IDLE PERIODS FOR THE FINITE G/M/1 QUEUE AND THE DEFICIT AT RUIN FOR A CASH RISK MODEL WITH CONSTANT DIVIDEND BARRIER

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Abstract

We consider a G/M/1 queue with restricted accessibility in the sense that the maximal workload is bounded by 1. If the current workload V_t of the queue plus the service time of an arriving customer exceeds 1, only $1-V_t$ of the service requirement is accepted. We are interested in the distribution of the idle period, which can be interpreted as the deficit at ruin for a risk reserve process R_t in the compound Poisson risk model. For this risk process a special dividend strategy applies, where the insurance company pays out all the income whenever R_t reaches level 1. In the queueing context we further introduce a set-up time $a \in [0,1]$. After every idle period, when the queue is empty, an arriving customer has to wait for a time units until the server is ready to serve the customer.

1 Introduction

Queues with workload restrictions have been studied extensively and appear under various settings and synonyms in the mathematical literature: "queues with restricted accessibility", "finite-buffer queues", "uniformly bounded virtual waiting time", "limited queueing waiting time", "finite dam", etc. Fundamental results can be found in [4, 8, 10, 11, 14, 17, 18, 19, 20, 21, 23, 27]. The current paper can be seen as a continuation of [1], with the extra feature that the sojourn times might be truncated.

We investigate a G/M/1 queue with restricted accessibility in the sense that the maximal workload is bounded by 1. If the current workload V_t of the queue plus the service time of an arriving customer exceeds 1, only $1 - V_t$ of the service requirement is accepted. The paper focusses on the study of I, the idle period of the finite queue.

We present two methods to derive relations for the Laplace transform and probability distribution function of I. For the case with no setup time we utilize the fact that a second G/M/1 queue, constructed from the original queue by means of collecting successive overshoots over the level 1, has the same idle periods as the original one.

For the case with setup-time the proposed method is based on a duality argument, representing the idle period as the overshoot in a finite M/G/1 queue.

Let V_t be the workload process (virtual waiting time process) of the queue with restricted accessibility at level 1. As shown in Figure 1 the quantity I can be interpreted as the deficit at ruin of a modified risk reserve process R_t in the compound Poisson regime with a constant barrier strategy (see Figure 1). When the risk reserve process reaches level 1, dividends are paid out with constant rate equal to 1, so that R_t is constant until the next claim occurs. Such strategies have been studied for instance in [2] and [13]. More references and expressions for the moments of I can be found in [16].

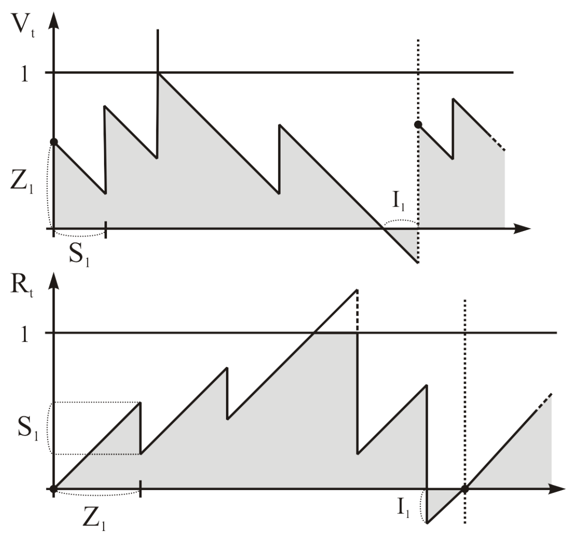

Figure 1: The workload process V_t and the associated risk reserve process R_t

The workload process of the finite queue can also be interpreted as the content of a finite dam, which is instantaneously filled up with a random level of water until the critical amount 1 is reached. As long as the content is larger than 1 no further water is fed into the dam. The water is released continuously until the dam is empty.

Let $S_1, S_2, \ldots$ denote the i.i.d. interarrival times and F_S the distribution function of S_1, with $1/\mu = ES_1$ being the mean of S_1. Let $Z_1, Z_2, \ldots$ denote the exponential service times with mean $1/\lambda$. We let $\rho = \lambda/\mu$. Since we are concerned with the finite queue, one can ignore all stability issues and investigate both $\rho > 1$, when the unrestricted G/M/1 queue is stable, and $\rho < 1$, when the dual M/G/1 queue is stable. Here the so called dual queue is obtained by interchanging the inter-arrival and service times, so that the exponentially distributed random variables $Z_1, Z_2, \ldots$ denote the inter-arrival times and the variables $S_1, S_2, \ldots$ become the successive service times of a Markovian M/G/1 queue.

It is instructive to first review the standard G/M/1 queue and introduce some known relevant results for the unrestricted case with no setup-time (the case $a = 0$). Let $\widehat{V}_t$ denote the workload process of the standard G/M/1 queue and let $\widehat{I}_1$ denote the first idle period. The Laplace transform of $\widehat{I}_1$ is then given by

$$\phi_{\widehat{I}}(s) = \lambda \cdot \frac{z - \phi_S(s)}{s - \lambda(1-z)} \ , \tag{1}$$

where z is the smallest positive root of $z = \phi_S(\lambda(1-z))$ (see [22], p.35, and [1]) and ϕ_S is the Laplace-Stieltjes transform of S. An inversion is possible for $\rho \geq 1$, when Lagrange's theorem yields (see [25])

$$z = \sum_{j=1}^{\infty} \frac{\lambda^{j-1}}{j!} \int_0^{\infty} x^{j-1} e^{-\lambda x} \, dF_S^{j*}(x),$$

where F_S^{j*} is the j-fold convolution of F_S with itself. In case that $\rho < 1$ we have $z = 1$, so that (1) reduces to

$$\phi_{\widehat{I}}(s) = \lambda \cdot \frac{1 - \phi_S(s)}{s}. \tag{2}$$

The distribution function of $\widehat{I}$ is given by

$$F_{\widehat{I}}(x) = \lambda \int_0^x (1 - G(u)) \, du,$$

(c.f. equation (4) in [12] for the risk process context). Note that $\frac{1-\phi_S(s)}{\mu s}$ is the Laplace transform of the asymptotic residual lifetime in a renewal process with epochs having distribution F_S and that $\phi_{\widehat{I}}(s)$ is a transform of a defective probability distribution function; in particular, $P(\widehat{I} < \infty) = \phi_{\widehat{I}}(0) = \rho$.

When $\rho < 1$ the dual M/G/1 queue is stable and its workload process $\widehat{W}$ has a stationary distribution $\widehat{F}$ with Laplace transform

$$\phi_{\widehat{F}}(s) = \frac{1-\rho}{1 - \phi_{\widehat{I}}(s)}, \tag{3}$$

which is the transform version of the Pollaczek-Khintchine formula.

Next, let π be the probability that $\widehat{V}$ up-crosses level 1 during a busy period and let η be the probability that starting in 1, the process hits level 0 before it returns to 1. It has also been shown (see [1, 7, 10, 18, 24, 26]) that if $\rho < 1$

$$\pi = \frac{1-\rho}{\widehat{F}(1)}$$

and

$$\eta = 1 - \frac{\widehat{F} * F_S(1)}{\widehat{F}(1)} = \frac{\widehat{f}(1)}{\lambda \widehat{F}(1)}, \tag{4}$$

where $\widehat{f}(x)$ is the density of $\widehat{F}(x)$ for $x > 0$ and the second step of (4) follows from the Pollaczek-Khintchine formula. For $\rho > 1$ the probability π is given in Theorem 4 of [1].

More results and references about the standard G/M/1 queue can be found in [1, 3, 10, 15].

2 Conditional idle period

We derive a formula for the conditional distribution of the idle period, given the event that the workload process exceeds level 1 during a busy period, or equivalently: we present an expression for the distribution of the deficit at ruin of the risk reserve process R_t, given that a dividend is paid out.

Theorem 1. *Let $\overline{V}$ be the maximum of V_t during the first busy cycle. The conditional distribution H of the idle period $\widehat{I}_1$, given the event $\{\overline{V} \geq 1\}$, is equal to the residual lifetime distribution at time $t = 1$ of a renewal process with renewal epochs having the same distribution as the idle period $\widehat{I}_1$.*

Proof. By tracing figure 2 for a typical sample path of $\widehat{V}_t$, let T^* and T_1 denote the last up- and down-crossing times of level 1 before the idle period starts. T_1 is the endpoint of an excess period of $\widehat{V}_t$ over level 1. The time X_1 from T_1 to the next arrival has the same distribution as that of the G/M/1 idle period, since the epoch $T_1 - T^*$ can be seen as the busy period of a G/M/1 queue (indicated by a grey area). At time $T_1 + X_1$ another busy period of a G/M/1 queue starts; it ends at time T_2. Again, the distribution of X_2 is the same as that of an idle period and we see that this property also holds for $X_3, X_4, \ldots$. Thus, the sequence $X_1, X_2, \ldots$ forms a renewal process. Since $J_1 = X_1 + X_2 + \ldots + X_\kappa - 1$ where $\kappa = \inf\{k | X_1 + \ldots + X_k > 1\}$, $\widehat{I}_1$ can be seen as its residual lifetime at time $t = 1$ of the renewal process. $\square$

Expressions for the distribution of the residual lifetime of a renewal process can be found in [5] and [10]. We note that H is a solution of the renewal equation ([3], p.143)

$$H(x) = (F_{\widehat{I}}(1 + x) - F_{\widehat{I}}(1)) + F_{\widehat{I}} * H(x).$$

3 The idle period without setup-time

In this section we let the setup time $a = 0$. We are interested in the Laplace transform of the idle period. To derive it we make use of a sample path analysis, based on the comparison of the idle periods of two related queues.

Theorem 2. *The Laplace transform of I_1 is given by*

$$\phi_I(s) = \frac{\phi_L(s)}{\pi + (1 - \pi)\phi_L(s)}, \tag{5}$$

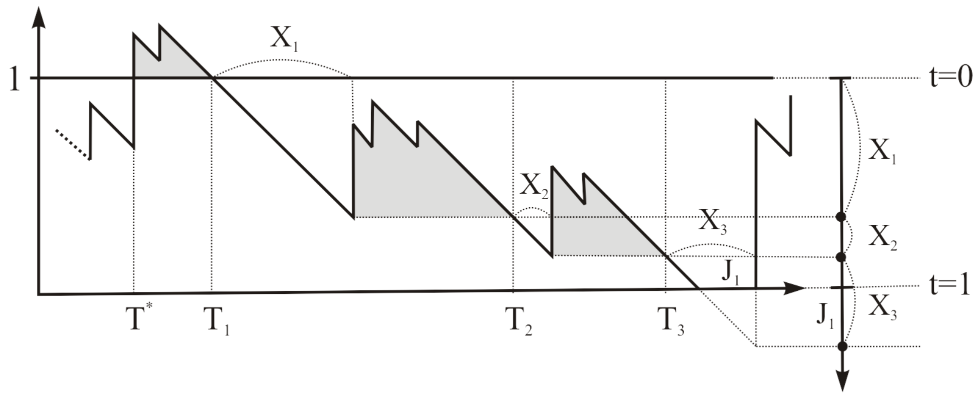

Figure 2: Visualization of the proof of Theorem 1

with

$$\phi_L(s) \quad = \quad \begin{cases} z_0 - \left(z_0 - 1 + \frac{s}{\lambda\eta}\right)\phi_H(s) & ,\rho \geq 1 \\ 1 - \frac{s}{\lambda\eta}\phi_H(s) & ,\rho < 1 \end{cases} \tag{6}$$

where z_0 *is the smallest positive root of* $z_0 = \phi_L(\lambda\eta(1-z_0))$ *and*

$$\phi_H(s) \quad = \quad \frac{\pi\phi^*(s)}{1-\phi_{\hat{I}}(s)+\pi\phi^*(s)}. \tag{7}$$

Here $\phi^*(s) = \int_0^\infty e^{-su}\, dH(u)$ *is the conditional Laplace transform of* H*.*

Before we prove Theorem 2, we note that the Laplace transform $\phi_{\hat{I}}$ is given in (1) and Theorem 1. For $\rho \geq 1$ equation (6) is implicit in the sense that $\phi_L(s)$ is given in terms of the root z_0, which itself can be determined only when ϕ_L is known.

Proof. Let K_1 be the number of overshoots $U_{1,1}, U_{1,2}, \ldots, U_{1,K_1}$ of level 1 during the first busy period R_1 of the restricted queue, see figure 3. Note that $\pi = P(K_1 > 0)$ and $\eta = P(K_1 = 1 | K_1 > 0)$.

The dashed line in Figure 3(a) shows the standard G/M/1 workload process and the solid line shows the restricted queue. We construct a modified queueing process by collecting all time intervals with $V_t = 0$ from the original G/M/1 workload process (see shaded areas in Figure 3(a)). Let L_k, A_k and H_k denote the interarrival times, service times and idle periods of this new queue, as shown in 3(b). Note that all three variables can be represented as geometric sums. Indeed, we have

$$L_1 = \sum_{i=1}^{M} I_i, \tag{8}$$

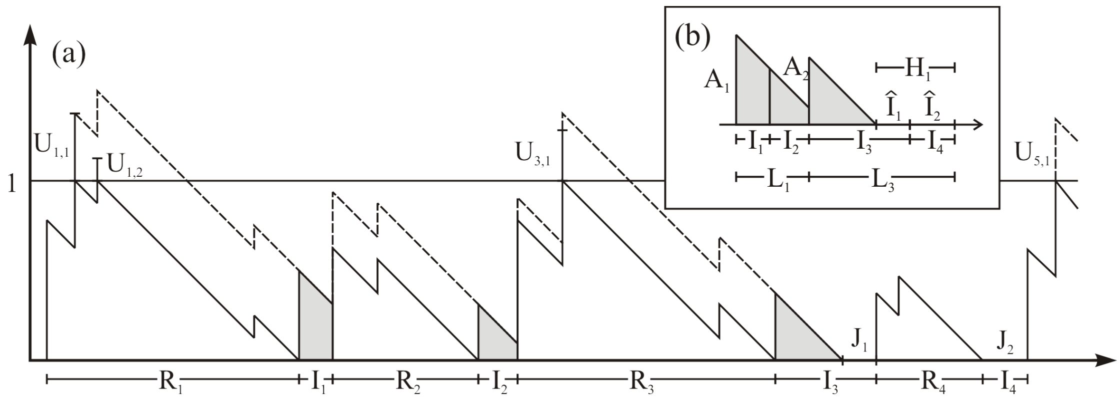

Figure 3: Construction used in the proof of Theorem 2.

where M is a geometrically distributed random variable with support $M \geq 1$ and $P(M = 1) = \pi$, representing the number of idle periods of the finite queue during a cycle of the unrestricted queue. H_1 is the sum of the idle periods of the restricted queue,

$$H_1 = \widehat{I}_1 + \sum_{i=1}^{M-1} \widehat{I}_{i+1}, \tag{9}$$

where we separated $\widehat{I}_1$ from the sum because it has a different distribution than the other. Finally A_1 consists of the cumulated overshoots during the first busy period,

$$A_1 = \sum_{i=1}^{K} U_{1,i}, \tag{10}$$

where the variable K denotes the number of overshoots of level 1 during a cycle of the finite queue. It is also geometric with $K \geq 1$ and $P(K = 1) = \eta$. From (8) and the fact that

$$\phi_L(s) \quad = \quad \sum_{i=1}^{\infty} \phi_I(s)^i \pi (1-\pi)^{i-1} = \frac{\pi \phi_I(s)}{1-(1-\pi)\phi_I(s)},$$

the relation (5) follows immediately .

Since the U_i are exponential with Laplace transform $\phi_U(s) = \lambda/(\lambda + s)$ we obtain

$$\phi_A(s) = \frac{\lambda}{\lambda + s} \frac{\eta}{1-(1-\eta)\frac{\lambda}{\lambda+s}} = \frac{\lambda\eta}{s+\lambda\eta},$$

so that the A_i are exponentially distributed with rate $\lambda\eta$. The new queue is thus again a G/M/1 queue with service rate $\lambda\eta$ and we can apply (1) for the law of its idle period. Hence, replacing λ by $\lambda\eta$, z by z_0 and ϕ_S by ϕ_L in (1), we obtain

$$\phi_{\widehat{I}}(s) = \lambda\eta \cdot \frac{z_0 - \phi_L(s)}{s - \lambda\eta(1-z_0)},$$

and (6) follows.

According to Theorem 1, J_1 and Z are stochastically equal and $J_2, J_3, \ldots$ are i.i.d. with Laplace transform ϕ_Y (the conditional transform of the idle period given $K_1 = 0$). Hence

$$\phi_H(s) = \phi^*(s) \cdot \frac{\pi}{1 - (1-\pi)\phi_Y(s)}.$$

Finally (7) follows by applying the law of total probability, $\phi_{\widehat{I}}(s) = (1-\pi)\phi_Y(s) + \pi\phi^*(s)$. □

As an example we verify the above theorem by applying the M/M/1 special case, where the interarrival distribution is exponential with mean $1/\mu$ and we assume that $\rho < 1$.

Then $\phi_S(s) = \mu/(\mu + s)$ and the smallest positive root of $z = \phi_S(\lambda(1-z))$ is simply $z = 1$. Consequently we have

$$\phi_{\widehat{I}}(s) = \lambda \cdot \frac{1 - \phi_S(s)}{s} = \rho \frac{\mu}{\mu + s}.$$

According to (3), the Laplace transform of the stationary distribution of $\widehat{W}$ is given by

$$\phi_{\widehat{F}}(s) = \frac{1-\rho}{1 - \phi_{\widehat{I}}(s)} = \frac{1-\rho}{1 - \rho\frac{\mu}{\mu+s}}$$

which is the transform of $\widehat{F}(x) = 1 - \frac{\lambda}{\mu} e^{-(\mu-\lambda)x}$. Hence it follows that the taboo probabilities η and π are given by

$$\eta = \frac{\mu - \lambda}{\mu e^{(\mu-\lambda)} - \lambda} = \pi e^{-(\mu-\lambda)} \quad \text{and} \quad \pi = \frac{\mu-\lambda}{\mu - \lambda e^{-(\mu-\lambda)}} = \eta e^{(\mu-\lambda)}.$$

To find the transform ϕ^* of Z, the residual lifetime from Theorem 1, let ℓ be the number of finite renewals in a renewal process $X_1, X_2, \ldots$ with defective interarrival distribution $F_{\widehat{I}}$ and let $S = \sum_{i=1}^{\ell} X_i$. The random variable ℓ has a geometric distribution with $P(\ell = 0) = 1 - \rho$, so that S is the sum of ℓ exponential random variables with rate μ and thus exponentially distributed with rate $\mu(1-\rho) = \mu - \lambda$. It follows that

$$P(Z \le x) \;=\; P(S > 1, S - 1 \le x, \ell > 0) = P(S > 1, X \le x, \ell > 0)$$

where X is exponential with mean $1/\mu$, independent of S. Hence

$$P(Z \le x) \;=\; \rho P(S > 1 | \ell > 0)(1 - e^{-\mu x}) = \rho e^{-(\mu-\lambda)}(1 - e^{-\mu x})$$

with transform $\phi^*(s) = e^{-(\mu-\lambda)} \frac{\lambda}{\mu+s}$. It follows from (7) that

$$\phi_H(s) = \frac{\pi\phi^*(s)}{1 - \phi_{\widehat{I}}(s) + \pi\phi^*(s)} = \frac{\lambda\eta}{s + \mu\pi}.$$

Equation (5) yields

$$\phi_L(s) = 1 - \frac{s}{\lambda\eta}\phi_H(s) = \frac{\mu\pi}{\mu\pi + s}.$$

After applying (5) we obtain

$$\phi_I(s)\frac{\phi_L(s)}{\pi + (1-\pi)\phi_L(s)} = \frac{\mu}{s+\mu},$$

so that I is exponentially distributed, as expected by the lack of memory property.

4 Idle period with setup-time

Now assume that we have a setup-time $a \in (0, 1]$, i.e. each busy period starts with $V_t = Z_1 + a$. The upper left diagram in Figure 4 shows the workload process V_t of the restricted G/M/1 queue with setup-time a, together with the first cycle C_1 of that queue.

From V_t we construct a new process $\overline{R}_t$, representing the time elapsed since the arrival of the customer being served. $\overline{R}_t$ is obtained from the risk reserve process R_t (Figure 1) by removing the time intervals where $R_t = 1$.

Next we define the process $W_t = 1 - U_t$ by flipping the process $\overline{R}_t$ vertically.

By construction, the idle-periods of the restricted G/M/1 queue are identical with the overflows of the dual M/G/1 queue, i.e. the overshoots of the workload process W_t over level 1. Since the overflows occur only once in a cycle, it follows that the I_i are i.i.d. with common distribution function F_I.

Before we proceed with determining the distribution of $\widehat{I}$, we show that the stationary density f fulfills a certain integral equation.

Lemma 3. *The density function of the stationary distribution of W_t is given by*

$$f(x) \quad = \quad \begin{cases} ch(x) + \rho\, h * f(x), & 0 \le x < 1-a \\ ch(x) - d + \rho\, h * f(x), & 1-a \le x < 1 \\ ch(x) + \rho \int_0^1 h(x-y)f(y)\,dy, & x \ge 1, \end{cases} \tag{11}$$

where $c = \frac{f(0)}{\mu}$, $d = ch(1) + \rho\int_0^1 h(1-y)f(y)\ dy$ and $h(x) = \mu\big(1 - F_S(x)\big)$ is the equilibrium density of the service time distribution.

Proof. Let $D_x(C)$ and $U_x(C)$ denote the number of down- and upcrossings of level x by W_t during the first cycle C. By level crossing theory ([9, 18, 6]) the long-run average number of downcrossings is given by $E(D_x(C))/E(C) = f(x)$, for all $x \ge 0$. By a similar reasoning as in [18] we find, that for $x < 1-a$ the average number of upcrossings is given by

$$\frac{E(U_x(C))}{E(C)} = f(0)\,(1 - F_S(x)) + \lambda\int_0^x (1 - F_S(x-u))\,f(u)\,du = ch(x) + \rho\, h * f(x)$$

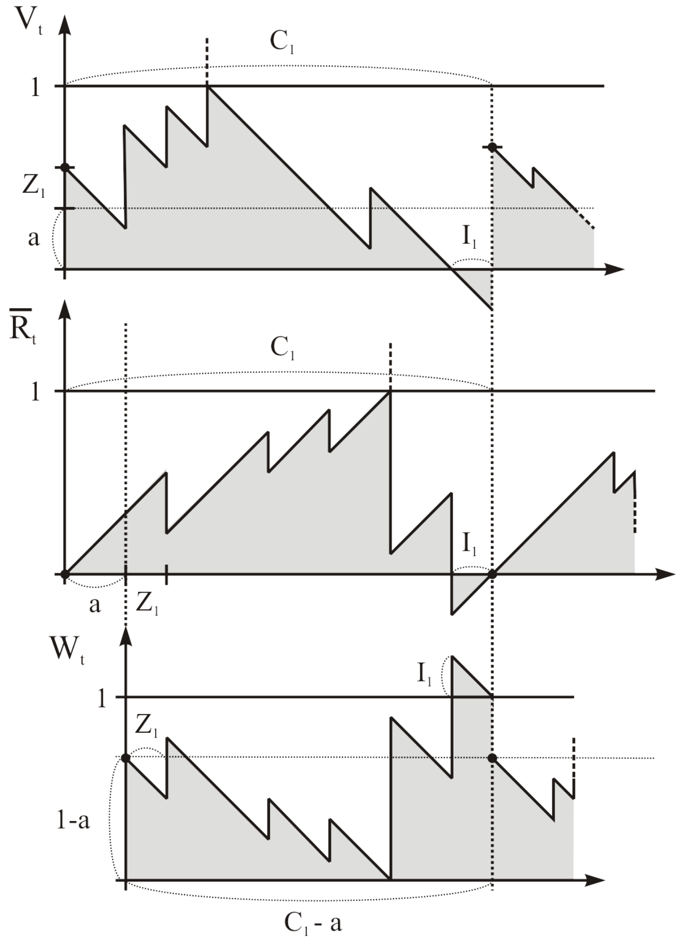

Figure 4: Construction of the processes $\overline{R}$ and W

and equating the two averages leads to the first line in (11).

For $x \geq 1$ the number of up- and downcrossings is again equal and the average number of upcrossings is $f(0)\,(1 - F_S(x)) + \lambda \int_0^1 (1 - F_S(x-u))\,f(u)\,du$, since there are no jumps from above level 1.

If $x \in [1-a, 1)$ then $D_x(C) = U_x(C) - 1$, since after crossing level 1 the process never returns to $[0, x)$ during the cycle. Hence

$$\begin{aligned} \frac{E(D_x(C))}{E(C)} &= \frac{E(U_x(C))}{E(C)} - \frac{1}{E(C)} \\ &= f(0)\,(1 - F_S(x)) + \lambda \int_0^x (1 - F_S(x-u))\,f(u)\,du - \frac{1}{E(C)}. \end{aligned}$$

Now, since W_t crosses level 1 exactly once every cycle, we have $f(1+) = \frac{1}{E(C)}$. But $f(1+) = d$ and hence the second equation in (11) follows. $\square$

Solving for the renewal equation $f(x) = ch(x) + \rho\, h(x) * f$ in $[0, 1-a)$ we obtain

$$f(x) \quad = \quad ch * H(x),$$

where $H(x) = \sum_{n=0}^{\infty} \rho^n h^{*n}(x)$ and h^{*n} is the n-fold convolution of h with itself. Similarly for $[1-a, 1)$, we have the renewal equation $f(x) = (ch(x) - d) + \rho\, h * f(x)$. Letting $\hat{d}(x) = d\mathbb{1}_{\{x \geq 1\}}$ we get

$$\begin{aligned} f(x) &= (c\,h - \hat{d}) * H(x) = c\,h * H(x) - d \int_0^x \mathbb{1}_{\{x-u \geq 1\}}\, dH(u) \\ &= c\,h * H(x) - d\, H((x-1)-). \end{aligned} \tag{12}$$

Finally for the interval $[1, \infty)$ we have

$$f(x) \quad = \quad ch(x) + \rho \int_0^1 h(x-y) f(y)\, dy,$$

where $f(y)$ is already known for $y \in [0, 1)$. To find the two constants c and d note that from $d = c\,h(1) + \rho\, h * f(1-)$ and (11) it follows that $f(1-) = 0$. By using (12) we obtain $c\,h * H(1) = d\, H(0)$ and since $H(0) = 1$,

$$\frac{d}{c} = h * H(1).$$

Now the constants can then be determined from the normalizing condition

$$\int_0^\infty f(u)\, du = 1.$$

We now prove a result, that relates the distribution of the idle period $\widehat{I}$ to the stationary density f.

Theorem 4. *The distribution of I_1 is given by*

$$F_I(x) \quad = \quad 1 - \frac{f(1+x)}{f(1)}, \tag{13}$$

where f and F denote the equilibrium density and distribution of $W_\infty = \lim_{t\to\infty} W_t$ (the latter limit is defined in terms of weak convergence).

Proof. The conditional density f_W^c of $W_\infty - 1$, given that $W_\infty > 1$ is given by

$$f_W^c(x) = \frac{f(1+x)}{1 - F(1)}.$$

By looking at the renewal process $I_1, I_2, \ldots$ we conclude that f_W^c is also the density of the (equilibrium) forward recurrence times of that process. Hence $f_W^c(x) = (1 - F_I(x))/EI_1$, and

$$F_I(x) = 1 - EI_1 \cdot f_W^c(x) = 1 - f(1+x)/\left(\tfrac{1-F(1)}{EI_1}\right).$$

By Lemma 3 $f(1) = 1/E(C)$ and by renewal theory $1 - F(1) = EI_1/E(C)$. Consequently $\frac{1-F(1)}{EI_1} = f(1)$ and thus (13) follows. □

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