

A Markov modulated growth collapse model

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Abstract

We consider a growth collapse model in a random environment where the input rates may depend on the state of an underlying irreducible Markov chain and at state change epochs there is a possible downward jump to a level which is a random fraction of the level just before the jump. The distributions of these jumps are allowed to depend on both the originating and target states. Under a very weak assumption we develop an explicit formula for the conditional moments (of all orders) of the time stationary distribution. We then consider special cases and show how to use this result to study a growth collapse process in which the times between collapses have a phase type distribution.

Keywords: Growth collapse process, Markov modulated, phase type distribution, Markov process

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1 Introduction

In this paper we consider a Markov modulated growth collapse model, represented by a continuous time Markov process $\{(W_t, J_t)\}_{t \geq 0}$. Here $J = \{J_t | t \geq 0\}$ is a right continuous irreducible continuous time Markov chain with state space $\{1, \dots, K\}$, rate transition matrix $Q = (q_{ij})$ with $q_i = -q_{ii}$ and stationary distribution $\pi = (\pi_i)$. As long as J is in state i , the right continuous process $W(t)$ increases at a linear rate $r_i \geq 0$. Whenever there is a state change from, say, i to j , the process is reduced to a fraction of what it was before the jump. This fraction is independent of the history and is distributed like some random variable X_{ij} satisfying $\mathbb{P}[0 \leq X_{ij} \leq 1] = 1$. Namely, if at time t there is a state change from i to j then W_{t-}/W_t is distributed like X_{ij} .

Our primary concern is the derivation of formulas for the moments associated with (W_t, J_t) and its stationary version (W_*, J_*) . The present paper is divided as follows. In the second section we first give general (necessary and sufficient) conditions that ensure convergence of the process (W_t, J_t) to a stationary limit. In section three we show how the moments of W_* can be calculated, after which we apply similar ideas to the derivation of the transient moments $\mathbb{E}(W_t^n)$ in the fourth section. Finally we describe the special case, where the time between the jumps has a phase type distribution and the jumps are all of the same type.

The monograph [4] provides an overview and a toolbox for studying piecewise deterministic Markov processes (PDMPs), a class of stochastic models to which our process belongs. In [2] some results for a general type of a Markovian growth collapse model are given, including a Markov modulated case different from the one investigated here. More general processes are considered in [6]. In [9] formulas for the non-modulated case are given, including results also for non-integer and transient moments. Further results, mainly focussing on the stationary case, can also be found in [1, 5, 7] and [10]. The latter articles use growth collapse processes with multiplicative jumps to describe the window size process for congestion avoidance in the TCP data transmission protocol. Instead of looking at the continuous time process W_t one may also study the behavior of the embedded processes just before and immediately after a jump. Related to this approach are stochastic relations that have been investigated by [11, 3]. In the recent paper [8] criteria for stability and formulas

for moments for stationary growth collapse models in a more general setting are developed.

2 Some preliminaries

Consider the following.

Condition 1 $\exists i, j$ such that $q_{ij} > 0$ and $\mathbb{P}[X_{ij} = 1] < 1$.

Theorem 1 *Under Condition 1 the process (W_t, J_t) has a well defined time stationary distribution which is also the limiting distribution, independent of initial conditions.*

Proof: According to Condition 1 there are i and j , such that $q_{ij} > 0$ and $\mathbb{P}[X_{ij} = 1] < 1$. Consider a growth collapse process $(\bar{W}_t, J_t)_{t \geq 0}$ with the following properties. For $t = 0$ we have $\bar{W}_0 = W_0$ and the process increases with rate $r = \max\{r_1, \dots, r_K\}$. The process jumps at times $(\bar{T}_n)_{n=1,2,\dots}$, when J_t moves from state i to state j . At such a jump time $t = \bar{T}_n$ the process jumps from \bar{W}_{t-} to $\bar{W}_t = X_n \cdot \bar{W}_{t-}$, where $X_n = W_{t-}/W_t$.

The sequence $(Y_n)_{n=1,2,\dots}$, defined by $Y_n = \bar{W}_{\bar{T}_n} - \bar{W}_{\bar{T}_n-} = r(\bar{T}_n - \bar{T}_{n-1})$, is a sequence of i.i.d. random variables independent of $(X_n)_{n=1,2,\dots}$, in fact having a phase type distribution. It follows that $(X_n, Y_n)_{n=1,2,\dots}$ is i.i.d. and hence stationary and ergodic and we may apply Corollary 1 in [8] to deduce the existence of a stationary/limiting distribution for \bar{W}_t .

Since $W_t \leq \bar{W}_t$, it follows that the (Markov) process $\{(W_t, J_t) \mid t \geq 0\}$ is tight. It is easy to check that the difference between a process that starts with $W_0 = 0$ and that of a process that starts with $W_0 = x$ converges almost surely to zero as $t \rightarrow \infty$, since it is less than or equal to $x \prod_{n=1}^{N_{ij}(t)} X_n$ where $N_{ij}(t)$ is the number of (i, j) transitions that occurred by time t . Also, it is clear that any set of the form $[0, \epsilon]$ is accessible by \bar{W}_t and thus any set of the form $[0, \epsilon] \times \{j\}$ is accessible by (W_t, J_t) and thus a unique stationary/limiting distribution for the process $\{(W_t, J_t) \mid t \geq 0\}$ exists. ■

Note that Condition 1 is the weakest possible, in the sense that as soon as it is not fulfilled (and $r_i > 0$ at least for one i), W_t will not jump at all and hence $W_t \rightarrow \infty$.

The extended generator (in the sense of [4]) of the Markov process (W, J) is of the form

$$\mathcal{A}f(x, i) = r_i f'(x, i) + \sum_{\substack{j=1 \\ j \neq i}}^K q_{ij} (\mathbb{E}f(xX_{ij}, j) - f(x, i))$$

with the convention that $X_{ii} \equiv 1$ and assuming that $f(\cdot, \cdot)$ is in the domain of \mathcal{A} . Bearing in mind that $\sum_{j=1}^K q_{ij} = 0$, we can write this as

$$\mathcal{A}f(x, i) = r_i f'(x, i) + \sum_{j=1}^K q_{ij} \mathbb{E}f(xX_{ij}, j).$$

Theorem 2 For every $\alpha \geq 0$ the function $f(x, i) = c_i x^\alpha$ is in the domain of \mathcal{A} and thus, with $a_{ij}(\alpha) = \mathbb{E}X_{ij}^\alpha$, we have that

$$\mathcal{A}f(x, i) = \alpha r_i c_i x^{\alpha-1} + x^\alpha \sum_{j=1}^K q_{ij} a_{ij}(\alpha) c_j \quad (1)$$

If we let $A(\alpha) = (a_{ij}(\alpha))$ and $A \circ B = (a_{ij} b_{ij})$ for any two matrices A, B and $D_r = \text{diag}(r_1, \dots, r_K)$ as well as $c = (c_i)$, then we can write the generator in vector form as

$$\mathcal{A}f(x) = \alpha x^{\alpha-1} D_r c + x^\alpha Q \circ A(\alpha) c, \quad (2)$$

where $f(x) = (f(x, 1), \dots, f(x, K))^T$ and \mathcal{A} acts componentwise.

Proof: Following the general theory of PDMPs (see [4]) a measurable function $f : [0, \infty) \times \mathbb{K} := \{1, \dots, K\} \rightarrow \mathbb{R}$ belongs to the domain of \mathcal{A} if $t \mapsto f(x + r_i t, i)$ is absolutely continuous for all $x \geq 0$ and $i \in \mathbb{K}$, and the integrability condition

$$\mathbb{E} \left[\sum_{k=1}^{N_t} |f(W_{T_k-}, J_{T_k-}) - f(W_{T_k}, J_{T_k})| \middle| W_0 = w, J_0 = j \right] < \infty \quad (3)$$

holds for all w, j and all $t \geq 0$, where N_t denotes the number of jumps until time t and the T_i are the successive jump times.

The function $t \mapsto c_i(x + r_it)^\alpha$ is clearly absolutely continuous. Given that $W_0 = x$ we have $W_t \leq x + rt$, with $r = \max\{r_1, \dots, r_K\}$. Let $z(x, t)$ denote the left side of (3), then

$$z(x, t) \leq 2c\mathbb{E}\left[\sum_{k=1}^{N_t} W_{T_k^-}^\alpha \mid W_0 = w, J_0 = j\right] \leq 2c(x + rt)^\alpha \mathbb{E}[N_t],$$

where $c = \max\{c_1, \dots, c_K\}$. The last term is certainly finite for all $x \geq 0$ and all $t \geq 0$.

3 Moments

We define the vector of moments $\xi^n = (\xi_1^n, \xi_2^n, \dots, \xi_K^n)$ by $\xi_i^n = \mathbb{E}[W_*^n 1_{\{J^*=i\}}]$. Recall that the matrix A is given by $a_{ij}(\alpha) = \mathbb{E}X_{ij}^\alpha$.

Lemma 1 *Let $D_q = \text{diag}(q_1, \dots, q_K)$. Under Condition 1 the matrix $Q \circ A(\alpha)$ is nonsingular for every positive α and $(-Q \circ A(\alpha))^{-1} \geq D_q^{-1}$.*

Proof: Denoting $p_{ij} = q_{ij}/q_i$ for $i \neq j$, $p_{ii} = 0$, then $Q = D_q(P - I)$ where I is the identity matrix. Also, $Q \circ A(\alpha) = D_q(P \circ A(\alpha) - I)$, as $a_{ii}(\alpha) = 1$ and $p_{ii} = 0$. For a fixed α , let us define a new Markovian transition matrix with the extra absorbing state σ as follows.

$$\tilde{p}_{ij} = \begin{cases} p_{ij}a_{ij}(\alpha) & 1 \leq i, j \leq K \\ 1 - \sum_{j=1}^K p_{ij}a_{ij}(\alpha) & 1 \leq i \leq K, j = \sigma \\ 0 & i = \sigma, 1 \leq j \leq K \\ 1 & i = j = \sigma \end{cases} \quad (4)$$

If we show that for the discrete Markov chain with transition matrix $\tilde{P} = (\tilde{p}_{ij})$ the states $1, \dots, K$ are all transient, then if we denote the i, j th coordinate of \tilde{P}^k by \tilde{p}_{ij}^k , then for every $(i, j) \neq (\sigma, \sigma)$ we have that $\tilde{p}_{ij}^k \rightarrow 0$ as $k \rightarrow \infty$. Since the states $1, \dots, K$ cannot be reached from σ it readily

follows that for $1 \leq i, j \leq K$ we have that $(P \circ A(\alpha))^k = (\tilde{p}_{ij}^k)$ and thus $(P \circ A(\alpha))^k \rightarrow 0$ as $k \rightarrow \infty$. Therefore $I - P \circ A(\alpha)$ is nonsingular with

$$(I - P \circ A(\alpha))^{-1} = \sum_{k=0}^{\infty} (P \circ A(\alpha))^k \geq I \quad (5)$$

and thus $-Q \circ A(\alpha)$ is also nonsingular with

$$(-Q \circ A(\alpha))^{-1} = (I - P \circ A(\alpha))^{-1} D_q^{-1} \geq D_q^{-1}. \quad (6)$$

To argue that indeed the states $1, \dots, K$ are transient in the chain with transition matrix \tilde{P} we observe that since $\tilde{p}_{ij} \leq p_{ij}$ for $1 \leq i, j \leq K$, and since the chain with transition matrix P is irreducible, the submatrix of \tilde{P} associate with any strict subset of $1, \dots, K$ cannot be stochastic, since this would imply that there are states that cannot be reached in the original chain which contradicts irreducibility. Since there exists a pair $1 \leq i, j \leq K$ for which $q_{ij} > 0$, hence $p_{ij} > 0$, for which $P[X_{ij} = 1] < 1$, it follows that for all $\alpha > 0$, $a_{ij}(\alpha) = \mathbb{E}X_{ij}^\alpha < 1$ and thus $\tilde{p}_{ij} < p_{ij}$. Thus, $P \circ A(\alpha)$ which is the submatrix of \tilde{P} associate with the states $1, \dots, K$ cannot be stochastic either. This means that for \tilde{P} every state other than σ must belong to an open class and is thus transient. ■

Theorem 3 *If Condition 1 is fulfilled, then*

$$(\xi^n)^T = n! \pi^T \prod_{k=1}^n D_r (-Q \circ A(k))^{-1}, n \geq 1. \quad (7)$$

Proof: If we denote by (W_*, J_*) a random pair having the stationary distribution, then it is well known that for every f in the domain of \mathcal{A} we have that $\mathbb{E}[\mathcal{A}f(W_*, J_*)] = 0$. Since $\xi_i^0 = \pi_i$ it immediately follows from Theorem 2 and (2), that for $f(x, i) = c_i x^n$ where n is a nonnegative integer and c_i are arbitrary, that

$$0 = \mathbb{E}[\mathcal{A}f(W_*, J_*)] = n(\xi^{n-1})^T D_r c + (\xi^n)^T Q \circ A(n) c. \quad (8)$$

Thus, if we take any square matrix C , then clearly the following system of equations is also valid

$$n(\xi^{n-1})^T D_r C + (\xi^n)^T Q \circ A(n) C = 0. \quad (9)$$

Since, by Lemma 1, $Q \circ A(n)$ is nonsingular, it follows by taking $C = (-Q \circ A(n))^{-1}$ that the following recursion is valid.

$$(\xi^n)^T = n(\xi^{n-1})^T D_r (-Q \circ A(n))^{-1} \quad (10)$$

from which (7) immediately follows. ■

Remark 1 *We note that one could generalize the setup in a way that allows a geometric number of jumps down while in state i , but assuming that the jumps rate is some $\lambda_i < q_i$ with $p_{ii} = 1 - \lambda_i/q_i$. However this effect can also be achieved in the above framework as well by simply allowing two different states, say i, k to have $q_{ij} = q_{kj}$ and $r_i = r_k$ and $q_{ik} = q_{ki} = q_i - \lambda_i$ where $\lambda_i = \lambda_k$.*

4 Transient moments

As in the stationary case we define the vector $\xi^n(t) = (\xi_1^n(t), \dots, \xi_K^n(t))$ of transient moments, with components $\xi_i^n(t) = \mathbb{E}[W_t^n 1_{\{J_t=i\}}]$. It follows as in (8) that

$$\mathbb{E}[\mathcal{A}f(W_t, J_t)] = n(\xi^{n-1}(t))^T D_r c + (\xi^n(t))^T Q \circ A(n) c. \quad (11)$$

Dynkin's martingale is given by $f(W_t, J_t) - f(W_0, J_0) - \int_0^t \mathcal{A}f(W_s, J_s) ds$ for functions f in the domain of \mathcal{A} (c.f. [4]). In particular if $f(x, i) = c_i x^n$ then

$$\mathbb{E}[c_{J_t} W_t^n] = \mathbb{E}[c_{J_0} W_0^n] + \int_0^t \mathbb{E}[\mathcal{A}f(W_s, J_s)] ds.$$

It follows that for any K -vector c

$$\xi^n(t)^T c = \xi^n(0)^T c + \int_0^t (n(\xi^{n-1}(s))^T D_r c + (\xi^n(s))^T Q \circ A(n) c) ds.$$

Letting C be an invertible $K \times K$ square matrix, we obtain after differentiation w.r.t. t ,

$$\left(\frac{d}{dt} \xi^n(t)\right)^T C = n(\xi^{n-1}(t))^T D_r C + (\xi^n(t))^T Q \circ A(n) C,$$

and after a multiplication by C^{-1} and transposition we arrive at the following system of linear equations,

$$\frac{d}{dt}\xi^n(t) = nD_r\xi^{n-1}(t) + (Q \circ A(n))^T\xi^n(t).$$

A solution is given by

$$\xi^n(t) = e^{(Q \circ A(n))^T t}\xi^n(0) + n \int_0^t e^{(Q \circ A(n))^T(t-s)} D_r \xi^{n-1}(s) ds. \quad (12)$$

Letting $R_n(t) = e^{(Q \circ A(n))^T t} D_r$, we obtain

$$\xi^n(t) = R_n(t) D_r^{-1} \xi^n(0) + n R_n * \xi^{n-1}(t),$$

where $*$ denotes convolution, and then iteratively, with $R^{*n} = R_n * R^{*(n-1)}$ we have that

$$\xi^n(t) = n! \left(\sum_{k=1}^n \frac{D_r^{-1}}{(n-k+1)!} \xi^{n-k+1}(0) R^{*k}(t) + \int_0^t R^{*n}(s) \pi ds \right).$$

5 The case of phase type inter-jump times

In this section we would like to consider a growth-collapse process for which the time between jumps has a phase type distribution and the jump is only of one type. To define a phase type distribution we need to define a continuous time Markov chain with states, say, $0, \dots, K$ such that 0 is accessible from any other state. Given some initial distribution β on $1, \dots, K$, a phase type distribution is the distribution of time until 0 is visited for the first time. Thus, we define a continuous time Markov chain with the following transition matrix.

$$Q = \begin{pmatrix} -1 & \beta^T \\ -R\mathbf{1} & R \end{pmatrix} \quad (13)$$

where the first row and column correspond to state 0 and the rest to states $1, \dots, K$ and $\mathbf{1}$ denotes a K -vector of ones.

From the assumption it follows that the distribution of time it takes to reach 0 given the initial distribution β on $1, \dots, K$ is given by $\beta^T e^{Rt} \mathbf{1}$ where

e^A is a matrix exponential. If the defined Markov chain is not irreducible, then, starting from state 0, some states will never be visited and thus can be removed. Thus, without loss of generality, irreducibility is assumed.

We now define the upward rates of our process as $r_0 = 0$ and $r_1 = \dots = r_K = r$ and

$$X_{ij} = \begin{cases} X & 1 \leq i \leq K, j = 0 \\ 1 & \text{otherwise,} \end{cases} \quad (14)$$

where $\mathbb{P}[X = 1] < 1$. Hence, the jumps when visiting state 0 are i.i.d. and independent of the originating state. We also observe that after 0 is visited, the growth collapse process remains constant until the modulating Markov chain makes a transition to a different state. This is not quite the structure of a growth collapse process with phase type interarrival times. However if we remove the time intervals where the modulating Markov chain is in zero, then the process does become one. Hence with (W^*, J^*) denoting a random pair having the stationary distribution of our Markov modulated growth collapse process, then the distribution of the growth collapse process with phase type interarrival times is given by

$$F(t) = \frac{\sum_{i=1}^K \mathbb{P}[W^* \leq t, J^* = i]}{1 - \pi_0}$$

and in particular the moments are given by

$$\mu^n = \int_{[0,t]} t^n dF(t) = \frac{\sum_{i=1}^K \xi_i^n}{1 - \pi_0} \quad (15)$$

Clearly, Condition 1 is satisfied and with $a(\alpha) = \mathbb{E}X^\alpha$ we have that $a_{i0}(\alpha) = a(\alpha)$ for $i = 1, \dots, K$ and $a_{ij}(\alpha) = 1$ for all other pairs. Hence

$$Q \circ A(\alpha) = \begin{pmatrix} -1 & \beta^T \\ -a(\alpha)R\mathbf{1} & R \end{pmatrix} \quad (16)$$

Observe that since $\beta^T \mathbf{1} = 1$ it is straightforward to check that

$$\begin{aligned} (Q \circ A(\alpha))^{-1} &= \begin{pmatrix} -\frac{1}{1-a(\alpha)} & \frac{1}{1-a(\alpha)} \beta^T R^{-1} \\ -\frac{a(\alpha)}{1-a(\alpha)} \mathbf{1} & \left(I + \frac{a(\alpha)}{1-a(\alpha)} \mathbf{1} \beta^T \right) R^{-1} \end{pmatrix} \\ &= \begin{pmatrix} 0 & \mathbf{0}^T \\ \mathbf{0} & R^{-1} \end{pmatrix} + \frac{1}{1-a(\alpha)} \begin{pmatrix} -1 & \beta^T R^{-1} \\ -a(\alpha) \mathbf{1} & a(\alpha) \mathbf{1} \beta^T R^{-1} \end{pmatrix}, \end{aligned} \quad (17)$$

where $\mathbf{0}$ is a K -vector of zeros. Recall that we are assuming here that

$$D_r = \begin{pmatrix} 0 & \mathbf{0}^T \\ \mathbf{0} & rI \end{pmatrix}, \quad (18)$$

so that

$$D_r(-Q \circ A(\alpha))^{-1} = r \begin{pmatrix} 0 & \mathbf{0}^T \\ \mathbf{0} & -R^{-1} \end{pmatrix} + \frac{ra(\alpha)}{1-a(\alpha)} \begin{pmatrix} 0 & \mathbf{0}^T \\ \mathbf{1} & \mathbf{1} \beta^T (-R^{-1}) \end{pmatrix}. \quad (19)$$

Finally noting that

$$\begin{pmatrix} 0 & \mathbf{0} \\ \mathbf{a} & A \end{pmatrix} \begin{pmatrix} 0 & \mathbf{0} \\ \mathbf{b} & B \end{pmatrix} = \begin{pmatrix} 0 & \mathbf{0} \\ A\mathbf{a} & AB \end{pmatrix} \quad (20)$$

where \mathbf{a}, \mathbf{b} are some K -vectors and observing that from $\pi^T Q = 0$ we have that

$$p^T = (\pi_1, \dots, \pi_K) = \pi_0 \beta^T (-R^{-1}),$$

we now have that

$$\begin{aligned} \mu^n &= \frac{1}{1-\pi_0} (\xi^n)^T \begin{pmatrix} 0 \\ \mathbf{1} \end{pmatrix} = \frac{n!}{1-\pi_0} \pi^T \prod_{k=1}^n D_r(-Q \circ A(k))^{-1} \begin{pmatrix} 0 \\ \mathbf{1} \end{pmatrix} \\ &= \frac{n! r^n \pi_0}{1-\pi_0} \beta^T (-R^{-1}) \prod_{k=1}^n \left[\left(I + \frac{a(k)}{1-a(k)} \mathbf{1} \beta^T \right) (-R^{-1}) \right] \mathbf{1}. \end{aligned} \quad (21)$$

Finally, since $\pi^T = \pi_0(1, \beta^T(-R^{-1}))$ and since the coordinates of π sum up to one, it follows that $\pi_0(1 + \beta^T(-R^{-1})\mathbf{1}) = 1$ so that

$$\frac{\pi_0}{1 - \pi_0} = \frac{1}{\beta^T(-R^{-1})\mathbf{1}}. \quad (22)$$

This gives the following.

Theorem 4 *For a growth collapse model with linear increase with rate $r > 0$, remaining proportion after a jump with distribution not concentrated at one, with n th moment $a(n)$ and with i.i.d. inter-collapse times having the phase type distribution $F(t) = \beta^T e^{Rt} \mathbf{1}$, a stationary distribution exists and has the following n th moment:*

$$\mu^n = n!r^n \frac{\beta^T(-R^{-1})}{\beta^T(-R^{-1})\mathbf{1}} \prod_{k=1}^n \left[\left(I + \frac{a(k)}{1 - a(k)} \mathbf{1}\beta^T \right) (-R^{-1}) \right] \mathbf{1}. \quad (23)$$

Corollary 1 *If in Theorem 4, in addition the remaining proportion after a jump is zero, then the growth collapse model becomes a clearing process and the corresponding moments are:*

$$\mu^n = n!r^n \frac{\beta^T(-R^{-1})^{n+1} \mathbf{1}}{\beta^T(-R^{-1})\mathbf{1}}. \quad (24)$$

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