

# Nonparametric estimation of the characteristic triplet of a discretely observed Lévy process

Shota Gugushvili  
EURANDOM  
Technische Universiteit Eindhoven  
P.O. Box 513  
5600 MB Eindhoven  
The Netherlands  
gugushvili@eurandom.tue.nl

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## Abstract

Given a discrete time sample  $X_1, \dots, X_n$  from a Lévy process  $X = (X_t)_{t \geq 0}$  of a finite jump activity, we study the problem of nonparametric estimation of the characteristic triplet  $(\gamma, \sigma^2, \rho)$  corresponding to the process  $X$ . Based on Fourier inversion and kernel smoothing, we propose estimators of  $\gamma, \sigma^2$  and  $\rho$  and study their asymptotic behaviour. The obtained results include derivation of upper bounds on the mean square error of the estimators of  $\gamma$  and  $\sigma^2$  and an upper bound on the mean integrated square error of an estimator of  $\rho$ .

*Keywords:* Characteristic triplet; Fourier inversion; kernel smoothing; Lévy density; Lévy process; mean integrated square error; mean square error.

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# 1 Introduction

Lévy processes are stochastic processes with stationary independent increments. The class of such processes is extremely rich, the best known representatives being Poisson and compound Poisson processes, Brownian motion, Cauchy process and, more generally, stable processes. Though the basic properties of Lévy processes have been well-studied and understood since a long time, see e.g. [29], during the last years there has been a renaissance of interest in Lévy processes. This revival of interest is mainly due to the fact that Lévy processes found numerous applications in practice and proved to be useful in a broad range of fields, including finance, insurance, queueing, telecommunications, quantum theory, extreme value theory and many others, see e.g. [3] for an overview. [13] provides a thorough treatment of applications of Lévy processes in finance. Comprehensive modern texts on fundamentals of Lévy processes are [6, 23, 27], and we refer to those for precise definitions and more details concerning properties of Lévy processes.

Already from the outset an intimate relation of Lévy processes with infinitely divisible distributions was discovered. For a detailed exposition of infinitely divisible distributions see e.g. [30]. In fact there is a one-to-one correspondence between Lévy processes and infinitely divisible distributions: if  $X = (X_t)_{t \geq 0}$  is a Lévy process, then its marginal distributions are all infinitely divisible and are determined by the distribution of  $X_1$ . Conversely, given an infinitely divisible distribution  $\mu$ , one can construct a Lévy process, such that  $P_{X_1} = \mu$ . The celebrated Lévy-Khintchine formula for infinitely divisible distributions provides us with an expression for the characteristic function of  $X_1$ , which can be written as

$$\phi_{X_1}(z) = \exp \left[ i\gamma z - \frac{1}{2}\sigma^2 z^2 + \int_{\mathbb{R}} (e^{izx} - 1 - izx1_{[-1,1]}(x))\nu(dx) \right], \quad (1)$$

where  $\gamma \in \mathbb{R}$ ,  $\sigma \geq 0$  and  $\nu$  is a measure concentrated on  $\mathbb{R} \setminus \{0\}$ , such that  $\int_{\mathbb{R}} (1 \wedge x^2)\nu(dx) < \infty$ . This measure is called the Lévy measure corresponding to the Lévy process  $X$ , while the triple  $(\gamma, \sigma^2, \nu)$  is referred to as the characteristic or Lévy triplet of  $X$ . The representation in (1) in terms of the triplet  $(\gamma, \sigma^2, \nu)$  is unique. Thus the Lévy triplet provides us with means for unique characterisation of a law of any Lévy process. Bearing this in mind, the statistical inference for Lévy processes can be reduced to inference on the characteristic triplet. There are several ways to approach estimation problems for Lévy processes: parametric, nonparametric and semiparametric approaches. These approaches depend on whether one decides to parametrise the Lévy measure (or its density, in case it exists) with a Euclidean parameter, or to work in a nonparametric setting. A semiparametric approach to parametrisation of the Lévy measure is also possible. Most of the existing literature dealing with estimation problems for Lévy processes is concerned with parametric estimation of the Lévy measure (or its density, in case it

exists), see e.g. [1, 2], where a fairly general setting is considered. There are relatively few papers that study nonparametric inference procedures for Lévy processes, and the majority of them assume that high frequency data are available, i.e. either a Lévy process is observed continuously over a time interval  $[0, T]$  with  $T \rightarrow \infty$ , or it is observed at equidistant time points  $\Delta_n, \dots, n\Delta_n$  and  $\lim_{n \rightarrow \infty} \Delta_n = 0$ ,  $\lim_{n \rightarrow \infty} n\Delta_n = \infty$ , see e.g. [4, 21, 26]. On the other hand it is equally interesting to study estimation problems for the case when the high frequency data are not available, i.e. when  $\Delta_n = \Delta$  is kept fixed. The latter case is more involved due to the fact that the information on the Lévy measure is contained in jumps of the process  $X$  and impossibility to observe them directly as in the case of a continuous record of observations, or to ‘disentangle’ them from the Brownian motion as in the high frequency data setting, makes the estimation problem rather difficult. In the particular context of a compound Poisson process we mention [7, 8, 18], where given a sample  $Y_1, \dots, Y_n$  from a compound Poisson process  $Y = (Y_t)_{t \geq 0}$ , nonparametric estimators of the jump size distribution function  $F$  (see [7, 8]) and its density  $f$  (see [18]) were proposed and their asymptotics were studied as  $n \rightarrow \infty$ . This problem is referred to as decompounding. Nonparametric estimation of the Lévy measure  $\nu$  based on low frequency observations from a general Lévy process  $X$  was studied in [25, 35]. However, these papers treat the case of estimation of the Lévy measure only (or of the canonical function  $K$  in case of [35]) and not of its density. Moreover, the rates of convergence of the proposed estimators are studied under the strong moment condition  $E[|X_1|^{4+\delta}] < \infty$ , where  $\delta$  is some strictly positive number. This condition automatically excludes distributions with heavy tails. Nonparametric estimation of the Lévy density of a pure jump Lévy process (i.e. a Lévy process without a drift and a Brownian component) was considered in [12]. We refer to those papers for additional details.

In the present work we concentrate on nonparametric inference for Lévy processes that are of finite jump activity and have absolutely continuous Lévy measures. In essence this means that we consider a superposition of a compound Poisson process and an independent Brownian motion. The Lévy-Khintchine formula in our case takes the form

$$\phi_{X_1}(z) = \exp \left[ i\gamma z - \frac{1}{2}\sigma^2 z^2 + \int_{\mathbb{R}} (e^{izx} - 1)\rho(x)dx \right], \quad (2)$$

where the Lévy density  $\rho$  is such that  $\lambda := \int_{-\infty}^{\infty} \rho(x)dx < \infty$ . To keep the notation compact, we again use  $\gamma$  to denote the drift coefficient in (2), even though it is in general different from  $\gamma$  in (1). Observe that the process  $X$  is related to Merton’s jump-diffusion model of an asset price, see [24]. Additional details on exponential Lévy models, of which Merton’s model is a particular case, can be found e.g. in [13].

Suppose that we dispose a sample  $X_\Delta, X_{2\Delta}, \dots, X_{n\Delta}$  from the process  $X$ . By a rescaling argument, without loss of generality, we may take  $\Delta = 1$ . Based on this sample, our goal is to infer the characteristic triplet  $(\gamma, \sigma^2, \rho)$ , corresponding to (2). At this point we mention that a problem related to ours was studied in [5]. There an exponential of the process  $X$  (this exponential models evolution of an asset price over time) was considered and inference was drawn on parameters  $\sigma, \lambda$  and  $\gamma$  and the functional parameter, the Lévy density  $\rho$ , based on observations on prices of vanilla options on this asset. The difference of our estimation problem with this problem is the observation scheme, since we observe directly the process  $X$ . Moreover, existence of an exponential moment of  $X$  was assumed in [5] (this is unavoidable in the financial setting, because otherwise one cannot price financial derivatives).

Our estimators of  $\gamma, \lambda$  and  $\sigma^2$  will be based on (2) and the use of a plug-in device. To estimate  $\rho$ , we will use methods developed in nonparametric density estimation based on i.i.d. observations, in particular we will employ the Fourier inversion approach and kernel smoothing, see e.g. Sections 6.3 and 10.1 in [34] for an overview. In fact by the stationary independent increments property of a Lévy process, see Definition 1.6 in [27], the problem of estimating  $(\gamma, \sigma^2, \rho)$  from a discrete time sample  $X_1, \dots, X_n$  from the process  $X$  is equivalent to the following one (to keep the notation compact, we again use  $X$ 's to denote our observations): let  $X_1, \dots, X_n$  be i.i.d. copies of a random variable  $X$  with characteristic function given by (2) (in the sequel we will use  $X$  to denote a generic observation). Based on these observations, the problem is to construct estimators of  $\gamma, \sigma^2$  and  $\rho$ . We henceforth will concentrate on this equivalent problem.

The rest of the paper is organised as follows: in Section 2 we construct consistent estimators of parameters  $\sigma^2, \lambda$  and  $\gamma$ . In Section 3, using the estimators of  $\sigma^2, \lambda$  and  $\gamma$ , we propose a plug-in type estimator for  $\rho$  and study the behaviour of its mean integrated square error. In Section 4 we derive a lower bound for estimation of  $\rho$ . All the proofs are collected in Section 5.

## 2 Estimation of $\sigma, \lambda$ and $\gamma$

In the sequel we will find it convenient to use the jump size density  $f(x) := \rho(x)/\lambda$ . We first formulate conditions on  $\rho, \sigma$  and  $\gamma$ , that will be used throughout the paper.

**Condition 2.1.** Let the unknown density  $\rho$  belong to the class

$$W(\beta, L, \Lambda, K) = \left\{ \rho : \rho(x) = \lambda f(x), f \text{ is a density, } \int_{-\infty}^{\infty} x^2 f(x) dx \leq K, \int_{-\infty}^{\infty} |t|^\beta |\phi_f(t)| dt \leq L, \lambda \in (0, \Lambda] \right\},$$

where  $\beta, L, \Lambda$  and  $K$  are strictly positive numbers.

This condition implies in particular that the Fourier transform  $\phi_\rho(t) = \lambda \phi_f(t)$  of  $\rho$  is integrable. The latter is natural in light of the fact that our estimation procedure for  $\rho$  will be based on Fourier inversion, see Section 3. The integrability of  $\phi_\rho$  implies that  $\rho$  is bounded and continuous. It follows that  $f$  is bounded and continuous, and hence, being a probability density, it is also square integrable. Therefore  $\rho(x) = \lambda f(x)$  is square integrable as well. This again is a natural assumption, because we will select the mean integrated square error as a performance criterion for our estimator of  $\rho$ . The condition  $\lambda > 0$  ensures that the process  $X$  has a compound Poisson component. Restriction of the class of densities  $f$  to those densities that have the finite second moment is needed to ensure that  $E[X^2]$  is bounded from above uniformly in  $\rho, \gamma$  and  $\sigma$ . The latter is a technical condition used in the proofs.

**Condition 2.2.** Let  $\sigma$  be such that  $\sigma \in (0, \Sigma]$ , where  $\Sigma$  is a strictly positive number.

This is not a restrictive assumption in many applications, since for instance in the financial context  $\sigma$ , which models volatility, typically belongs to some bounded set, e.g. a compact  $[0, \Sigma]$  as in [5]. The condition  $\sigma > 0$  in our case ensures that  $X$  has a Brownian component. If  $\sigma = 0$ , then our problem in essence reduces to the one studied in [18].

**Condition 2.3.** Let  $\gamma$  be such that  $|\gamma| \leq \Gamma$ , where  $\Gamma$  denotes a positive number.

Remarks similar to those we made after Condition 2.2 apply in this case as well.

Next we turn to the construction of estimators of  $\sigma^2, \lambda$  and  $\gamma$ . The ideas we use resemble those in [5]. Let  $\Re(z)$  and  $\Im(z)$  denote the real and the imaginary parts of a complex number  $z$ , respectively. From (2) we have

$$\log(|\phi_X(t)|) = -\lambda + \lambda \Re(\phi_f(t)) - \frac{\sigma^2 t^2}{2}. \quad (3)$$

Here we used the fact that

$$\log\left(\left|e^{\lambda \phi_f(t)}\right|\right) = \log\left(e^{\lambda \Re(\phi_f(t))}\right) + \log\left(\left|e^{i\lambda \Im(\phi_f(t))}\right|\right) = \lambda \Re(\phi_f(t)).$$

Let  $v^h$  be a kernel that depends on a bandwidth  $h$  and is such that

$$\int_{-1/h}^{1/h} v^h(t) dt = 0, \quad \int_{-1/h}^{1/h} \left(-\frac{t^2}{2}\right) v^h(t) dt = 1.$$

Observe that unlike kernels in kernel density estimation, see e.g. Definition 1.3 in [31], the function  $v^h$  does not integrate to one and by calling it a kernel we abuse the terminology. In view of (3)

$$\int_{-1/h}^{1/h} \log(|\phi_X(t)|) v^h(t) dt = \lambda \int_{-1/h}^{1/h} \Re(\phi_f(t)) v^h(t) dt + \sigma^2. \quad (4)$$

Provided enough assumptions on  $v^h$ , one can achieve that the right-hand side of (4) tends to  $\sigma^2$  as  $h \rightarrow 0$ . A natural way to construct an estimator of  $\sigma^2$  then is to replace in (4)  $\log(|\phi_X(t)|)$  by its estimator  $\log(|\phi_{emp}(t)|)$ . Consequently, we propose

$$\tilde{\sigma}_n^2 = \int_{-1/h}^{1/h} \max\{\min\{M_n, \log(|\phi_{emp}(t)|)\}, -M_n\} v^h(t) dt \quad (5)$$

as an estimator of  $\sigma^2$ . Here  $M_n$  denotes a sequence of positive numbers diverging to infinity at a suitable rate. The truncation in (5) is introduced due to technical reasons in order to obtain a consistent estimator.

We now state our assumptions on the kernel  $v_h$ , the bandwidth  $h$  and the sequence  $M = (M_n)_{n \geq 1}$ .

**Condition 2.4.** *Let the kernel  $v^h(t) = h^3 v(ht)$ , where the function  $v$  is continuous and real-valued, has a support on  $[-1, 1]$  and is such that*

$$\int_{-1}^1 v(t) dt = 0, \quad \int_{-1}^1 \left(-\frac{t^2}{2}\right) v(t) dt = 1, \quad v(t) = O(t^\beta) \text{ as } t \rightarrow 0.$$

Here  $\beta$  is the same as in Condition 2.1.

**Condition 2.5.** *Let the bandwidth  $h$  depend on  $n$  and be such that  $h_n = (\eta \log n)^{-1/2}$  with  $0 < \eta < \Sigma^{-2}$ .*

Using a default convention in kernel density estimation, we will suppress the index  $n$  when writing  $h_n$ , since no ambiguity will arise. Condition 2.5 implies that  $ne^{-\Sigma^2/h^2} \rightarrow \infty$ , since the logarithm of the left-hand side of this expression diverges to minus infinity. Condition 2.5 is required to establish consistency of estimators of  $\sigma^2, \lambda, \gamma$  and  $\rho$ . Hence it is of the asymptotic nature. For finite samples of moderate size, however, it might lead to unsatisfactory estimates. A separate simulation study in the spirit of [17] is needed to study possible bandwidth selection methods in practical problems.

**Condition 2.6.** Let the truncating sequence  $M = (M_n)_{n \geq 1}$  be such that  $M_n = m_n h^{-2}$ , where  $m_n$  is a sequence of real numbers diverging to plus infinity at a slower rate than  $\log n$ , for instance  $m_n = \log \log n$ .

Other restrictions on  $M$  are also possible.

In the sequel we will frequently employ the symbol  $\lesssim$  and  $\gtrsim$ , meaning ‘less or equal up to a universal constant’, or ‘greater or equal up to a universal constant’, respectively. The following theorem establishes consistency of  $\tilde{\sigma}_n^2$ .

**Theorem 2.1.** Let Conditions 2.1–2.6 be satisfied and let the estimator  $\tilde{\sigma}_n^2$  be defined by (5). Then

$$\sup_{|\gamma| \leq \Gamma} \sup_{\sigma \in (0, \Sigma]} \sup_{\rho \in W(\beta, L, \Lambda, K)} \mathbf{E} [(\tilde{\sigma}_n^2 - \sigma^2)^2] \lesssim (\log n)^{-\beta-3}.$$

To construct an estimator of the jump intensity  $\lambda$ , we will again use (2), but now in a different way. Let  $u^h$  denote a kernel that depends on  $h$  and is such that

$$\int_{-1/h}^{1/h} u^h(t) dt = -1, \quad \int_{-1/h}^{1/h} t^2 u^h(t) dt = 0.$$

Then

$$\int_{-1/h}^{1/h} \log(|\phi_X(t)|) u^h(t) dt = \lambda + \lambda \int_{-1/h}^{1/h} \Re(\phi_f(t)) u^h(t) dt. \quad (6)$$

With a proper selection of  $u^h$  one can ensure that (6) converges to  $\lambda$  as  $h \rightarrow 0$ . Using a plug-in device, we therefore propose the following estimator of  $\lambda$ :

$$\tilde{\lambda}_n = \int_{-1/h}^{1/h} \max\{\min\{M_n, \log(|\phi_{emp}(t)|)\}, -M_n\} u^h(t) dt.$$

Now we state a condition on the kernel  $u^h$ .

**Condition 2.7.** Let the kernel  $u^h(t) = hu(ht)$ , where the function  $u$  is continuous and real-valued, has a support on  $[-1, 1]$  and is such that

$$\int_{-1}^1 u(t) dt = -1, \quad \int_{-1}^1 t^2 u(t) dt = 0, \quad u(t) = O(t^\beta) \text{ as } t \rightarrow 0.$$

Here  $\beta$  is the same as in Condition 2.1.

The following theorem deals with asymptotics of the estimator  $\tilde{\lambda}_n$ .

**Theorem 2.2.** Let Conditions 2.1–2.3 and 2.5–2.7 be satisfied and let the estimator  $\tilde{\lambda}_n$  be defined by (6). Then

$$\sup_{|\gamma| \leq \Gamma} \sup_{\sigma \in (0, \Sigma]} \sup_{\rho \in W(\beta, L, \Lambda, K)} \mathbf{E} [(\tilde{\lambda}_n - \lambda)^2] \lesssim (\log n)^{-\beta-1}.$$

Finally, we consider estimation of the drift coefficient  $\gamma$ . By (2) we have

$$\mathfrak{S}(\text{Log}(\phi_X(t))) = \gamma t + \lambda \mathfrak{S}(\phi_f(t)),$$

where  $\text{Log}(\phi_X(t))$  denotes the distinguished logarithm of the characteristic function  $\phi_X(t)$ , i.e. a logarithm that is a single-valued and continuous function of  $t$ , such that  $\text{Log}(\phi_X(0)) = 0$ , see Theorem 7.6.2 in [11] for details of its construction. Let  $w^h$  denote a kernel that depends on  $h$  and is such that

$$\int_{-1/h}^{1/h} t w^h(t) dt = 1.$$

Then

$$\int_{-1/h}^{1/h} \mathfrak{S}(\text{Log}(\phi_X(t))) w^h(t) dt = \gamma + \lambda \int_{-1/h}^{1/h} \mathfrak{S}(\phi_f(t)) w^h(t) dt.$$

With an appropriate choice of  $w^h$  the right-hand side will converge to  $\gamma$ . Therefore, by a plug-in device, for those  $\omega$ 's from the underlying sample space  $\Omega$  for which the distinguished logarithm can be defined, we define an estimator of  $\gamma$  as

$$\tilde{\gamma}_n = \int_{-1/h}^{1/h} \max\{\min\{\mathfrak{S}(\text{Log}(\phi_{emp}(t))), M_n\}, -M_n\} w^h(t) dt, \quad (7)$$

while for those  $\omega$ 's for which it cannot be defined, we assign an arbitrary value to the distinguished logarithm in (7), e.g. zero. The distinguished logarithm in (7) can be defined only for those  $\omega$ 's for which  $\phi_{emp}(t)$  as a function of  $t$  does not vanish on  $[-h^{-1}, h^{-1}]$ , see Theorem 7.6.2 in [11]. In fact the probability of the exceptional set, where the distinguished logarithm is undefined, tends to zero as  $n \rightarrow \infty$ . We will show this by finding a set  $B_n$ , such that on this set the distinguished logarithm might be undefined, while on its complement  $B_n^c$  it is necessarily well-defined. We have

$$\inf_{t \in [-h^{-1}, h^{-1}]} |\phi_X(t)| \geq e^{-2\lambda - \sigma^2/(2h^2)} \geq e^{-2\Lambda - \Sigma^2/(2h^2)}. \quad (8)$$

Define

$$B_n = \left\{ \sup_{t \in [-h^{-1}, h^{-1}]} |\phi_{emp}(t) - \phi_X(t)| > \delta \right\}, \quad (9)$$

$$B_n^c = \left\{ \sup_{t \in [-h^{-1}, h^{-1}]} |\phi_{emp}(t) - \phi_X(t)| \leq \delta \right\},$$

with  $\delta = (1/2)e^{-2\Lambda - \Sigma^2/(2h^2)}$ . From (8), (9) and Theorem 7.6.2 of [11] it follows that on the set  $B_n^c$  the distinguished logarithm is well-defined (with



$t$  restricted to  $[-h^{-1}, h^{-1}]$ , since on this set  $\phi_{emp}$  cannot take the value zero. Notice that given our conditions on  $\rho$  and  $\sigma$ , our choice of  $\delta$  is suitable whatever  $\rho, \sigma$  and  $\gamma$  are. All we need to show is that  $P(B_n) \rightarrow 0$ . The following theorem holds true.

**Theorem 2.3.** *Let  $W(\beta, L, \Lambda, K)$  be defined as in Condition 2.1. Then*

$$\sup_{|\gamma| \leq \Gamma} \sup_{\sigma \in (0, \Sigma]} \sup_{\rho \in W(\beta, L, \Lambda, K)} P(B_n) \lesssim \frac{e^{\Sigma^2/h^2}}{nh^2}.$$

Notice that by Condition 2.5 we have  $P(B_n) \rightarrow 0$ . We now state a condition on the kernel  $w^h$ .

**Condition 2.8.** *Let the kernel  $w^h(t) = h^2 w(ht)$ , where the function  $w$  is continuous and real-valued, has a support on  $[-1, 1]$  and is such that*

$$\int_{-1}^1 tw(t)dt = 1, \quad w(t) = O(t^\beta) \text{ as } t \rightarrow 0.$$

Here  $\beta$  is the same as in Condition 2.1.

The following result holds.

**Theorem 2.4.** *Let Conditions 2.1–2.3, 2.5–2.6 and 2.8 be satisfied and let the estimator  $\tilde{\gamma}_n$  be defined by (7). Then*

$$\sup_{|\gamma| \leq \Gamma} \sup_{\sigma \in (0, \Sigma]} \sup_{\rho \in W_{sym}(\beta, L, \Lambda, K)} \mathbb{E}[(\tilde{\gamma}_n - \gamma)^2] \lesssim (\log n)^{-\beta-2},$$

where  $W_{sym}(\beta, L, \Lambda, K)$  denotes the class of symmetric Lévy densities that belong to  $W(\beta, L, \Lambda, K)$ .

The reason why we restrict ourselves to the class of symmetric Lévy densities is that we would like to obtain a uniformly consistent estimator of  $\gamma$  (and eventually of  $\rho$ , see Section 3). The main technical difficulty in this respect is the (uniform) control of the argument (i.e. of the imaginary part) of the distinguished logarithm in (7), see the proofs of Theorems 2.4 and 2.5. For transparency purposes we restrict ourselves to the class of symmetric  $\rho$ 's. If we are only interested in the consistency of the estimator for a fixed  $\rho$ , then the above restriction is not needed and the result holds without it. We formulate the corresponding theorem below.

**Theorem 2.5.** *Let Conditions 2.5–2.6 and 2.8 be satisfied. Furthermore, let  $\gamma \in \mathbb{R}$ ,  $\sigma^2 > 0$  and let  $\rho$  be such that*

$$0 < \lambda < \infty; \quad \int_{-\infty}^{\infty} x^2 f(x) dx < \infty; \quad \int_{-\infty}^{\infty} |t|^\beta |\phi_f(t)| dt < \infty. \quad (10)$$

Let the estimator  $\tilde{\gamma}_n$  be defined by (7). Then

$$\mathbb{E}[(\tilde{\gamma}_n - \gamma)^2] \lesssim (\log n)^{-\beta-2}.$$

Now that we obtained uniformly consistent estimators of  $\sigma^2$ ,  $\lambda$  and  $\gamma$ , we can move to the construction of an estimator of  $\rho$ .

### 3 Estimation of $\rho$

The method that will be used to construct an estimator of  $\rho$  is based on Fourier inversion and is similar to the approach in [18]. Solving for  $\phi_\rho$  in (2), we get

$$\phi_\rho(t) = \text{Log} \left( \frac{\phi_X(t)}{e^{i\gamma t} e^{-\lambda} e^{-\sigma^2 t^2/2}} \right). \quad (11)$$

Here Log again denotes the distinguished logarithm, which can be constructed as in Theorem 7.6.2 of [11] taking into account an obvious difference that in our case the function  $e^{\phi_\rho(t)}$  equals  $e^\lambda$  at  $t = 0$ .

By Fourier inversion we have

$$\rho(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-itx} \text{Log} \left( \frac{\phi_X(t)}{e^{i\gamma t} e^{-\lambda} e^{-\sigma^2 t^2/2}} \right) dt.$$

This expression will be used as the basis for construction of an estimator of  $\rho$ . Let  $k$  be a symmetric kernel with Fourier transform  $\phi_k$  supported on  $[-1, 1]$  and nonzero there, and let  $h > 0$  be a bandwidth. Since the characteristic function  $\phi_X$  is integrable, there exists a density  $q$  of  $X$ , and moreover, it is continuous and bounded. This density can be estimated by a kernel density estimator

$$q_n(x) = \frac{1}{nh} \sum_{j=1}^n k \left( \frac{x - X_j}{h} \right),$$

see e.g. [31, 34] for an introduction to kernel density estimation. Its characteristic function  $\phi_{emp}(t)\phi_k(ht)$  will then serve as an estimator of  $\phi_X(t)$ . For those  $\omega$ 's from the sample space  $\Omega$ , for which the distinguished logarithm in the integral below is well-defined,  $\rho$  can be estimated by the plug-in type estimator,

$$\rho_n(x) = \frac{1}{2\pi} \int_{-1/h}^{1/h} e^{-itx} \text{Log} \left( \frac{\phi_{emp}(t)\phi_k(ht)}{e^{i\tilde{\gamma}_n t} e^{-\tilde{\lambda}_n} e^{-\tilde{\sigma}_n^2 t^2/2}} \right) dt, \quad (12)$$

while for those  $\omega$ 's, for which the distinguished logarithm cannot be defined, we can assign an arbitrary value to  $\rho_n(x)$ , e.g. zero. Notice that the estimator (12) is real-valued, which can be seen by changing the integration variable from  $t$  into  $-t$ .

Our definition of the estimator is quite intuitive, however in order to investigate its asymptotic behaviour, some modifications are due: we need

to introduce truncation in the definition of  $\rho_n$  and consequently, we propose

$$\begin{aligned} \hat{\rho}_n(x) &= -i\tilde{\gamma}_n \frac{1}{2\pi} \int_{-1/h}^{1/h} e^{-itx} t dt + \tilde{\lambda}_n \frac{1}{2\pi} \int_{-1/h}^{1/h} e^{-itx} dt + \frac{\tilde{\sigma}_n^2}{2} \frac{1}{2\pi} \int_{-1/h}^{1/h} e^{-itx} t^2 dt \\ &\quad + \frac{1}{2\pi} \int_{-1/h}^{1/h} e^{-itx} \max\{\min\{M_n, \log(|\phi_{emp}(t)\phi_k(ht)|)\}, -M_n\} dt \\ &\quad + i \frac{1}{2\pi} \int_{-1/h}^{1/h} e^{-itx} \max\{\min\{M_n, \arg(\phi_{emp}(t)\phi_k(ht))\}, -M_n\} dt \end{aligned} \tag{13}$$

as an estimator of  $\rho(x)$ . Here  $M = (M_n)_{n \geq 1}$  denotes a sequence of positive numbers satisfying Condition 2.6, while  $\log$  and  $\arg$  are the real and imaginary parts of the distinguished logarithm, respectively. Notice that in (13) we essentially truncate the real and imaginary parts of the distinguished logarithm from above and from below. The truncation is only necessary to make asymptotic arguments work and in practice we do not need to employ it. Observe that  $|\hat{\rho}_n(x)|^2$  is integrable, since by Parseval's identity each summand in (13) is square integrable. Furthermore, by Theorem 2.3 the probability of the set, where the distinguished logarithm in (13) can be defined, tends to one as the sample size  $n$  tends to infinity.

We now state a condition on the kernel  $k$  that will be used when studying asymptotics of  $\hat{\rho}_n$ .

**Condition 3.1.** *Let the kernel  $k$  be the sinc kernel,  $k(x) = \sin x / (\pi x)$ .*

The Fourier transform of the sinc kernel is given by  $\phi_k(t) = 1_{[-1,1]}(t)$ . The use of the sinc kernel in our problem is equivalent to the use of the spectral cut-off method in [5] in a problem similar to ours. The sinc kernel has been used successfully in kernel density estimation since a long time, see e.g. [15, 16]. An attractive feature of the sinc kernel in ordinary kernel density estimation is that it is asymptotically optimal when one selects the mean square error or the mean integrated square error as the criterion of the performance of an estimator. Notice that the sinc kernel is not Lebesgue integrable, but its square is.

Now we will study the asymptotics of  $\hat{\rho}_n$ . As a criterion of performance of the estimator  $\hat{\rho}_n$  we select the mean integrated square error

$$\text{MISE}[\hat{\rho}_n] = \text{E} \left[ \int_{-\infty}^{\infty} |\hat{\rho}_n(x) - \rho(x)|^2 dx \right].$$

Other possible choices include, for instance, the mean square error and the mean integrated error of the estimator. These are not discussed here. The theorem given below constitutes the main result of the paper. It provides an order bound on  $\text{MISE}[\hat{\rho}_n]$  over an appropriate class of characteristic triplets and demonstrates that the estimator  $\hat{\rho}_n$  is consistent in the MISE sense.

**Theorem 3.1.** *Assume that assumptions of Theorems 2.1–2.4 hold. Let the estimator  $\hat{\rho}_n$  be defined by (13). Then*

$$\sup_{|\gamma| \leq \Gamma} \sup_{\sigma \in (0, \Sigma]} \sup_{\rho \in W_{sym}^*(\beta, L, C, \Lambda, K)} \text{MISE}[\hat{\rho}_n] \lesssim (\log n)^{-\beta},$$

where  $W_{sym}^*(\beta, L, C, \Lambda, K)$  denotes the class of Lévy densities  $\rho$ , such that  $\rho \in W_{sym}(\beta, L, \Lambda, K)$  and additionally

$$\int_{-\infty}^{\infty} |t|^{2\beta} |\phi_f(t)|^2 dt \leq C.$$

The remark that we made after Theorem 2.4 applies in this case as well: if we are willing to abandon the uniform convergence requirement, the similar upper bound as in Theorem 3.1 can be established for a fixed target density  $\rho$  without an assumption that it is necessarily symmetric. We state the corresponding theorem below.

**Theorem 3.2.** *Assume that Conditions 2.4–3.1 hold. Let  $\lambda > 0, \sigma > 0$  and let  $\rho$  satisfy (10). In addition, suppose that*

$$\int_{-\infty}^{\infty} |t|^{2\beta} |\phi_f(t)|^2 dt < \infty.$$

Let the estimator  $\hat{\rho}_n$  be defined by (13). Then

$$\text{MISE}[\hat{\rho}_n] \lesssim (\log n)^{-\beta}.$$

## 4 Lower bound for estimation of $\rho$

In the previous section we showed that under certain smoothness assumptions on the class of target densities  $\rho$ , the convergence rate of our estimator  $\hat{\rho}_n$  is logarithmic. This convergence rate can be easily understood on an intuitive level when comparing our problem to a deconvolution problem, see e.g. Section 10.1 of [34] for an introduction to deconvolution problems. A deconvolution problem consists of estimation of a density (or a distribution function) of a directly unobservable random variable  $Y$  based on i.i.d. copies  $X_1, \dots, X_n$  of a random variable  $X = Y + Z$ . The  $X$ 's can be thought of as repetitive measurements of  $Y$ , which are corrupted by an additive measurement error  $Z$ . It is well-known that if the distribution of  $Z$  is normal, and if the class of the target densities is sufficiently large, e.g. some Hölder class (see Definition 1.2 in [31]), the minimax convergence rate will be logarithmic for both the mean squared error and mean integrated squared error as measures of risk, see [19, 20]. We will prove a similar result for a problem of estimation of a Lévy density  $\rho$ .

**Theorem 4.1.** Denote by  $T$  an arbitrary Lévy triplet  $(\gamma, \sigma^2, \rho)$ , such that  $|\gamma| \leq \Gamma, \sigma \in (0, \Sigma], \lambda \in (0, \Lambda]$ . Furthermore, let

$$\int_{-\infty}^{\infty} |t|^{2\beta} |\phi_f(t)|^2 dt \leq C \quad (14)$$

for  $\beta \geq 1/2$ . Let  $\mathcal{T}$  be a collection of all such triplets. Then

$$\inf_{\tilde{\rho}_n} \sup_{\mathcal{T}} \text{MISE}[\tilde{\rho}_n] \gtrsim (\log n)^{-\beta},$$

where the infimum is taken over all estimators  $\tilde{\rho}_n$  based on observations  $X_1, \dots, X_n$ .

Using similar techniques, it is expected that lower bounds of the logarithmic order can be obtained for estimation of  $\gamma, \sigma^2$  and  $\lambda$  as well. Such a result is not surprising e.g. for  $\sigma^2$ , if one recalls comparable results from [9] for estimation of the error variance in the supersmooth deconvolution problem. Another paper containing examples of the breakdown of the usual root  $n$  convergence rate for estimation of a finite-dimensional parameter is [22]. We do not pursue this question any further. We also notice that the logarithmic lower bounds for estimation of the components of a characteristic triplet (under a different observation scheme) were obtained in [5].

Our estimation procedure for  $\rho$  in Section 3 relies on the assumption that the random variable  $X$  has a density (the latter is ensured by the condition  $\sigma > 0$ ). If  $\sigma = 0$ , then an approach of [18] may be used for estimation of  $\rho$ . For completeness purposes, however, we will show that the lower bound for the minimax risk in this case is not logarithmic as in Theorem 4.1, but polynomial.

**Theorem 4.2.** Let  $\mathcal{T}$  denote a collection of Lévy triplets  $T = (\gamma, 0, \rho)$ , such that  $|\gamma| \leq \Gamma$  and  $\lambda \in (0, \Lambda]$ . Furthermore, let  $\phi_f$  satisfy (14) for  $\beta \geq 1/2$ . Then

$$\inf_{\tilde{\rho}_n} \sup_{\mathcal{T}} \text{MISE}[\tilde{\rho}_n] \gtrsim n^{-2\beta/(2\beta+1)},$$

where the infimum is taken over all estimators  $\tilde{\rho}_n$  based on observations  $X_1, \dots, X_n$ .

This theorem in essence says that estimation of the Lévy density  $\rho$  in the case  $\sigma = 0$  seems to be as difficult as e.g. nonparametric density estimation based on i.i.d. observations coming from the target density itself, see e.g. Section 24.3 in [32]. This result has a parallel in [5]. In absence of the corresponding upper bound for estimation of  $\rho$  nothing can be said about how sharp the lower bound in Theorem 4.2 is, but in any case the polynomial minimax convergence rate seems to be natural. An upper bound of order  $n^{-\beta/(2\beta+1)}$  has been obtained in the compound Poisson model in [12] for the mean integrated squared error when estimating  $x\rho(x)$  under the condition that the class of Lévy densities is a Sobolev class  $\Sigma(\beta, C)$ .

## 5 Proofs

We first prove the following technical lemma.

**Lemma 5.1.** *Let the sets  $B_n$  and  $B_n^c$  be defined by (9). Suppose Conditions 2.5 and 2.6 hold. Then there exists an integer  $n_0$ , such that on the set  $B_n^c$  for all  $n \geq n_0$  we have*

$$\max\{\min\{M_n, \log(|\phi_{emp}(t)|)\}, -M_n\} = \log(|\phi_{emp}(t)|) \quad (15)$$

for  $t$  restricted to the interval  $[-h^{-1}, h^{-1}]$  and for all  $\rho \in W(\beta, L, \Lambda, K)$ ,  $\sigma \in (0, \Sigma]$  and  $|\gamma| \leq \Gamma$ . Furthermore,

$$\max\{\min\{M_n, \arg(\phi_{emp}(t))\}, -M_n\} = \arg(\phi_{emp}(t)) \quad (16)$$

for  $t$  restricted to the interval  $[-h^{-1}, h^{-1}]$  and for all  $\rho \in W_{sym}(\beta, L, \Lambda, K)$ ,  $\sigma \in (0, \Sigma]$  and  $|\gamma| \leq \Gamma$ . Here  $\arg$  denotes the imaginary part of the distinguished logarithm of  $\phi_{emp}(t)$ , i.e. a continuous version of its argument, such that  $\arg \phi_{emp}(0) = 0$ .

*Proof.* Formula (15) can be seen as follows:

$$\begin{aligned} |\log(|\phi_{emp}(t)|)| &\leq |\log(|\phi_X(t)|)| + \left| \log \left( \left| \frac{\phi_{emp}(t)}{\phi_X(t)} \right| \right) \right| \\ &\leq |\log(|\phi_X(t)|)| + \left| \frac{\phi_{emp}(t)}{\phi_X(t)} - 1 \right| + \left| \frac{\phi_{emp}(t)}{\phi_X(t)} - 1 \right|^2 \\ &\leq |\log(|\phi_X(t)|)| + \frac{3}{4} \\ &\leq 2\Lambda + \frac{\Sigma^2}{2h^2} + \frac{3}{4}. \end{aligned} \quad (17)$$

Here in the third line we used an elementary inequality  $|\log(1+z) - z| \leq |z|^2$  valid for  $|z| < 1/2$  and the fact that on the set  $B_n^c$  we have

$$\left| \left| \frac{\phi_{emp}(t)}{\phi_X(t)} \right| - 1 \right| \leq \left| \frac{\phi_{emp}(t)}{\phi_X(t)} - 1 \right| < \frac{1}{2}, \quad (18)$$

while in the last line we used the bound  $|\log |\phi_X(t)|| \leq 2\Lambda + \Sigma^2/(2h^2)$ . The equality (15) now is immediate from Conditions 2.5 and 2.6, because the upper bound for  $|\log(|\phi_{emp}(t)|)|$  grows slower than  $M_n$ . Next we prove (16). The symmetry of  $\rho$  implies that  $\phi_\rho$  is real-valued and hence  $\arg(\phi_X(t)) = 0$ . On the set  $B_n^c$  we have  $|\arg(\phi_{emp}(t))| \leq 2\pi$ , because the path  $\phi_{emp}(t)$  cannot make a turn around zero on this set. This proves (16), since  $M_n$  diverges to infinity.  $\square$

Now we are ready to prove Theorems 2.1–3.1.

*Proof of Theorem 2.1.* Write

$$\mathbb{E} [(\tilde{\sigma}_n^2 - \sigma^2)^2] = \mathbb{E} [(\tilde{\sigma}_n^2 - \sigma^2)^2 1_{B_n}] + \mathbb{E} [(\tilde{\sigma}_n^2 - \sigma^2)^2 1_{B_n^c}] = I + II,$$

where the set  $B_n$  is defined as in (9). For  $I$  we have

$$\begin{aligned} I &\lesssim \left( M_n^2 \left( \int_{-1/h}^{1/h} |v^h(t)| dt \right)^2 + \Sigma^4 \right) \mathbb{P}(B_n) \\ &\lesssim \left( M_n^2 \left( \int_{-1/h}^{1/h} |v^h(t)| dt \right)^2 + \Sigma^4 \right) \frac{e^{\Sigma^2/h^2}}{nh^2} \\ &= \left( M_n^2 h^4 \left( \int_{-1}^1 |v(t)| dt \right)^2 + \Sigma^4 \right) \frac{e^{\Sigma^2/h^2}}{nh^2}, \end{aligned}$$

where we used Theorem 2.3 to see the second line. Observe that under Conditions 2.5 and 2.6 the last term in the above chain of inequalities converges to zero faster than  $h^{2\beta+6}$ . Now we turn to  $II$ . On the set  $B_n^c$ , for  $n$  large enough, truncation in the definition of  $\tilde{\sigma}_n^2$  becomes unimportant, see Lemma 5.1, and we have

$$\begin{aligned} II &= \mathbb{E} \left[ \left( \int_{-1/h}^{1/h} \log(|\phi_{emp}(t)|) v^h(t) dt - \sigma^2 \right)^2 1_{B_n^c} \right] \\ &= \mathbb{E} \left[ \left( \int_{-1/h}^{1/h} \log \left( \left| \frac{\phi_{emp}(t)}{\phi_X(t)} \right| \right) v^h(t) dt + \int_{-1/h}^{1/h} \log(|\phi_X(t)|) v^h(t) dt - \sigma^2 \right)^2 1_{B_n^c} \right]. \end{aligned}$$

Using this fact, (4) and an elementary inequality  $(a+b)^2 \leq 2(a^2 + b^2)$ , we obtain that

$$\begin{aligned} II &\lesssim \Lambda^2 \left( \int_{-1/h}^{1/h} \Re(\phi_f(t)) v^h(t) dt \right)^2 \\ &\quad + \mathbb{E} \left[ \left( \int_{-1/h}^{1/h} \log \left( \left| \frac{\phi_{emp}(t)}{\phi_X(t)} \right| \right) v^h(t) dt \right)^2 1_{B_n^c} \right] \\ &= III + IV. \end{aligned}$$

For *III* we have

$$\begin{aligned}
III &\lesssim h^{2\beta} \left( \int_{-1/h}^{1/h} t^\beta \Re(\phi_f(t)) \frac{v^h(t)}{(ht)^\beta} dt \right)^2 \\
&\lesssim h^{2\beta+6} \left( \int_{-\infty}^{\infty} |t^\beta| |\Re(\phi_f(t))| dt \right)^2 \\
&\lesssim h^{2\beta+6} \left( \int_{-\infty}^{\infty} |t^\beta| |\phi_f(t)| dt \right)^2 \\
&\lesssim h^{2\beta+6} \\
&\lesssim (\log n)^{-\beta-3},
\end{aligned}$$

where in the second line we used Condition 2.4, to obtain the third line we used the fact that  $|\Re(\phi_f(t))| \leq |\phi_f(t)| + |\phi_f(-t)|$ , while the fourth line follows from Condition 2.1. We turn to *IV*. We have

$$\begin{aligned}
IV &\lesssim \mathbb{E} \left[ \left( \int_{-1/h}^{1/h} \left| \frac{\phi_{emp}(t)}{\phi_X(t)} - 1 \right| v^h(t) dt \right)^2 \mathbf{1}_{B_n^c} \right] \\
&\quad + \mathbb{E} \left[ \left( \int_{-1/h}^{1/h} \left\{ \log \left( \left| \frac{\phi_{emp}(t)}{\phi_X(t)} \right| \right) - \left( \left| \frac{\phi_{emp}(t)}{\phi_X(t)} \right| - 1 \right) \right\} v^h(t) dt \right)^2 \mathbf{1}_{B_n^c} \right] \\
&= V + VI.
\end{aligned}$$

Some further bounding and an application of the Cauchy-Schwarz inequality give

$$V \lesssim e^{4\Lambda + \Sigma^2/h^2} \int_{-1/h}^{1/h} (v^h(t))^2 dt \mathbb{E} \left[ \int_{-1/h}^{1/h} |\phi_{emp}(t) - \phi_X(t)|^2 dt \right].$$

Parseval's identity and Proposition 1.7 of [31] applied to the sinc kernel then yield

$$\mathbb{E} \left[ \int_{-1/h}^{1/h} |\phi_{emp}(t) - \phi_X(t)|^2 dt \right] = 2\pi \mathbb{E} \left[ \int_{-1/h}^{1/h} (q_n(x) - \mathbb{E}[q_n(x)])^2 dx \right] \lesssim \frac{1}{nh},$$

whence

$$V \lesssim e^{\Sigma^2/h^2} h^4 \frac{1}{n}.$$

As far as *VI* is concerned, using (18), an elementary inequality  $|\log(1+z) - z| \leq |z|^2$ , valid for  $|z| < 1/2$ , and the Cauchy-Schwarz inequality, we obtain



that

$$\begin{aligned}
VI &\lesssim \int_{-1/h}^{1/h} (v^h(t))^2 dt \mathbf{E} \left[ \int_{-1/h}^{1/h} \left| \frac{\phi_{emp}(t)}{\phi_X(t)} - 1 \right|^4 dt 1_{B_n^c} \right] \\
&\leq \frac{1}{4} \int_{-1/h}^{1/h} (v^h(t))^2 dt \mathbf{E} \left[ \int_{-1/h}^{1/h} \left| \frac{\phi_{emp}(t)}{\phi_X(t)} - 1 \right|^2 dt \right] \\
&\lesssim e^{\Sigma^2/h^2} \int_{-1/h}^{1/h} (v^h(t))^2 dt \mathbf{E} \left[ \int_{-1/h}^{1/h} |\phi_{emp}(t) - \phi_X(t)|^2 dt \right].
\end{aligned}$$

Hence  $VI$  can be analysed in the same way as  $V$ . From the above bounds on  $V$  and  $VI$  it also follows that  $IV$  is negligible in comparison to  $III$ . Combination of all these intermediate results completes the proof of the theorem.  $\square$

*Proof of Theorem 2.2.* The proof is quite similar to that of Theorem 2.1. Write

$$\mathbf{E}[(\tilde{\lambda}_n - \lambda)^2] = \mathbf{E}[(\tilde{\lambda}_n - \lambda)^2 1_{B_n}] + \mathbf{E}[(\tilde{\lambda}_n - \lambda)^2 1_{B_n^c}] = I + II.$$

By an argument similar to that in the proof of Theorem 2.1,

$$I \lesssim (M_n^2 \left( \int_{-1}^1 |u(t)| dt \right)^2 + \Lambda^2) \frac{e^{\Sigma^2/h^2}}{nh^2}.$$

This is negligible compared to  $h^{2\beta+2}$ . Now we turn to  $II$ . We have

$$\begin{aligned}
II &= \mathbf{E} \left[ \left( \int_{-1/h}^{1/h} \log(|\phi_{emp}(t)|) u^h(t) dt - \lambda \right)^2 1_{B_n^c} \right] \\
&= \mathbf{E} \left[ \left( \int_{-1/h}^{1/h} \left\{ \log \left( \left| \frac{\phi_{emp}(t)}{\phi_X(t)} \right| \right) + \log(|\phi_X(t)|) \right\} u^h(t) dt - \lambda \right)^2 1_{B_n^c} \right] \\
&\lesssim \Lambda^2 \left( \int_{-1/h}^{1/h} \Re(\phi_f(t)) u^h(t) dt \right)^2 \\
&+ \mathbf{E} \left[ \left( \int_{-1/h}^{1/h} \log \left( \left| \frac{\phi_{emp}(t)}{\phi_X(t)} \right| \right) u^h(t) dt \right)^2 1_{B_n^c} \right] \\
&= III + IV.
\end{aligned}$$

Here in the third line we used (6). Similar as we did it for  $III$  in the proof of Theorem 2.1, one can check that in this case as well  $III \lesssim h^{2\beta+2}$ . As far as  $IV$  is concerned, it is of order  $e^{\Sigma^2/h^2} n^{-1}$ , which can be seen by exactly the same reasoning as in the proof of Theorem 2.1. Combination of these results completes the proof of the theorem, because under Condition 2.5 the dominating term is  $III$ .  $\square$

*Proof of Theorem 2.3.* By Chebyshev's inequality

$$P(B_n) \leq \frac{1}{\delta^2} \mathbb{E} \left[ \left( \sup_{t \in [-h^{-1}, h^{-1}]} |\phi_{emp}(t) - \phi_X(t)| \right)^2 \right].$$

Thus we need to bound the expectation on the right-hand side. This will be done via reasoning similar to that on pp. 326–327 in [9]. For all unexplained terminology and notation used in the sequel we refer to Chapter 2 of [33]. Notice that

$$\mathbb{E} \left[ \left( \sup_{t \in [-h^{-1}, h^{-1}]} |\phi_{emp}(t) - \phi_X(t)| \right)^2 \right] = \frac{1}{n} \mathbb{E} \left[ \left( \sup_{t \in [-h^{-1}, h^{-1}]} |G_n v_t| \right)^2 \right].$$

Here  $G_n v_t$  denotes an empirical process defined by

$$G_n v_t = \frac{1}{\sqrt{n}} \sum_{j=1}^n (v_t(X_j) - \mathbb{E} v_t(X_j)),$$

where the function  $v_t : x \mapsto e^{itx}$ . Introduce the functions  $v_t^1 : x \mapsto \cos(tx)$  and  $v_t^2 : x \mapsto \sin(tx)$ . Then

$$\begin{aligned} \mathbb{E} \left[ \left( \sup_{t \in [-h^{-1}, h^{-1}]} |G_n v_t| \right)^2 \right] &\lesssim \mathbb{E} \left[ \left( \sup_{t \in [-h^{-1}, h^{-1}]} |G_n v_t^1| \right)^2 \right] \\ &\quad + \mathbb{E} \left[ \left( \sup_{t \in [-h^{-1}, h^{-1}]} |G_n v_t^2| \right)^2 \right]. \end{aligned}$$

As it will turn out below, both terms on the right-hand side can be treated in the same manner. Observe that the mean value theorem implies

$$|v_t^i(x) - v_s^i(x)| \leq |x| |t - s| \tag{19}$$

for  $i = 1, 2$ , i.e.  $v_t^i$  is Lipschitz in  $t$ . Theorem 2.7.11 of [33] applies and gives that the bracketing number  $N_{[]}^i$  of the class of functions  $\mathbb{F}_n$  (this refers either to  $v_t^1$  or  $v_t^2$  for  $|t| \leq h^{-1}$ ) is bounded by the covering number  $N$  of the interval  $I_n = [-h^{-1}, h^{-1}]$ , i.e.

$$N_{[]}^i(2\epsilon \|x\|_{\mathbb{L}_2(Q)}; \mathbb{F}_n; \mathbb{L}_2(Q)) \leq N(\epsilon; I_n; |\cdot|).$$

Here  $Q$  is any discrete probability measure, such that  $\|x\|_{\mathbb{L}_2(Q)} > 0$ . Since

$$N(\epsilon \|x\|_{\mathbb{L}_2(Q)}; \mathbb{F}_n; \mathbb{L}_2(Q)) \leq N_{[]}^i(2\epsilon \|x\|_{\mathbb{L}_2(Q)}; \mathbb{F}_n; \mathbb{L}_2(Q)),$$

see p. 84 in [33], and trivially

$$N(\epsilon; I_n; |\cdot|) \leq \frac{2}{\epsilon} \frac{1}{h},$$

we obtain that

$$N(\epsilon \|x\|_{\mathbb{L}_2(Q)}; \mathbb{F}_n; \mathbb{L}_2(Q)) \leq \frac{2}{\epsilon} \frac{1}{h}. \quad (20)$$

Define  $J(1, \mathbb{F}_n)$ , the entropy of the class  $\mathbb{F}_n$ , as

$$J(1, \mathbb{F}_n) = \sup_Q \int_0^1 \{1 + \log(N(\epsilon \|x\|_{\mathbb{L}_2(Q)}; \mathbb{F}_n; \mathbb{L}_2(Q)))\}^{1/2} d\epsilon,$$

where the supremum is taken over all discrete probability measures  $Q$ , such that  $\|x\|_{\mathbb{L}_2(Q)} > 0$ . Since  $\mathbb{F}_n$  is a measurable class of functions with a measurable envelope (the latter follows from (19)), by Theorem 2.14.1 in [33] we obtain that

$$\mathbb{E} \left[ \left( \sup_{t \in [-h^{-1}, h^{-1}]} |G_n v_t^i| \right)^2 \right] \lesssim \|x\|_{\mathbb{L}_2(\mathbb{P})}^2 (J(1, \mathbb{F}_n))^2,$$

where the probability  $\mathbb{P}$  refers to  $\mathbb{P}_{\gamma, \sigma^2, \rho}$ . Now notice that

$$\|x\|_{\mathbb{L}_2(\mathbb{P})}^2 = \mathbb{E}[(\gamma + Y + \sigma Z)^2] \lesssim \gamma^2 + \mathbb{E}[Y^2] + \sigma^2,$$

where  $Y := \sum_{j=1}^{N(\lambda)} W_j$  denotes the Poisson sum of i.i.d. random variables  $W_j$  with density  $f$ , while  $Z$  is a standard normal variable. Under conditions of the theorem the term

$$\mathbb{E}[Y^2] = \lambda^2 \left( \int_{-\infty}^{\infty} x f(x) dx \right)^2 + \lambda \int_{-\infty}^{\infty} x^2 f(x) dx,$$

is bounded uniformly in  $\rho$ . Hence  $\|x\|_{\mathbb{L}_2(\mathbb{P})}^2$  is also bounded uniformly in  $\rho, \sigma$  and  $\gamma$ . Using (20), the entropy can be further bounded as

$$J(1, \mathbb{F}_n) \leq \int_0^1 \left\{ 1 + \log \left( \frac{2}{\epsilon} \frac{1}{h} \right) \right\}^{1/2} d\epsilon.$$

Here we implicitly assume that  $n$  is large enough, so that we take a square root of a positive number. Working out the integral, it is not difficult to check that  $J(1, \mathbb{F}_n) = O(h^{-1})$ . Combination of these results yields the statement of the theorem.  $\square$

*Proof of Theorem 2.4.* Again, the proof is quite similar to that of Theorem 2.1. Write

$$\mathbb{E}[(\tilde{\gamma}_n - \gamma)^2] = \mathbb{E}[(\tilde{\gamma}_n - \gamma)^2 1_{B_n}] + \mathbb{E}[(\tilde{\gamma}_n - \gamma)^2 1_{B_n^c}] = I + II.$$

For  $I$  we have

$$I \lesssim \left( M_n^2 h^2 \left( \int_{-1}^1 |w(t)| dt \right)^2 + \Gamma^2 \right) \mathbb{P}(B_n).$$

Thanks to Theorem 2.3 the right-hand side converges to zero as  $n \rightarrow \infty$ . Moreover, it is negligible compared to  $h^{2\beta+4}$ . Next we turn to  $II$ . By Lemma 5.1 on the set  $B_n^c$  for  $n$  large enough truncation in the definition of  $\tilde{\gamma}_n$  becomes unimportant and we have

$$\begin{aligned}
II &= \mathbb{E} \left[ \left( \int_{-1/h}^{1/h} \Im(\text{Log}(\phi_{emp}(t))) w^h(t) dt - \gamma \right)^2 1_{B_n^c} \right] \\
&\lesssim \Lambda^2 \mathbb{E} \left[ \left( \int_{-1/h}^{1/h} \Im(\phi_f(t)) w^h(t) dt \right)^2 1_{B_n^c} \right] \\
&+ \mathbb{E} \left[ \left( \int_{-1/h}^{1/h} \Im \left( \text{Log} \left( \frac{\phi_{emp}(t)}{\phi_X(t)} \right) \right) w^h(t) dt \right)^2 1_{B_n^c} \right] \\
&= III + IV.
\end{aligned}$$

The same reasoning as in Theorem 2.1 shows that here as well  $III$  is of order  $h^{2\beta+4}$ . As far as  $IV$  is concerned, the inequality  $|\Im(z)| \leq |z|$  implies that

$$IV \lesssim \mathbb{E} \left[ \left( \int_{-1/h}^{1/h} \left| \text{Log} \left( \frac{\phi_{emp}(t)}{\phi_X(t)} \right) \right| w^h(t) dt \right)^2 1_{B_n^c} \right].$$

Now notice that on the set  $B_n^c$  the inequality

$$\left| \text{Log} \left( \frac{\phi_{emp}(t)}{\phi_X(t)} \right) - \left( \frac{\phi_{emp}(t)}{\phi_X(t)} - 1 \right) \right| \leq \left| \frac{\phi_{emp}(t)}{\phi_X(t)} - 1 \right|^2 \quad (21)$$

holds, cf. formula (4.8) in [18]. Therefore

$$\begin{aligned}
IV &\lesssim \mathbb{E} \left[ \left( \int_{-1/h}^{1/h} \left| \frac{\phi_{emp}(t)}{\phi_X(t)} - 1 \right| w^h(t) dt \right)^2 1_{B_n^c} \right] \\
&+ \mathbb{E} \left[ \left( \int_{-1/h}^{1/h} \left| \frac{\phi_{emp}(t)}{\phi_X(t)} - 1 \right|^2 w^h(t) dt \right)^2 1_{B_n^c} \right].
\end{aligned}$$

Just as for  $IV$  in the proof of Theorem 2.1, one can check that in this case as well  $IV$  is negligible in comparison to  $III$ . Combination of these results completes the proof of the theorem.  $\square$

*Proof of Theorem 2.5.* The proof follows essentially the same steps as the proof of Theorem 2.4. The only significant difference is that we have to verify that there exists an integer  $n_0$ , such that on the set  $B_{nh}^c$  for all  $n \geq n_0$  truncation in the definition of  $\tilde{\gamma}_n$  is unimportant for an arbitrary  $\rho$  satisfying

conditions of the theorem, and not necessarily for a symmetric  $\rho$  as in Lemma 5.1. To see this, first notice that

$$\begin{aligned}\Im(\text{Log}(\phi_{emp}(t))) &= \Im(\text{Log}(e^\lambda e^{\sigma^2 t^2/2} \phi_{emp}(t))), \\ \Im(\text{Log}(\phi_X(t))) &= \Im(\text{Log}(e^\lambda e^{\sigma^2 t^2/2} \phi_X(t))) = \Im(e^{\lambda \phi_f(t)}).\end{aligned}$$

Let  $\psi : \mathbb{R} \rightarrow \mathbb{C}$ , where

$$\psi(t) = \phi_X(t) e^\lambda e^{\sigma^2 t^2/2} = e^{\lambda \phi_f(t)}.$$

By the Riemann-Lebesgue theorem  $\psi(t)$  converges to 1 as  $|t| \rightarrow \infty$  and hence there exists  $t^* > 0$ , such that

$$|\psi(t) - 1| < \frac{e^{-\lambda}}{2}, \quad |t| > t^*. \quad (22)$$

Furthermore, we have

$$|\psi(t)| \geq e^{-\lambda}, \quad t \in \mathbb{R}. \quad (23)$$

Since  $f$  has a finite second moment, by Theorem 1 on p. 182 of [28] the characteristic function  $\phi_f$  is continuously differentiable. Consequently, so is the exponent  $\psi$ . Therefore the path  $\psi : [-t^*, t^*] \rightarrow \mathbb{C}$  is rectifiable, i.e. has a finite length. In view of this fact and (23),  $\psi : [-t^*, t^*] \rightarrow \mathbb{C}$  cannot spiral infinitely many times around zero (because otherwise it would have an infinite length) and for  $|t| > t^*$  it cannot make a turn around zero at all because of (22). Since  $M_n$  diverges to infinity, it follows that for every  $\omega \in B_{nh}^c$  there exists  $n_0(\omega)$ , such that  $h_{n_0}^{-1} \geq t^*$  and for all  $n \geq n_0(\omega)$

$$\max\{\min\{M_n, \Im(\text{Log}(\phi_{emp}(t)))\}, -M_n\} = \Im(\text{Log}(\phi_{emp}(t))). \quad (24)$$

However, it is easy to see that in fact there exist a universal integer  $n_0$ , such that (24) holds for all  $\omega \in B_{nh}^c$ : just notice that for each  $\omega$  the number of turns that  $\phi_{emp}(t)$  makes around zero is determined by the number of turns  $m$  that  $\psi(t)$  makes around zero and cannot be greater than  $2m$ , say. Consequently, there exists a universal bound  $4m\pi$  on  $\Im(\text{Log}(\phi_{emp}(t)))$  valid for all  $\omega \in B_{nh}^c$ . This concludes the proof of the theorem.  $\square$

*Proof of Theorem 3.1.* We have

$$\begin{aligned}\mathbb{E} \left[ \int_{-\infty}^{\infty} |\hat{\rho}_n(x) - \rho(x)|^2 dx \right] &= \mathbb{E} \left[ \int_{-\infty}^{\infty} |\hat{\rho}_n(x) - \rho(x)|^2 dx 1_{B_n} \right] \\ &\quad + \mathbb{E} \left[ \int_{-\infty}^{\infty} |\hat{\rho}_n(x) - \rho(x)|^2 dx 1_{B_n^c} \right] \\ &= I + II,\end{aligned}$$

where  $B_n$  and  $B_n^c$  are defined by (9). Notice that

$$\int_{-\infty}^{\infty} |\hat{\rho}_n(x) - \rho(x)|^2 dx \lesssim \int_{-\infty}^{\infty} |\hat{\rho}_n(x)|^2 dx + \int_{-\infty}^{\infty} |\rho(x)|^2 dx.$$

By Parseval's identity and Condition 2.1

$$\int_{-\infty}^{\infty} |\rho(x)|^2 dx \lesssim 1.$$

For the Fourier transform of  $\hat{\rho}_n$  we have

$$|\phi_{\hat{\rho}_n}(t)| \lesssim M_n 1_{[-h^{-1}, h^{-1}]}(t).$$

Hence by Parseval's identity

$$\int_{-\infty}^{\infty} |\hat{\rho}_n(x)|^2 dx \lesssim M_n^2 \frac{1}{h}.$$

Using this and Theorem 2.3, we get that

$$I \lesssim \left\{ M_n^2 \frac{1}{h} + 1 \right\} \frac{e^{\Sigma^2/h^2}}{nh^2}.$$

Under Conditions 2.5 and 2.6 the latter is negligible in comparison to  $h^{2\beta}$ .

Now we turn to *II*. By Parseval's identity

$$\begin{aligned} II &= \frac{1}{2\pi} \mathbb{E} \left[ \int_{-\infty}^{\infty} |\phi_{\hat{\rho}_n}(t) - \phi_{\rho}(t)|^2 dt 1_{B_n^c} \right] \\ &= \frac{1}{2\pi} \mathbb{E} \left[ \int_{-1/h}^{1/h} |\phi_{\hat{\rho}_n}(t) - \phi_{\rho}(t)|^2 dt 1_{B_n^c} \right] + \frac{1}{2\pi} \int_{\mathbb{R} \setminus (-h^{-1}, h^{-1})} |\phi_{\rho}(t)|^2 dt \mathbb{P}(B_n^c) \\ &= III + IV. \end{aligned}$$

For *IV* we have

$$\begin{aligned} IV &\leq \int_{\mathbb{R} \setminus (-h^{-1}, h^{-1})} |\phi_{\rho}(t)|^2 dt = \lambda^2 \int_{\mathbb{R} \setminus (-h^{-1}, h^{-1})} |t^{2\beta}| \frac{|\phi_{\rho}(t)|^2}{|t^{2\beta}|} dt \\ &\leq \lambda^2 h^{2\beta} \int_{-\infty}^{\infty} |t^{2\beta}| |\phi_f(t)|^2 dt \\ &\leq C \Lambda^2 h^{2\beta}, \end{aligned}$$

where the last inequality follows from the definition of the class  $W_{sym}^*(\beta, L, C, \Lambda, K)$ .

Next we turn to *III*. With (15) and (16) we have that

$$III = \frac{1}{2\pi} \mathbb{E} \left[ \int_{-1/h}^{1/h} |\phi_{\rho_n}(t) - \phi_{\rho}(t)|^2 dt 1_{B_n^c} \right]$$

for all  $n$  large enough. Consequently,

$$\begin{aligned}
III &\lesssim \mathbb{E} \left[ (\tilde{\sigma}_n^2 - \sigma^2)^2 \int_{-1/h}^{1/h} t^4 dt 1_{B_n^c} \right] \\
&+ \mathbb{E} \left[ \int_{-1/h}^{1/h} |\text{Log}(\phi_{emp}(t)) - \text{Log}(\phi_X(t))|^2 1_{B_n^c} \right] \\
&+ \mathbb{E} \left[ (\tilde{\gamma}_n - \gamma)^2 \int_{-1/h}^{1/h} t^2 dt 1_{B_n^c} \right] \\
&+ \mathbb{E} \left[ (\tilde{\lambda}_n - \lambda)^2 \int_{-1/h}^{1/h} dt 1_{B_n^c} \right] \\
&= IV + V + VI + VII.
\end{aligned}$$

For  $IV$  we have by Theorem 2.1 that

$$IV \lesssim \frac{1}{h^5} \mathbb{E} \left[ (\tilde{\sigma}_n^2 - \sigma^2)^2 1_{B_n^c} \right] = O(h^{2\beta+1}).$$

As far as  $V$  is concerned, by the inequality (21)

$$V \lesssim \mathbb{E} \left[ \int_{-1/h}^{1/h} \left| \frac{\phi_{emp}(t)}{\phi_X(t)} - 1 \right|^2 dt 1_{B_n^c} \right] + \mathbb{E} \left[ \int_{-1/h}^{1/h} \left| \frac{\phi_{emp}(t)}{\phi_X(t)} - 1 \right|^4 dt 1_{B_n^c} \right].$$

The right-hand side can be analysed similar to  $V$  in the proof of Theorem 2.1 and in fact it is negligible in comparison to  $h^{2\beta}$ . Furthermore, by Theorem 2.4  $VI$  is of order  $h^{2\beta+1}$ . Also  $VII$  is of order  $h^{2\beta+1}$  by Theorem 2.2. Combination of all the intermediate results completes the proof of the theorem.  $\square$

*Proof of Theorem 3.2.* The proof uses the same type of arguments as that of Theorem 3.1. The only essential difference is to show that there exists  $n_0$ , such that on the set  $B_{nh}^c$  for all  $n \geq n_0$  we have  $\hat{\rho}_n(x) = \rho_n(x)$ . We therefore consider in detail only this part of the proof. For  $\arg(\phi_{emp}(t))$  the corresponding argument was already given in the proof of Theorem 2.5. Thus we only have to prove that

$$\max\{\min\{M_n, \log(|\phi_{emp}(t)|)\}, -M_n\} 1_{B_{nh}^c} = \log(|\phi_{emp}(t)|) 1_{B_{nh}^c}.$$

The latter can be shown by exactly the same arguments that were used in the proof of (15) in Lemma 5.1.  $\square$

*Proof of Theorem 4.1.* The proof makes use of some of the ideas found in [10, 19]. Consider two Lévy triplets  $T_1 = (0, \sigma^2, \rho_1)$  and  $T_2 = (0, \sigma^2, \rho_2)$ , where  $\rho_i(x) = \lambda f_i(x)$ ,  $i = 1, 2$  and  $\lambda < \Lambda$ . Let

$$f_1(x) = \frac{1}{2}(r_1(x) + r_2(x)),$$

where the probability densities  $r_1$  and  $r_2$  are defined via their characteristic functions,

$$r_1(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-itx} \frac{1}{(1+t^2/\beta_1^2)^{(\beta_2+1)/2}} dt; \quad r_2(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-itx} e^{-\alpha_1|t|^{\alpha_2}} dt.$$

With a proper selection of  $\beta_1, \beta_2, \alpha_1$  and  $\alpha_2$  one can achieve that  $f_1$  satisfies (14) with a constant  $C/4$  (instead of  $C$ ). We also assume that  $1 < \alpha_2 < 2$ . Notice that  $r_1$  is a bilateral gamma density, while  $r_2$  is a stable density. To define  $f_2$ , we perturb  $f_1$  as follows:

$$f_2(x) = f_1(x) + \delta_n^{\beta-1/2} H(x/\delta_n),$$

where  $\delta_n \rightarrow 0$  as  $n \rightarrow \infty$ , and the function  $H$  satisfies the following conditions:

1.  $\int_{-\infty}^{\infty} |t|^{2\beta} |\phi_H(t)|^2 dt \leq C/4$ ;
2.  $\int_{-\infty}^{\infty} H(x) dx = 0$ ;
3.  $\int_{-\infty}^0 H(x) dx \neq 0$ ;
4.  $\phi_H(t) = 0$  for  $t$  outside  $[1, 2]$ ;
5.  $\phi_H(t)$  is twice continuously differentiable.

To see why such a function exists, see e.g. p. 1268 in [19]. It is also obvious, that there are many functions  $H$  with an appropriate tail behaviour, such that  $f_2(x) \geq 0$  for all  $x \in \mathbb{R}$ , at least for small enough  $\delta_n$ . With such an  $H$  and small enough  $\delta_n$ , the function  $f_2$  will be a probability density satisfying (14). Notice that

$$\int_{-\infty}^{\infty} (\rho_2(x) - \rho_1(x))^2 dx \asymp \delta_n^{2\beta}. \quad (25)$$

Here the symbol  $\asymp$  means ‘asymptotically of the same order’. Denote by  $q_i$  a density of a random variable  $X$  corresponding to a triplet  $T_i, i = 1, 2$ . The statement of the theorem will follow from (25) and Lemma 8 of [10], if we prove that the  $\chi^2$ -divergence (see p. 72 in [31] for a definition) between  $q_2$  and  $q_1$  satisfies

$$n\chi^2(q_2, q_1) = n \int_{-\infty}^{\infty} \frac{(q_2(x) - q_1(x))^2}{q_1(x)} dx \leq c, \quad (26)$$

where a positive constant  $c < 1$  is independent of  $n$ .

Let  $g_i$  be a density of a Poisson sum  $Y$  conditional on the fact that its number of summands  $N(\lambda) > 0$ . Here the index  $i$  refers to a triplet  $T_i, i = 1, 2$ . Since

$$\phi_Y(t) = e^{-\lambda} + (1 - e^{-\lambda}) \frac{1}{e^\lambda - 1} \left( e^{\lambda \phi_{f_i}(t)} - 1 \right), \quad (27)$$



it follows that

$$\phi_{g_i}(t) = \frac{1}{e^\lambda - 1} \left( e^{\lambda \phi_{f_i}(t)} - 1 \right).$$

We also have

$$g_i(x) = \sum_{n=1}^{\infty} f_i^{*n}(x) P(N(\lambda) = n | N(\lambda) > 0). \quad (28)$$

From (27) we obtain

$$q_1(x) \geq (1 - e^{-\lambda}) \phi_{0, \sigma^2} * g_1(x),$$

where  $\phi_{0, \sigma^2}$  denotes a normal density with mean zero and variance  $\sigma^2$ . Moreover, by Lemma 2 of [9], there exists a large enough constant  $A$ , such that the right-hand side of the above display is not less than  $(1 - e^{-\lambda}) g_1(|x| + A)$ . Hence

$$n\chi^2(q_2, q_1) \lesssim n \int_{-\infty}^{\infty} \frac{(q_2(x) - q_1(x))^2}{g_1(|x| + A)} dx \lesssim n \int_{-\infty}^{\infty} \frac{(q_2(x) - q_1(x))^2}{f_1(|x| + A)} dx,$$

where the last inequality follows from (28). Splitting the integration region into two parts, we then get that

$$\begin{aligned} n\chi^2(q_2, q_1) &\lesssim n \int_{|x| \leq A} (q_2(x) - q_1(x))^2 dx + n \int_{|x| > A} x^4 (q_2(x) - q_1(x))^2 dx \\ &= I + II. \end{aligned}$$

Here we used the fact that  $f_1(x)$  behaves as  $|x|^{-1-\alpha_2}$  at plus and minus infinity, see e.g. formula (14.37) in [27], and that  $1 < \alpha_2 < 2$ . Since

$$\delta_n^{\beta-1/2} \int_{-\infty}^{\infty} e^{itx} H(x/\delta_n) dx = \delta_n^{\beta+1/2} \phi_H(\delta_n t),$$

by Parseval's identity it holds that

$$\begin{aligned} I &\leq n \frac{1}{2\pi} \int_{-\infty}^{\infty} |\phi_{q_2}(t) - \phi_{q_1}(t)|^2 dt \\ &= n \frac{(1 - e^{-\lambda})^2}{2\pi} \int_{-\infty}^{\infty} |\phi_{g_2}(t) - \phi_{g_1}(t)|^2 e^{-\sigma^2 t^2} dt \\ &= n \frac{(1 - e^{-\lambda})^2}{(e^\lambda - 1)^2} \frac{1}{2\pi} \int_{-\infty}^{\infty} |e^{\lambda \phi_{f_2}(t)} - e^{\lambda \phi_{f_1}(t)}|^2 e^{-\sigma^2 t^2} dt \\ &\lesssim n \int_{-\infty}^{\infty} |\phi_{f_2}(t) - \phi_{f_1}(t)|^2 e^{-\sigma^2 t^2} dt, \end{aligned}$$

where the last inequality follows from the mean-value theorem applied to the function  $e^x$  and the fact that  $|\lambda \phi_{f_i}(t)| \leq \lambda$ . By definition of  $f_1$  and  $f_2$

we then get that

$$\begin{aligned}
I &\lesssim n\delta_n^{2\beta+1} \int_{-\infty}^{\infty} |\phi_H(\delta_n t)|^2 e^{-\sigma^2 t^2} dt \\
&= n\delta_n^{2\beta} \int_{-\infty}^{\infty} |\phi_H(s)|^2 e^{-\sigma^2 s^2 / \delta_n^2} ds \\
&= O\left(n\delta_n^{2\beta} e^{-\sigma^2 / \delta_n^2}\right).
\end{aligned}$$

The choice  $\delta_n \asymp (\log n)^{-1/2}$  with small enough constant will now imply that  $I \rightarrow 0$  as  $n \rightarrow \infty$ . Next we turn to  $II$ . By Parseval's identity

$$II \leq n \frac{1}{2\pi} \int_{-\infty}^{\infty} |(\phi_{q_2}(t) - \phi_{q_1}(t))'|^2 dt.$$

Here we use the fact that even though  $\phi_{f_1}$  and  $\phi_{f_2}$  are not twice differentiable at zero, the difference  $\phi_{q_2}(t) - \phi_{q_1}(t)$  still is, because  $\phi_H$  is identically zero outside the interval  $[1, 2]$ . By exactly the same type of arguments as we used for  $I$ , one can show that  $II \rightarrow 0$  as  $n \rightarrow \infty$ , provided  $\delta_n \asymp (\log n)^{-1/2}$ . Hence (26) is satisfied and the statement of the theorem follows.  $\square$

*Proof of Theorem 4.2.* The proof is similar to the proof of Theorem 4.1. Let  $\rho_1(x) = \lambda f_1(x)$  with  $f_1$  as in the proof of Theorem 4.1. Consider a perturbation of  $\rho_1$ , say  $\rho_2(x) = \lambda f_2(x)$ , where  $f_2$  is defined as in Theorem 4.1. Assume that the function  $H$  in the definition of  $f_2$  has a compact support on  $[-1, 1]$  and that it satisfies Conditions 1–3 in the proof of Theorem 4.1. This implies that  $f_2(x) \geq 0$  for  $\delta_n$  small enough. Therefore  $\rho_2$  is a Lévy density satisfying (14), provided  $\delta_n$  is small enough. Denote by  $\mathbb{P}_{1n}$  and  $\mathbb{P}_{2n}$  the laws of a Lévy process  $X = (X)_{t \geq 0}$  restricted to the time interval  $[0, n]$  and corresponding to the characteristic triplets  $T_1 = (0, 0, \rho_1)$  and  $T_2 = (0, 0, \rho_2)$ , respectively. Notice that

$$\inf_{\tilde{\rho}_n} \sup_{\mathcal{T}} \mathbb{E} \left[ \int_{-\infty}^{\infty} (\tilde{\rho}_n(x) - \rho(x))^2 dx \right] \geq \inf_{\rho_n} \sup_{\mathcal{T}} \mathbb{E} \left[ \int_{-\infty}^{\infty} (\rho_n(x) - \rho(x))^2 dx \right], \quad (29)$$

where  $\rho_n$  denotes an arbitrary estimator based on a continuous record of observations of  $X$  over  $[0, n]$ . Let  $K(P, Q)$  denote the Kullback-Leibler divergence between the probability measures  $P$  and  $Q$ ,

$$K(P, Q) = \begin{cases} \int \log \frac{dP}{dQ} dP & \text{if } P \ll Q, \\ +\infty & \text{if otherwise,} \end{cases}$$

see Definition 2.5 in [31]. In view of (25), the result will follow from formula (29) above, the arguments of Section 2.2 of [31] combined with Theorem 2.2 (iii) of [31], provided the Kullback-Leibler divergence  $K(\mathbb{P}_{2n}, \mathbb{P}_{1n})$  between the measures  $\mathbb{P}_{2n}$  and  $\mathbb{P}_{1n}$  remains bounded for all  $n$  by a constant

independent of  $n$ . The Kullback-Leibler divergence between  $\mathbb{P}_{2n}$  and  $\mathbb{P}_{1n}$  can be easily computed via Theorem A.1 of [14], which in our case gives that  $K(\mathbb{P}_{2n}, \mathbb{P}_{1n}) = nK(\rho_2, \rho_1)$ , because both  $\rho_1$  and  $\rho_2$  have the same total mass. Let  $\chi^2(\rho_2, \rho_1)$  denote the  $\chi^2$ -divergence between the densities  $\rho_2$  and  $\rho_1$ . It is not difficult to see that  $K(\rho_2, \rho_1) \leq \chi^2(\rho_2, \rho_1)$ , cf. formula (2.20) in [31]. It follows that in order to prove the theorem, it suffices to show that  $\chi^2(\rho_2, \rho_1) = O(n^{-1})$ . By definition of  $\rho_1, \rho_2, H$  and a change of the integration variable we have that

$$\chi^2(\rho_2, \rho_1) \lesssim \delta_n^{2\beta+1} \int_{-1}^1 \frac{(H(u))^2}{f_1(\delta_n u)} du. \quad (30)$$

The dominated convergence theorem implies that the right-hand side of the above equation is of order  $\delta_n^{2\beta+1}$ . Taking  $\delta_n \asymp n^{-1/(2\beta+1)}$  gives that (30) is of order  $n^{-1}$ . This yields the statement of the theorem.  $\square$

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