# A Characterization Related to the Equilibrium Distribution Associated with a Polynomial Structure

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September 21, 2009

**Abstract**

Let $f$ be a probability density function on $(a, b) \subset (0, \infty)$ and consider the class $\mathcal{C}_f$ of all probability density functions of the form $Pf$ where $P$ is a polynomial. Assume that if $X$ has its density in $\mathcal{C}_f$ then the equilibrium probability density $x \mapsto P(X > x)/\mathbb{E}(X)$ also belongs to $\mathcal{C}_f$ : this happens for instance when $f(x) = Ce^{-\lambda x}$ or $f(x) = C(b - x)^{\lambda - 1}$. The present paper shows that actually they are the only possible two cases. This surprising result is achieved with an unusual tool in renewal theory, by using ideals of polynomials.

*Keywords:* Renewal theory; excess life time; polynomial densities; ideals of polynomials.

## 1 Introduction: equilibrium distribution

Let $X_1, X_2, \ldots$ be a sequence of non-negative independent random variables with a common distribution $F$, with probability density function (pdf) $f$

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and Laplace-Stieltjes transform (LST) $\phi$. Letting $\mu = \mathbb{E}(X_i)$, it is assumed that $0 < \mu < \infty$. The random variable $X_i$ denotes the interoccurrence time between the $(i-1)$th and $i$th event in some probability problem. The counting process $\{N(t),\ t \geq 0\}$, where $N(t)$ is the largest integer $n \geq 0$ such that $X_1 + \cdots + X_n \leq t$, is called the renewal process generated by the interoccurrence times $X_1, X_2, \ldots$ (cf. the classical textbooks [3, 4, 5]). An important role in renewal theory is played by the backward recurrence time $A_t$ (the time since the last renewal before $t$) and the forward recurrence time $B_t$ (the time until the first renewal after $t$). If the $X_i$ are interpreted as life times, then $A_t$ is the past life time at $t$ and $B_t$ the residual or excess life time at $t$. It is well-known that the limiting distributions of $A_t$ and $B_t$ for $t \to \infty$ are given by (with $X$ a generic random variable with distribution $F$):

$$\lim_{t\to\infty} P(A_t \leq x) = \lim_{t\to\infty} P(B_t \leq x) = \int_{y=0}^{x} \frac{P(X > y)}{\mu} \mathrm{d}y. \tag{1}$$

Denote this limiting or equilibrium excess life time distribution by $F_e$, and its pdf by $f_e(x) = P(X > x)/\mu = \int_x^\infty \frac{f(y)}{\mu} \mathrm{d}y$. Its LST is given by

$$\varphi_e(s) = \int_{x=0}^{\infty} \mathrm{e}^{-sx} \mathrm{d}F_e(x) = \frac{1 - \varphi(s)}{s\mu}. \tag{2}$$

A delayed renewal process is defined just like an ordinary renewal process, except that $X_1$ has a different distribution. If $X_1$ has distribution $F_e$, then the process begins at time 0 in equilibrium, and then the excess life time at any time $t \geq 0$ has distribution $F_e$; cf. Section 3.5 of Ross [4].

The equilibrium excess life time distribution can be given the following interpretation (Tijms [5] p. 11). Suppose that an outside person observes the state of the process at an arbitrarily chosen point in time when the process has been in operation since a very long time. Assuming that the outside observer has no information about the history of the process, the best prediction the person can give about the residual time until the next renewal is according to the equilibrium excess life time distribution.

Excess life times play an extremely important role in applied probability. They arise in a host of real-life problems, ranging from reliability theory to inventory and queueing theory. For example, the celebrated Pollaczek-Khintchine formula, arguably the most important formula in queueing theory, gives an expression for the LST of the distribution of the steady-state waiting time $W$ in an $M/G/1$ queue, i.e., a single server queue with Poisson arrivals of customers and generally distributed service times $X_1, X_2, \ldots$, with service in First-Come-First-Served (FCFS) order ([1], Ch. VIII):

$$\int_{x=0}^{\infty} \mathrm{e}^{-sx} \mathrm{d}P(W \leq x) = \frac{1-\rho}{1-\rho\varphi_e(s)}. \tag{3}$$

Here $\rho$ denotes the load, i.e., the product of arrival rate and mean service requirement. Inversion shows that

$$P(W \leq x) = \sum_{n=0}^{\infty} (1-\rho)\rho^n P(\tilde{X}_1 + \cdots + \tilde{X}_n \leq x), \tag{4}$$

where $\tilde{X}_i \sim F_e$, $i = 1, 2, \ldots, n$. Cooper and Niu [2] interpret this formula by explaining that (1) the waiting time distribution in $M/G/1$ FCFS equals the distribution of the workload in this queue, and that (2) by work conservation this equals the distribution of the workload in the $M/G/1$ queue with the service discipline Last-Come-First-Served Preemptive-Resume (i.e., immediately take the newest arrival into service; notice that each waiting customer has already received some service), and that (3) in the latter system, the number of customers is geometrically distributed with parameter $\rho$ while all residual service times are independent and identically distributed with distribution $F_e$.

More generally speaking, in many queueing problems one needs to know the time until the next arrival (residual interarrival time) or the time until completion of the ongoing service (residual service time). And in reliability and maintenance problems, one needs to know the time until breakdown of a machine, or until an ongoing repair is completed; etc. We refer to Chapter 1 of [5] for a host of other examples, which confirm the importance of obtaining insight into the characteristics of the distribution of the residual life time.

A related important random variable is $X_{N(t)+1}$, the length of the renewal interval seen by an outside observer at $t$. Denote by $\hat{X}$ a random variable with distribution the limiting distribution of $X_{N(t)+1}$. Its steady-state pdf is $yf(y)/\mu$ and $P(\hat{X} > x) = \int_x^\infty \frac{yf(y)}{\mu}\mathrm{d}y$.

In Section 2 we shall see that, for the classes of exponential, Erlang and hyperexponential distributions, the pdf $f_e$, and also the pdf of $\hat{X}$, are again exponential, Erlang, hyperexponential, or mixtures of those. The beta distribution has a similar closure property. That has led us to study a much more general question: which pdf's have the property that, for any polynomial

$P$, we have that $\int_x^b P(t)f(t)\mathrm{d}t$ can be written in the form of a product of another polynomial and $f(x)$? This question is answered in our main result, Proposition 3.1 in Section 3. But first, in Section 2, we provide several examples where we demonstrate the property of Proposition 3.1. In considering these examples, it should be realized that $P(t)f(t)$ is also a pdf, up to a multiplicative constant.

# 2 Examples

In this section we consider two examples. One is related to the exponential distribution, the other to the beta distribution.

1. A. If $X \sim \exp(\lambda)$, i.e., $\varphi(s) = \lambda/(\lambda + s)$, then $\varphi_e(s) = \varphi(s)$, and $f_e(x) = f(x)$: the residual life time is again exponential. Of course, this is the familiar memoryless property.

   B. If $\varphi$ is the LST of a hyper-exponential distribution (i.e., of a mixture of exponential distributions) in the form

$$\varphi(s) = \sum_{i=1}^{k} p_i \varphi_i(s), \tag{5}$$

   with

$$\varphi_i(s) = \lambda_i/(\lambda_i + s),\ p_i > 0,\ i = 1, ..., k,\ \sum_{i=1}^{k} p_i = 1, \tag{6}$$

   then $F_e$ is also hyper-exponential with different weights $p_i^*$: $f_e(x) = \frac{\sum_{i=1}^k p_i \mathrm{e}^{-\lambda_i x}}{\sum_{i=1}^k p_i/\lambda_i}$.

   C. If $F$ is $Erlang(n)$, i.e., $f(x) = \lambda^n \frac{x^{n-1}}{(n-1)!}\mathrm{e}^{-\lambda x}$, $x > 0$ and $\varphi(s) = [\lambda/(\lambda + s)]^n$, $n \in \mathbb{N}$, then $F_e$ is a mixture of $Erlang(i)$ with weights $p_i = 1/n, i = 1, ..., n$, i.e.,

$$\varphi_e(s) = \sum_{i=1}^{n} p_i \varphi_i(s),\ \varphi_i(s) = [\lambda/(\lambda + s)]^i\ ,\ i = 1, ..., n. \tag{7}$$

   D. If $F$ is a mixture of $Erlang(i), i = 1, ..., n$, i.e.,

$$\varphi(s) = \sum_{i=1}^{k} p_i \varphi_i(s), \tag{8}$$

with

$$\varphi_i(s) = [\lambda/(\lambda+s)]^i, \; p_i > 0, \; i = 1, ..., n, \; \sum_{i=1}^{n} p_i = 1, \tag{9}$$

then $F_e$ is also a mixture of $Erlang(i), i = 1, ..., n$ with different weights $p_i^*$ given by

$$p_i^* = \sum_{j=i}^{n} \frac{p_j}{\sum\limits_{k=1}^{n} k p_k}, \; i = 1, ..., n.$$

Indeed, note that $\sum\limits_{i=1}^{n} p_i^* = 1$.

In all of the above examples $F_e$ is a mixture of exponential distributions or a mixture of convolutions of exponential distributions; or, equivalently, the related pdf $f_e$ has the form

$$f_e(x) = \sum_{i=1}^{n} P_i(x) e^{-\lambda_i x}, \tag{10}$$

where $n \in \mathbb{N}$, $P_i(x)$ is a polynomial in $x$ and $\lambda_i > 0$. A similar statement holds for $P(\hat{X} > x) = \frac{1}{\mu}\int_x^\infty t f(t) \mathrm{d}t$. It should further be noticed that in each of the examples 1A,...,1D, we have a pdf of the form $P(t)f(t)$ with $P$ a polynomial and $f$ an exponential; and $\int_x^b P(t)f(t)\mathrm{d}t$ has the form of the product of another polynomial and $f$.

2. Now consider the beta pdf $f(x) = (\frac{x-a}{b-a})^{\zeta-1}(\frac{b-x}{b-a})^{\lambda-1}\frac{1}{(b-a)B(\zeta,\lambda)}$, where $B(\zeta,\lambda) = \int_0^1 x^{\zeta-1}(1-x)^{\lambda-1}\mathrm{d}x$ is the beta function. If $\zeta = 1$, then $f_e(x) = \frac{(b-x)^\lambda}{(b-a)^\lambda}$, which is again a beta pdf with $\zeta = 1$. We see here a similar closure property as in the previous example. We could also have taken a weighted sum of beta pdf's multiplied by polynomials, and it is easily seen that taking the integration $\int_x^b$ w.r.t. such a sum results in other polynomials multiplied by beta pdf's.

This raises the following question. For which pdf's $f$ (or, equivalently, LST's $\varphi$) is the equilibrium pdf in (1) a pdf in the same 'class' of pdf's as $f$, or a polynomial multiplied with $f$? In the next section we introduce such a closure property in a more general setting, and we prove a characterization result: If $f$ is concentrated on $0 \leq a < b \leq \infty$, then which pdf's $f$ have the property that, for any polynomial $P$, we have that $\int_x^b P(t)f(t)\mathrm{d}t$ can be written in the form of a product of another polynomial and $f(x)$? We show the following in Proposition 3.1 in the next section: A necessary and sufficient condition for this to hold is that either $b = \infty$ and $f(x) = Ce^{-\lambda x}$ where $\lambda > 0$ and $1/C = \int_a^b (t-a)f(t)\mathrm{d}t$, or that $b$ is finite and $f(x) = C(b-x)^{\lambda-1}$, i.e., $f$ is either exponential or of a beta type.

# 3 The main result

Let $f$ be a pdf on $(a, b)$ with $0 \leq a < b \leq \infty$ such that $1/C = \int_a^b (t - a)f(t)\mathrm{d}t < \infty$. Consider the new pdf on $(a, b)$ defined by $T(f)(x) = C\int_x^b f(t)\mathrm{d}t$. Notice that, up to a multiplicative constant, this is $f_e(x)$. For instance if $(a, b) = (0, \infty)$ consider the class $\mathcal{F}$ of the pdf's of the form

$$f(x) = \sum_{i=1}^{n} P_i(x)e^{-\lambda_i x}$$

where $P_i(x)$ is a polynomial and $\lambda_i > 0$. Because of the formula

$$\int_x^\infty \lambda^n \frac{t^{n-1}}{(n-1)!}e^{-\lambda t}\mathrm{d}t = \sum_{k=0}^{n-1} \lambda^k \frac{x^k}{k!}e^{-\lambda x}, \tag{11}$$

clearly $T(f)$ is also in $\mathcal{F}$. A similar situation occurs when considering a bounded interval $(a, b)$ and the class $\mathcal{G}$ of pdf's on $(a, b)$ which are polynomials $P$ multiplied by the function $f(x) = (b-x)^{\lambda-1}$ where $\lambda > 0$. Here, $\mathcal{G}$ is stable by $T$, meaning that $T(\mathcal{G}) \subset \mathcal{G}$ (write $P(x)f(x)$ in the form $\sum_{k=0}^n p_k(b - x)^{k+\lambda-1}$ to be convinced of this fact). Of course, choosing a class $\mathcal{C}$ of pdf's on $(a, b)$ having all their moments implies that the class of pdf's defined by

$$\mathcal{C}_1 = \bigcup_{n=0}^{\infty} T^n(\mathcal{C})$$

is stable by $T$. But we are going to show that the classes $\mathcal{F}$ and $\mathcal{G}$ above are unique in the following sense:

**Proposition 3.1.** Let $f$ be a positive function on $(a, b)$ with $0 \leq a < b \leq \infty$ such that $\int_a^b t^n f(t)\mathrm{d}t < \infty$ for any non-negative integer $n$. Suppose that for any polynomial $P$ there exists a polynomial $A(P)$ such that for all $x \in (a,b)$ we have

$$\int_x^b P(t)f(t)\mathrm{d}t = A(P)(x)f(x). \tag{12}$$

Then there exist $C, \lambda > 0$ such that either $b$ is infinite and $f(x) = Ce^{-\lambda x}$, or $b$ is finite and $f(x) = C(b-x)^{\lambda-1}$.

**Comments.** Thus this statement describes the few functions $f$ on $(a,b)$ such that the class $\mathcal{C}_f$ of pdf's of the form $P(x)f(x)$ is stable by the operation $T$ described above, with $T(Pf) = A(P)f$. Note that in both cases $a$ is not necessarily 0. For instance, if $f(x) = e^{-\lambda x}$ if $(a,b) = (a,\infty)$ and $P(x) = \lambda^n x^{n-1}/(n-1)!$, we have (cf. (11)):

$$A(P)(x) = \sum_{k=0}^{n-1} \lambda^k x^k/k!. \tag{13}$$

Note that $A(1) = 1/\lambda$. Since $A$ is a linear operator, these formulas describe $A$ completely. Similarly if $f$ is $(b-x)^{\lambda-1}$ on the bounded interval $(a,b)$ and if $P(x) = (b-x)^n$ we have

$$A(P)(x) = (b-x)^{n+1}/(n+\lambda). \tag{14}$$

For instance $A(1) = (b-x)/\lambda$.

Let us also insist on the fact that the proposition describes really the only two possible cases. One could be tempted if $f$ satisfies (12) to coin the new function $f_1(x) = R(x)f(x)$ where $R$ is a nonconstant polynomial which is positive on $(a,b)$ and to observe that for all polynomials $P$ we have

$$\int_x^b P(t)f_1(t)\mathrm{d}t = \frac{A(PR)(x)}{R(x)} f_1(x).$$

A consequence of the proposition is that it is impossible that $R$ divides $A(PR)$ for all polynomials $P$.

**Proof of Proposition 3.1.** For $P \equiv 1$ we denote $Q(x) = A(1)(x)$. Writing $\int_x^b f(t)\mathrm{d}t = Q(x)f(x)$ shows that the polynomial $Q$ must be positive on $(a,b)$. Since $f$ is integrable, thus writing $f(x) = \frac{1}{Q(x)}\int_x^b f(t)\mathrm{d}t$ shows that $f$ must be continuous, thus differentiable, thus infinitely differentiable. Now taking derivative in $x$ of $\int_x^b P(t)f(t)\mathrm{d}t = A(P)(x)f(x)$ gives the differential equation

$$-P(x)f(x) = A(P)'(x)f(x) + A(P)(x)f'(x)$$

that we rewrite as

$$\frac{f'(x)}{f(x)} = -\frac{P(x) + A(P)'(x)}{A(P)(x)}.$$

Note that since the left hand side of this equation does not depend on $P$, we can get information on $A(P)$ by replacing $P$ by 1, getting the following differential equation in $A(P)$

$$\frac{P(x) + A(P)'(x)}{A(P)(x)} = \frac{1 + Q'(x)}{Q(x)}.$$

As a consequence all information on $f$ and $A(P)$ is actually given by the polynomial $Q$.

The case of $Q$ of degree 0. If $Q$ is the nonzero constant $1/\lambda$, the equation $f'/f = -(1 + Q')/Q$ gives $f(x) = e^{-\lambda x}$ on $(a, b)$. If $b = \infty$, we have already seen that if $\lambda > 0$ the identity $\int_x^b P(t)f(t)\mathrm{d}t = A(P)(x)f(x)$ holds for a suitable operator $A$ defined by (13). If $\lambda \leq 0$, the condition $\int_a^b t^n f(t)\mathrm{d}t < \infty$ is not fulfilled. If $b < \infty$ then $\int_x^b P(t)f(t)\mathrm{d}t = A(P)(x)f(x)$ does not hold since for $P = \lambda$ we get

$$\int_x^b \lambda e^{-\lambda t}\mathrm{d}t = e^{-\lambda x} - e^{-\lambda b}$$

which is not of the desired form of a polynomial multiplied by $e^{-\lambda x}$.

The case of $Q$ of degree 1. If $Q$ is a first degree polynomial we write it as $Q(x) = (b_1 - x)/\lambda$ where $b_1$ is a real number and $\lambda$ is a nonzero number. From the equation $f'/f = -(1 + Q')/Q$ on $(a, b)$ we get that $f(x) = C|b_1 - x|^{\lambda-1}$ for some positive number $C$. Suppose that $b = \infty$. Clearly $\int_a^\infty t^n f(t)\mathrm{d}t < \infty$ is impossible if $n$ is big enough. Thus $b < \infty$. Now for all $x$ in $(a, b)$ we have $\int_x^b |b_1 - t|^{\lambda-1}\mathrm{d}t = |b_1 - x|^\lambda/|\lambda|$. Since the left hand side must converge to zero when $x \to b$, this would imply that $b = b_1$ and that $\lambda > 0$.

The case of $Q$ of degree $\geq 2$. We now claim that $Q$ has necessarily degree $\leq 1$, a more difficult part of the proof. Suppose that $Q$ has degree $\geq 2$ and however suppose that the differential equation $QP = (1+Q')Y - QY'$ has always a polynomial solution $Y = A(P)$ for any polynomial $P$.

To reach a contradiction, we introduce three notations: we denote by $\mathcal{A}$ the ring of polynomials with real coefficients. If $A \in \mathcal{A}$ we denote by $\mathcal{I}_A$ the ideal generated by $A$, that is the set of polynomials divisible by $A$:

$$\mathcal{I}_A = \{AP \, ; P \in \mathcal{A}\}.$$

Finally we consider the endomorphism $\varphi$ of $\mathcal{A}$ defined by

$$Y \mapsto \varphi(Y) = (1 + Q')Y - QY'.$$

Assuming that $QP = \varphi(Y)$ has a solution $Y$ in $\mathcal{A}$ for each $P \in \mathcal{A}$ is saying that the image $\varphi(\mathcal{A})$ of $\varphi$ contains the ideal $\mathcal{I}_Q$.

**Lemma 3.2.** Let $B_0$ and $C_0$ be two polynomials and consider the endomorphism $\varphi_0$ of $\mathcal{A}$ defined by

$$\varphi_0(Y) = B_0 Y - C_0 Y'.$$

One assumes that $\varphi_0(\mathcal{A}) \supset \mathcal{I}_{C_0}$. Then there exists $A_1, B_1, C_1 \in \mathcal{A}$ such that $\varphi_0(\mathcal{A}) = \mathcal{I}_{A_1}$, $B_0 = A_1 B_1$ and $C_0 = A_1 C_1$. Furthermore if $\varphi_1(Y) = B_1 Y - C_1 Y'$ we have $\varphi_1(\mathcal{A}) \supset \mathcal{I}_{C_1}$.

**Proof.** We show that $\varphi_0(\mathcal{A})$ is an ideal: Choose $Y_0$ and $P$ in $\mathcal{A}$. Since $\varphi_0(\mathcal{A}) \supset \mathcal{I}_{C_0}$, there exists $Y_1 \in \mathcal{A}$ such that $\varphi_0(Y_1) = C_0 P' Y_0$. Thus

$$\varphi_0(PY_0 + Y_1) = \varphi_0(PY_0) + \varphi_0(Y_1) = B_0 P Y_0 - C_0 P Y_0' - C_0 P' Y_0 + \varphi_0(Y_1)$$

$$= B_0 P Y_0 - C_0 P Y_0' = P \varphi_0(Y_0)$$

which shows that $\varphi_0(\mathcal{A})$ is an ideal of $\mathcal{A}$. Since any ideal in $\mathcal{A}$ is principal, there exists $A_1$ such that $\varphi_0(\mathcal{A}) = \mathcal{I}_{A_1}$. Since $\mathcal{I}_{A_1} \supset \mathcal{I}_{C_0}$ we get that $A_1$ divides $C_0$. Thus, there exists $C_1$ such that $C_0 = A_1 C_1$. Since $\varphi_0(Y) = B_0 Y - C_0 Y' = B_0 Y - A_1 C_1 Y'$ is a multiple of $A_1$ for any $Y$, we get that $B_0 = \varphi_0(1) = A_1 B_1$ is also a multiple of $A_1$. Finally, since for each $P$ there exists $Y$ such that $\varphi_0(Y) = B_0 Y - C_0 Y' = C_0 P$, this implies that the same $Y$ satisfies $\varphi_1(Y) = B_1 Y - C_1 Y' = C_1 P$, which shows $\varphi_1(\mathcal{A}) \supset \mathcal{I}_{C_1}$. □

We now iterate Lemma 3.2 : For each $n = 1, 2, \dots$ there exist $A_n, B_n, C_n$ such that

$$B_0 = A_1 A_2 \dots A_n B_n, \quad C_0 = A_1 A_2 \dots A_n C_n$$

and such that if we denote $\varphi_n(Y) = B_n Y - C_n Y'$ we have $\varphi_n(\mathcal{A}) = \mathcal{I}_{A_{n+1}} \supset \mathcal{I}_{C_n}$. Thus for $n$ big enough, $A_{n+1}$ must be a constant polynomial, which is equivalent to saying that $\varphi_n$ is *surjective.*

We now apply the above considerations to the particular case where $B_0 = 1 + Q'$ and $C_0 = Q$, where $Q$ is a polynomial of degree $d_0 \geq 2$. Thus $\varphi = \varphi_0$ in the lemma. With this choice of $(B_0, C_0)$ we show that whatever $n$ is, the map $\varphi_n$ cannot be surjective when the degree of $Q$ is $\geq 2$. Write

$$Q(x) = C_0(x) = c_0 x^{d_0} + \text{terms of lower degree},$$

and more generally

$$C_n(x) = c_n x^{d_n} + \text{terms of lower degree}, \ B_n(x) = b_n x^{d_n - 1} + \text{terms of lower degree}.$$

We show by induction on $n$ that $b_n = d_0 c_n$. This is obvious for $n = 0$ since $B_0 = 1 + Q'$ and since $d_0 \geq 2$. Suppose that it is true for $n - 1$. Since $B_{n-1} = A_n B_n$ and $C_{n-1} = A_n C_n$ and if the term of maximum degree of $A_n$ is $a_n x^m$ then $d_{n-1} = d_n + m$, $b_{n-1} = a_n b_n$ and $c_{n-1} = a_n c_n$. Since by definition $a_n \neq 0$, the equality $b_n = d_0 c_n$ holds.

We finally use this fact to prove that $\varphi_n$ cannot be surjective by showing that there is no $Y$ such that

$$\varphi_n(Y)(x) = B_n(x)Y(x) - C_n(x)Y'(x) = x^{d_0 + d_n - 1}$$

holds. For suppose that there exists such a $Y$, with highest degree term $\alpha x^m$. The highest degree term of $B_n Y - C_n Y'$ is $(d_0 - m)\alpha c_n x^{d_n + m - 1}$ if $m \neq d_0$ and cannot be equal to $x^{d_0 + d_n - 1}$. If $m = d_0$ the highest degree term of $B_n Y - C_n Y'$ has degree less than $d_0 + d_n - 1$. We get the desired contradiction. □

# References


[1] S. Asmussen (2003). *Applied Probability and Queues*, 2nd. ed. Springer, New York.

[2] R.B. Cooper and S.-C. Niu (1986). Benes's formula for $M/G/1$-FIFO 'explained' by preemptive-resume LIFO. *J. Appl. Probab.* **23**, 550-554.

[3] D.R. Cox (1962). *Renewal Theory.* Methuen & Co., London.

[4] S.M. Ross (1996). *Stochastic Processes*, 2nd ed. Wiley, New York.

[5] H.C. Tijms (1994). *Stochastic Models - An Algorithmic Approach.* Wiley, New York.