An alternating risk reserve process – Part I

O.J. Boxma[a,b], H. Jönsson[a], J.A.C. Resing[b], S. Shneer[a,b]

[a]EURANDOM
P.O. Box 513, 5600 MB Eindhoven, The Netherlands

[b]Eindhoven University of Technology
P.O. Box 513, 5600 MB Eindhoven, The Netherlands

Abstract

We consider an alternating risk reserve process with a threshold dividend strategy. The process can be in two different states and the state of the process can only change just after claim arrival instants. If at such an instant the capital is below the threshold, the system is set to state 1 (paying no dividend), and if the capital is above the threshold, the system is set to state 2 (paying dividend). Our interest is in the survival probabilities. In the case of exponentially distributed claim sizes, survival probabilities are found by solving a system of integro-differential equations. In the case of generally distributed claim sizes, they are expressed in the survival probabilities of the corresponding standard risk reserve processes.

1 Introduction and model description

In this paper we consider an alternating risk reserve process with a threshold dividend strategy. Dividend strategies for insurance risk models were first proposed by De Finetti [10]. There are many different types of barrier dividend strategies. In the classical *constant barrier dividend strategy*, where dividend is paid out as soon as the surplus of the insurance company reaches a constant barrier, the whole premium amount collected above the barrier is paid out as dividend, that is, the dividend intensity is equal to the premium intensity. Some recent studies of the constant barrier strategies can be found in, for example, Lin et al. [18] and Frostig [13]. A second type of dividend strategies is the *threshold strategy* where dividend is paid out with smaller intensity than the premium intensity as soon as the surplus is above a barrier. This strategy can be viewed as a generalization of the constant barrier strategy. The threshold strategy has been studied for the classical compound Poisson risk process in, among others, Asmussen [3], Lin et al. [18], Lin and Pavlova [17], and Gerber and Shiu [16]. A third type of barrier dividend strategies is the *linear barrier strategy*, where the barrier grows linearly in time and dividends are paid out with a fixed intensity whenever the surplus reaches the barrier, see, for example, Gerber [14], Albrecher et al [1], and Albrecher et al [2]. We refer the reader to Avanzi [6] for a recent review on dividend strategies.

In the model under consideration in this paper, the decision to start (or to cancel) dividend payments only takes place at claim arrival instants. Moreover, at these instants also claim size distribution and claim arrival rate can be changed. To be more precise, the risk reserve process can be either in state 1 or in state 2. If the process is in state 1 it is described by the net premium rate r_1, generic claim size C_1 with distribution $C_1(\cdot)$ and a Poisson claim arrival process with rate λ_1. If the process is in state 2 it is described by the net premium rate r_2, generic claim size C_2 with distribution $C_2(\cdot)$ and a Poisson claim arrival process with rate λ_2. For future use we define the parameters $\rho_i := \lambda_i E(C_i)/r_i$, $i = 1, 2$. Note that our analysis does not depend on the assumption that $r_2 < r_1$, although this is the case when assuming that there is dividend paid out (with rate $r_1 - r_2$) above the threshold.

If after a claim arrival the risk process is in state 1 and above the barrier $K > 0$, then it is put into state 2 (cf. τ_2 and τ_4 in Figure 1). If after a claim arrival the risk process is in state 2 and below K, then it is put into state 1 (cf. τ_1, τ_3 and τ_5 in Figure 1). Otherwise the state of the risk process is unchanged even after a claim arrival (cf. other claim arrival instants in Figure 1).

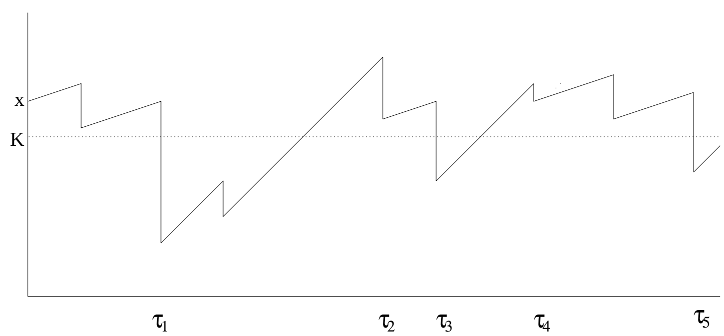

Figure 1: Sample path of the risk reserve process in the case that $r_2 < r_1$. The claim arrival instants resulting in state changes are indicated by $\tau_1, \tau_2, \ldots$

The focus of the paper is on the *survival probabilities* starting at some level x. We have to distinguish between the cases $x < K$ and $x \geq K$. If the input quantities r_i, $C_i(\cdot)$ and λ_i are the same for $i = 1$ and $i = 2$, then we are in the setting of the classical risk reserve model, for which the survival probability (1 minus the ruin probability) is well studied; see, e.g., Asmussen [4]. In particular, the following is well known (cf. [4], pp. 30-32) if $\rho_1 < 1$: the survival probability starting at x equals the steady-state probability that, in an $M/G/1$ queue with arrival rate λ_1, service time distribution $C_1(\cdot)$ and service speed r_1, the workload is less than x. This result has been extended by Asmussen and Schock Petersen [5] to a case in which the premium rate is not constant but some function of the present risk reserve. A special case thereof is that the premium rate *instantaneously* changes when the level K is crossed, the rate otherwise being constant.

We shall not restrict ourselves to the case $\rho_1 < 1$. However, we do assume that $\rho_2 < 1$. For $\rho_2 \geq 1$, the risk process is guaranteed to return below K, and eventually to go below 0; hence the survival probability is zero.

While equality between survival probability in the risk model and workload distribution in the corresponding queueing model (as in the classical risk reserve case mentioned above) does not hold in our case, we do identify strong relations between our risk reserve model and the corresponding queueing model. We refer to [7] for a detailed analysis of an $M/G/1$ queue with a switching level K and in which the service speed may be adapted right after customer arrivals. In [8] Lévy processes without negative jumps are considered, with reflection at the origin. Such processes contain as special case the compound Poisson process with negative drift, that corresponds to the workload process in the $M/G/1$ queue. In the model of [8], the Lévy exponent may be changed at arrival instants, or at Poisson observer instants, depending on the level of the process w.r.t. a barrier. It would be interesting to extend the analysis of the present paper to the case of Lévy processes without *positive* jumps, and with adaptable Lévy exponent; however, that falls outside the scope of this paper.

One motivation for studying the present model is that it is quite natural to have different premium rates in risk reserve processes below and above a certain threshold. Moreover, it is often not realistic that a change of such rates occurs instantaneously when the threshold is crossed. Furthermore, such instantaneous changes might lead to a very large number of changes per time unit, which is undesirable. Remark that we not only allow different premium rates, but also different claim arrival rates and claim size distributions. This gives much additional modeling flexibility. For example, in this way we can also model the situation in which part of the claims or part of the sizes of the claims are paid by others (due to a reinsurance contract) whenever the reserve process is below the threshold. Finally, we would like to emphasize that our model may be applied to a quite large variety of practical applications, and may also be used for, e.g., studying various storage models (where claims correspond to orders).

In a companion paper [9], we consider the same risk reserve process, with one essential difference: the state of the process may only change at arrival instants of an independent observer. Paper [9] also contains numerical results comparing the two models.

The paper is organized as follows. In Section 2 we obtain a system of integro-differential equations for the survival probabilities starting at level x. Next, in Section 3, we obtain the solution of this system in the special case that claim sizes are exponentially distributed. In Section 4 we show how the survival probabilities can be obtained in the case of general claim sizes. We do that by relating the survival probabilities in the alternating risk reserve process to the survival probabilities in the standard risk reserve process (with only one underlying state). In Section 5 we discuss the structure of the solution in the case that the claim sizes are distributed according to a mixture of exponentials. Section 6 concludes.

2 Integro-differential equations

Denote by $F_i(x)$, $i = 1, 2$, the survival probability when initially the risk process is in state i and the risk reserve is equal to x. We begin by analysing the survival probability when the risk reserve is below K. Assume that the initial capital is $x-\Delta x<K$. If the risk reserve is in state 1, analysing the survival probability over the time interval $[0,\frac{\Delta x}{r_1}]$ we obtain the following expression:

$$F_1(x-\Delta x)=(1-\frac{\lambda_1\Delta x}{r_1})F_1(x)+\frac{\lambda_1\Delta x}{r_1}\int_0^x F_1(x-y)dC_1(y)+o(\Delta x). \tag{1}$$

Here, the first term on the right-hand side corresponds to the case when no claim arrives in the interval $[0,\frac{\Delta x}{r_1}]$ while the second term corresponds to the case when a claim arrives of size smaller than x.

Dividing both sides of (1) by $-\Delta x$, rearranging the terms, and letting Δx tend to zero we get the following integro-differential equation for the survival probability $F_1(x)$ for $0<x<K$:

$$\frac{dF_1(x)}{dx}=\frac{\lambda_1}{r_1}F_1(x)-\frac{\lambda_1}{r_1}\int_0^x F_1(x-y)dC_1(y). \tag{2}$$

Equivalently, if the risk process is in state 2, analysing the survival probability over the time interval $[0,\frac{\Delta x}{r_2}]$ yields the following expression

$$F_2(x-\Delta x)=(1-\frac{\lambda_2\Delta x}{r_2})F_2(x)+\frac{\lambda_2\Delta x}{r_2}\int_0^x F_1(x-y)dC_2(y)+o(\Delta x), \tag{3}$$

which gives the following integro-differential equation for the survival probability $F_2(x)$ for $0<x<K$:

$$\frac{dF_2(x)}{dx}=\frac{\lambda_2}{r_2}F_2(x)-\frac{\lambda_2}{r_2}\int_0^x F_1(x-y)dC_2(y). \tag{4}$$

Analysing the survival probability when the initial risk reserve is above K gives in a similar manner:

$$\frac{dF_1(x)}{dx}=\frac{\lambda_1}{r_1}F_1(x)-\frac{\lambda_1}{r_1}\int_0^{x-K} F_2(x-y)dC_1(y)-\frac{\lambda_1}{r_1}\int_{x-K}^x F_1(x-y)dC_1(y), \tag{5}$$

and

$$\frac{dF_2(x)}{dx}=\frac{\lambda_2}{r_2}F_2(x)-\frac{\lambda_2}{r_2}\int_0^{x-K} F_2(x-y)dC_2(y)-\frac{\lambda_2}{r_2}\int_{x-K}^x F_1(x-y)dC_2(y). \tag{6}$$

3 Exponentially distributed claim sizes

In this section we solve the system of integro-differential equations in the special case that the claim sizes are exponentially distributed, i.e., $C_1(x)=1-e^{-\mu_1 x}$ and $C_2(x)=1-e^{-\mu_2 x}$. Let us begin with the survival probability when $0<x<K$. Integro-differential equation (2) can be written as

$$\frac{dF_1(x)}{dx}=\frac{\lambda_1}{r_1}F_1(x)-\frac{\lambda_1}{r_1}\int_0^x F_1(y)\mu_1 e^{-\mu_1(x-y)}dy. \tag{7}$$

Multiplying both sides of (7) with $e^{\mu_1 x}$ and using the fact that

$$\frac{d}{dx}(e^{\mu_1 x}F_1(x)) = \mu_1 e^{\mu_1 x}F_1(x) + e^{\mu_1 x}\frac{dF_1(x)}{dx},$$

we get

$$\frac{d}{dx}(e^{\mu_1 x}F_1(x)) = (\mu_1 + \frac{\lambda_1}{r_1})e^{\mu_1 x}F_1(x) - \frac{\lambda_1\mu_1}{r_1}\int_0^x e^{\mu_1 y}F_1(y)dy. \tag{8}$$

Let $G_1(x) := e^{\mu_1 x}F_1(x)$. We can now rewrite (8) using $G_1(x)$

$$\frac{dG_1(x)}{dx} = (\mu_1 + \frac{\lambda_1}{r_1})G_1(x) - \frac{\lambda_1\mu_1}{r_1}\int_0^x G_1(y)dy. \tag{9}$$

Differentiating (9) once yields

$$\frac{d^2G_1(x)}{dx^2} = (\mu_1 + \frac{\lambda_1}{r_1})\frac{dG_1(x)}{dx} - \frac{\lambda_1\mu_1}{r_1}G_1(x). \tag{10}$$

Thus we have a second order linear equation with constant coefficients. The characteristic equation

$$k^2 - (\mu_1 + \frac{\lambda_1}{r_1})k + \frac{\lambda_1\mu_1}{r_1} = 0$$

has two distinct roots $k_1 = \mu_1$ and $k_2 = \frac{\lambda_1}{r_1}$, and the solution to (10) is given by

$$G_1(x) = C_{11}e^{\mu_1 x} + C_{12}e^{\frac{\lambda_1}{r_1}x}, \quad 0 < x < K.$$

Consequently the solution to (7) is

$$F_1(x) = C_{11} + C_{12}e^{-(\mu_1 - \frac{\lambda_1}{r_1})x}, \quad 0 < x < K. \tag{11}$$

To find the survival probability $F_2(x)$ for $0 < x < K$ we need to solve integro-differential equation (4) with $C_2(x) = 1 - e^{-\mu_2 x}$:

$$\begin{aligned} \frac{dF_2(x)}{dx} &= \frac{\lambda_2}{r_2}F_2(x) - \frac{\lambda_2}{r_2}\int_0^x F_1(x-y)\mu_2 e^{-\mu_2 y}dy \qquad (12) \\ &= \frac{\lambda_2}{r_2}F_2(x) - \frac{\lambda_2}{r_2}\left[C_{11} + (A_1 - C_{11})e^{-\mu_2 x} - A_1 e^{-(\mu_1 - \frac{\lambda_1}{r_1})x}\right], \end{aligned}$$

with

$$A_1 = \frac{\mu_2}{\mu_1 - \mu_2 - \frac{\lambda_1}{r_1}}C_{12}. \tag{13}$$

Thus we have a first order inhomogeneous linear equation with constant coefficients. The solution of this equation is given by

$$F_2(x) = C_{21}e^{\frac{\lambda_2}{r_2}x} + C_{11} + A_2 e^{-\mu_2 x} + A_3 e^{-(\mu_1 - \frac{\lambda_1}{r_1})x}, \quad 0 < x < K, \tag{14}$$

with

$$A_2 = \frac{\frac{\lambda_2}{r_2}(A_1 - C_{11})}{\frac{\lambda_2}{r_2} + \mu_2}, \quad A_3 = \frac{\frac{\lambda_2}{r_2} A_1}{\frac{\lambda_1}{r_1} - \mu_1 - \frac{\lambda_2}{r_2}}. \tag{15}$$

The constants C_{11}, C_{12} and C_{21} should be determined by the boundary conditions.

Let us now continue with the survival probabilities for initial values x above K described by differential equation (5) and (6). We begin with (6). Using the assumption that the claims are exponentially distributed we get

$$\frac{dF_2(x)}{dx} = \frac{\lambda_2}{r_2} F_2(x) - \frac{\lambda_2}{r_2} \int_K^x F_2(y) \mu_2 e^{-\mu_2(x-y)} dy - \frac{\lambda_2}{r_2} \int_0^K F_1(y) \mu_2 e^{-\mu_2(x-y)} dy, \tag{16}$$

where $F_1(x)$ in the second integral on the right hand side is the survival probability for $0 < x < K$ given in (11). Multiply both sides of (16) by $e^{\mu_2 x}$ and define $G_2(x) := e^{\mu_2 x} F_2(x)$ so that we get

$$\frac{dG_2(x)}{dx} = (\mu_2 + \frac{\lambda_2}{r_2}) G_2(x) - \frac{\lambda_2 \mu_2}{r_2} \int_K^x G_2(y) dy - \frac{\lambda_2 \mu_2}{r_2} \int_0^K F_1(y) e^{\mu_2 y} dy. \tag{17}$$

Differentiating (17) once we get a second order linear equation

$$\frac{d^2 G_2(x)}{dx^2} = (\mu_2 + \frac{\lambda_2}{r_2}) \frac{dG_2(x)}{dx} - \frac{\lambda_2 \mu_2}{r_2} G_2(x). \tag{18}$$

This equation has the same structure as (10) and we can immediately write down the solution

$$G_2(x) = D_{21} e^{\mu_2 x} + D_{22} e^{\frac{\lambda_2}{r_2} x} \quad x > K.$$

Consequently the solution to (16) is

$$F_2(x) = D_{21} + D_{22} e^{-(\mu_2 - \frac{\lambda_2}{r_2})x} \quad x > K. \tag{19}$$

For the survival probability $F_1(x)$ we rewrite the integro-differential equation (5) as

$$\frac{dF_1(x)}{dx} = \frac{\lambda_1}{r_1} F_1(x) - \frac{\lambda_1}{r_1} \int_K^x F_2(y) \mu_1 e^{-\mu_1(x-y)} dy - \frac{\lambda_1}{r_1} \int_0^K F_1(y) \mu_1 e^{-\mu_1(x-y)} dy, \tag{20}$$

where we now can use the fact that we know $F_2(y)$ for $y > K$ via (19) and $F_1(y)$ for $0 < y < K$ via (11). Thus we have a first order inhomogeneous linear equation with constant coefficients. The solution of this equation is given by

$$F_1(x) = D_{11} e^{\frac{\lambda_1}{r_1} x} + B_1 + B_2 e^{-(\mu_2 - \frac{\lambda_2}{r_2})x} + B_3 e^{-\mu_1 x}, \quad x > K. \tag{21}$$

The constants B_1, B_2 and B_3 are given by

$$B_1 = D_{21},$$

$$B_2 = \frac{\mu_1 \frac{\lambda_1}{r_1}}{\left(\frac{\lambda_1}{r_1} - \frac{\lambda_2}{r_2} + \mu_2\right)\left(\mu_1 + \frac{\lambda_2}{r_2} - \mu_2\right)} D_{22},$$

$$B_3 = \frac{\frac{\lambda_1}{r_1} C_{11}\left(e^{\mu_1 K} - 1\right) + \mu_1 C_{12}\left(e^{\frac{\lambda_1}{r_1} K} - 1\right) - \frac{\lambda_1}{r_1} D_{21} e^{\mu_1 K} - \frac{\frac{\lambda_1}{r_1}\mu_1 D_{22}}{\mu_1 + \frac{\lambda_2}{r_2} - \mu_2} e^{\left(\mu_1 + \frac{\lambda_2}{r_2} - \mu_2\right) K}}{\frac{\lambda_1}{r_1} + \mu_1}.$$

Again, the constants D_{11}, D_{21} and D_{22} should be determined by the boundary conditions.

3.1 Boundary Conditions

In the four expressions for the survival probabilities (11), (14), (19), and (21) we have in total six unknown constants. To find these constants we set up six boundary conditions. The first two are

B1 $\lim_{x\to\infty} F_1(x) = 1$,

B2 $\lim_{x\to\infty} F_2(x) = 1$.

The next two boundary conditions follow from the fact that at the barrier K the probabilities of surviving should be continuous, that is

B3 $F_1(K^-) = F_1(K^+)$,

B4 $F_2(K^-) = F_2(K^+)$.

The fifth boundary condition comes from the behaviour of the derivative of $F_1(x)$ at $x = 0$. From (2) we get

B5 $F_1'(0^+) = \frac{\lambda_1}{r_1} F_1(0^+)$.

Finally, the sixth boundary condition comes from the behaviour of the derivative of $F_2(x)$ at $x = K$. From (4), (6) and condition **B4** we get

B6 $F_2'(K^-) = F_2'(K^+)$.

Conditions **B5** for the derivative of $F_1(x)$ at $x = 0$ and **B6** for the derivative of $F_2(x)$ at $x = K$ are required since (7) for $F_1(x)$, $x < K$, and (16) for $F_2(x)$, $x > K$, give rise to *second* order differential equations.

Conditions **B1** and **B2** imply that $D_{11} = 0$ and $D_{21} = 1$. Condition **B5** implies that $C_{12} = -\frac{\lambda_1}{r_1 \mu_1} C_{11}$. Conditions **B3**, **B4** and **B6** can be used to determine the three remaining constants C_{11}, C_{21} and D_{22}.

4 Generally distributed claim sizes

In this section, we shall call the process under consideration "original process" and a standard risk process - with premium rate equal to r_i, claim arrival process with parameter λ_i and distribution of claims C_i - is called "standard process in state i", for $i = 1, 2$. In the case of generally distributed claim sizes, we shall relate the survival probabilities for the original process to the survival probabilities of the standard processes in state 1 and 2. As mentioned in the introduction, these latter survival probabilities are extensively studied in the literature.

4.1 $F_1(x)$ for $x < K$.

To find the survival probability in this case, note that the process starts in state 1 at a level below K and its state will not change at least until it reaches level K. Taking this into account, one can apply arguments similar to those used in the proof of [3, Chapter VII, Proposition 1.10] to obtain

$$F_1(x) = \widetilde{F}_1(x)\frac{F_1(K)}{\widetilde{F}_1(K)}, \tag{22}$$

where $\widetilde{F}_1(y)$ denotes the survival probability for the standard process in state 1 starting at level y. Hence, it remains to find the value of $F_1(K)$, which is done in Subsection 4.4. We refer to Section 4 of the companion paper [9] for a more detailed discussion of (2) and its solution (22). Formula (2) also appears in the model studied there, with state changes at Poisson observer epochs.

4.2 $F_2(x)$ for $x \geq K$.

We exploit here the following idea: the original process starts at a level $x \geq K$ at state 2 and will stay in this state before the time it will first cross level K. Until this time, the process follows the trajectory of a standard process in state 2 starting at level $x - K$. Hence, the original process "survives" in either of the two following cases: (i) the standard process in state 2 starting at level $x - K$ never hits the negative half-line, or (ii) the standard process in state 2 starting at level $x - K$ hits the negative half-line, but its first negative value (we will denote it by $-T_{x-K}$ so that T_{x-K} is the quantity that is usually referred to in the literature as the deficit at ruin) is in the interval $(-K, 0)$, and the original process starting at the level $K - T_{x-K}$ "survives". Note that in the second scenario the probability of survival after the standard process in state 2 has reached the negative half-line is equal to $F_1(K - T_{x-K})$ (since only a claim arrival can cause the process to go downwards, and as soon as the standard process in state 2 goes below the level 0, the state of the original process changes to 1) and is a known quantity due to the previous subsection.

Formally,

$$F_2(x) = \widetilde{F}_2(x - K) + (1 - \widetilde{F}_2(x - K)) \int_0^K F_1(K - y) d\mathsf{P}(T_{x-K} \leq y), \tag{23}$$

where $x \geq K$ and $\widetilde{F}_2(y)$ denotes the survival probability in the standard risk model starting at level y in state 2.

Remark 1. *The random variable T_y was investigated in a number of papers. Its Laplace transform may be found in [15]. In some particular cases it is also possible to find an explicit formula for its distribution (see, e.g. [11] and [12]).*

Remark 2. *In the case $x = K$ Formula (23) may be simplified with the use of the following well-known fact (see, e.g. [3, Chapter III, Theorem 2.2]):*

$$\mathsf{P}(T_0 < y) = \frac{1}{\mathsf{E}C_2} \int_{z=0}^{y} (1 - C_2(z))dz \text{ for all } y > 0.$$

Using this, we obtain

$$\begin{aligned} F_2(K) &= \widetilde{F}_2(0) + \frac{1}{\mathsf{E}C_2}(1 - \widetilde{F}_2(0)) \int_0^K (1 - C_2(y))F_1(K - y)dy \\ &= 1 - \frac{\lambda_2 \mathsf{E}C_2}{r_2} + \frac{\lambda_2}{r_2} \int_0^K (1 - C_2(y))F_1(K - y)dy. \end{aligned}$$

Note that in the last equality we also used the fact that $\widetilde{F}_2(0) = 1 - \frac{\lambda_2 \mathsf{E}C_2}{r_2}$ (see, e.g. [3, Chapter III, Corollary 3.1]

4.3 $F_2(x)$ for $x < K$.

Consider (4):

$$\frac{dF_2(x)}{dx} = \frac{\lambda_2}{r_2} F_2(x) - \frac{\lambda_2}{r_2} \int_0^x F_1(x - y)dC_2(y).$$

In the second term on the RHS the range of $x-y$ is $(0, x)$ and since the function F_1 in this range is known, one can use here the standard method of solving an inhomogeneous first-order differential equation. We represent $F_2(x)$ as $C(x)e^{\lambda_2 x/r_2}$ and obtain a differential equation satisfied by $C(x)$:

$$C'(x) = -\frac{\lambda_2}{r_2} e^{-\lambda_2 x/r_2} \int_0^x F_1(x - y)dC_2(y),$$

and hence,

$$C(x) = C_1 - \frac{\lambda_2}{r_2} \int_0^x e^{-\lambda_2 z/r_2} \int_0^z F_1(z - y)dC_2(y)dz,$$

where C_1 is a constant. This implies that

$$F_2(x) = C_1 e^{\lambda_2 x/r_2} - \frac{\lambda_2}{r_2} \int_0^x e^{\lambda_2 (x-z)/r_2} \int_0^z F_1(z - y)dC_2(y)dz.$$

To find C_1, use the continuity of the function F_2 at point K ($F_2(K^-) = F_2(K^+)$):

$$C_1 = e^{-\lambda_2 K/r_2} F_2(K) + \frac{\lambda_2}{r_2} \int_0^K e^{-\lambda_2 z/r_2} \int_0^z F_1(z - y)dC_2(y)dz,$$

and, finally, after re-arranging terms,

$$F_2(x) = e^{-\lambda_2(K-x)/r_2} F_2(K) + \frac{\lambda_2}{r_2} \int_x^K e^{\lambda_2(x-z)/r_2} \int_0^z F_1(z-y) dC_2(y) dz.$$

Note here that $F_2(K)$ has already been obtained (see Remark 2). An intuitive interpretation of the formula above is as follows. The first term corresponds to the case when there is no claim before the process starting at level x reaches level K (the probability of this event is equal to $e^{-\lambda_2(K-x)/r_2}$, since the slope in this case is always equal to r_2, and the probability of survival conditioned on this event happening is equal to $F_2(K)$). The second term corresponds to the case when a claim arrives when the process is at a level z between x and K.

4.4 $F_1(x)$ for $x \geq K$.

Consider (5):

$$\frac{dF_1(x)}{dx} = \frac{\lambda_1}{r_1} F_1(x) - \frac{\lambda_1}{r_1} \int_0^{x-K} F_2(x-y) dC_1(y) - \frac{\lambda_1}{r_1} \int_{x-K}^x F_1(x-y) dC_1(y)$$

for $x \geq K$. Note that the range of $x-y$ in the first integral on the RHS is (K, x) (the function F_2 is known in this range), and the range of $x-y$ in the second integral on the RHS is $(0, K)$ (the function F_1 is known in this range). Hence, one can use the same methods as in the previous subsection to solve an inhomogeneous first-order differential equation:

$$\begin{aligned} F_1(x) = D_1 e^{\lambda_1 x/r_1} &- \frac{\lambda_1}{r_1} \int_0^x e^{\lambda_1(x-z)/r_1} \int_0^{z-K} F_2(z-y) dC_1(y) dz \\ &- \frac{\lambda_1}{r_1} \int_0^x e^{\lambda_1(x-z)/r_1} \int_{z-K}^z F_1(z-y) dC_1(y) dz, \end{aligned}$$

where D_1 is a constant. To find it, first re-write the expression above as

$$\begin{aligned} F_1(x) = e^{\lambda_1 x/r_1} \Bigg(D_1 &- \frac{\lambda_1}{r_1} \int_0^x e^{-\lambda_1 z/r_1} \int_0^{z-K} F_2(z-y) dC_1(y) dz \\ &- \frac{\lambda_1}{r_1} \int_0^x e^{-\lambda_1 z/r_1} \int_{z-K}^z F_1(z-y) dC_1(y) dz \Bigg) \end{aligned}$$

and then use the fact that $F_1(x) \to 1$ as $x \to \infty$:

$$D_1 = \frac{\lambda_1}{r_1} \int_0^\infty e^{-\lambda_1 z/r_1} \int_0^{z-K} F_2(z-y) dC_1(y) dz + \frac{\lambda_1}{r_1} \int_0^\infty e^{-\lambda_1 z/r_1} \int_{z-K}^z F_1(z-y) dC_1(y) dz,$$

and hence,

$$F_1(x) = \frac{\lambda_1}{r_1} \int_x^\infty e^{\lambda_1(x-z)/r_1} \int_0^{z-K} F_2(z-y) dC_1(y) dz + \frac{\lambda_1}{r_1} \int_x^\infty e^{\lambda_1(x-z)/r_1} \int_{z-K}^z F_1(z-y) dC_1(y) dz. \tag{24}$$

As in the previous subsection, the formula above has an intuitive explanation. One should just think of conditioning on the level z (between x and ∞) at which the first claim arrives.

Remark 3. *The formula above may be used for finding $F_1(K)$, which will complete the task of finding the survival probabilities in the general case. To find $F_1(K)$, one should use formulas for $F_1(x)$ for $x < K$ from subsection 4.1 and for $F_2(x)$ for $x \geq K$ from subsection 4.2. Once these values are plugged into (24), all but one terms in the obtained equation depend on $F_1(K)$, and the remaining term is a constant, hence, the value of $F_1(K)$ may be computed.*

5 Claim sizes distributed as mixtures of exponentials

In this section we assume that the claim size distributions have the form

$$C_j(x) = 1 - \sum_{i=1}^{L_j} c_{j,i} e^{s_{j,i}x}, \quad j = 1, 2. \tag{25}$$

In (25), $s_{j,i} < 0$ for all j and i, and $\sum_{i=1}^{L_j} c_{j,i} = 1$ for $j = 1, 2$. We will show how to use the results of the previous section to find ruin probabilities in this case. Hereby, we will use the results of [15].

- $F_1(x)$ for $x < K$.

  From, e.g., Formula (18) in [15] it follows that in this case

  $$\widetilde{F}_1(x) = 1 - \sum_{i=1}^{L_1} D_{1,i} e^{q_{1,i}x},$$

  where $q_{1,i}$ are solutions of the equation

  $$\frac{\lambda_1}{r_1} \sum_{i=1}^{L_1} \frac{c_{1,i}}{q - s_{j,i}} = 1$$

  (see (14) in [15]), and $D_{1,i}$ may be found using Formula (15) from [15].

  It is now clear from (22) that $F_1(x)$ for $x < K$ is a constant plus a mixture of L_1 exponentials.

- $F_2(x)$ for $x \geq K$. We will use here (23):

$$\begin{aligned} F_2(x) &= \widetilde{F}_2(x-K) + (1 - \widetilde{F}_2(x-K)) \int_0^K F_1(K-y) d\mathsf{P}(T_{x-K} \leq y) \\ &= \widetilde{F}_2(x-K) + \int_0^K F_1(K-y) g(x-K, y) dy, \end{aligned} \tag{26}$$

where function g is defined via (2) and (3) in [15]. Now we need to make use of the facts that, analogously to the previous case,

$$\widetilde{F}_2(x-K) = 1 - \sum_{i=1}^{L_2} D_{2,i} e^{q_{2,i}(x-K)}$$

and that (from Formula (16) in [15])

$$g(x-K, y) = \sum_{i=1}^{L_2} \sum_{k=1}^{L_2} D_{2,i,k} e^{s_{2,i} y} e^{q_{2,k}(x-K)}.$$

Plugging the two expressions above into (26) (and taking into account the expression for $F_1(x)$ for $x < K$), yields that in our case $F_2(x)$ for $x \geq K$ is a constant plus a mixture of L_2 exponentials.

- Remaining cases.

  In both remaining cases ($F_2(x)$ for $x < K$ and $F_1(x)$ for $x \geq K$) one should use results of the cases already considered and the last expressions from Sections 4.3 and 4.4, respectively. After some tedious computations, one obtains that $F_2(x)$ for $x < K$ is a constant plus a mixture of $L_1 + L_2 + 1$ exponentials, and $F_1(x)$ for $x \geq K$ is a constant plus a mixture of $L_1 + L_2$ exponentials.

6 Conclusion

We considered a risk reserve process with a threshold dividend strategy which can be in two different states and which can only change state at claim arrival instants. In the companion paper [9] we will look at a similar model in which the process can only change state at the arrival instants of an independent Poisson observer. In [9] we will also present numerical results for both models.

References

[1] H. Albrecher, J. Hartinger, R. F. Tichy (2005). On the distribution of dividend payments and the discounted penalty function in a risk model with linear dividend barrier. *Scandinavian Actuarial Journal*, 103-126.

[2] H. Albrecher, J. Hartinger, S.Thonhauser (2007). On exact solutions for dividend strategies of threshold and linear barrier type in a Sparre Andersen model. *ASTIN Bulletin,* **37**(2), 203-233.

[3] S. Asmussen (2000). *Ruin Probabilities.* World Scientific Publishing Company, Singapore.

[4] S. Asmussen (2001). *Applied Probability and Queues.* Springer, New York, 2nd ed.

[5] S. Asmussen, S. Schock Petersen (1988). Ruin probabilities expressed in terms of storage processes. *Advances in Applied Probability* **20**, 913-916.

[6] B. Avanzi (2008). Strategies for dividend distribution: a review. To appear in: *North American Actuarial Journal*.

[7] R. Bekker, O.J. Boxma (2007). An $M/G/1$ queue with adaptable service speed. *Stochastic Models* **23**, 373-396.

[8] R. Bekker, O.J. Boxma, J.A.C. Resing (2009). Lévy processes with adaptable exponent. *Advances in Applied Probability* **41**, 177-205.

[9] O.J. Boxma, H. Jönsson, J.A.C. Resing, S. Shneer. An alternating risk reserve process – Part II. *EURANDOM report 2009-010*.

[10] B. De Finetti (1957). Su unimpostazione alternative della teoria collective del risicho. *Proceedings of Transactions of the XV International Congress of Actuaries*, **2**, 433-443.

[11] S. Drekic, D. Dickson, D. Stanford, G. Willmot (2004). On the distribution of the deficit at ruin when claims are phase-type. *Scandinavian Actuarial Journal*, 105-120.

[12] F. Dufresne, H. U. Gerber (1988). The probability and severity of ruin for combinations of exponential claim amount distributions and their translations. *Insurance: Mathematics and Economics* **7**, 75-80.

[13] E. Frostig (2005). The expected time to ruin in a risk process with constant barrier via martingales. *Insurance: Mathematics and Economics*, **37**, 216-228.

[14] H. U. Gerber (1974). The dilemma between dividends and safety and a generalization of the Lundberg-Cramér formulas. *Scandinavian Actuarial Journal*, 46-57.

[15] H. U. Gerber, M.J. Goovaerts, R. Kaas (1987). On the probability and severity of ruin. *Astin Bulletin* **17**, 151-163.

[16] H. U. Gerber, E. S. W. Shiu (2006). On optimal dividend strategies in the compound Poisson model. *North American Actuarial Journal*, **10**, Iss. 2, 76-93.

[17] X. S. Lin, K. P. Pavlova (2006). The compound Poisson risk model with a threshold dividend strategy. *Insurance: Mathematics and Economics*, **38**, 57-80.

[18] X. S. Lin, G. E. Willmot, S. Drekic (2003). The classical risk model with a constant dividend barrier: analysis of the Gerber-Shiu discounted penalty function. *Insurance: Mathematics and Economics*, **33**, 551-566.