An alternating risk reserve process – Part II

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Abstract

We consider an alternating risk reserve process with a threshold dividend strategy. The process can be in two different states and the state of the process can only change at the arrival instants of an independent Poisson observer. Whether or not a change then occurs depends on the value of the risk reserve w.r.t. the barrier. If at such an instant the capital is below the threshold, the system is set to state 1 (paying no dividend), and if the capital is above the threshold, the system is set to state 2 (paying dividend). In each of the two states, the process is described by different premium rates, Poisson claim arrival intensities, and claim size distributions. For this model we determine the survival probabilities, distinguishing between the initial state being 1 or 2, and the process starting below or above the barrier. In the case of exponentially distributed claim sizes, survival probabilities are found by solving a system of integro-differential equations. In the case of generally distributed claim sizes, they are expressed in the survival probabilities of the corresponding standard risk reserve processes. We perform several numerical experiments, including a comparison with the case in which state changes can only occur just after claim arrival instants; that case is treated in Part I.

1 Independent Poisson Observer

In this paper we consider the alternating risk reserve process with a threshold dividend strategy. In the classical threshold dividend strategy model(s), dividend is paid out with a rate less than or equal to the premium rate as soon as the risk reserve process is above a certain threshold or barrier. In our model, the decision to start (or to cancel) dividend payments only takes place at so-called observer instants; these instants occur according to a Poisson process. Moreover, at these observer instants also claim size distribution and claim arrival rate can be changed.

To be more precise, the risk reserve process can be either in state 1 or in state 2. If the process is in state 1 it is described by the premium rate r_1, the generic claim size C_1 with distribution $C_1(\cdot)$ and a Poisson claim arrival process with rate λ_1. If the process is in state 2 it is described by the premium rate r_2, the generic claim size C_2 with distribution $C_2(\cdot)$ and a Poisson claim arrival process with rate λ_2. For future use we define the parameters $\rho_i := \lambda_i E(C_i)/r_i$, $i = 1, 2$. Note that our analysis does not depend on the assumption that $r_2 < r_1$, although this is the case when assuming that there is dividend paid out above the threshold.

If the observer finds the risk process in state 1 and above the barrier $K > 0$ the risk process is put into state 2 (cf. τ_4 and τ_6 in Figure 1). If the observer finds the risk process in state 2 and below K the risk process is put into state 1 (cf. τ_2 and τ_5 in Figure 1). Otherwise the risk process is unchanged by the observer (cf. τ_1 and τ_3 in Figure 1). We further assume that the observation instants occur at rate ν_1 or ν_2 depending on whether the risk process was below or above K at the last observation.

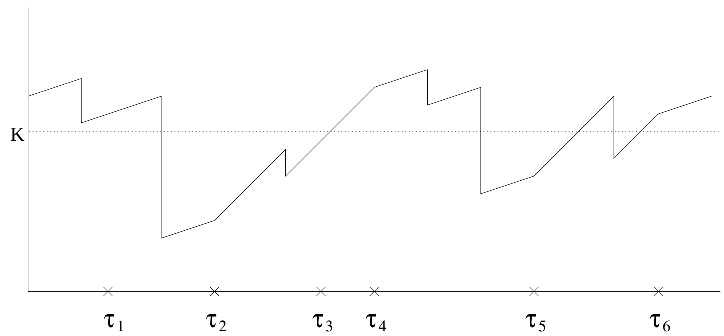

Figure 1: Sample path of the risk reserve process in the case that $r_2 < r_1$. The observer instants are indicated as $\tau_1, \tau_2, \ldots$.

The focus of the paper is on the *survival probabilities* starting at some level x. We have to distinguish between the cases $x < K$ and $x \geq K$. We do not assume that $\rho_1 < 1$. However, we do assume that $\rho_2 < 1$ since for $\rho_2 \geq 1$, the risk process is guaranteed to return below K, and eventually to go below 0; hence the survival probability is zero.

In a companion paper [5], we have considered the same risk reserve process, with one essential difference: a change can only occur right after claim arrival instants. We refer the reader to the introduction in [5] for a literature account on dividend strategies. We shall sometimes refer to the model with changes at claim arrival instants as the ACAI model, whereas the independent Poisson observer model is referred to as IPO model. Both models may arise quite naturally, and together they offer much modelling flexibility. If the arrival rate of the Poisson observer in IPO is taken infinitely large, the classical model with instantaneous change of premium rate is retrieved. The feature of different claim arrival rates and different claim size distributions in states 1 and 2 allows us, e.g., to represent reinsurance whenever the reserve process is below a certain threshold.

The approach we have taken in [5] is more probabilistic, whereas in the present paper we rely more heavily on analytic methods and Laplace-Stieltjes transforms. On the one

hand we felt it was more interesting to use another approach, and on the other hand the IPO model seems slightly more complicated than the ACAI model, and the probabilistic approach seems harder than for ACAI.
The paper is organized as follows. In Section 2 we derive integro-differential equations for the survival probabilities when the initial risk reserve equals x below, respectively above, K. These equations are solved in Section 3 for the case of exponential claim size distributions. Subsection 3.2 contains numerical results via which the survival probabilities in our model are compared with those in the model with state changes at claim arrival instants [5]. The case of general claim size distributions is studied in Section 4. The structure of the survival probabilities is considered in more detail, in Section 5, for the case that the claim sizes have a rational Laplace-Stieltjes transform, i.e., claim sizes are distributed according to a mixture of exponentials. Section 6 contains some suggestions for future research.

2 Integro-Differential Equations of the Survival Probabilities

Denote by $F_i(x)$, $i = 1, 2$, the probability of survival (i.e., no ruin) if the initial risk reserve is x and the risk process is in state i. It is our goal in this section to determine these survival probabilities.

We begin by analysing the survival probability when the risk reserve is below K. Assume that the initial capital is $x - \Delta x < K$. If the risk reserve is in state 1, analysing the survival probability over the small time interval $[0, \frac{\Delta x}{r_1}]$ we have the following expression:

$$\begin{aligned} F_1(x-\Delta x) &= (1-\tfrac{\lambda_1 \Delta x}{r_1})(1-\tfrac{\nu_1 \Delta x}{r_1})F_1(x) \\ &+\tfrac{\lambda_1 \Delta x}{r_1}\int_0^x F_1(x-y)dC_1(y) \\ &+\tfrac{\nu_1 \Delta x}{r_1}F_1(x) + o(\Delta x). \end{aligned} \tag{1}$$

The first term on the right hand side corresponds to the case when no claim arrives and the observer does not arrive either. The second term corresponds to the case when there is a claim arriving but no observer. And the third term corresponds to the case when there is no claim but the observer arrives.

Subtracting $F_1(x)$ from both sides, subsequently dividing both sides of (1) by $-\Delta x$, rearranging the terms, and letting Δx tend to zero we get the following integro-differential equation for the survival probability $F_1(x)$ for $0 < x < K$:

$$\frac{dF_1(x)}{dx} = \frac{\lambda_1}{r_1}F_1(x) - \frac{\lambda_1}{r_1}\int_0^x F_1(x-y)dC_1(y). \tag{2}$$

Equivalently, if the risk process is in state 2, analysing the survival probability over the time interval $[0, \frac{\Delta x}{r_2}]$ yields the integro-differential equation for the survival probability $F_2(x)$ for $0 < x < K$:

$$\frac{dF_2(x)}{dx} = \frac{\lambda_2+\nu_2}{r_2}F_2(x) - \frac{\lambda_2}{r_2}\int_0^x F_2(x-y)dC_2(y) - \frac{\nu_2}{r_2}F_1(x). \tag{3}$$

Analysing the survival probability when the initial risk reserve is above K gives in a similar manner:

$$\frac{dF_1(x)}{dx} = \frac{\lambda_1+\nu_1}{r_1}F_1(x) - \frac{\lambda_1}{r_1}\int_0^x F_1(x-y)dC_1(y) - \frac{\nu_1}{r_1}F_2(x), \tag{4}$$

and

$$\frac{dF_2(x)}{dx} = \frac{\lambda_2}{r_2}F_2(x) - \frac{\lambda_2}{r_2}\int_0^x F_2(x-y)dC_2(y). \tag{5}$$

3 Exponential Claims

In this section we assume that the claims are exponentially distributed, that is, $C_1(x) = 1 - e^{-\mu_1 x}$ and $C_2(x) = 1 - e^{-\mu_2 x}$. For this case we are able to derive explicit expressions for the survival probabilities $F_1(x)$ and $F_2(x)$. Furthermore, the insight obtained in this section will be helpful in tackling the case of generally distributed claims in the next section.

Let us begin with the survival probability when $0 < x < K$. Integro-differential Equation (2) can be written as

$$\frac{dF_1(x)}{dx} = \frac{\lambda_1}{r_1}F_1(x) - \frac{\lambda_1}{r_1}\int_0^x F_1(y)\mu_1 e^{-\mu_1(x-y)}dy. \tag{6}$$

The solution to (6) is derived in exactly the same way as for the corresponding survival probability in the model described in [5] (cf. Equation (7) in [5]) and is

$$F_1(x) = C_{11} + C_{12}e^{-(\mu_1 - \frac{\lambda_1}{r_1})x}, \quad 0 < x < K. \tag{7}$$

To find the survival probability $F_2(x)$ for $0 < x < K$ we need to solve integro-differential Equation (3) with $C_2(x) = 1 - e^{-\mu_2 x}$:

$$\frac{dF_2(x)}{dx} = \frac{\lambda_2+\nu_2}{r_2}F_2(x) - \frac{\lambda_2}{r_2}\int_0^x F_2(y)\mu_2 e^{-\mu_2(x-y)}dy - \frac{\nu_2}{r_2}F_1(x). \tag{8}$$

Multiplying both sides of (8) with $e^{\mu_2 x}$ and letting $G_2(x) := e^{\mu_2 x}F_2(x)$ we get

$$\frac{dG_2(x)}{dx} = (\mu_2 + \frac{\lambda_2+\nu_2}{r_2})G_2(x) - \frac{\lambda_2\mu_2}{r_2}\int_0^x G_2(y)dy - \frac{\nu_2}{r_2}e^{\mu_2 x}F_1(x). \tag{9}$$

Differentiating (9) once gives

$$\frac{d^2G_2(x)}{dx^2} = (\mu_2 + \frac{\lambda_2+\nu_2}{r_2})\frac{dG_2(x)}{dx} - \frac{\lambda_2\mu_2}{r_2}G_2(x) - \frac{\nu_2}{r_2}\frac{d}{dx}(e^{\mu_2 x}F_1(x)). \tag{10}$$

Thus we have a second order inhomogeneous linear equation with constant coefficients.

For the homogeneous solution of (10) we have the characteristic equation

$$k^2 - (\mu_2 + \frac{\lambda_2 + \nu_2}{r_2})k + \frac{\lambda_2\mu_2}{r_2} = 0, \tag{11}$$

with two distinct roots

$$k_1 = \frac{1}{2}(\mu_2 + \frac{\lambda_2 + \nu_2}{r_2}) + \frac{1}{2}\sqrt{(\mu_2 + \frac{\lambda_2 + \nu_2}{r_2})^2 - 4\frac{\lambda_2\mu_2}{r_2}}, \tag{12}$$

and

$$k_2 = \frac{1}{2}(\mu_2 + \frac{\lambda_2 + \nu_2}{r_2}) - \frac{1}{2}\sqrt{(\mu_2 + \frac{\lambda_2 + \nu_2}{r_2})^2 - 4\frac{\lambda_2\mu_2}{r_2}}. \tag{13}$$

So the homogeneous solution of (10) is

$$G_{2h}(x) = C_{21}e^{k_1 x} + C_{22}e^{k_2 x}, \quad 0 < x < K. \tag{14}$$

To find the particular solution of (10) we first note that

$$\frac{d}{dx}(e^{\mu_2 x}F_1(x)) = \mu_2 C_{11}e^{\mu_2 x} + C_{12}(\mu_2 - \mu_1 + \frac{\lambda_1}{r_1})e^{(\mu_2 - \mu_1 + \frac{\lambda_1}{r_1})x}.$$

Since this is a sum with two terms we can find a particular solution to (10) for each term, that is, we want to find particular solutions to

$$\frac{d^2G_2(x)}{dx^2} - (\mu_2 + \frac{\lambda_2 + \nu_2}{r_2})\frac{dG_2(x)}{dx} + \frac{\lambda_2\mu_2}{r_2}G_2(x) = -\frac{\nu_2\mu_2}{r_2}C_{11}e^{\mu_2 x}, \tag{15}$$

respectively,

$$\frac{d^2G_2(x)}{dx^2} - (\mu_2 + \frac{\lambda_2 + \nu_2}{r_2})\frac{dG_2(x)}{dx} + \frac{\lambda_2\mu_2}{r_2}G_2(x) = -\frac{\nu_2}{r_2}C_{12}(\mu_2 - \mu_1 + \frac{\lambda_1}{r_1})e^{(\mu_2 - \mu_1 + \frac{\lambda_1}{r_1})x}, \tag{16}$$

for $0 < x < K$.

For Equation (15) we try a solution of the form $G_{2p}(x) = A_1 e^{\mu_2 x}$. Inserting this into (15),

$$[\mu_2^2 - (\mu_2 + \frac{\lambda_2 + \nu_2}{r_2})\mu_2 + \frac{\lambda_2\mu_2}{r_2}]A_1 e^{\mu_2 x} = -\frac{\nu_2\mu_2}{r_2}C_{11}e^{\mu_2 x}.$$

Hence $A_1 = C_{11}$.

For Equation (16) we try a solution of the form $G_{2p}(x) = A_2 e^{(\mu_2 - \mu_1 + \frac{\lambda_1}{r_1})x}$. Inserting this into (16) gives $A_2 = -AC_{12}$, where

$$A = \frac{\frac{\nu_2}{r_2}(\mu_2 - \mu_1 + \frac{\lambda_1}{r_1})}{(\mu_2 - \mu_1 + \frac{\lambda_1}{r_1})^2 - (\mu_2 + \frac{\lambda_2 + \nu_2}{r_2})(\mu_2 - \mu_1 + \frac{\lambda_1}{r_1}) + \frac{\lambda_2\mu_2}{r_2}}. \tag{17}$$

The solution of (10) is therefore

$$G_2(x) = C_{21}e^{k_1 x} + C_{22}e^{k_2 x} + C_{11}e^{\mu_2 x} - AC_{12}e^{(\mu_2 - \mu_1 + \frac{\lambda_1}{r_1})x}, \quad 0 < x < K. \tag{18}$$

Thus

$$F_2(x) = C_{21}e^{(k_1-\mu_2)x} + C_{22}e^{(k_2-\mu_2)x} - AC_{12}e^{-(\mu_1-\frac{\lambda_1}{r_1})x} + C_{11}, \quad 0 < x < K, \tag{19}$$

where k_1 and k_2 are the two roots of the characteristic Equation (11) and A is the constant given in (17). The constants C_{11}, C_{12}, C_{21} and C_{22} should be determined by boundary conditions.

Let us now continue with the survival probabilities for initial values x *above* K as described by differential Equations (4) and (5).

We begin with (5). Using the assumption that the claims are exponentially distributed, we get

$$\frac{dF_2(x)}{dx} = \frac{\lambda_2}{r_2}F_2(x) - \frac{\lambda_2}{r_2}\int_K^x F_2(y)\mu_2 e^{-\mu_2(x-y)}dy - \frac{\lambda_2}{r_2}\int_0^K F_2(y)\mu_2 e^{-\mu_2(x-y)}dy, \tag{20}$$

where $F_2(x)$ in the second integral on the right hand side is the survival probability for $0 < x < K$ given in (19).

The solution to (20) is obtained in the same way as for Equation (6) (cf. the survival probability in the model described in [5] for $x > K$ and initial state 2) and is

$$F_2(x) = D_{21} + D_{22}e^{-(\mu_2-\frac{\lambda_2}{r_2})x}, \quad x > K. \tag{21}$$

For the survival probability $F_1(x)$ we can rewrite the integro-differential equation (4) as follows

$$\begin{aligned}\frac{dF_1(x)}{dx} &= \frac{\lambda_1+\nu_1}{r_1}F_1(x) - \frac{\lambda_1\mu_1}{r_1}\int_K^x F_1(y)e^{-\mu_1(x-y)}dy \\ &- \frac{\lambda_1\mu_1}{r_1}\int_0^K F_1(y)e^{-\mu_1(x-y)}dy - \frac{\nu_1}{r_1}F_2(x), \quad x > K,\end{aligned} \tag{22}$$

where we take into account that the survival probability $F_1(x)$ for $0 < x < K$ is now known.

The integro-differential equation for $G_1(x) := e^{\mu_1 x}F_1(x)$ is in this case

$$\begin{aligned}\tfrac{dG_1(x)}{dx} &= (\mu_1 + \tfrac{\lambda_1+\nu_1}{r_1})G_1(x) - \tfrac{\lambda_1\mu_1}{r_1}\int_K^x G_1(y)dy \\ &-\tfrac{\lambda_1\mu_1}{r_1}\int_0^K G_1(y)dy - \tfrac{\nu_1}{r_1}e^{\mu_1 x}F_2(x).\end{aligned} \tag{23}$$

Differentiating once gives

$$\frac{d^2G_1(x)}{dx^2} = (\mu_1 + \frac{\lambda_1+\nu_1}{r_1})\frac{dG_1(x)}{dx} - \frac{\lambda_1\mu_1}{r_1}G_1(x) - \frac{\nu_1}{r_1}\frac{d}{dx}(e^{\mu_1 x}F_2(x)). \tag{24}$$

This differential equation has the same structure as (10) so we can immediately write down the solution

$$G_1(x) = D_{11}e^{\hat{k}_1 x} + D_{12}e^{\hat{k}_2 x} + D_{21}e^{\mu_1 x} - BD_{22}e^{(\mu_1-\mu_2+\frac{\lambda_2}{r_2})x}, \quad x > K, \tag{25}$$

where

$$\hat{k}_1 = \frac{1}{2}(\mu_1 + \frac{\lambda_1 + \nu_1}{r_1}) + \frac{1}{2}\sqrt{(\mu_1 + \frac{\lambda_1 + \nu_1}{r_1})^2 - 4\frac{\lambda_1 \mu_1}{r_1}}, \tag{26}$$

$$\hat{k}_2 = \frac{1}{2}(\mu_1 + \frac{\lambda_1 + \nu_1}{r_1}) - \frac{1}{2}\sqrt{(\mu_1 + \frac{\lambda_1 + \nu_1}{r_1})^2 - 4\frac{\lambda_1 \mu_1}{r_1}}, \tag{27}$$

and

$$B = \frac{\frac{\nu_1}{r_1}(\mu_1 - \mu_2 + \frac{\lambda_2}{r_2})}{(\mu_1 - \mu_2 + \frac{\lambda_2}{r_2})^2 - (\mu_1 + \frac{\lambda_1+\nu_1}{r_1})(\mu_1 - \mu_2 + \frac{\lambda_2}{r_2}) + \frac{\lambda_1 \mu_1}{r_1}}. \tag{28}$$

Thus

$$F_1(x) = D_{11} e^{(\hat{k}_1 - \mu_1)x} + D_{12} e^{(\hat{k}_2 - \mu_1)x} - B D_{22} e^{-(\mu_2 - \frac{\lambda_2}{r_2})x} + D_{21}, \quad x > K. \tag{29}$$

The constants can be determined from boundary conditions.

3.1 Boundary Conditions

In the four expressions for the survival probabilities (7), (19), (21), and (29) we have in total eight unknown constants. To find these constants we set up eight boundary conditions. The first two are

B1 $\lim_{x\to\infty} F_1(x) = 1$.

B2 $\lim_{x\to\infty} F_2(x) = 1$.

The first condition implies that $D_{11} = 0$ since $\hat{k}_1 - \mu_1 > 0$. The second implies that $D_{21} = 1$.

The second pair of boundary conditions is given by the fact that the survival probabilities should be continuous at the barrier K, that is

B3 $F_1(K^-) = F_1(K^+)$.

B4 $F_2(K^-) = F_2(K^+)$.

The third pair of boundary conditions comes from the behaviour of the derivatives at 0. From Equations (2) and (3) we get

B5 $F_1'(0^+) = \frac{\lambda_1}{r_1} F_1(0^+)$.

B6 $F_2'(0^+) = \frac{\lambda_2+\nu_2}{r_2} F_2(0^+) - \frac{\nu_2}{r_2} F_1(0^+)$.

Condition **B5** implies that $C_{12} = -\rho_1 C_{11}$.

Finally, the last pair of boundary conditions comes from the behaviour of the derivatives of the survival probabilities at the barrier K. Subtracting Equation (2) from (4) with $x = K^-$ and $x = K^+$, respectively, we get using boundary condition **B3**

B7 $F_1'(K^+) - F_1'(K^-) = \frac{\nu_1}{r_1}(F_1(K^+) - F_2(K^+))$.

A similar expression can be found for F_2 by subtracting Equation (5) from (3) and applying boundary condition **B4**:

B8 $F_2'(K^-) - F_2'(K^+) = \frac{\nu_2}{r_2}(F_2(K^-) - F_1(K^-))$.

We have five unknown constants left to determine, C_{11}, C_{21}, C_{22}, D_{12}, and D_{22}, which can be found by boundary conditions **B3**, **B4**, **B6**, **B7**, and **B8**.

3.2 Numerical Examples

The differences between $F_1(x)$ and $F_2(x)$, for the parameter settings $r_1 = 0.9, r_2 = 0.3, \lambda_1 = \lambda_2 = 1, \mu_1 = \mu_2 = 4$ and $K = 1$ are shown for three different cases: the case $\nu_1 = \nu_2 = 0.1$ in Figure 2, the case $\nu_1 = \nu_2 = 1$ in Figure 3 and the case $\nu_1 = \nu_2 = 10$ in Figure 4. The figures appear at the end of the paper. They nicely illustrate that the current state of the system highly influences the survival probability if the arrival rate of the observer is low, while it hardly has any influence on the survival probability if the arrival rate of the observer is high. The dependence of $F_1(x)$ on the value of ν_1 is illustrated in Figure 5. The differences between the survival probabilities in the IPO and the ACAI model are illustrated in Figure 6. We have taken the arrival rates of the Poisson observer equal to the arrival rates of claims. In the ACAI model, a claim that crosses the $K = 1$ threshold from above immediately gives rise to a high premium rate r_1; apparently, this causes the survival probability to be larger than in IPO.

3.3 The Always Active Observer

If we let the arrival rate of the observer tend to infinity in both states, then we should get the classical result for the constant barrier when the premium rate is changed as soon as the barrier is crossed, see [1].

Indeed, from (12), (13) and (17) we have that $k_1 \to \infty$, $k_2 \to 0$, and $A \to -1$ as $\nu_2 \to \infty$ and from (26), (27) and (28) we have that $\hat{k}_1 \to \infty$, $\hat{k}_2 \to 0$, and $B \to -1$ as $\nu_1 \to \infty$.

In this case we have the survival probabilities (cf. p. 195 of [1])

$$F_1(x) = F_2(x) = \begin{cases} C_{11}(1 - \rho_1 e^{-(\mu_1 - \frac{\lambda_1}{r_1})x}), & 0 < x < K, \\ 1 + D_{22} e^{-(\mu_2 - \frac{\lambda_2}{r_2})x}, & x \geq K. \end{cases} \tag{30}$$

While these are obvious from (7) and (21), they can also be obtained from (19) and (29) using the boundary conditions.

4 General claims

In this section we no longer restrict ourselves to exponentially distributed claim sizes.

Part 1: $F_1(x)$ for $x < K$.
Let us start the analysis of the survival probabilities for the case of generally distributed

claim sizes with Equation (2). This equation coincides with Equation (2) of our companion paper [5], that is formally solved in Section 4 of that paper. In the present paper we'd like to pay some more attention to this formula, giving additional analytic and probabilistic insight into its solution. It should first be observed that Equation (2) is also the equation for the steady-state buffer content distribution $P(V^{(K)} < x)$ in an $M/G/1$ queue with finite workload capacity K and rejection of customers whose service requirement would cause the workload to exceed level K. It is well-known, and was rigorously proved in [9], that, if $\rho_1 = \lambda_1 EC_1/r_1 < 1$,

$$P(V^{(K)} < x) = \frac{P(V^{(\infty)} < x)}{P(V^{(\infty)} < K)}, \quad x \leq K. \tag{31}$$

It is also well-known (cf. [2], p. 281) that

$$P(V^{(\infty)} < x) = (1-\rho_1) \sum_{k=0}^{\infty} \rho_1^k P(C_1^{res} + \cdots + C_k^{res} < x), \tag{32}$$

where the i.i.d. random variables C_i^{res} denote residual claim sizes (or rather residual service times) with distribution $\int_{y=0}^{x} \frac{(1-C_i(y))}{EC_i} \mathrm{d}y$. The LST $\phi_1(s) := \int_0^\infty \mathrm{e}^{-sx} \mathrm{d}P(V^{(\infty)} < x)$ is given by

$$\phi_1(s) = \frac{(1-\rho_1)s}{s - \frac{\lambda_1}{r_1}(1-\gamma_1(s))}, \tag{33}$$

where $\gamma_1(s)$ is the LST of $C_1(x)$.
Concluding: if $\rho_1 < 1$ then $F_1(x)$ for $x < K$ is – up to a multiplicative constant – given by (31), where $P(V^{(\infty)} < x)$ is specified by (32) and by its LST (33). In the case of exponential claim sizes, this is easily seen to lead to Formula (7) (which is indeed also the expression for the workload distribution in the $M/M/1/K$ queue). If $\rho_1 \geq 1$ then $F_1(x)$ still coincides with the steady-state distribution $P(V^{(K)} < x)$ in the $M/G/1$ queue with finite capacity K, and we still have the truncated form $P(V^{(K)} < x) = \frac{V(x)}{V(K)}$, but now $V(\cdot)$ no longer is the steady-state workload distribution in the $M/G/1$ queue. See Cohen [7], pp. 72-73, for a detailed discussion of this function $V(\cdot)$.

Remark It is intuitively obvious that the solution $F_1(x)$ of Equation (2), that holds for all $x < K$, is a truncated version of the solution of that same equation holding for all non-negative x (as confirmed in (31)). It becomes probabilistically obvious via the following reasoning. Split the survival event into two disjoint events; one in which K is never reached, and another in which K is reached. We can further ignore the first event, as survival while never reaching K has probability zero. The second event has probability:

$$P(survival \mid K \; is \; reached, X_0 = x, \; system \; at \; 0 \; in \; state \; 1)$$

$$\times P(K \; is \; reached \; | X_0 = x, \; system \; at \; 0 \; in \; state \; 1) =: A_1^{(K)}(x) A_2^{(K)}(x).$$

Once the level K has been upcrossed, it takes an $\exp(\lambda_1)$ distributed, hence memoryless, amount of time until a claim occurs. Therefore the initial condition x plays no role in $A_1^{(K)}(x) \equiv A_1^{(K)}$. Now observe that, for any $L > 0$:

$$A_2^{(K+L)}(x) = A_2^{(K)}(x) P(K{+}L \textit{ is reached} \mid K \textit{ is reached},\ X_0 = x,\ \textit{system at } 0 \textit{ in state } 1)$$

$$= A_2^{(K)}(x) P(K + L \textit{ is reached} \mid K \textit{ is reached}).$$

In the last step, the memoryless property is used once more. Hence $A_2^{(K+L)}(x)$ is proportional to $A_2^{(K)}(x)$ for all $L > 0$, including $L = \infty$, and consequently $F_1(x)$ is proportional to the survival probability for $K = \infty$.

Part 2: $F_2(x)$ for $x < K$.
Consider Equation (3) for $F_2(x)$, $x < K$. Again assume that this equation holds for *all* $x > 0$, and let V_2 be a random variable with distribution that satisfies (3) for all $x > 0$. The LST $\phi_2(s) := \int_0^\infty \mathrm{e}^{-sx} \mathrm{d}P(V_2 < x)$ satisfies the following equation:

$$[s - \frac{\lambda_2 + \nu_2}{r_2} + \frac{\lambda_2}{r_2}\gamma_2(s)]\phi_2(s) = -\frac{\nu_2}{r_2}\phi_1(s) + sF_2(0+). \tag{34}$$

Notice that if $\phi_1(s) = \phi_2(s)$ (which occurs if all parameters are the same in both states), then (34) coincides with (33).

We can rewrite (34) as

$$\phi_2(s) = \frac{sF_2(0+) - \frac{\nu_2}{r_2}\phi_1(s)}{s - \frac{\lambda_2}{r_2}(1 - \gamma_2(s)) - \frac{\nu_2}{r_2}}. \tag{35}$$

To invert this equation, it should be noticed that $\frac{s}{s - \frac{\lambda_2}{r_2}(1-\gamma_2(s)) - q}$ is the LST of the *scale* function $W^{(q)}(x)$ corresponding to the Lévy process with Lévy exponent $s - \frac{\lambda_2}{r_2}(1 - \gamma_2(s))$; see, e.g., [4, 12]. Accordingly, with $*$ denoting a convolution:

$$F_2(x) = F_2(0+) W^{(\nu_2/r_2)}(x) - \frac{\nu_2}{r_2} P(V_1 < x) * \int_0^x W^{(\nu_2/r_2)}(y)\mathrm{d}y, \quad x < K. \tag{36}$$

Remark In the case of $\exp(\mu_i)$ distributed claim sizes, $\phi_2(s)$ in (35) becomes a quotient of two polynomials, the denominator equalling $(s^2 + (\mu_2 - \frac{\lambda_2}{r_2} - \frac{\nu_2}{r_2})s - \frac{\nu_2}{r_2}\mu_2)\ (s + \mu_1(1-\rho_1))$. This denominator has three zeroes: $s_{1,2} = -\mu_2 + k_{1,2}$, $s_3 = -\mu_1(1 - \rho_1)$, cf. (12,13). Inversion of (35) thus yields the same three exponential terms as were found in (19).

Remark It should be noticed that the denominator of (35) also appears in Cohen's analysis ([8], Formula (4.91) on p. 259) of the transient workload behavior of the $M/G/1$ queue. As a special case of that Formula (4.91) (take arrival rate λ_2, service speed $r_2 = 1$, service requirement LST $\gamma_2(s)$),

$$\int_0^\infty a\mathrm{e}^{-at} E[\mathrm{e}^{-sV_t} | V_0 = 0]\mathrm{d}t = \frac{a}{a + \lambda_2(1 - \mu_2(a))} \frac{s - a - \lambda_2(1 - \mu_2(a))}{s - \lambda_2(1 - \gamma_2(s)) - a}. \tag{37}$$

Here $\mu_2(s)$ is the LST of the busy period length of the $M/G/1$ queue. It can be checked (cf. Theorem 46.3 of Sato [11]) that $a+\lambda_2(1-\mu_2(a))$ is the inverse of the Lévy exponent $s-\lambda_2(1-\gamma_2(s))$. It may be interesting to study this scale function $W^{(q)}(x)$ with its interpretation in terms of the transient $M/G/1$ workload behaviour more closely, the more so since there is a growing interest in scale functions of Lévy processes and since there are only a few concrete interpretations of scale functions. However, this is beyond the scope of the present paper. We end this remark with the observation that substitution of $a=\nu_2/r_2$ in (37) yields $E[\mathrm{e}^{-sV_Y}|V_0=0]$, where $Y\sim\exp(\nu_2/r_2)$: the $M/G/1$ workload observed an exponentially distributed amount of time after the system has become empty.

Part 3: $F_2(x)$ for $x\geq K$.
Rewrite Formula (5) into:

$$\begin{aligned}\frac{dF_2(x)}{dx} &= \frac{\lambda_2}{r_2}F_2(x)-\frac{\lambda_2}{r_2}\int_{0-}^{x}C_2(x-y)\mathrm{d}F_2(y)\\ &= \frac{\lambda_2}{r_2}F_2(x)-\frac{\lambda_2}{r_2}\int_{0-}^{K}C_2(x-y)\mathrm{d}F_2(y)-\frac{\lambda_2}{r_2}\int_{K}^{x}C_2(x-y)\mathrm{d}F_2(y).\end{aligned}\tag{38}$$

Introducing $\psi_2(s):=\int_K^\infty \mathrm{e}^{-sx}\mathrm{d}F_2(x)$, and taking transforms on both sides of (38), we obtain:

$$\begin{aligned}s\psi_2(s)-\mathrm{e}^{-sK}\frac{dF_2(x)}{dx}|_{x=K} &= \frac{\lambda_2}{r_2}\psi_2(s)-\frac{\lambda_2}{r_2}\int_K^\infty \mathrm{e}^{-sx}\mathrm{d}_x\int_{0-}^{K}C_2(x-y)\mathrm{d}F_2(y)\\ &- \frac{\lambda_2}{r_2}\psi_2(s)\gamma_2(s),\end{aligned}\tag{39}$$

and hence

$$\begin{aligned}&[s-\frac{\lambda_2}{r_2}(1-\gamma_2(s))]\psi_2(s)\\ &= \mathrm{e}^{-sK}\frac{dF_2(x)}{dx}|_{x=K}-\frac{\lambda_2}{r_2}\int_K^\infty \mathrm{e}^{-sx}\mathrm{d}_x\int_{0-}^{K}C_2(x-y)\mathrm{d}F_2(y).\end{aligned}\tag{40}$$

Introducing

$$H_2(s):=\int_K^\infty \mathrm{e}^{-sx}\mathrm{d}_x\int_{0-}^{K}C_2(x-y)\mathrm{d}F_2(y),\tag{41}$$

we can rewrite (40) as:

$$\psi_2(s)=\frac{(1-\frac{\lambda_2}{r_2}EC_2)s}{s-\frac{\lambda_2}{r_2}(1-\gamma_2(s))}\,\frac{\mathrm{e}^{-sK}\frac{dF_2(x)}{dx}|_{x=K}-\frac{\lambda_2}{r_2}H_2(s)}{(1-\frac{\lambda_2}{r_2}EC_2)s}.\tag{42}$$

Notice that $\frac{dF_2(x)}{dx}|_{x=K}=\frac{\lambda_2}{r_2}H_2(0)$ (cf. (40) for $s=0$).

The first term in the right-hand side of (42) is the LST of $P(W_2<x)$, the waiting time or workload in a standard $M/G/1$ queue with arrival rate λ_2, service speed r_2 and

service requirement C_2 – or, alternatively, the survival probability in the corresponding ruin model. Notice that the condition for a steady-state distribution of that waiting time to exist, or equivalently, for the corresponding survival probability to be positive, is $\lambda_2 EC_2/r_2 < 1$. Some thought makes it obvious that this is also the condition for the survival probability in our alternating risk reserve process to be positive.

The second term concerns random variables on $[K, \infty)$: e^{-sK} is the LST of the constant K, and (41) shows that $H_2(s)$ is the LST of the restriction to $[K, \infty)$ of the sum of C_2 and the random variable on $[0, K]$ with distribution $F_2(\cdot)$. Division by s amounts to integrating the distribution.

Part 4: $F_1(x)$ for $x \geq K$.
Finally we turn to (4), rewriting it into:

$$\frac{dF_1(x)}{dx} = \frac{\lambda_1+\nu_1}{r_1}F_1(x) - \frac{\lambda_1}{r_1}\int_{0-}^{K} C_1(x-y)dF_1(y) - \frac{\lambda_1}{r_1}\int_{K}^{x} C_1(x-y)dF_1(y) - \frac{\nu_1}{r_1}F_2(x), \quad x > K. \tag{43}$$

Notice that $F_2(x)$ is - by now - known for $x > K$. Introducing $\psi_1(s) := \int_K^\infty \mathrm{e}^{-sx}\mathrm{d}F_1(x)$, and taking transforms on both sides of (43), we obtain:

$$\begin{aligned} s\psi_1(s) - \mathrm{e}^{-sK}\frac{dF_1(x)}{dx}|_{x=K} &= \frac{\lambda_1+\nu_1}{r_1}\psi_1(s) - \frac{\lambda_1}{r_1}\gamma_1(s)\psi_1(s) \\ &- \frac{\lambda_1}{r_1}\int_K^\infty \mathrm{e}^{-sx}\mathrm{d}_x\int_{0-}^{K} C_1(x-y)\mathrm{d}F_1(y) - \frac{\nu_1}{r_1}\psi_2(s). \end{aligned} \tag{44}$$

Hence

$$\begin{aligned} &[s - \frac{\lambda_1}{r_1}(1-\gamma_1(s)) - \frac{\nu_1}{r_1}]\psi_1(s) \\ = \ &\mathrm{e}^{-sK}\frac{dF_1(x)}{dx}|_{x=K} - \frac{\lambda_1}{r_1}\int_K^\infty \mathrm{e}^{-sx}\mathrm{d}_x\int_{0-}^{K} C_1(x-y)\mathrm{d}F_1(y) - \frac{\nu_1}{r_1}\psi_2(s). \end{aligned} \tag{45}$$

Finally,

$$\psi_1(s) = \frac{\mathrm{e}^{-sK}\frac{dF_1(x)}{dx}|_{x=K} - \frac{\lambda_1}{r_1}\int_K^\infty \mathrm{e}^{-sx}\mathrm{d}_x\int_{0-}^{K} C_1(x-y)\mathrm{d}F_1(y) - \frac{\nu_1}{r_1}\psi_2(s)}{s - \frac{\lambda_1}{r_1}(1-\gamma_1(s)) - \frac{\nu_1}{r_1}}. \tag{46}$$

This transform can be inverted by observing the following:
1. The three terms in the numerator of (46) are the Laplace-Stieltjes transforms of (i) $I(x > K)\frac{dF_1(x)}{dx}|_{x=K}$, with $I(\cdot)$ denoting an indicator function; (ii) $-\frac{\lambda_1}{r_1}\int_0^K C_1(x-y)\mathrm{d}F_1(y)$ on $[K, \infty)$, and (iii) $-\frac{\nu_1}{r_1}F_2(x)$ on $[K, \infty)$.
2. The denominator of (46) is the Laplace-Stieltjes transform of $\int_0^x W^{(\nu_1/r_1)}(y)\mathrm{d}y$, where $W^{(q)}(x)$ is the scale function corresponding to the Lévy process with Lévy exponent $s - \frac{\lambda_1}{r_1}(1-\gamma_1(s))$, cf. the discussion above (36).

5 Claim sizes with a rational Laplace-Stieltjes transform

In this section we assume that the claim size distributions have rational Laplace-Stieltjes transforms (put differently: the claim sizes are distributed as mixtures of exponentials):

$$\gamma_j(s) = \frac{\gamma_{N,j}(s)}{\gamma_{D,j}(s)} = \frac{\gamma_{N,j}(s)}{\prod_{i=1}^{L_j}(s - s_{j,i})}, \quad j = 1, 2, \tag{47}$$

where $\gamma_{N,j}(s)$ are polynomials of at most degree $L_j - 1$, $j = 1, 2$. Accordingly, one can write:

$$C_j(x) = 1 - \sum_{i=1}^{L_j} c_{j,i} \mathrm{e}^{s_{j,i}x}, \quad j = 1, 2. \tag{48}$$

Here we have assumed that all $s_{j,i}$ are different; if they aren't, like in the case of Erlang distributed claim sizes, then minor adaptations are required.

We shall successively discuss the resulting form of $F_j(x)$, $j = 1, 2$, for $x < K$ and $x \geq K$, starting from Formulas (33), (35), (42) and (46). We shall also compare the results with those obtained for the special case of exponential claim sizes in Section 3.

Part 1: $F_1(x)$ for $x < K$.
It is easily seen that $\phi_1(s)$ in (33) is a quotient of two polynomials of degree L_1: $\phi_1(s) = \frac{\phi_{N,1}(s)}{\phi_{D,1}(s)}$. Accordingly, $F_1(x)$ is for $x < K$ a constant plus a mixture of L_1 exponentials. Since $\lambda_1 E(C_1)/r_1$ is not necessarily less than one, some of the poles of $\phi_1(s)$ may be positive, and hence some of the exponentials may have a positive exponent.

In case $C_1(x) = 1 - \mathrm{e}^{-\mu_1 x}$, as in Section 3, this exponential is the familiar $\mathrm{e}^{-(\mu_1 - \lambda_1/r_1)x}$ from the theory of the $M/M/1$ queue and the theory of ruin probabilities for exponential claim sizes; cf. Formula (7).

Part 2: $F_2(x)$ for $x < K$.
Using (35) and the above representation $\phi_1(s) = \frac{\phi_{N,1}(s)}{\phi_{D,1}(s)}$, in which $\phi_{N,1}(s)$ and $\phi_{D,1}(s)$ are polynomials of degree L_1, we can write:

$$\phi_2(s) = \frac{sF_2(0+)\phi_{D,1}(s) - \frac{\nu_2}{r_2}\phi_{N,1}(s)}{\phi_{D,1}(s)[s - \frac{\lambda_2}{r_2}(1 - \gamma_2(s)) - \frac{\nu_2}{r_2}]}. \tag{49}$$

It is known from $M/G/1$ theory (cf. Cohen [8], p. 548) that, for $\lambda_2 E(C_2)/r_2 < 1$, the function $s - \frac{\lambda_2}{r_2}(1 - \gamma_2(s)) - \frac{\nu_2}{r_2}$ has one zero with positive real part. In addition, there are L_2 zeroes with non-positive real part. Multiplying numerator and denominator with $\gamma_{D,2}(s)$ results in:

$$\phi_2(s) = \frac{sF_2(0+)\phi_{D,1}(s)\gamma_{D,2}(s) - \frac{\nu_2}{r_2}\phi_{N,1}(s)\gamma_{D,2}(s)}{\phi_{D,1}(s)[s\gamma_{D,2}(s) - \frac{\lambda_2}{r_2}(\gamma_{D,2}(s) - \gamma_{N,2}(s)) - \frac{\nu_2}{r_2}\gamma_{D,2}(s)]}. \tag{50}$$

It is now easily verified that both numerator and denominator of the expression for $\phi_2(s)$ are polynomials of degree $L_1 + L_2 + 1$. Hence $F_2(x)$ is on $[0, K)$ a constant plus a mixture of $L_1 + L_2 + 1$ exponentials.

For $L_1 = L_2 = 1$, as in Section 3, we indeed find a mixture of the three exponentials of (19), one term corresponding to $\phi_{D,1}(s) = s + \mu_1 - \frac{\lambda_1}{r_1}$.

Part 3: $F_2(x)$ for $x \geq K$.
Consider the expression (42) for $\psi_2(s) = \int_K^\infty \mathrm{e}^{-sx}\mathrm{d}F_2(x)$. It is a product of two terms, yielding after inversion a convolution of two inversions. The first one is the waiting time distribution in an $M/G/1$ queue with service time distribution $C_2(\cdot)$ – which here is a mixture of L_2 exponentials. Now consider the second term and its inversion. Using (48) and the identity (see below (42)) $\frac{dF_2(x)}{dx}|_{x=K} = \frac{\lambda_2}{r_2}H_2(0)$, and defining $c_{2,i}^* = c_{2,i}\int_0^K \mathrm{e}^{-s_{2,i}y}\mathrm{d}F_2(y)$, we can write the second term in the right-hand side of (42) as

$$\frac{\frac{\lambda_2}{r_2}}{1 - \frac{\lambda_2}{r_2}E(C_2)} \sum_{i=1}^{L_2} c_{2,i}^* \frac{1}{s - s_{2,i}} \mathrm{e}^{-(s-s_{2,i})K}, \tag{51}$$

which is easily inverted to a mixture of L_2 shifted exponentials, since

$$\frac{1}{s-a}\mathrm{e}^{-(s-a)K} = \int_{y=K}^\infty \mathrm{e}^{-sy}\mathrm{e}^{ay}\mathrm{d}y.$$

For $L_2 = 1$, the first term in the right-hand side of (42) reduces to the $M/M/1$ term $(1-\rho_2)(\mu_2+s)/(\mu_2(1-\rho_2)+s)$, and the second term reduces to $Z_1\mathrm{e}^{-(s-s_{2,1})K}/(s - s_{2,1})$ $= Z_2\mathrm{e}^{-sK}/(s + \mu_2)$, the Z_i denoting constants. The $(\mu_2 + s)$ terms cancel, yielding

$$\psi_2(s) = Z_3 \frac{\mathrm{e}^{-sK}}{\mu_2 - \frac{\lambda_2}{r_2} + s},$$

which after inversion indeed yields the form (21) (an exponential shifted over K).

Part 4: $F_1(x)$ for $x \geq K$.
In a similar fashion as in Part 3 above, we rewrite the numerator of the right-hand side of (46) as:

$$\frac{\lambda_1}{r_1}\sum_{i=1}^{L_1} c_{1,i}^* \frac{s}{s - s_{1,i}} \mathrm{e}^{-(s-s_{1,i})K} + \frac{\nu_1}{r_1}[(F_1(K) - F_2(K))\mathrm{e}^{-sK} - \psi_2(s)],$$

where $c_{1,i}^* = c_{1,i}\int_0^K \mathrm{e}^{s_{1,i}y}\mathrm{d}F_1(y)$. This leads to the following expression for $\psi_1(s)$:

$$\psi_1(s) = \mathrm{e}^{-sK} \frac{\frac{\lambda_1}{r_1}\sum_{i=1}^{L_1} c_{1,i}^* \frac{s}{s-s_{1,i}}\mathrm{e}^{s_{1,i}K} + \frac{\nu_1}{r_1}[F_1(K) - F_2(K) - \frac{\frac{\lambda_2}{r_2}s}{s - \frac{\lambda_2}{r_2}(1-\gamma_2(s))}\sum_{i=1}^{L_2} c_{2,i}^* \frac{1}{s-s_{2,i}}\mathrm{e}^{s_{2,i}K}]}{s - \frac{\lambda_1}{r_1}(1-\gamma_1(s)) - \frac{\nu_1}{r_1}}. \tag{52}$$

Inversion again yields a mixture of exponentials, shifted over K. The exponents are determined by the zeroes of the denominator of (52), and by the $s_{1,i}$, $s_{2,i}$ and the zeroes of $s - \frac{\lambda_2}{r_2}(1-\gamma_2(s))$. Some of these cancel; e.g., the $1/(s-s_{1,i})$ in the numerator of (52) disappear when we multiply the denominator and numerator of (52) by $\gamma_{D,1}(s)$.

Now consider the case $L_1 = L_2 = 1$, as in Section 3. The denominator of (52) becomes $(s^2 - s(\frac{\lambda_1+\nu_1}{r_1} - \mu_1) - \nu_1\mu_1/r_1)/(s+\mu_1)$. The factor $s+\mu_1$ takes care of the term $1/(s-s_{1,i}) = 1/(s+\mu_1)$ in the numerator. After partial fraction expansion and inversion the factor $\frac{1}{s^2-s(\frac{\lambda_1+\nu_1}{r_1}-\mu_1)-\nu_1\mu_1/r_1}$ yields two exponentials, which are easily verified to be the first two exponentials in (29). The third exponential in (29) is retrieved by considering the last term of the numerator of (52):

$$Z_4 \frac{s}{s - \frac{\lambda_2}{r_2}(1-\gamma_2(s))} \frac{1}{s - s_{2,1}} = \frac{Z_4}{\mu_2 - \frac{\lambda_2}{r_2} + s}.$$

6 Conclusion

In this paper, and in [5], we have studied a risk reserve process with a threshold dividend strategy which can be in two different states, and which can change only at particular epochs. We have determined the survival probability when starting at a particular level and in a particular state. Topics for future research may include other performance measures (like the time until ruin, if ruin is certain), and generalization to the case that the claim process is a spectrally negative Lévy process (a Lévy process with only negative jumps), instead of a compound Poisson process. We would then consider different Lévy exponents above and below a barrier, and an independent Poisson observer. In [3] a *queueing* model with such adaptable Lévy exponents has been analysed. See also Bratiychuk and Derfla [6] for a risk process with a two-step premium function, with a perturbation by a Brownian motion, and see Kyprianou and Loeffen [10] for a study of Lévy processes without positive jumps and with state-dependent exponents.

References

[1] S. Asmussen (2000). *Ruin Probabilities.* World Scientific Publishing Company, Singapore.

[2] S. Asmussen (2001). *Applied Probability and Queues.* Springer, New York, 2nd ed.

[3] R. Bekker, O.J. Boxma, J.A.C. Resing (2009). Lévy processes with adaptable exponent. *Advances in Applied Probability* **41**, 177-205.

[4] J. Bertoin (1996). *Lévy Processes.* Cambridge University Press, Cambridge.

[5] O.J. Boxma, H. Jönsson, J.A.C. Resing, S. Shneer (2009). An alternating risk reserve process – Part I. *EURANDOM* report 2009-009.

[6] M.S. Bratiychuk, D. Derfla (2007). On a modification of the classical risk process. *Insurance: Mathematics and Economics* **41**, 156-162.

[7] J.W. Cohen (1976). *On Regenerative Processes in Queueing Theory*. Springer, Berlin.

[8] J.W. Cohen (1982). *The Single Server Queue*. North-Holland Publ. Cy., Amsterdam, 2nd ed.

[9] G. Hooghiemstra (1987). A path construction for the virtual waiting time of an $M/G/1$ queue. *Statistica Neerlandica* **41**, 175-181.

[10] A.E. Kyprianou, R.L. Loeffen (2008). Refracted Lévy processes. Preprint. Available at http://arxiv.org/abs/0801.4655.

[11] K. Sato (1999). *Lévy processes and Infinitely Divisible Distributions*. Cambridge University Press, Cambridge.

[12] L. Nguyen-Ngoc and M. Yor (2005). Some martingales associated to reflected Lévy processes. In: Séminaire de Probabilités XXXVIII (Lecture Notes Math. 1857), Springer, Berlin, pp. 42-69.

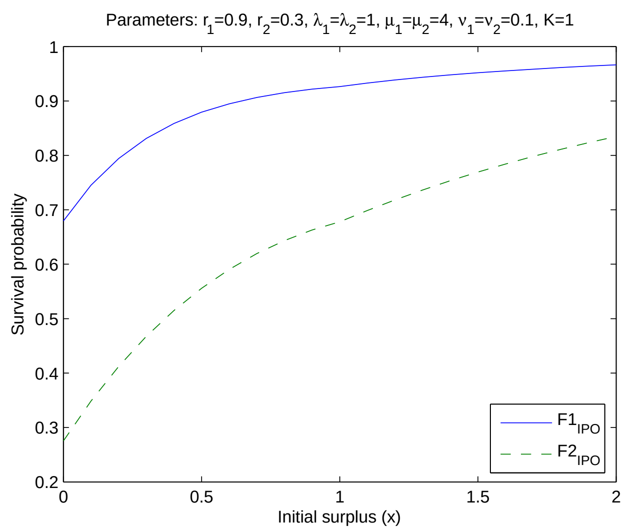

Figure 2: Difference between $F_1(x)$ and $F_2(x)$ in the case $\nu_1 = \nu_2 = 0.1$. The other parameters are fixed to $r_1 = 0.9$, $r_2 = 0.3$, $\lambda_1 = \lambda_2 = 1$, $\mu_1 = \mu_2 = 4$, and $K = 1$.

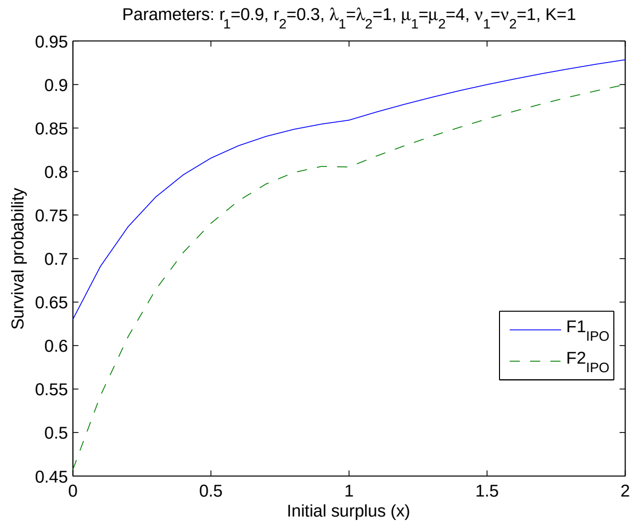

Figure 3: Difference between $F_1(x)$ and $F_2(x)$ in the case $\nu_1 = \nu_2 = 1$. The other parameters are fixed to $r_1 = 0.9$, $r_2 = 0.3$, $\lambda_1 = \lambda_2 = 1$, $\mu_1 = \mu_2 = 4$, and $K = 1$.

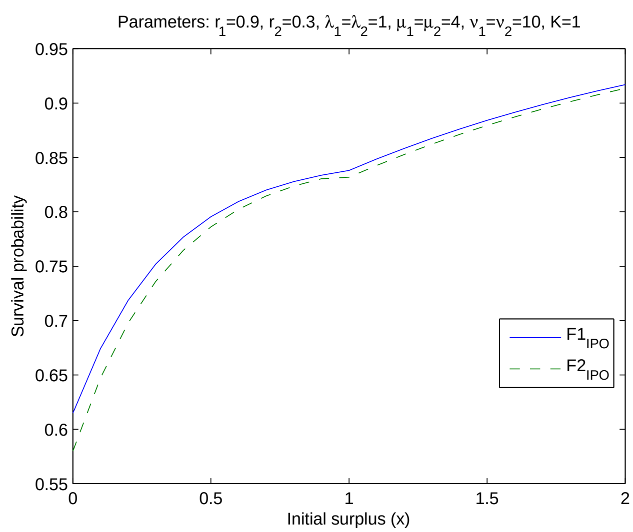

Figure 4: Difference between $F_1(x)$ and $F_2(x)$ in the case $\nu_1 = \nu_2 = 10$. The other parameters are fixed to $r_1 = 0.9$, $r_2 = 0.3$, $\lambda_1 = \lambda_2 = 1$, $\mu_1 = \mu_2 = 4$, and $K = 1$.

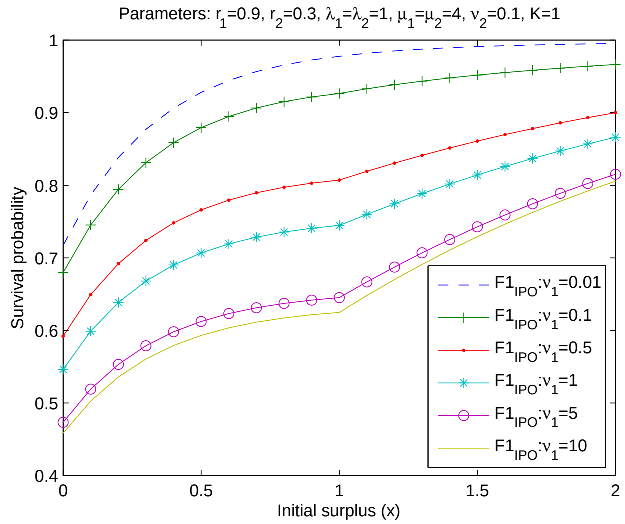

Figure 5: The function $F_1(x)$ in the independent Poisson observer model (IPO) for different values of ν_1. The other parameters are fixed to $r_1 = 0.9$, $r_2 = 0.3$, $\lambda_1 = \lambda_2 = 1$, $\mu_1 = \mu_2 = 4$, $\nu_2 = 0.1$, and $K = 1$.

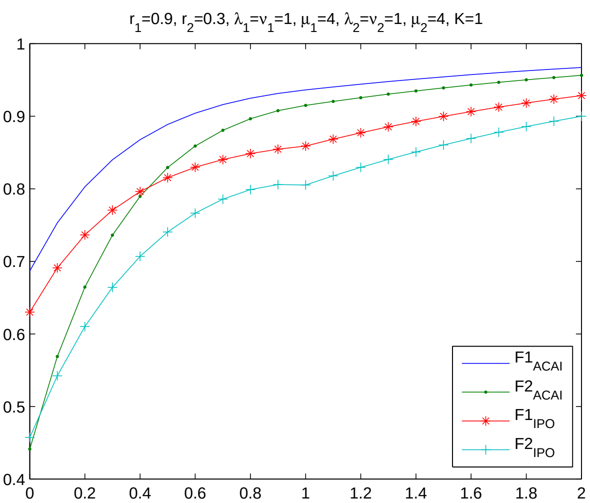

Figure 6: The functions $F_1(x)$ and $F_2(x)$ in the independent Poisson observer model (IPO) and the model where state changes can occur at claim arrivals (ACAI). The parameters are fixed to $r_1 = 0.9$, $r_2 = 0.3$, $\lambda_1 = \lambda_2 = 1$, $\mu_1 = \mu_2 = 4$, $\nu_1 = \nu_2 = 1$, and $K = 1$.