $1 < \alpha < 2$), that by the Post-Widder (1) lies below 20 %, that by the Widder (2) is up to 30 %, and that by the Gaver-Stehfest (5) overshoots 100%. The largest error by Shohat-Tamarkin formula corresponds to the small values of the α that are not integers. If we considered only the integer values, as e.g. in Figure 5, then the relative errors would lie below 1%.

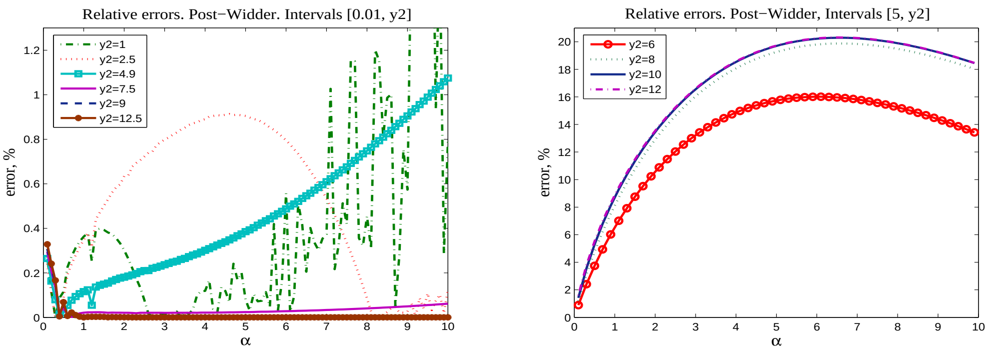


Figure 1: Relative errors on the intervals $[0.01,\, y_2]$ and $[5,\, y_2]$. Post-Widder (P-W, 1).

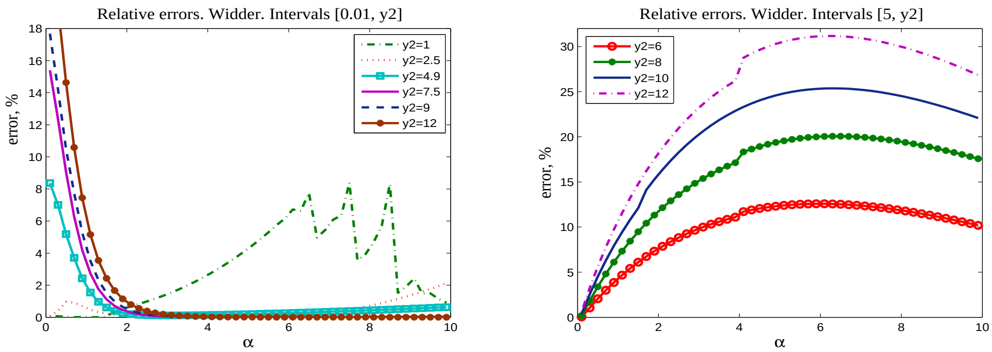


Figure 2: Relative errors on the intervals $[0.01,\, y_2]$ and $[5,\, y_2]$. Widder (W, 2).

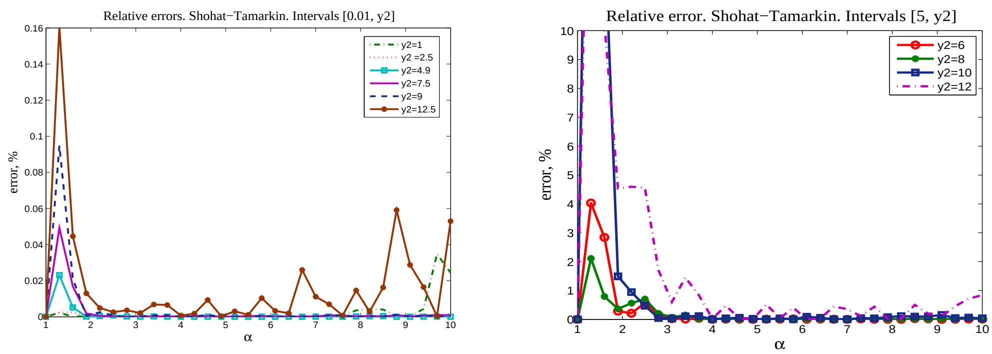


Figure 3: Relative errors on the intervals $[0.01,\, y_2]$ and $[5,\, y_2]$. Shohat-Tamarkin (S-T, 3).

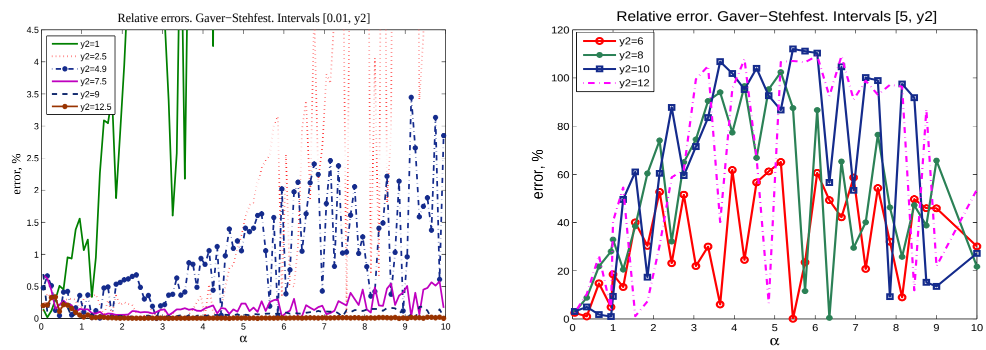


Figure 4: Relative errors on the intervals $[0.01,\, y_2]$ and $[5,\, y_2]$. Gaver-Stehfest (G-S, 5).

The Shohat-Tamarkin formula is inferior to the others only in the following two cases:

1. The measure of the interval is approaching one (i.e., $1-(F_G(y_2)-F_G(y_1)) < 10^{-5}$), which corresponds to

   - $y_1 = 0$ or y_1 is very close to the origin, and
   - y_2 is large enough, the order of magnitude of the y_2 being determined by the value of α (the larger the α, the smaller the y_2).

2. α is small, $1 < \alpha < 2$, and non-integer. Compare Figure 3 and Figure 5.

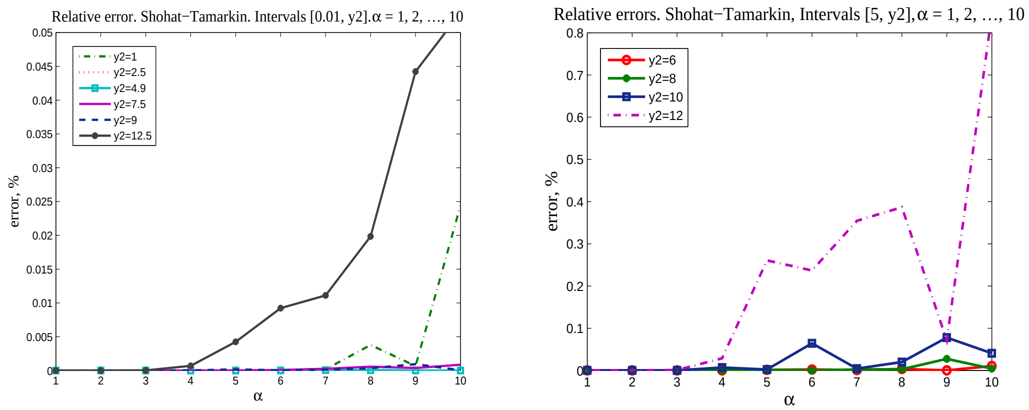


Figure 5: Relative errors on the intervals $[0.01, y_2]$ and $[5, y_2]$; α is integer. Shohat-Tamarkin (S-T, 3).

In the first case, it is, as a rule, the Post-Widder formula (1) that allows to get the most accurate numerical inversion. Note however that (P-W, 1) is not applicable for $y_1 = 0$, and in that case (G-S, 5) would be the best choice. Despite the fact that the Shohat-Tamarkin formula yields to the other formulas on the intervals whose measure is close to one, it still provides relatively good precision with the largest relative error being less than 1%. Compare this to the largest relative errors by the Post-Widder and Gaver-Stehfest formulas that are equal to 0.3% and 0.4% respectively (see Figure 1 (left) and Figure 4 (left)).

3.2 The heavy tail case $\alpha < 1$

It turns out that the inversion by (S-T, 3) is no longer recommended. As α decreases and/or we move the interval $[y_1, y_2]$ to the right, the error becomes very large. For example, the relative error overshoots 100% when $\alpha = 0.5$ and the formula is applied to invert the measure on the interval $[1, 10]$. Here, again, the Post-Widder approach (2) is, as a rule, the best alternative.

3.3 Global comparison

In Table 1 we give the average relative errors of inverting the Gamma(α) probability distribution function on the compact intervals $[y_1, y_2]$, $y_1 = 0.01$, $y_2 = 1 + 0.1 \cdot j$, $j = 1, 2, \ldots, 110$, by the four formulas and for 11 different values of α. The errors are quoted in per cent. The precision of the Shohat-Tamarkin and Gaver-Stehfest formulas does not depend much on whether or not we take $y_1 = 0$ or $y_1 = 0.01$. The precision of the Widder approach however does depend on this fact as on average it gives better accuracy when the origin is not included. The Post-Widder approach, as already mentioned, cannot be tested for $y_1 = 0$. Taking this into account we calculated the average errors on the intervals that start at 0.01 and let y_2 go from 1 to 12. As mentioned before, the Shohat-Tamarkin formula may perform differently for values of α, integral or not. Compare, for example, the relative errors for $\alpha = 1; 1.3; 1.8; 2; 2.1$ in Table 1.

α	0.05	0.5	1	1.3	1.8	2	2.1	4	5.5	7	9
P-W	0.33	0.05	0.12	0.14	0.18	0.18	0.19	0.22	0.24	0.29	0.47
W	12	7.2	2.8	1.5	0.5	0.4	0.3	0.2	0.3	0.4	0.4
S-T	170	4	$8{\cdot}10^{-9}$	0.09	0.016	$2{\cdot}10^{-7}$	$4{\cdot}10^{-3}$	$4 \cdot 10^{-5}$	$7 \cdot 10^{-4}$	$3 \cdot 10^{-4}$	$2 \cdot 10^{-3}$
G-S	0.4	0.27	0.23	0.18	0.22	0.27	0.32	0.8	1.8	2.4	6.7

Table 1: Average relative errors of inversion formulas on compact intervals $[0.01, y_2]$, $1 \le y_2 \le 12$, for given α.

Table 2 presents the average by α relative errors of inverting the same probability measure on 10 different intervals. Averaging is done over the values $\alpha = 0.05 + 0.3 \cdot j$, $j = 0, 1, \ldots, 30$. As the performance of the Shohat-Tamarkin and Gaver-Stehfest formulas depends significantly on whether or not the tail of the distribution to invert is heavier than exponential, we give the average relative errors on compact intervals for $\alpha > 1$ and $\alpha < 1$ separately in Tables 3 and 4.

Interval	$[0, 0.5]$	$[0, 1]$	$[0, 2.5]$	$[0, 4.9]$	$[0, 7.5]$	$[0, 9]$	$[0, 12.5]$	$[1, 5.5]$	$[2.5, 7.5]$	[5, 9]
P-W								0.3	1.2	17
W	16.4	6.5	3.7	4.1	4.3	4.5	4.9	1.6	0.8	17.5
S-T	0.07	0.05	0.02	0.18	0.64	0.68	14	0.2	0.5	4.7
G-S	78	39	3.2	1	0.2	0.06	0.01	2.3	2.8	52

Table 2: Average relative errors of inversion formulas on given compact interval, $0 < \alpha < 10$.

Interval	$[0, 0.5]$	$[0, 1]$	$[0, 2.5]$	$[0, 4.9]$	$[0, 7.5]$	$[0, 9]$	$[0, 12.5]$
S-T	0.16	0.4	0.14	1.38	4.8	5.05	105
G-S	0.8	0.4	0.08	0.1	0.04	0.02	0.01

Table 3: Average relative errors of Shohat-Tamarkin and Gaver-Stehfest inversion formulas on given compact interval, $\alpha < 1$.

We consider now each of the formulas separately and point out their pros and cons.

Post-Widder formula. The formula can be recommended for small vales of α, $0 < \alpha < 2$, especially when y_1 is close (but not equal) to 0. Then the relative error does not exceed 0.4%. It is often possible to obtain a higher precision by choosing larger N, however the upper limit

Interval	$[0, 0.5]$	$[0, 1]$	$[0, 2.5]$	$[0, 4.9]$	$[0, 7.5]$	$[0, 9]$	$[0, 12.5]$
S-T	0.057	$6 \cdot 10^{-4}$	$3 \cdot 10^{-4}$	$6 \cdot 10^{-4}$	$1.4 \cdot 10^{-3}$	0.02	0.62
G-S	80	40	3.4	1.3	0.22	0.07	0.006

Table 4: Average relative errors of Shohat-Tamarkin and Gaver-Stehfest inversion formulas on given compact interval, $\alpha > 1$.

for N is 143 for computer (Matlab) implementations.

Widder formula. This formula can be recommended

- to invert a distribution with very light tail, $\alpha \geq 10$, on the $[0, y_2]$ where y_2 is large, e.g. $y_2 \geq 10$; the relative error is then in the range of $[10^{-6}\%, 10^{-4}\%]$; in this case the estimates are obtained with small values of N such as $8 \leq N \leq 16$;
- to invert a heavy-tailed distribution, $\alpha < 0.5$, on short intervals to the right from the mean, the relative error being less than 1%; here $N \geq 20$.

In general, the choice of N depends on y_2 (the larger the y_2 the smaller the N); this is caused by the limited capacity of the computer to manage factorials of the kind $[N \cdot y_2]!$ appearing in the summation. As in the Post-Widder case, the maximal possible value of N is 143.

For $\alpha < 2$ the formula exhibits different behavior depending on whether or not the value of 0 is included in the interval. In Figure 6 one can see that the error may be reduced if we exclude zero; this fact holds for $\alpha < 2$ and all values of y_2. However from $\alpha \geq 2$ the estimates become undistinguishable.

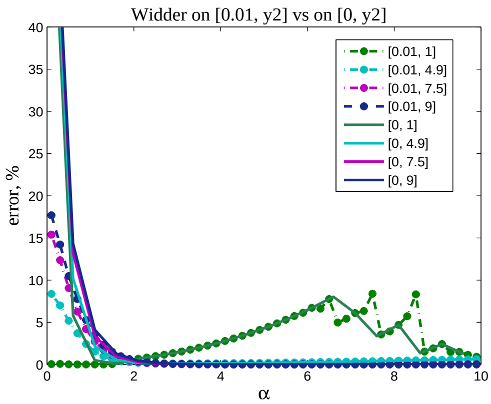


Figure 6: Relative errors on the intervals $[0,\ y_2]$ and $[0.01,\ y_2]$. Widder (W, 2).

Shohat-Tamarkin formula. The formula provides the best inversion in the majority of the considered cases. If we invert the Gamma(α) distribution function on a fixed interval $[y_1, y_2]$, then the error tends to increase as the Gamma parameter α increases, however the error growth is not monotone. In particular, the accuracy for integer values of α is typically higher than for non-integer values. For a fixed y_2, the error increases as y_1 increases, and this holds for all α.

For fixed y_1 and α, the error increases as the values of y_2 become larger.

Gaver-Stehfest formula. Similar to the Post-Widder approach (1), the Gaver-Stehfest formula allows to obtain rather precise inversion for any value of α when y_1 and y_2 are such that $1-(F_G(y_2)-F_G(y_1)) < 10^{-5}$. This is the case, for example, when y_1 is close to the origin while y_2 is large. The formula performs better when the distribution has a heavier than exponential tail, i.e. when $\alpha < 1$. However, in both cases one can obtain a higher accuracy of inversion using the Post-Widder approach when $y_1 > 0$. Therefore, the Gaver-Stehfest is only sensibly applicable when $y_1 = 0$.

For a fixed y_2, the error by Gaver-Stehfest increases as y_1 increases; this holds for any value of α. For a fixed y_1, the error decreases as we increase the length of the interval. However, the error as a function of α is again not monotone.

4 Refined Formulas

Let μ be a *probability* measure on $[0,\infty)$. Introduce the *integrated tail* of μ by the expression

$$\mu_1(y) := \int_0^y (1-\mu(x))\, dx.$$

Then its Laplace transform is given by

$$\hat{\mu}_1(s) = s^{-1}(1-\hat{\mu}(s)).$$

It is quite clear that the integral formulas given in Section 2 have a special version in the case when the measure μ has a derivative. For example, the Post-Widder case leads to

$$\frac{d\mu(y)}{dy} = \lim_{n\to\infty} \frac{(-n)^n}{\Gamma(n)} \hat{\mu}^{(n)}(\frac{n}{y})\ \frac{1}{y^{n+1}}\ .$$

But then these density formulas can be applied to recover the measure μ from the Laplace transform of μ_1. This then leads to a direct formula for the measure μ. Applying this procedure to the integral inversion formulas among (1)–(5) we obtain some refinements.

Post-Widder formula. From (1) for $y \geq 0$ we have

$$\mu(y) = \lim_{n\to\infty} \sum_{\ell=0}^{n} \frac{1}{\ell!} \left(-\frac{n}{y}\right)^{l} \hat{\mu}^{(\ell)}\left(\frac{n}{y}\right). \tag{8}$$

The latter formula has been discovered by Stadtmüller-Trautner in [7]. See also [8].

One can hardly compare the performance of the newly introduced formula to the original one because it inverts intervals starting at 0 while the original does not. Table 5 shows the average by α relative errors of inverting Gamma(α) probability function on the same intervals as in Table 2. One can immediately see that the refined Post-Widder formula outperforms all the others on large intervals and yields only to Shohat-Tamarkin on small intervals.

Shohat-Tamarkin formula. Starting from (3) we obtain for $y \geq 0$

$$1-\mu(y) = 1-\hat{\mu}(1) - \sum_{\ell=1}^{\infty} \frac{\hat{\mu}^{(\ell)}(1)}{\ell!} \sum_{n=\ell}^{\infty} \left\{ \sum_{k=\ell}^{n} (-1)^{k-\ell} \begin{pmatrix} n \\ k \end{pmatrix} \sum_{k=0}^{n} \begin{pmatrix} n \\ k \end{pmatrix} \frac{(-y)^k}{k!} \right\}. \tag{9}$$

Interval	$[0, 0.5]$	$[0, 1]$	$[0, 2.5]$	$[0, 4.9]$	$[0, 7.5]$	$[0, 9]$	$[0, 12.5]$
P-W	7.8	3	0.8	0.25	0.02	0.002	10^{-4}

Table 5: Average relative errors of the refined Post-Widder (P-W, 8) inversion formula on given compact interval, $0 < \alpha < 10$.

We apply expression (9) to numerically recover $1 - F_G(y)$ from its Laplace Transform, and compare the results to the inversion of $\mu\{y, \infty\}$ by the original Shohat-Tamarkin approach (3).

The newly introduced formula allows to improve the precision of the numerical inversion for $0 < \alpha < 1$, especially in the tail. The errors are further reduced at least by factor 3 (see Figure 7). As we increase α, say $1 < \alpha < 3$, the accuracy of (9) remains still higher than that of (3) for $y > 6$, whereas it is lower closer to the origin (see Figure 8(a)). For $\alpha > 3$, however the initial formula (3) works better (see Figure 8(b)).

Gaver-Stehfest formula. Let μ come from a probability measure without mass at the origin, then its Laplace transform takes on the values $0 = \hat{\mu}(\infty)$ and $1 = \hat{\mu(0)}$. We can take $y_2 = \infty$ in (5) to get the Gaver-Stehfest inversion formulas for the tail

$$1 - \mu(y) = \lim_{n\to\infty} \sum_{k=0}^{n} b_{n,k} \left[1 - \hat{\mu}\left(\frac{n+k}{y} \log 2 \right) \right], \quad y \geq 0, \tag{10}$$

where $b_{n,k}$ as defined in (6).

Alternatively we can take $y_1 = 0$ to get another version of this formula

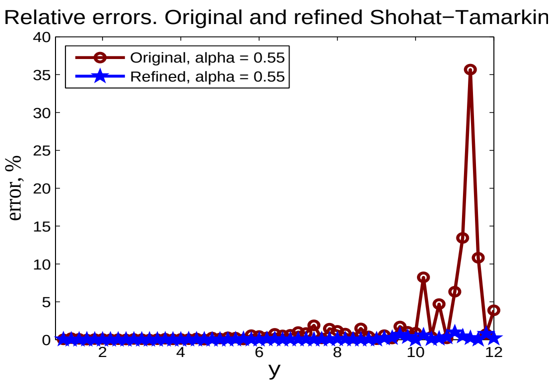
$$\mu(y) = \lim_{n\to\infty} \sum_{k=0}^{n} b_{n,k} \cdot \hat{\mu}\left(\frac{n+k}{y} \log 2 \right), \quad y \geq 0. \tag{11}$$

For given y and n both (10) and (11) provide, obviously, the same value. However, formula (11) allows in average to get more exact approximations of $\mu(y)$ due to another choice of n. The accuracy by formula (11) is slightly lower for $0 < \alpha < 1.25$ and significantly higher for $\alpha > 1.25$ than the accuracy by (10). In Figure 9(a) we plotted the Gamma(6.05) tail, $1 - F_G(y)$, together with its approximation by (8), (9), (10), and (11). The solid line corresponds to the approximation by formula (10) while the dash-dot line to the approximation by (11). One can immediately see that formula (11) provides higher accuracy than its counterpart (10) when $3 < y < 6$.

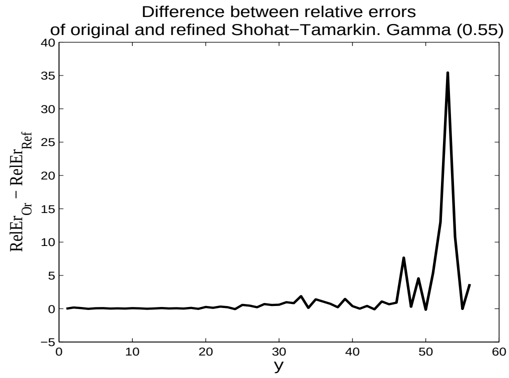
As in the former discussion for a compact interval, the refined Shohat-Tamarkin formula remains less accurate when $\alpha < 1$ (see e.g. Figure 9(b)).

5 Inversion Formulas on an Arbitrary Half-line

Let us now allow the measure to be spread over a half-line that is not necessarily the *positive* real line. Assume that ν is a bounded measure concentrated on the interval $[-a, \infty)$ where we

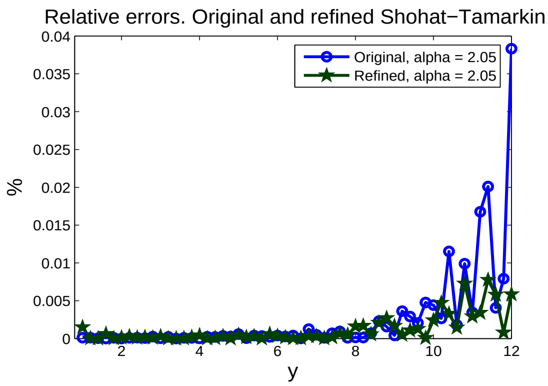


(a) Relative errors, $\alpha = 0.55$

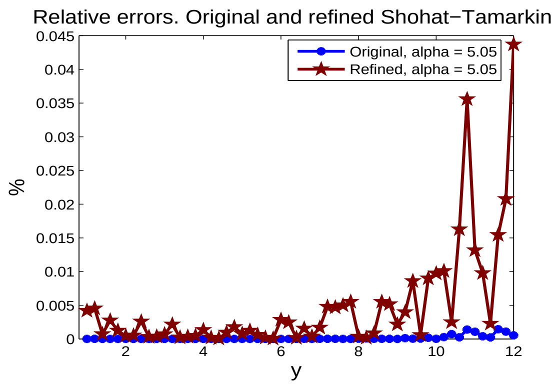


(b) Difference in relative errors, $\alpha = 0.55$

Figure 7: Original (S-T, 3) vs refined (S-T, 9) Shohat-Tamarkin formulas, $0 < \alpha < 1$

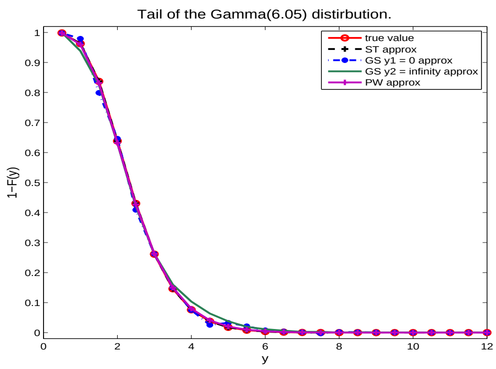


(a) $1 < \alpha < 3$

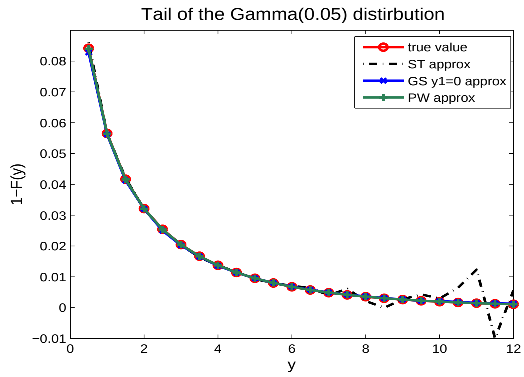


(b) $\alpha \geq 3$

Figure 8: Original (S-T, 3) vs refined (S-T, 9) Shohat-Tamarkin formulas, $\alpha > 1$.



(a) $\alpha = 6.05$



(b) $\alpha < 1$

Figure 9: Post-Wider, (P-W, 8), Shohat-Tamarkin (S-T, 9), and Gaver-Stehfest (G-S, 10) and (G-S, 11) refined formulas

suppose that $a \geq 0$. The *Laplace transform* is then given by

$$\hat{\nu}(u) := \int_{-a}^{\infty} e^{-ux}\, d\nu(x) = e^{au} \int_{0}^{\infty} e^{-uy} \mu(dy) =: e^{au} \hat{\mu}(u) , \tag{12}$$

where

$$\mu(dy) := \nu(-a + dy) ,$$

is a measure concentrated on the non-negative real line. We can recover the measure $\nu(.)$ once we know the Laplace transform $\hat{\mu}(.)$. Following the notations introduced in Section 2, the measure ν on $[\omega_1, \omega_2]$, $-a \leq \omega_1 < \omega_2$, is given by

$$\nu\{\omega_1; \omega_2\} = \mu\{a + \omega_1; a + \omega_2\}. \tag{13}$$

Now we can apply any inversion formula on a compact interval to the right-hand side of (13).

5.1 Post-Widder Formula

From [8] we have for $-a \leq \omega_1 < \omega_2$

$$\nu\{\omega_1; \omega_2\} = \lim_{n\to\infty} \int_{\omega_1}^{\omega_2} \frac{(-n)^n}{\Gamma(n)}\, \hat{\mu}^{(n)} \left(\frac{n}{a+v}\right) \frac{dv}{(a+v)^{n+1}} . \tag{14}$$

The link between the derivatives of $\hat{\mu}$ and those of $\hat{\nu}$ is given by the following expression

$$\hat{\mu}^{(k)}(u) = e^{-au} \sum_{\ell=0}^{k} \binom{k}{\ell} (-a)^{k-\ell}\, \hat{\nu}^{(\ell)}(u) \tag{15}$$

that can be obtained from (12) by applying Leinitz's differentiation formula.

5.2 Widder Formula

From [8] we have for $-a < \omega$

$$\nu\{-a; \omega\} = \lim_{n\to\infty} \sum_{\ell=0}^{[n(a+\omega)]} \frac{(-n)^{\ell}}{\ell!} \hat{\mu}^{(\ell)}(n) \sum_{r=0}^{[n(\omega+a)]-\ell} e^{-an} \frac{(an)^r}{r!} . \tag{16}$$

5.3 Shohat-Tamarkin Formula

The extension of the Shohat-Tamarkin formula (3) for $-a \leq \omega_1 < \omega_2$ is

$$\nu\{\omega_1; \omega_2\} = \int_{\omega_1}^{\omega_2} \sum_{n=0}^{\infty} \frac{\hat{\mu}^{(n)}(1)}{n!}\, t_n(a, v)\, dv, \tag{17}$$

where

$$t_n(a, v) = \sum_{m=0}^{\infty} e^{-a} \frac{(-a)^m}{m!} \frac{c_{n+m}(a+v)}{(n+m)!},$$

and where in turn

$$c_k(u) = \sum_{n=k}^{\infty} \frac{n!}{(n-k)!}\, L_n(u) ,$$

with $L_n(u)$ as defined in (4).

For numerical implementation it is convenient to rearrange the terms in the right-hand side of (17) in such a way that there remains only one unlimited sum:

$$\nu\{\omega_1;\omega_2\} = \int_{\omega_1}^{\omega_2} \sum_{\ell=0}^{\infty}\sum_{n=0}^{\ell} \frac{\ell!\,\hat{\mu}^{(n)}(1)}{n!} L_n(a+v) \sum_{m=0}^{l-n} e^{-a} \frac{(-a)^m}{m!(n+m)!} \frac{l!}{(l-n-m)!} dv.$$

5.4 Gaver-Stehfest Formula

We have for $-a \le \omega_1 < \omega_2$

$$\nu\{\omega_1;\omega_2\} = \lim_{n\to\infty} \sum_{k=0}^{n} b_{n,k} \left[2^{-\frac{a(n+k)}{a+\omega_2}} \hat{\nu}\left(\frac{n+k}{a+\omega_2} \log 2 \right) - 2^{-\frac{a(n+k)}{a+\omega_1}} \hat{\nu}\left(\frac{n+k}{a+\omega_1} \log 2 \right) \right], \qquad (18)$$

where $b_{n,k}$ has been defined in (6).

5.5 Stadtmüller-Trautner Formula [7]

If $\nu(x)$ is a probability measure on $[-a,\infty)$ then for $-a < x$ we have

$$\nu(x) = \lim_{n\to\infty} \sum_{m=0}^{n} \frac{\hat{\mu}^{(m)}(\frac{n}{x})}{m!} (-\frac{n}{x})^m \sum_{r=0}^{n-m} e^{-\frac{na}{x}} \frac{(\frac{na}{x})^r}{r!} . \qquad (19)$$

5.6 Numerical Results

As an example of a bounded measure on $[-a,\infty)$, $a \ge 0$, we take the *shifted* Gamma(α) distribution whose Laplace transform is

$$\hat{\nu}(u) = e^{au}\hat{\mu}(u) = e^{au} \cdot \left(1 + \frac{u}{\sqrt{\alpha}}\right)^{-\alpha},$$

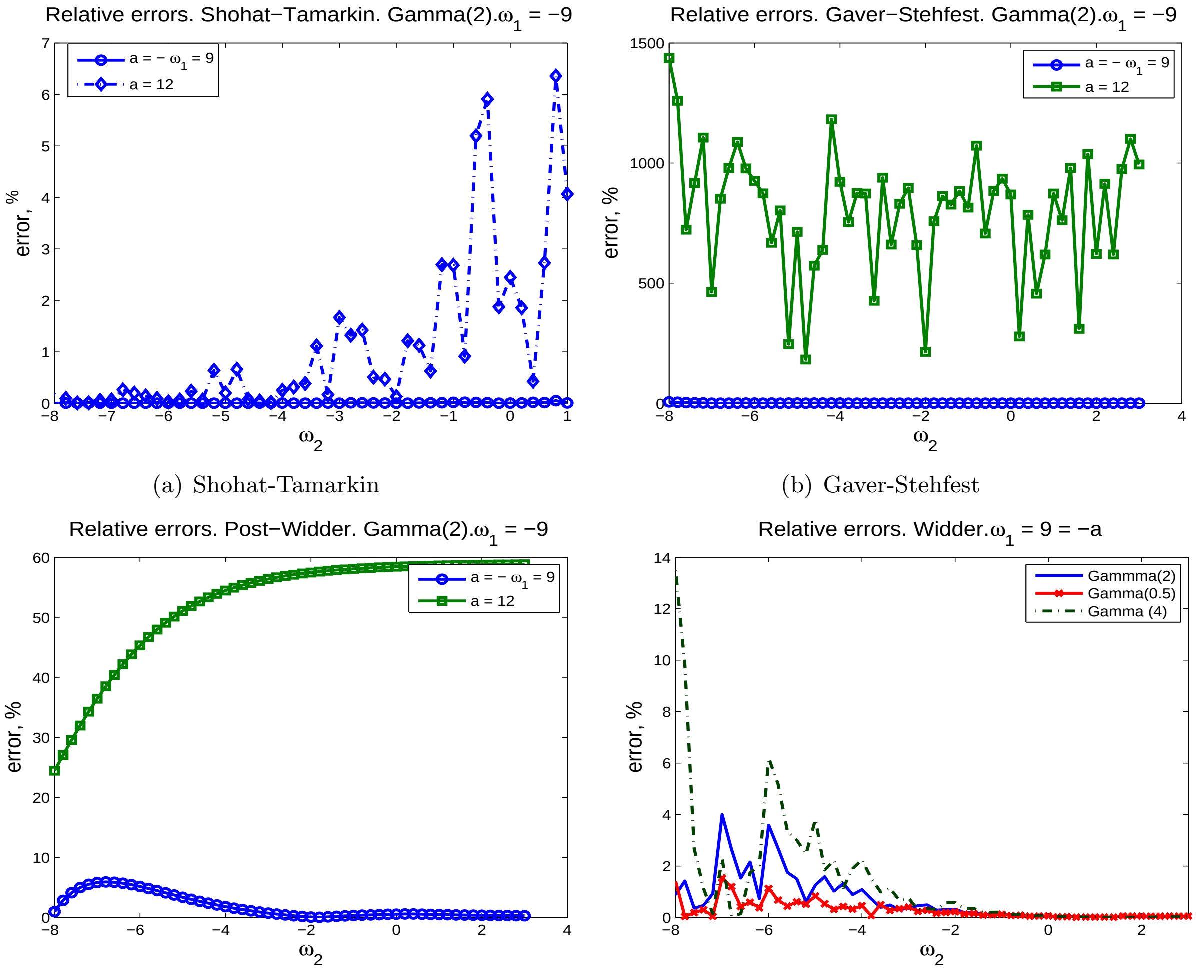
where $\hat{\mu}$ is the Laplace transform of the *Gamma* distribution as defined in (7). Then the derivatives of $\hat{\mu}$ can be calculated as

$$\hat{\mu}^{(k)}(u) = e^{-au} \sum_{l=0}^{k} \binom{k}{l} (-a)^{k-l} \hat{\nu}^{(l)}(u) =$$

$$e^{-au} \sum_{l=0}^{k} \binom{k}{l} (-a)^{k-l} e^{au} \sum_{j=0}^{l} \binom{l}{j} a^{l-j} \left(\left(1 + \frac{u}{\sqrt{\alpha}}\right)^{-\alpha} \right)^{(j)} .$$

We invert the Laplace transform on the intervals $[\omega_1,\omega_2]$, where $-a \le \omega_1 < \omega_2$. There are two formulas, the Shohat-Tamarkin and Gaver-Stehfest, that can be used not only for $-a = \omega_1$ but also for $-a < \omega_1$. Both of them, however, provide higher accuracy when $\omega_1 = -a$. We illustrate this in Figures 10(a) and 10(b) where we plotted the absolute errors of the Shohat-Tamarkin and Gaver-Stehfest numerical inversion on $[\omega_1,\omega_2]$ for the fixed $\omega_1 = -9$, varying $\omega_2 = -8.5, -8, \ldots, 2$, two different values of a: $a = -\omega_1 = 9$ and $a = 12 > -\omega_1$. One can see on the graph that the errors are significantly larger when $\omega_1 > -a$. It implies that it is preferable to take $a = -\omega_1$ in order to recover a (probability) measure on the interval $[\omega_1,\omega_2]$.

As was the case for $a = 0$, the Post-Widder formula (14) it is not applicable for $\omega_1 = -a$, in analogy to (1) where it was not applicable for $y_1 = 0$. Taking this into account, the inversion



(a) Shohat-Tamarkin (b) Gaver-Stehfest

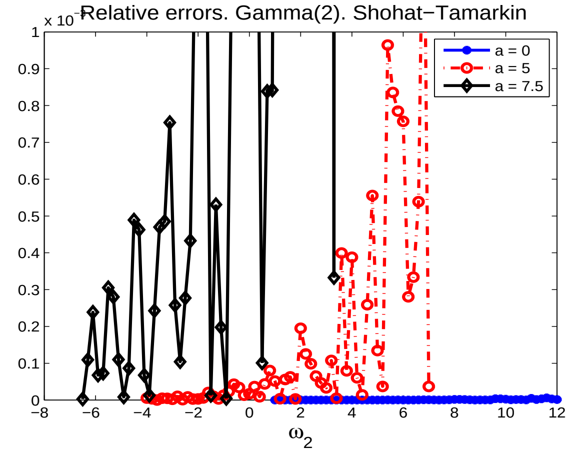
(c) Post-Widder (d) Widder

Figure 10: Absolut errors depending on whether $\omega_1 = -a$ or $\omega_1 > -a$.

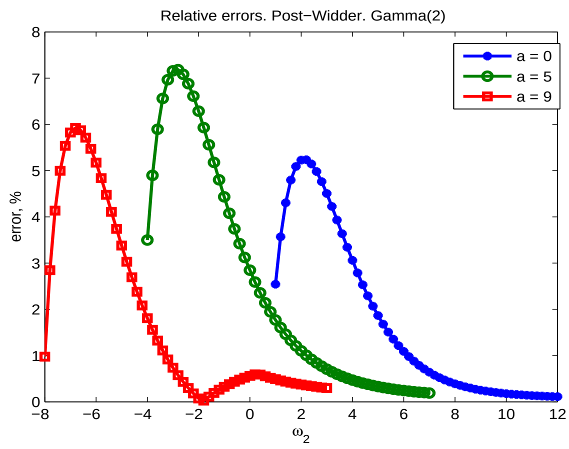
by (14) is done on $(-a, \omega_2]$. The relative errors of the inversion for the fixed $\omega_1 = -9$ and two values of a: $a = 9.01 \approx -\omega_1$ and $a = 12 > -\omega_1$ are plotted in Figure 10(c). The inversion with $a \approx -\omega_1$ is obviously more accurate.

The Widder formula (16) is derived for $a = -\omega_1$ only. In Figure 10(d) we plot the relative errors of recovering the probability measure for different values of α. Clearly, the accuracy decreases as α decreases.

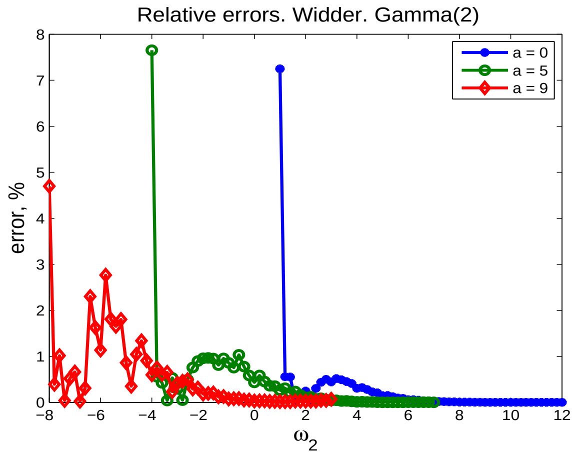
Finally, we study the performance of the inversion formulas for varying a. We take $\omega_1 = -a$ for convenience. We illustrate how the increase of a influences the precision of the formulas by taking three arbitrary values of a, say $a = 0$, $a = 5$, and $a = 9$, and comparing the corresponding inversion errors. For the Shohat-Tamarkin formula, the dependency of the precision on the value of a is obvious (see Figure 11(a)) while for the other formulas it is not so clear. As seen in Figure 11, larger errors correspond typically to the larger values of a, but there are also exceptions. In particular, one can see in Figure 11(e) that the errors by Gaver-Stehfest formula are of the same order of magnitude for the three considered values of the parameter a.



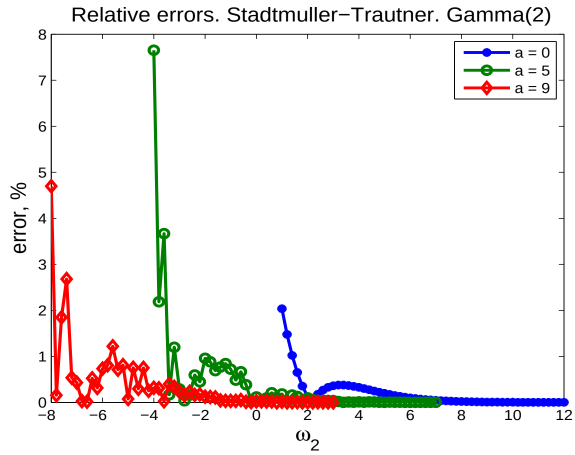
(a) Shohat-Tamarkin



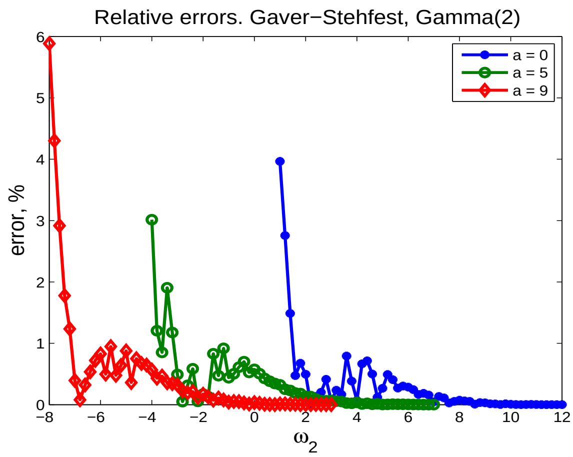
(b) Post-Widder



(c) Widder



(d) Stadtmuller-Trautner



(e) Gaver-Stehfest

Figure 11: Dependence of the inversion errors on the value of a.

6 Inversion of measures in the entire real line

It seems tempting to adapt the above procedures to the case where the measure μ is no longer restricted to a half-line, i.e. when

$$\hat{\mu}(u) := \int_{-\infty}^{\infty} e^{-ux} \, d\mu(x) \; .$$

However, a number of comments need to be made.

- None of the given procedures seems adaptable for application on the entire real line. In principle one could imagine that it should be possible to let a tend to ∞ in the formulas of Section 5. For an early example of such an approach, see [3].
- An alternative would be to find direct formulas that immediately apply to inversion for measures on the entire real line. An example of this kind has been recently derived by Yabukovich [9]. However, a number of numerical experiments suggest that the approximations show very large errors.
- It is well known that the $\hat{\mu}(s)$ exists only in a strip $(\sigma_- \, , \, \sigma_+)$ where the value of σ_- (σ_+) depends on the exponential decay of the right (left) tail of the measure μ. In many practical cases both quantities will be finite implying that one needs to look for inversion formulas that only use the function $\hat{\mu}(u)$ in values of u that satisfy $\sigma_- < \Re u < \sigma_+$. This simple observation suggests that only a potential formula of the Shohat-Tamarkin type is feasible.

It remains a challenging problem to construct accurate real inversion formulas for the two-sided Laplace transform.

7 Conclusions

In the above we have compared the performance of a number of real inversion formulas for the Laplace transform of measures concentrated on a half-line. Overall, the inversion by the Shohat-Tamarkin formula seems to perform best for compact intervals while the Post-Widder and Gaver-Stehfest are preferable for the tail behavior. We have not compared the used inversion formulas with other common inversion techniques that apply approximations by functions of a special type. For a survey of the latter we refer to [1].

8 Acknowledgments

The work was partly done when the first author was a Postdoctoral Fellow of the Fund for Scientific Research – Flanders, Belgium (FWO - Vlaanderen). Thanks are due to Wim Schoutens for many fruitful discussions on the content of this paper.

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