

Optimal lateral transshipment policy for a two location inventory problem

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Abstract

We consider an inventory model for spare parts with two stockpoints, providing repairable parts for a critical component of advanced technical systems. As downtime costs for these systems are huge, ready-for-use spare parts are kept on stock, to be able to quickly respond to a breakdown of a system. We allow for lateral transshipments of parts between the stockpoints upon a demand arrival for a spare part. We are interested in the optimal lateral transshipment policy.

We consider a continuous review setting, where the initial number of spare parts at each location is given. We assume Poisson demand processes, and allow for asymmetric demand rates and asymmetric costs structures at the two locations. Defective parts are replaced, and returned to the stockpoint for repair. Each location has ample repair capacity, and repair times are exponentially distributed, with the same mean repair time for both locations. Demands are satisfied from own stock, via a lateral transshipment, or via an emergency procedure.

Using dynamic programming, we completely characterize and prove the structure of the optimal lateral transshipment policy, that is, the policy for satisfying demands, minimizing the long-run average costs of the system. This optimal policy is a threshold type policy. In addition, we derive conditions under which the so-called *hold back* and *complete pooling* policies are optimal, which are both policies that are often assumed in the literature.

Keywords: inventory control, spare parts, lateral transshipments, pooling, optimal policy.

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1 Introduction

We study an inventory model with two stockpoints, which provide spare parts for advanced technical systems. As down-time of these systems is highly costly, ready-for-use spare parts are kept on stock for the critical component of these systems. We focus on a single, repairable part, for which a repair-by-replacement strategy is executed: upon failure of a system, the defective part is replaced by a part from inventory. The defective part is returned to the stockpoint, where it is repaired and added to the inventory. We assume ample repair capacity at both stockpoints, and exponentially distributed repair lead times.

The stockpoints service two groups of technical systems, where each group is assigned to one stockpoint. In case of a breakdown of a system, the system demands for a spare part at its dedicated stockpoint. If a demand for a spare part is directly met at the stockpoint, we refer to this as a demand that is directly fulfilled from own stock. When a demand is not met directly, there are two other possibilities to fulfill it. The first option is a *lateral transshipment*, in which case a part is shipped from the other stockpoint. The system is down while it is waiting for the part, and extra transportation costs are incurred. The second option is an *emergency repair procedure*: the defective part is repaired in a fast repair procedure, for which high costs are incurred, and the system is down for a longer period of time. Backordering of demands is not allowed. The demands are observed continuously, and form a Poisson process at each stockpoint, but the demand rates at the two stockpoints might be different. We assume the initial number of spare parts on hand at each location to be given.

By allowing for lateral transshipments, significant costs can be saved, as the costs for a lateral transshipment (transportation costs and costs for a short downtime of the system) are much smaller than the costs for an emergency procedure (cost for emergency repair of a part, extra transportation costs and costs for a long downtime of the system). Because of this cost difference, it is beneficial to apply lateral transshipments. Kranenburg and Van Houtum [17] show that the company ASML, an original equipment manufacturer in the semiconductor industry, can reduce their spare parts provisioning costs by up to 50% by the efficient use of lateral transshipments, while keeping the service at the same level. Robinson [26] shows, that substantial costs savings can be realized by the use of lateral transshipment, even when the transportation costs are high. Based on two case studies, in the computer and automobile industry, Cohen and Lee [8] address stock pooling as an effective way to improve the service levels, even with less on-hand inventory. Also, Cohen et al. [7] point out the pooling of spare parts as one of the best way for companies to realize costs reductions.

In this paper, we focus on the optimization of the lateral transshipment policy. That is, we determine the optimal decision on how to fulfill a demand, in order to minimize the long-run average running costs of the system: (i) directly from own stock, (ii) via a lateral transshipment, or (iii) via an emergency procedure. When is it beneficial to apply a lateral transshipment, and when is it better to apply an emergency procedure? A straightforward strategy would be to always fulfill demands from the own stockpoint if possible, and otherwise via a lateral transshipment, if possible. This strategy is known as *complete pooling* (or *full pooling*) of inventory. But, this strategy will turn out to be suboptimal in certain cases, which mainly depends on the cost parameters. If a stockpoint has, for example, only one part left on stock, it could be beneficial to hold this one back, even if the other stockpoint

requests for a lateral transshipment. This situation can occur in case the cost parameters for both stockpoints are equal (i.e. symmetric), but its effect may be even larger in the case of asymmetric costs parameters. In this case it can even be better to keep one or a few items on stock in case of a demand at the stockpoint itself, to be able to respond to a future lateral transshipment request of the other stockpoint. This situation, where stockpoints can hold back some inventory, is known as *partial pooling*.

Under the given repair strategy, we have a system with circulating stock, where the sum of the number of parts on stock and parts in repair is constant at a stockpoint, and equal to the initial number of parts. Besides, the model is equivalent to an inventory system with a given base stock level (equal to the initial number of parts on hand), executing a continuous review strategy with one-for-one replenishments (or productions), and lost sales. The repair times in the current model resemble the replenishment, c.q. production lead times, and the emergency procedures resemble lost sales. This would be more suitable for a setting in which the parts are consumables.

A considerable amount of work has already been done on the use of lateral transshipments in various settings, see e.g. Wong et al. [32] or Paterson et al. [24] for an overview. We can distinguish between two types of work, depending on the lateral transshipment rule. It can either be a predetermined and fixed rule, or it is subject to optimization. In most of the literature, a given predetermined fixed rule is assumed, and performance characteristics of the system are evaluated, either exactly or approximately, (e.g. Lee [19], Axsäter [4], Sherbrooke [27], Tagaras and Cohen [30] and Kukreja et al. [18]), or optimal reorder policies are derived (e.g. Robinson[26] and Olsson [23]).

More relevant in relation to our work, is the literature on the optimization of the lateral transshipment rule. For the periodic-review case, we distinguish the following results. For a system with two locations, Archibald et al. [3] prove the optimal transshipment policy in case of stockouts. It states when it is optimal to apply a lateral transshipment in case of a demand at a location with zero stock. The decision is based both on the stock level of the other location, and on the time left until the next replenishment opportunity, where it is assumed that the replenishments occur instantaneously (i.e., if we would let the period length go to zero, we would get a model with zero replenishment leadtime). The costs consist of the regular and emergency ordering costs, transshipments costs and holding costs. For multiple locations, Archibald et al. [2] comes up with heuristics for the lateral transshipment rule. Herer et al. [14] approximate the optimal transshipment rule for multiple locations. In Herer and Tzur [13], an optimal transshipment policy is derived, but here lateral transshipments are not used in reaction to stock outs, but only to balance stock because of different holding and replenishment costs at the locations. These are so-called pro-active transshipments. Wee and Dada [31] study the decision for a single period in a system with multiple locations and one central warehouse, and give five protocols for a transshipment attempt in case of a stock out. They prove that the optimal transshipment policy is described by exactly one of them, which can be determined by evaluating a set of conditions.

For optimal lateral transshipments rules in a continuous time review setting, hardly any results are available. Zhao et al. [34, 35] prove the optimal transshipment policy for so-called decentralized networks, where the locations are independently owned and operated. They find a policy where two, respectively three parameters determine when to send and when to

accept a transshipment request. The latter resembles work of Xu et al. [33], which consider a hold-back parameter, that limits the amount of outgoing transshipments, however, they work with an (Q, R) replenishment policy. For the case of compound Poisson processes, Axsäter [5] comes up with a heuristic rule determining which part of a given demand should be covered by a lateral transshipment. Evers [9] provides two heuristics giving critical values for on-hand inventory, above which a stock transfers should be applied. Minner et al. [22] improve these heuristics, using an approach based on net present value.

In Zhao et al. [36] a two location make-to-stock system is considered. The demands arrive according to two independent Poisson processes, and at each location the production is modelled by a single-server make-to-stock queue, with exponentially distributed production times. The optimal production and optimal transshipment policy are derived, which both can be described by a switching curve, e.g. the optimal production decision is described by a state-dependent stock level. In their model, demands are backordered, and lateral transshipments can be applied both at the moment of a demand and at the moment of a production completion. However, they do not allow inventory to be held back at a location in order to be able to respond to future lateral transshipment requests of the other location, if it is, or is having a large risk of, facing a stock out. The optimization of the lateral transshipment policy in the current setting, however, has not been done before.

Our main contribution is as follows. For the described model, (a) we completely characterize and prove the structure of the optimal lateral transshipment policy, that is, the optimal policy for fulfilling demands at both stockpoints. This optimal policy is a threshold type policy. Next to this, we give conditions under which the optimal policy simplifies: (b.1) a condition under which it is optimal to always fulfill a demand directly from own stock (option (i)), if possible, which is called a *hold back policy*, cf. [33]; (b.2) a similar condition under which it is optimal to always apply a lateral transshipment (option (ii)) in case of a stock-out, if possible. Under the latter condition, which implies the first one, (b.3) the optimal policy applies option (i) if possible, otherwise option (ii) if possible, and otherwise option (iii). This policy is known as a *complete pooling policy*, and this strategy is often assumed in the literature about lateral transshipments, see e.g. [1, 4, 10, 18, 19, 27, 29, 32, 36]. So, we present conditions on the cost parameters under which this policy is indeed optimal.

We model the inventory problem as a Markov decision problem, see e.g. Puterman [25] for an extensive overview on this. Based on the inventory levels, a decision has to be taken each time a demand arises at one of the two stockpoints. The technique we use for determining the optimal decisions, is based on Event Based Dynamic Programming, see Koole [15, 16]. Each event changing the inventory levels, i.e., a demand for a spare part or a repair of a broken part, is represented by an operator. The n -period minimal cost function (the so-called *value function*) is then composed of these operators. Proving structural properties of the value function, such as monotonicity and multimodularity, can then be done by considering each of these operators separately, hence reducing the complexity of the problem. From the properties of the value function the optimal lateral transshipment policy is derived, as well as conditions under which it simplifies.

The inventory model we consider is closely related to a queueing system. By viewing the stockpoints as multi-server queues, demands as arriving customers and repairs as service times, the problem translates into a routing problem in a queueing model with two parallel queues.

These problems are often modelled as a Markov decision problem, as is the case in the current paper. For an overview on these on related problems for the control of (networks of) queues, see e.g. Stidham and Weber [28]. For example, Menich and Serfozo [21] show optimality of a join-the-shortest-queue routing policy, and Brouns [6] gives a partial characterization of the optimal routing policy to two two parallel multiserver queues with no buffers, which is related to our work. The main difference is however, that in these kind of problems from the queueing literature, no costs are incurred for the routing of customers.

The outline of this paper is as follows. In Section 2 we describe the model in more detail and we introduce the notation. We model it as a Markov decision problem, and introduce the technique of Event Based Dynamic Programming. In Section 3 we give the structural properties of the event operators and of the value function. This leads to the characterization of the structure of the optimal policy, which is a threshold type policy. Conditions are given under which certain simple policies are optimal, and some some examples are shown. In Section 4 we consider an extension to the model, namely we limit the repair capacity. Finally we summarize the results, and indicate possibilities for further research in Section 5.

2 Model and notation

In Section 2.1 we introduce the problem, followed by its modelling as a Markov decision problem in Section 2.2. We introduce the *value function* (the n -period minimal cost function) and two types of event operators (for the demands and for the repairs), by which the value function can be recursively expressed.

2.1 Problem description

We consider a spare part inventory system consisting of two stockpoints, which provision a single spare part for the critical component of an advanced technical system. Initially, each stockpoint has a predetermined number of ready-for-use spare parts on stock of a given stockkeeping unit (SKU), $S_i \in \mathbb{N} \cup \{0\}$ at stockpoint i , $i = 1, 2$. There are two groups of systems, where each group is assigned to one stockpoint. When a system breaks down, the critical component has to be replaced by a spare part, i.e., the system demands for a spare part at its designated stock point. The demands arrive continuously, according to two independent Poisson processes with arrival rate $\lambda_i \geq 0$ at stockpoint i , such that $\lambda_1 + \lambda_2 > 0$. A demand can be fulfilled in one of the following three ways: (i) directly from own stock, (ii) via a lateral transshipment, or (iii) via an emergency procedure. In either of the three cases, the defective part is returned to the stockpoint the spare part originated from. The repair times of broken parts are exponentially distributed with mean $1/\mu_i$, where $\mu_i > 0$, and both stockpoints have ample repair capacity. In order to derive the structural results, it will appear that we have to require $\mu_1 = \mu_2$. We assume that parts can be repaired an unlimited number of times and repaired parts attain their original quality. The interarrival and replenishment times are all mutually independent.

Our goal is to minimize the long-run average costs. The costs are composed of the costs for the lateral transshipments, emergency procedures and the downtimes of the systems. We are

only interested in the influence of the decisions on the costs, i.e., in the *extra* costs a lateral transshipment or an emergency procedure causes, compared to a fulfillment of a demand directly from stock. The number of spare parts S_1 and S_2 is given, and we do not take into account the acquisition costs of these. Neither do we take into account holding costs, as we have circulating stock. We set the costs when a demand is met directly from own stock to zero. These would be the costs for the downtime of the machine and for the shipment of the spare part to the system, and replacing, shipping back and repairing the broken part. But these costs are made in any case, independently of the chosen action. If a lateral transshipment is applied, higher transportation costs are incurred and as the system is down during the transportation time, extra costs for loss of production are incurred too. All these extra costs for applying a lateral transshipment from the other stockpoint to stockpoint i , are put together in the penalty costs for a lateral transshipment *to* stockpoint i , denoted by P_{LT_i} . The third option for fulfilling a demand, is an emergency procedure. The broken part is repaired in a fast repair procedure, during which the machine is down, which can be a considerable amount of time. These extra costs form the penalty costs for an emergency procedure for a demand at stockpoint i , denoted by P_{EP_i} . We assume $P_{EP_i} \geq P_{LT_i} \geq 0$, $i = 1, 2$.

Under the given repair strategy, we have a system with circulating stock. The inventory position (the total number of parts on stock and parts in repair) is constant at each stockpoint, and equal to the initial amount of spare parts, which is S_i at stockpoint i .

The model is equivalent to an inventory system consisting of two stockpoints executing a continuous review base stock strategy with basestock levels S_i , the possibility of lateral transshipments, and lost sales for unmet demands.

2.2 Dynamic Programming formulation

The state x of the system is given by the inventory levels at both stockpoints: $x = (x_1, x_2)$, where $x_i \in \{0, 1, \dots, S_i\}$ is the on-hand stock at stockpoint i . The state space \mathcal{S} is given by all possible inventory levels, $\mathcal{S} = \{0, 1, \dots, S_1\} \times \{0, 1, \dots, S_2\}$. Upon a demand at stockpoint i , a decision has to be taken how to fulfill it, in one of the following three ways: (0) directly from own stock, (1) via a lateral transshipment or (2) via an emergency procedure. The action taken for a demand at i when in state x , is denoted by $a_i(x) \in \{0, 1, 2\}$, respectively, and an optimal action is denoted by $a_i^*(x)$. Backorders are not allowed. Hence the decision space of each $a_i(x)$ consist of the decisions under which x_1 and x_2 remain greater than or equal to zero.

As the interarrival times of demands as well as the replenishment times are independent exponentially distributed random variables, we can apply uniformization (cf. Lippman [20]) to convert the semi-Markov decision problem into an equivalent Markov decision problem (MDP).

The existence of a long-run average optimal policy is given by Puterman [25, Theorem 8.4.5a], which states, that if the state space and action space for every state are finite, the costs are bounded, and the model is *unichain*, then there exists a stationary average optimal policy. A model is said to be unichain, if the transition matrix of every (deterministic) stationary policy is unichain, that is, it consists of a single recurrent class plus a, possibly empty, set of

transient states. The current model is unichain, as the state (S_1, S_2) is accessible from every state $(x_1, x_2) \in \mathcal{S}$ for every stationary policy.

When facing a decision, we should take into account the direct costs for a decision, as well as the future expected costs this decision brings along. For the expected costs from a state, we introduce the *value function* (see e.g. Puterman [25]) $V_n : \mathcal{S} \mapsto \mathbb{R}^+$. $V_n(x_1, x_2)$ is the minimum expected total costs when there are n events (demands or repairs) left, starting in state $(x_1, x_2) \in \mathcal{S}$. This V_n can be recursively expressed. The two types of operators it consists of (G_i for the repairs at a stockpoint, and H_i for the demands) are defined below.

$$V_{n+1}(x_1, x_2) = \frac{1}{\nu} \left(\sum_{i=1}^2 \mu_i G_i V_n(x_1, x_2) + \sum_{i=1}^2 \lambda_i H_i V_n(x_1, x_2) \right), \text{ for } (x_1, x_2) \in \mathcal{S}, n \geq 0, \quad (1)$$

starting with $V_0 \equiv 0$, and $\nu = \lambda_1 + \lambda_2 + S_1\mu_1 + S_2\mu_2$ is the uniformization rate. Decisions are only made in the way of fulfilling demands (in the operator H_i). The decision is taken each time a demand arrives, and is based on the inventory levels. For the repairs no decisions are taken.

The operator G_1 models the repairs at stockpoint 1, and is defined by

$$G_1 f(x_1, x_2) = \begin{cases} (S_1 - x_1)f(x_1 + 1, x_2) + x_1 f(x_1, x_2) & \text{if } x_1 < S_1, \\ S_1 f(x_1, x_2) & \text{if } x_1 = S_1, \end{cases} \quad (2)$$

where f is an arbitrary function $f : \mathcal{S} \mapsto \mathbb{R}^+$. G_2 is defined analogously. If the inventory level is x_1 , there are $S_1 - x_1$ outstanding repairs, hence the repairs occur at a rate proportional to $S_1 - x_1$. The term $x_1 f(x_1, x_2)$ corresponds to *fictitious* transitions. In this way, we assure that the total rate at which G_1 occurs is always equal to S_1 .

The operator H_1 models the demands at stockpoint 1, and is defined by

$$H_1 f(x_1, x_2) = \begin{cases} P_{EP_1} + f(x_1, x_2) & \text{if } x_1 = 0, x_2 = 0, \\ \min\{f(x_1 - 1, x_2), P_{EP_1} + f(x_1, x_2)\} & \text{if } x_1 > 0, x_2 = 0, \\ \min\{P_{LT_1} + f(x_1, x_2 - 1), P_{EP_1} + f(x_1, x_2)\} & \text{if } x_1 = 0, x_2 > 0, \\ \min\{f(x_1 - 1, x_2), P_{LT_1} + f(x_1, x_2 - 1), P_{EP_1} + f(x_1, x_2)\} & \text{if } x_1 > 0, x_2 > 0. \end{cases} \quad (3)$$

H_2 is defined analogously. If a demand occurs, it has to be decided how to fulfill it. There are three options for this: directly from stock, via a lateral transshipment, or via an emergency procedure. H_i takes the costs-minimizing action, where the costs consist of the direct costs for an action and the expected remaining costs from the state the system is in after taking that action. Depending on the stock levels x_1 and x_2 , four cases are distinguished over which the minimization is carried out, as stock levels cannot become negative.

3 Structural results

In this section we prove our main result: the structure of the optimal policy. For this we first prove that the value function V_n satisfies certain structural properties, like monotonicity and

multimodularity. We show that the operators V_n is composed of, all preserve these properties. Then, as V_0 satisfies them, it follows directly by induction, that the property holds for V_n for all $n \geq 0$. A framework for this was introduced by Koole [15] (see also [16]), as *Event Based Dynamic Programming*. The main advantage of this approach is that one can prove the propagation of properties for each of the event operators separately, reducing the complexity of the problem. Changes or extensions to the model can easily be made by replacing or adding events.

In Section 3.1 we introduce the properties and prove that G_1, G_2, H_1 and H_2 preserve these properties. It then follows that V_n , for all $n \geq 0$, satisfies them as well. From this we derive in Section 3.2 the structure of the optimal lateral transshipment policy, which is a threshold type policy. We give conditions under which it reduces to a simple policy, such as a hold back or a complete pooling strategy, in Section 3.3. Some examples are given in Section 3.4, and the special case with symmetric system parameters is considered in Section 3.5. All proofs are given in the appendix.

3.1 Properties of operators and value function

Consider the following properties of a function f , defined for all x such that the states appearing in the right-hand and left-hand side of the inequalities exist in \mathcal{S} :

$$\text{Decr}(1) : f(x_1, x_2) \geq f(x_1 + 1, x_2), \quad (4)$$

$$\text{Decr}(2) : f(x_1, x_2) \geq f(x_1, x_2 + 1), \quad (5)$$

$$\text{Conv}(1) : f(x_1, x_2) + f(x_1 + 2, x_2) \geq 2f(x_1 + 1, x_2), \quad (6)$$

$$\text{Conv}(2) : f(x_1, x_2) + f(x_1, x_2 + 2) \geq 2f(x_1, x_2 + 1), \quad (7)$$

$$\text{Supermod} : f(x_1, x_2) + f(x_1 + 1, x_2 + 1) \geq f(x_1 + 1, x_2) + f(x_1, x_2 + 1), \quad (8)$$

$$\text{SuperC}(1, 2) : f(x_1 + 2, x_2) + f(x_1, x_2 + 1) \geq f(x_1 + 1, x_2) + f(x_1 + 1, x_2 + 1), \quad (9)$$

$$\text{SuperC}(2, 1) : f(x_1, x_2 + 2) + f(x_1 + 1, x_2) \geq f(x_1, x_2 + 1) + f(x_1 + 1, x_2 + 1). \quad (10)$$

$\text{Decr}(i)$ stands for (non-strict) decreasingness of f in x_i . $\text{Conv}(i)$ stands for convexity of f in x_i , that is the difference $f(x) - f(x + e_i)$ is decreasing in x_i , where e_i denotes the unit vector consisting of all zeros except for a 1 at position i . Supermod is supermodularity, the definition of which is symmetric in x_1 and x_2 . SuperC(i, j) stands for superconvexity, adopting the terminology of [16]. It is straightforward that Supermod and SuperC(i, j) imply Conv(i), by adding the respective inequalities and cancelling identical terms.

Decr stands for the combination of Decr(1) and Decr(2), i.e., $\text{Decr} = \text{Decr}(1) \cap \text{Decr}(2)$. Similarly, Conv = Conv(1) \cap Conv(2) and SuperC = SuperC(1, 2) \cap SuperC(2, 1).

Multimodularity (MM), introduced by [12], is, for the case of a two-dimensional domain, equal to the combination of Supermod and SuperC:

$$\text{MM} = \text{Supermod} \cap \text{SuperC}. \quad (11)$$

The following two lemmas give useful properties of the operators G_i and H_i , which enable us to derive the structure of the optimal policy.

Lemma 3.1. *a) Operator G_i , $i = 1, 2$, preserves each of the following properties: (i) Decr; (ii) Conv; (iii) Supermod. b) The sum of the operators $G_1 + G_2$ preserves each of the following properties: (i) Decr; (ii) Conv; (iii) Supermod; (iv) SuperC; (v) MM.*

Note that SuperC (and hence MM), is only preserved by the sum of the operators $G_1 + G_2$, and not by G_1 and G_2 separately. When $G_1 + G_2$ is applied to (9) and (10) respectively, some terms introduced by G_1 cancel out against terms introduced by G_2 .

Lemma 3.2. *Operator H_i , $i = 1, 2$, preserves each of the following properties: (i) Decr, (ii) MM.*

By induction on n , and using the results of Lemmas 3.1 and 3.2, the next theorem immediately follows.

Theorem 3.3. *If $\mu_1 = \mu_2$, then V_n satisfies (4)–(10) for all $n \geq 0$.*

The properties (4)–(10) of V_n are the key in classifying the structure of the optimal policy. The condition of equal μ_i 's is required, because of the fact that only $G_1 + G_2$ preserves MM. Hence in the sequel we make this assumption. This is a sufficient assumption, but not a necessary one, as we can easily construct examples with $\mu_1 \neq \mu_2$ for which the structural results, proven in the next section, do hold. However, there are also examples with $\mu_1 \neq \mu_2$ for which the structural results fail to hold, see Section 3.4, Example 3.

Assumption. *The repair rates are identical: $\mu_1 = \mu_2 =: \mu$.*

3.2 Structure of optimal policy

We now characterize the structure of the optimal policy in the following two theorems. We state the optimal policy for fulfilling a demand at stockpoint 1; for stockpoint 2, analogous results hold. First we give the result for x_2 fixed, next for x_1 fixed.

Theorem 3.4. *The optimal policy for fulfilling a demand at stockpoint 1 for fixed x_2 is a threshold type policy: for each $x_2 \in \{0, 1, \dots, S_2\}$, there exist thresholds $T_1^{lt}(x_2) \in \{0, 1, \dots, S_1 + 1\}$ and $T_1^{di}(x_2) \in \{1, \dots, S_1 + 1\}$, with $T_1^{lt}(x_2) \leq T_1^{di}(x_2)$, such that:*

$$\begin{aligned} a_1^*(x) &= 2 \text{ (emergency procedure), for } 0 \leq x_1 \leq T_1^{lt}(x_2) - 1; \\ a_1^*(x) &= 1 \text{ (lateral transshipment), for } T_1^{lt}(x_2) \leq x_1 \leq T_1^{di}(x_2) - 1; \\ a_1^*(x) &= 0 \text{ (directly from own stock), for } T_1^{di}(x_2) \leq x_1 \leq S_1, \end{aligned}$$

where $T_1^{lt}(0) = T_1^{di}(0) \geq 1$.

The analogous result holds for demands at stockpoint 2 under a fixed $x_1 \in \{0, 1, \dots, S_1\}$.

This structure is graphically represented in Figure 1. For each x_2 , the thresholds divide the set $\{0, \dots, S_1\}$ into (at most) three subsets. In the first subset, where x_1 is small, an emergency procedure is optimal, in the second one a lateral transshipment, and in the third

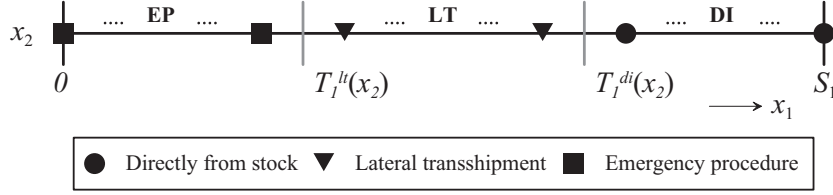


Figure 1: Optimal policy for a demand at stockpoint 1 for fixed x_2 .

one, where x_1 is large, it is optimal to take a part from stock. A threshold can be equal to $S_1 + 1$, hence implying that for a given x_2 taking parts from stock or applying lateral transshipments is never optimal.

A special case is $x_2 = 0$: as lateral transshipments are not possible at stockpoint 1, we have $T_1^{lt}(0) = T_1^{di}(0)$, where $T_1^{di}(0) \geq 1$. In this case, there are (at most) two subsets: an emergency procedure is applied for $0 \leq x_1 < T_1^{di}(0)$, and demand is directly delivered from stock for $T_1^{di}(0) \leq x_1 \leq S_1$.

A similar characterization can be made for fixed x_1 .

Theorem 3.5. *For the optimal policy for fulfilling a demand at stockpoint 1 for fixed $x_1 \in \{0, 1, \dots, S_1\}$, there exist $\hat{T}_1^{di}(x_1) \in \{0, 1, \dots, S_2 + 1\}$ and $\hat{T}_1^{lt}(x_1) \in \{1, \dots, S_2 + 1\}$, with $\hat{T}_1^{di}(x_1) \leq \hat{T}_1^{lt}(x_1)$, such that:*

$$\begin{aligned} a_1^*(x) &= 2 \text{ (emergency procedure), for } 0 \leq x_2 \leq \hat{T}_1^{di}(x_1) - 1; \\ a_1^*(x) &= 0 \text{ (direct from own stock), for } \hat{T}_1^{di}(x_1) \leq x_2 \leq \hat{T}_1^{lt}(x_1) - 1; \\ a_1^*(x) &= 1 \text{ (lateral transshipment), for } \hat{T}_1^{lt}(x_1) \leq x_2 \leq S_2, \end{aligned}$$

where $\hat{T}_1^{di}(0) = \hat{T}_1^{lt}(0) \geq 1$.

The analogous result holds for demands at stockpoint 2 under a fixed $x_2 \in \{0, 1, \dots, S_2\}$.

This structure is graphically represented in Figure 2. Now for a given x_1 the set $\{0, \dots, S_2\}$ is divided into subsets, such that in each subset one of the three decisions is optimal. Again, a $\hat{T}_1^{di}(x_1)$, $\hat{T}_1^{lt}(x_1)$ larger than the maximum stock level, indicates that a certain subset is empty, and hence that decision is never optimal. A special case is $x_1 = 0$, when it is not possible to deliver a demand directly from stock. Hence $\hat{T}_1^{di}(0) = \hat{T}_1^{lt}(0)$, where $\hat{T}_1^{lt}(0) \geq 1$.

Combining Theorem 3.4 and Theorem 3.5 restricts the possibilities for the optimal policy significantly. The states where an emergency procedure is an optimal action for a demand at stockpoint 1, i.e. the subset $EP = \{x \in \mathcal{S} \mid a_1^*(x) = 2\}$, form a connected part of the state space, located in the lower left corner. This follows as given that $a_1^*(\tilde{x}) = 2$ for some \tilde{x} , we have $a_1^*(x) = 2$ for all x with $x_1 \leq \tilde{x}_1$ (by Theorem 3.4), and all x with $x_2 \leq \tilde{x}_2$ (by Theorem 3.5). For the remaining states, a lateral transshipment or a delivery from stock is optimal. The curve dividing these two subsets, $LT = \{x \in \mathcal{S} \mid a_1^*(x) = 1\}$ respectively $DI = \{x \in \mathcal{S} \mid a_1^*(x) = 0\}$, is non-decreasing in x_1 . This implies that the general structure of the optimal policy is as given in Figure 3.

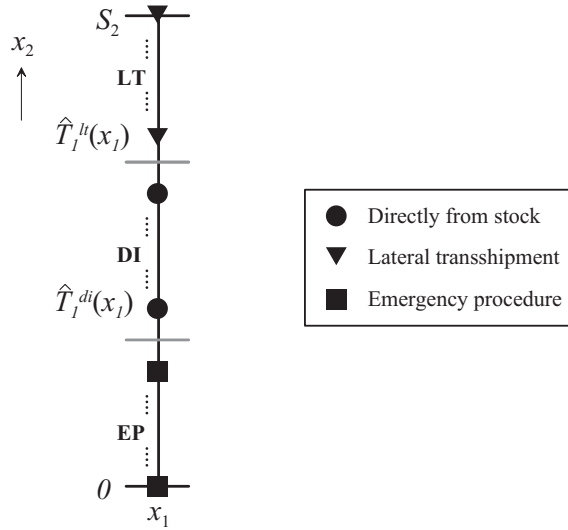


Figure 2: Optimal policy for a demand at stockpoint 1 for fixed x_1

3.3 Conditions simplifying the optimal policy

Under simple, sufficient conditions for the cost parameters, the structure of the optimal policy simplifies. We give two such conditions: under the first one, (i) it is optimal to fulfill a demand directly from own stock, whenever possible, but with a parameter limiting the amount of outgoing lateral transshipment. We refer to this as a *hold back policy*, see e.g. [33], as the parameters indicate the amount of stock that is hold back from a transshipment request. Hence we refer to these as the *hold back levels*. Next to this, under the second one, (ii) it is optimal to fulfill a demand directly from own stock, whenever possible, and otherwise to apply a lateral transshipment, whenever possible. This is called the *complete pooling* policy, in which no stock is hold back in any case. We refer to a policy both for an individual stockpoint, as well as for the whole system.

The following theorem states conditions under which it is optimal to always fulfill a demand directly from own stock:

Theorem 3.6. 1a) If

$$P_{EP_2} \leq P_{LT_2} + \left(1 + \frac{\mu}{\lambda_2}\right) P_{EP_1}, \quad (12)$$

then $T_1^{di}(x_2) = 1$ for all $x_2 \in \{0, 1, \dots, S_2\}$, i.e. a hold back policy is optimal at stockpoint 1.
b) If

$$P_{EP_1} \leq P_{LT_1} + \left(1 + \frac{\mu}{\lambda_1}\right) P_{EP_2}, \quad (13)$$

then $T_2^{di}(x_1) = 1$ for all $x_1 \in \{0, 1, \dots, S_1\}$, i.e. a hold back policy is optimal at stockpoint 2.
2) If (12) and (13) hold, then it is optimal for both stockpoints to execute a hold back policy.

Under condition (12), whenever there are items on stock at stockpoint 1, they should always be used in case of a demand at stockpoint 1, see Figure 4(a). However, stock can

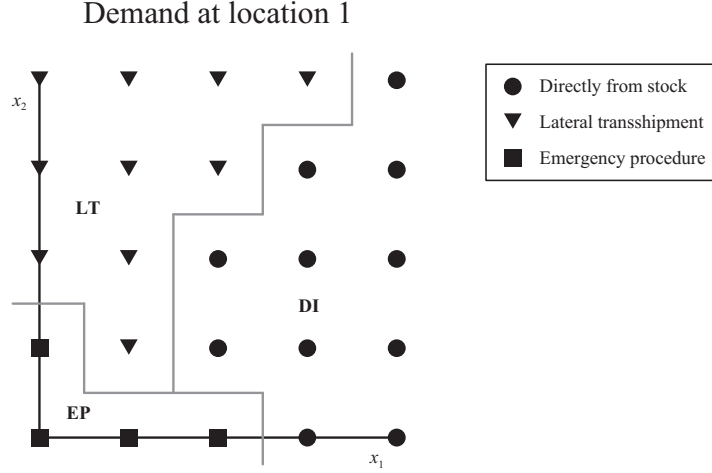


Figure 3: General structure of the optimal policy for a demand at location 1.

possibly be hold back from lateral transshipment requests. If both stockpoints execute a hold back policy, the entire policy is prescribed by only 2 parameters ($\hat{T}_1^{lt}(0)$ and $\hat{T}_2^{lt}(0)$). The case of symmetric costs at both stockpoints, i.e. $P_{LT_1} = P_{LT_2}$ and $P_{EP_1} = P_{EP_2}$, clearly satisfies conditions (12) and (13).

Next we give conditions under which the application of a lateral transshipments, when possible, is optimal, in case of a stock-out:

Theorem 3.7. 1a) If

$$P_{LT_1} + \frac{\lambda_2}{\lambda_2 + \mu} P_{EP_2} \leq P_{EP_1}, \quad (14)$$

then $\hat{T}_1^{lt}(0) = 1$, i.e. a complete pooling policy is optimal at stockpoint 1.

1b) If

$$P_{LT_2} + \frac{\lambda_1}{\lambda_1 + \mu} P_{EP_1} \leq P_{EP_2}, \quad (15)$$

then $\hat{T}_2^{lt}(0) = 1$, i.e. a complete pooling policy is optimal at stockpoint 2.

2) If (14) and (15) hold, then it is optimal for both stockpoints to execute a complete pooling policy.

Under condition (14), stockpoint 2 should not hold back stock if stockpoint 1 requests for a lateral transshipment when it is out-of-stock, see Figure 4(b). As condition (14) is stronger than condition (12), i.e. (14) implies (12), it follows that under condition (14) a complete pooling policy is optimal, see Figure 4(c). Under a complete pooling policy, a demand is directly met from own stock if possible, and otherwise always via a lateral transshipment, if possible. No stock is hold back in any case. A complete pooling policy is often assumed in the literature, see e.g. [1, 4, 10, 18, 19, 27, 29, 32, 36], to mention only a few. Theorem 3.7 gives sufficient conditions under which such a policy is indeed optimal.

The implication of (12) by (14) can be seen as follows. Rewriting (14) gives $\frac{\lambda_2}{\lambda_2 + \mu} P_{EP_2} \leq P_{EP_1} - P_{LT_1}$, but this implies $\frac{\lambda_2}{\lambda_2 + \mu} P_{EP_2} \leq P_{EP_1} + \frac{\lambda_2}{\lambda_2 + \mu} P_{LT_2}$ (as both P_{LT_1} and $\frac{\lambda_2}{\lambda_2 + \mu} P_{LT_2}$ are nonnegative), which is equivalent to (12). Analogously, (15) implies (13).

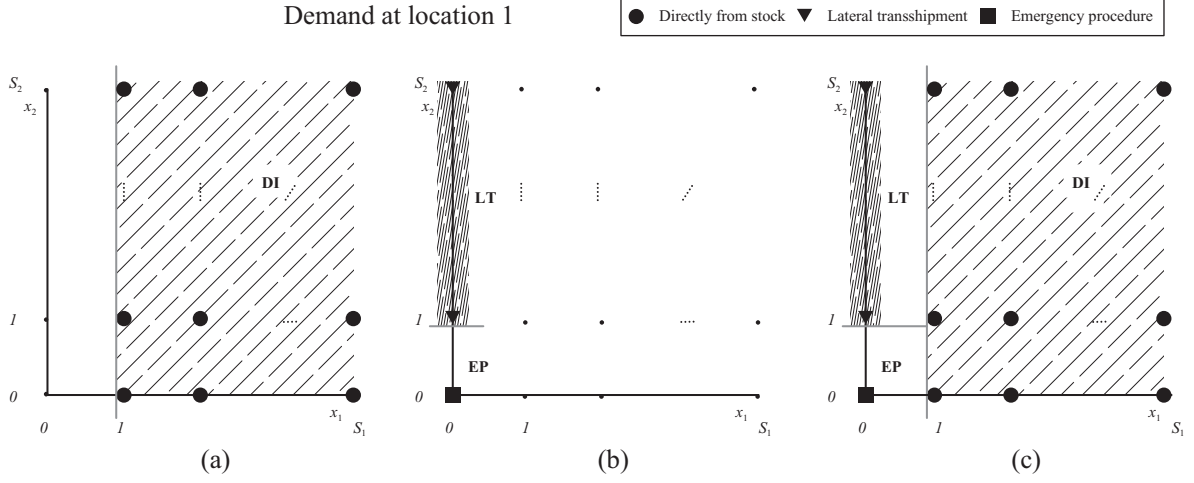


Figure 4: (a) Always DI (hold back policy, Theorem 3.6); (b) Always LT if out-of-stock (Theorem 3.7); (c) Complete pooling (again Theorem 3.7, as it implies Theorem 3.6).

The given conditions in Theorem 3.6 and Theorem 3.7 are, in general, sufficient, but not necessary. For the cases $S_1 = 1, S_2 = 0$, respectively $S_1 = 0, S_2 = 1$, the conditions are necessary and sufficient. There exist examples *not* satisfying these conditions, in which case the optimal policy is neither a hold back nor a complete pooling policy (see Section 3.4, Example 1).

There is an interesting relation between the conditions, when considered for both stock-points: either condition (12), or condition (15) holds (or both hold); and either condition (13), or condition (14) holds (or both hold). These statements follow from the following (e.g. for the first one): (i) if condition (15) does *not* hold, then surely condition (12) holds; and (ii) if condition (15) does *not* hold, then surely condition (12) holds. This follows by rewriting the conditions, e.g., for (i) we have: if (12) does not hold, then $P_{EP_2} \geq P_{LT_2} + \frac{\lambda_2 + \mu}{\lambda_2} P_{EP_1}$, but this implies $P_{EP_2} \geq P_{LT_2} + \frac{\lambda_1}{\lambda_1 + \mu} P_{EP_1}$ (as $\frac{\lambda_2 + \mu}{\lambda_2} \geq 1$, but $\frac{\lambda_1}{\lambda_1 + \mu} \leq 1$), which is exactly (15). And (ii) follows as $\frac{\lambda_1}{\lambda_1 + \mu} \leq 1$ in (15), but $1 + \frac{\mu}{\lambda_2} \geq 1$ in (12). The analogous reasoning holds for conditions (13) and (14). Combined with the properties that (14) implies (12), and that (15) implies (13), this immediately leads to the following corollary.

Corollary 3.8. *The optimal lateral transshipment policy is*

- (i) *either (at least) a holdback policy at both locations;*
- (ii) *or a complete pooling policy for (at least) one location.*

Here, by ‘at least’ a holdback policy we mean a either a holdback policy or a complete pooling policy. In the second case, the optimal policy for one location is a complete pooling policy, and the optimal policy for the other location can be a holdback policy, a complete pooling policy, or neither of the two.

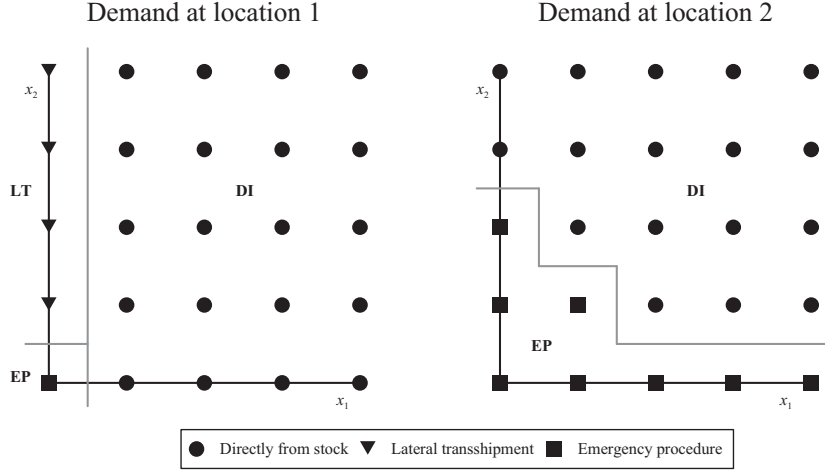


Figure 5: Optimal policy for the case with $S_1 = S_2 = 4$, $\lambda_1 = 2$, $\lambda_2 = 1$, $\mu_1 = \mu_2 = 1/3$ and penalty costs $P_{EP_1} = 25$, $P_{LT_1} = 5$, $P_{EP_2} = 10$, $P_{LT_2} = 2$.

3.4 Examples

We illustrate our results by two examples, and we give an example showing that if the assumption $\mu_1 = \mu_2$ is not satisfied, the structural results do not necessarily have to hold.

Example 1

Consider the following example: $S_1 = S_2 = 4$, and $\lambda_1 = 2$, $\lambda_2 = 1$, $\mu_1 = \mu_2 = 1/3$, and cost parameters given by $P_{EP_1} = 25$, $P_{LT_1} = 5$ and $P_{EP_2} = 10$, $P_{LT_2} = 2$. Hence an emergency procedure is five times as expensive as a lateral transshipment, and at location 1, the demand rate as well as the costs are higher. The optimal policy is given in Figure 5.

At stockpoint 1 a complete pooling policy is optimal: the demands are fulfilled directly from own stock if possible ($a_1^*(x_1, x_2) = 0$ for $x_1 > 0$), or via a lateral transshipment in case of a stock-out ($a_1^*(0, x_2) = 1$, for $x_2 > 0$). Only if stockpoint 2 is stocked-out as well, an emergency procedure is applied ($a_1^*(0, 0) = 2$). This structure is implied by the fact that the parameters satisfy condition (14), and hence part 1a) of Theorem 3.7 holds.

For stockpoint 2, no further conclusions can be drawn for the optimal policy. Demands are only fulfilled directly from stock if the sum of the inventory levels at both locations, is large enough, i.e. if $x_1 + x_2 \geq 3$ (and $x_2 > 0$), otherwise an emergency procedure is applied. This can be explained in the following way. The costs for lateral transshipments to and emergency procedures at stockpoint 1 are much higher than those at stockpoint 2. This results in the fact that stockpoint 2 will hold back parts, even when it faces a demand. By holding back parts, the expensive costs for an emergency procedure at stockpoint 1 are saved, at the expense of a lateral transshipments from 2 and possibly an emergency procedures at 2, but these being less costly.

The optimal policy at location 2 resembles a so-called *critical level* policy, which is common

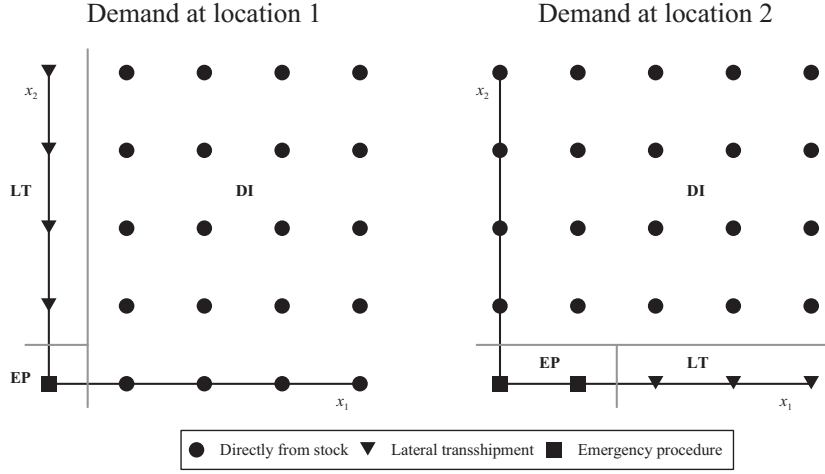


Figure 6: Optimal policy for example 1, now with $P_{EP_2} = 20$, $P_{LT_2} = 4$.

in stock rationing problems for single inventory point, see e.g. Ha [11]. One stockpoint satisfies demands of two (or more) types of customers, which differs in penalty costs for lost sales. In the typical optimal policy, demands from the most expensive customers are always satisfied, and there exists a threshold (called the *critical level*) for the inventory level, from which on demands for the less expensive customers are also satisfied. In this example we have such a critical level for the sum of the stocklevels at 1 and 2, which determines whether a demand at location 2 is directly satisfied, or ‘lost’ (i.e. satisfied by an emergency procedure).

The optimal policy gives expected average costs per time unit of 18.2. Without lateral transshipments, these costs would be 25.5, hence the optimal policy reduces this by almost 29%. A complete pooling policy has expected average costs per time unit of 20.0 in this case, so the optimal policy reduces this by 9.4%.

Example 2

In Example 1, condition (14) (and hence condition (12)) was satisfied for stockpoint 1, but not for stockpoint 2. By doubling the penalty costs at 2, into $P_{EP_2} = 20$ and $P_{LT_2} = 4$, condition (13) is satisfied as well, and hence, by Theorem 3.6, this results in the optimality of a hold back policy at both locations (with still complete pooling at 1). The optimal policy is given in Figure 6.

The two thresholds (the *hold back levels*), determine the entire policy, and are given by $\hat{T}_1^{lt}(0) = 1$ and $\hat{T}_2^{lt}(0) = 2$, the inventory levels from which on lateral transshipments ($a_i^* = 1$) are applied instead of emergency procedures ($a_i^* = 2$). The expected average costs per time unit in this case are 22.9. For a policy without lateral transshipments these would be 27.6 (almost 17% reduction for optimal policy), and complete pooling would give 23.2. This is only 1.4% reduction, but this policy differs from the optimal policy only in $a_2(1, 0)$.

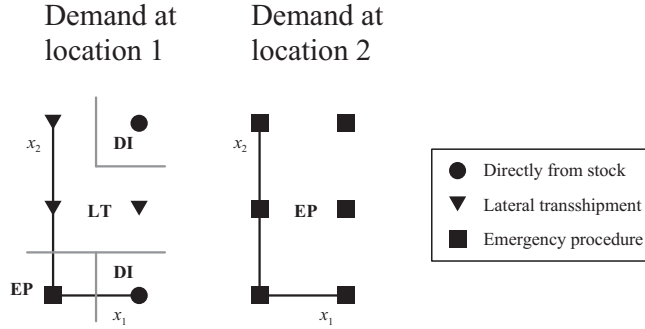


Figure 7: Example in which $\mu_1 \neq \mu_2$, and the structural properties of the optimal policy does not hold.

Example 3 ($\mu_1 \neq \mu_2$)

The following simple example illustrates that if the assumption $\mu_1 = \mu_2$ is not satisfied, the structural results do not necessarily have to hold. Let $S_1 = 1, S_2 = 2, \lambda_1 = \lambda_2 = 1$, and $\mu_1 = 1/3 \neq \mu_2 = 1$. Further, let $P_{EP_1} = 1000, P_{LT_1} = 175$ and $P_{EP_2} = P_{LT_2} = 10$. The (unique) optimal policy is given in Figure 7. Clearly, for demands at stockpoint 1 when $x_1 = 1$, the structure of the optimal policy is not the threshold type policy as described by Theorem 3.4.

3.5 Symmetric parameters

A special case is the system in which all parameters are symmetric, i.e. in which all parameters for both stockpoints are equal: $S_1 = S_2 =: S, \lambda_1 = \lambda_2 =: \lambda, \mu_1 = \mu_2 =: \mu, P_{LT_1} = P_{LT_2} =: P_{LT}, P_{EP_1} = P_{EP_2} =: P_{EP}$. It is straightforward that in this case there exists a symmetric optimal policy. As the conditions of Theorem 3.6 are clearly satisfied, it follows that for both stockpoints a hold back policy is optimal. The entire policy can now be described by a single (for both stockpoints equal) hold back level $\hat{T}_1^{lt}(0) = \hat{T}_2^{lt}(0) =: T \in \{1, 2, \dots, S + 1\}$, see Figure 8. $T = 1$ indicates complete pooling, $T = 2$ indicates that one part is hold back, and so on, and $T = S + 1$ indicates that no stock is shared in any way, i.e. there is no interaction between the stockpoints. Hence, there are only $S + 1$ possible optimal policies.

Given λ/μ , it turns out that the optimal policy is now determined by only the ratio P_{LT}/P_{EP} . For $S = 4$ it is indicated in Figure 9(a) when each of the five possible policies is optimal. These areas are determined by solving the steady state distribution of the Markov process for each of the policies, and deriving the average costs of a policy from this.

For the symmetric case Theorem 3.7 reduces to the following corollary, which also holds when $S_1 \neq S_2$.

Corollary 3.9. *In case of symmetric system parameters, and if*

$$P_{LT} \leq \frac{\mu}{\lambda + \mu} P_{EP}, \quad (16)$$

a complete pooling policy is optimal.

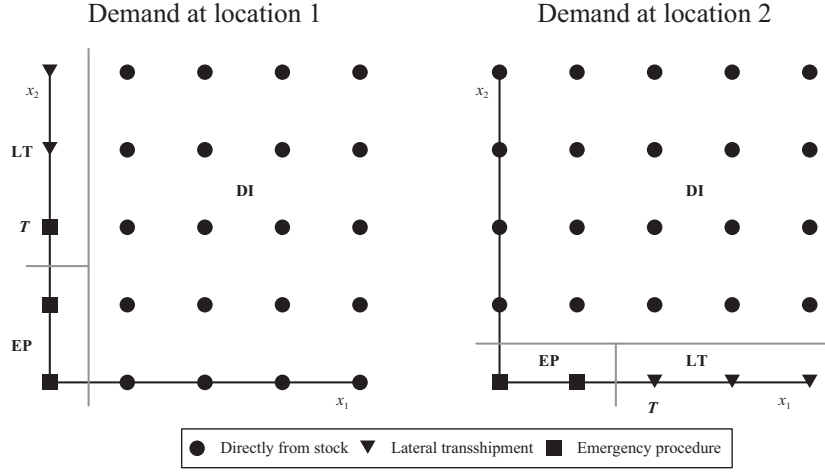


Figure 8: For symmetric system parameters, the optimal policy can be described by only one threshold T (e.g., here $T = 3$).

In Figure 9(a) the curve $\mu/(\lambda + \mu) = 1/(1 + \lambda/\mu)$ is plotted as well. The condition is a sufficient condition, however, from the figure it turns out that it covers a large part of the exact area in which complete pooling is optimal.

4 Limited repair capacity

A variant of the described system is a system in which there is limited repair capacity: at each stockpoint there is only one server to repair the parts. The repair times remain exponentially distributed with mean $1/\mu_i$ at stockpoint i , where we allow for non-identical repair rates at both locations. We only have to change the operator G_i into, say, \tilde{G}_i , where:

$$\tilde{G}_1 f(x_1, x_2) = \begin{cases} f(x_1 + 1, x_2) & \text{if } x_1 < S_1, \\ f(x_1, x_2) & \text{if } x_1 = S_1, \end{cases} \quad (17)$$

and \tilde{G}_2 analogously. The value function \tilde{V}_n becomes

$$\tilde{V}_{n+1}(x_1, x_2) = \frac{1}{\tilde{\nu}} \left(\sum_{i=1}^2 \mu_i \tilde{G}_i \tilde{V}_n(x_1, x_2) + \sum_{i=1}^2 \lambda_i H_i \tilde{V}_n(x_1, x_2) \right), \text{ for } (x_1, x_2) \in \mathcal{S}, n \geq 0,$$

with $\tilde{V}_0 \equiv 0$, $\tilde{\nu} = \lambda_1 + \lambda_2 + \mu_1 + \mu_2$, and \mathcal{S} and H_i , $i = 1, 2$, unchanged. The following holds for \tilde{G}_j :

Lemma 4.1. *The operator \tilde{G}_j , $j = 1, 2$, preserves each of the following properties: (i) Decr; (ii) Conv and Decr(j); (iii) Supermod; (iv) SuperC and Conv($j + 1$); (v) MM and Conv($j + 1$).*

Here $j + 1$ should be read as $j + 1 \bmod 2$, and e.g. (ii) states that if f is Conv and Decr(j), then $\tilde{G}_j f$ is so as well. However, it does *not* hold that f Conv implies $\tilde{G}_j f$ Conv.

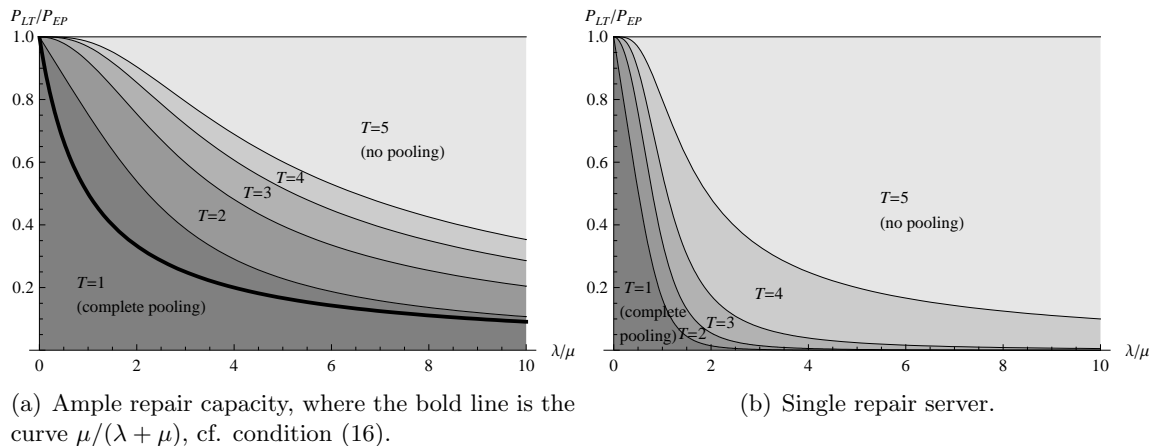


Figure 9: For $S = 4$ and symmetric system parameters, the $S + 1$ regions where each of the thresholds T is optimal.

Corollary 4.2. \tilde{V}_n satisfies (4)–(10) for all $n \geq 0$.

The following theorem is a direct consequence of this corollary.

Theorem 4.3. *In case of a single repair server at both locations, the same structural results for the optimal policy hold for this system, i.e. Theorem 3.4 and Theorem 3.5 still hold, even without the assumption of equal μ_i 's.*

For Theorem 4.3 we do not need equal μ_i 's, as \tilde{G}_1 and \tilde{G}_2 separately preserve MM, and not only the sum of both. For the symmetric case, we compare the optimal policy with the case of ample repair capacity, see Figure 9(b). By comparing the two graphs of Figure 9, it follows that the set of system parameters where one can benefit from lateral transshipments, is much smaller now.

5 Conclusion

In this paper, we proved that the structure of the optimal lateral transshipment policy is a threshold type policy, and we gave sufficient conditions under which a hold back policy or a complete pooling policy is optimal.

Interesting problems for further research would be the extension to three or more stock-points, variations in the repair time distribution (such as Erlang $_k$ distributed repair times, or state dependent repair rates), and the incorporation of so-called pro-active lateral transshipments, i.e. rebalancing of the stocklevels triggered by a replenishment.

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A Appendix

A.1 Proof of Lemma 3.1

Proof. a) We give the proofs for the operator G_1 . By interchanging the numbering of the locations, the results directly follow for the operator G_2 as well.

(i) It is straightforward to check that if f is Decr(1) (cf. (4)), then G_1f is Decr(1) as well, i.e. if $f(x_1, x_2) \geq f(x_1 + 1, x_2)$, then $G_1f(x_1, x_2) \geq G_1f(x_1 + 1, x_2)$, for all (x_1, x_2) such that the states appearing exists $\in \mathcal{S}$. Along the same lines it follows that if f is Decr(2) (cf. (5)), then G_1f is Decr(2) as well, i.e. then $G_1f(x_1, x_2) \geq G_1f(x_1, x_2 + 1)$. Combining this proves that the operator G_1 preserves Decr.

(ii) Assume that f is Conv(1) (cf. (6)), then we show that G_1f is Conv(1) as well. For $x_1 + 2 < S_1$:

$$\begin{aligned}
& G_1f(x_1, x_2) + G_1f(x_1 + 2, x_2) \\
&= (S_1 - x_1)f(x_1 + 1, x_2) + x_1f(x_1, x_2) \\
&\quad + (S_1 - x_1 - 2)f(x_1 + 3, x_2) + (x_1 + 2)f(x_1 + 2, x_2) \\
&= (S_1 - x_1 - 2)\left[f(x_1 + 1, x_2) + f(x_1 + 3, x_2)\right] + x_1\left[f(x_1, x_2) + f(x_1 + 2, x_2)\right] \\
&\quad + 2f(x_1 + 1, x_2) + 2f(x_1 + 2, x_2) \\
&\geq 2(S_1 - x_1 - 2)f(x_1 + 2, x_2) + 2x_1f(x_1 + 1, x_2) + 2f(x_1 + 1, x_2) + 2f(x_1 + 2, x_2) \\
&= 2(S_1 - x_1 - 1)f(x_1 + 2, x_2) + 2(x_1 + 1)f(x_1 + 1, x_2) \\
&= 2G_1f(x_1 + 1, x_2),
\end{aligned}$$

where the inequality holds by applying that f is Conv(1) on the parts between brackets. And for $x_1 + 2 = S_1$:

$$\begin{aligned}
& G_1f(S_1 - 2, x_2) + G_1f(S_1, x_2) \\
&= 2f(S_1 - 1, x_2) + (S_1 - 2)f(S_1 - 2, x_2) + S_1f(S_1, x_2) \\
&= 2f(S_1 - 1, x_2) + (S_1 - 2)[f(S_1 - 2, x_2) + f(S_1, x_2)] + 2f(S_1, x_2) \\
&\geq 2f(S_1 - 1, x_2) + 2(S_1 - 2)f(S_1 - 1, x_2) + 2f(S_1, x_2) \\
&= 2f(S_1, x_2) + 2(S_1 - 1)f(S_1 - 1, x_2) \\
&= 2G_1f(S_1 - 1, x_2),
\end{aligned}$$

where again the inequality holds by applying that f is Conv(1) on the part between brackets.

It is straightforward to check that if f is Conv(2) (cf. (7)), then G_1f is Conv(2) as well, i.e. then $G_1f(x_1, x_2) + G_1f(x_1, x_2 + 2) \geq 2G_1f(x_1, x_2 + 1)$. Combining this proves that the operator G_1 preserves Conv.

(iii) Along the same lines of the proof of (ii) one can prove that if f is Supermod (cf. (8)), then G_1f is Supermod as well. Hence the operator G_1 preserves Supermod.

b) (i)–(iii) trivially follow from part a).

(iv) We show that $G_1 + G_2$ preserves SuperC(1, 2); then SuperC(2, 1) follows by interchanging the numbering of the locations. Assume that f is SuperC(1, 2) (cf. (9)), then, for $x_1 + 2 < S_1$ and $x_2 + 1 < S_2$:

$$\begin{aligned}
& (G_1 + G_2)f(x_1, x_2 + 1) + (G_1 + G_2)f(x_1 + 2, x_2) \\
&= (S_1 - x_1 - 2)\left[f(x_1 + 1, x_2 + 1) + f(x_1 + 3, x_2)\right] + 2f(x_1 + 1, x_2 + 1) \\
&\quad + x_1\left[f(x_1, x_2 + 1) + f(x_1 + 2, x_2)\right] + 2f(x_1 + 2, x_2) \\
&\quad + (S_2 - x_2 - 1)\left[f(x_1, x_2 + 2) + f(x_1 + 2, x_2 + 1)\right] + f(x_1 + 2, x_2 + 1) \\
&\quad + (x_2 + 1)\left[f(x_1, x_2 + 1) + f(x_1 + 2, x_2)\right] - f(x_1 + 2, x_2).
\end{aligned}$$

Now we use that f is SuperC(1, 2), and apply this to the terms between brackets. This gives

$$\begin{aligned}
& (G_1 + G_2)f(x_1, x_2 + 1) + (G_1 + G_2)f(x_1 + 2, x_2) \\
& \geq (S_1 - x_1 - 2) \left[f(x_1 + 2, x_2) + f(x_1 + 2, x_2 + 1) \right] + 2f(x_1 + 1, x_2 + 1) \\
& \quad + x_1 \left[f(x_1 + 1, x_2) + f(x_1 + 1, x_2 + 1) \right] + 2f(x_1 + 2, x_2) \\
& \quad + (S_2 - x_2 - 1) \left[f(x_1 + 1, x_2 + 1) + f(x_1 + 1, x_2 + 2) \right] + f(x_1 + 2, x_2 + 1) \\
& \quad + (x_2 + 1) \left[f(x_1 + 1, x_2) + f(x_1 + 1, x_2 + 1) \right] - f(x_1 + 2, x_2) \\
& = (S_1 - x_1 - 1)f(x_1 + 2, x_2) + (x_1 + 1)f(x_1 + 1, x_2) \\
& \quad + (S_1 - x_1 - 1)f(x_1 + 2, x_2 + 1) + (x_1 + 1)f(x_1 + 1, x_2 + 1) \\
& \quad + (S_2 - x_2)f(x_1 + 1, x_2 + 1) + x_2f(x_1 + 1, x_2) \\
& \quad + (S_2 - x_2 - 1)f(x_1 + 1, x_2 + 2) + (x_2 + 1)f(x_1 + 1, x_2 + 1) \\
& = (G_1 + G_2)f(x_1 + 1, x_2) + (G_1 + G_2)f(x_1 + 1, x_2 + 1).
\end{aligned}$$

The cases $x_1 + 2 = S_1$ and/or $x_2 + 1 = S_2$ are along the same lines.

(v) As $\text{MM} = \text{Supermod} \cap \text{SuperC}$ (cf. (11)), it directly follows from parts (iii) and (iv) that $G_1 + G_2$ preserves MM. \square

A.2 Proof of Lemma 3.2

Proof. (i) It is straightforward to check that if f is $\text{Decr}(j)$, then $H_i f$ is $\text{Decr}(j)$, for $i, j = 1, 2$.

(ii) In order to prove that H_i preserves MM, we prove (cf. (11)) that it preserves Supermod, SuperC(1, 2) and SuperC(2, 1) (cf. (8)–(10)) together, that is, given that f is Supermod, SuperC(1, 2) and SuperC(2, 1), we show that $H_i f$ is Supermod, SuperC(1, 2) and SuperC(2, 1) as well. We show this for H_1 ; then for H_2 it follows by interchanging the numbering of the locations. Recall that Supermod and SuperC(i, j) imply Conv(i) (cf. (6) and (7)).

The proofs come down to case checking: applying H_1 to $f(x)$ introduces a minimization over three terms, so the sum of two gives a total of $3 \times 3 = 9$ possibilities, which we all check separately. For this we use the trivial result:

$$a \geq \min\{a, b\}, \quad \forall a, b \in \mathbb{R}.$$

The proofs are given for $x_1 > 0, x_2 > 0$, but it is straightforward to check that they also hold for the cases $x_1 = 0, x_2 > 0$, and $x_1 > 0, x_2 = 0$, and $x_1 = 0, x_2 = 0$.

Assume that f is Supermod, SuperC(1, 2) and SuperC(2, 1), which implies that f is also Conv(1) and Conv(2). Below we prove that H_1 preserves (i) Supermod, (ii) SuperC(1, 2), and (iii) SuperC(2, 1).

(i) Supermod

For $x_1 > 0, x_2 > 0$:

$$\begin{aligned}
& H_1 f(x_1, x_2) + H_1 f(x_1 + 1, x_2 + 1) \\
& = \min \left\{ f(x_1 - 1, x_2), f(x_1, x_2 - 1) + P_{LT_1}, f(x_1, x_2) + P_{EP_1} \right\} \\
& \quad + \min \left\{ f(x_1, x_2 + 1), f(x_1 + 1, x_2) + P_{LT_1}, f(x_1 + 1, x_2 + 1) + P_{EP_1} \right\} \\
& = \min \left\{ f(x_1 - 1, x_2) + f(x_1, x_2 + 1), f(x_1 - 1, x_2) + f(x_1 + 1, x_2) + P_{LT_1}, \right. \\
& \quad f(x_1 - 1, x_2) + f(x_1 + 1, x_2 + 1) + P_{EP_1}, f(x_1, x_2 - 1) + P_{LT_1} + f(x_1, x_2 + 1), \\
& \quad f(x_1, x_2 - 1) + P_{LT_1} + f(x_1 + 1, x_2) + P_{LT_1}, f(x_1, x_2 - 1) + P_{LT_1} + f(x_1 + 1, x_2 + 1) + P_{EP_1}, \\
& \quad f(x_1, x_2) + P_{EP_1} + f(x_1, x_2 + 1), f(x_1, x_2) + P_{EP_1} + f(x_1 + 1, x_2) + P_{LT_1}, \\
& \quad \left. f(x_1, x_2) + P_{EP_1} + f(x_1 + 1, x_2 + 1) + P_{EP_1} \right\}.
\end{aligned}$$

It holds that:

$$\begin{aligned}
& f(x_1 - 1, x_2) + f(x_1, x_2 + 1) \geq f(x_1, x_2) + f(x_1 - 1, x_2 + 1) \text{ (by (8))}, \\
& f(x_1 - 1, x_2) + f(x_1 + 1, x_2) + P_{LT_1} \geq 2f(x_1, x_2) + P_{LT_1} \text{ (by (6))}, \\
& f(x_1 - 1, x_2) + f(x_1 + 1, x_2 + 1) + P_{EP_1} \geq 2f(x_1, x_2) - f(x_1 + 1, x_2) + f(x_1 + 1, x_2 + 1) + P_{EP_1} \\
& \quad \geq f(x_1, x_2) + f(x_1, x_2 + 1) + P_{EP_1} \text{ (by (6), resp. (8))}, \\
& f(x_1, x_2 - 1) + P_{LT_1} + f(x_1, x_2 + 1) \geq 2f(x_1, x_2) + P_{LT_1} \text{ (by (7))}, \\
& f(x_1, x_2 - 1) + P_{LT_1} + f(x_1 + 1, x_2) + P_{LT_1} \geq f(x_1, x_2) + P_{LT_1} + f(x_1 + 1, x_2 - 1) + P_{LT_1} \text{ (by (8))}, \\
& f(x_1, x_2 - 1) + P_{LT_1} + f(x_1 + 1, x_2 + 1) + P_{EP_1} \geq 2f(x_1, x_2) - f(x_1, x_2 + 1) + P_{LT_1} + f(x_1 + 1, x_2 + 1) + P_{EP_1} \\
& \quad \geq f(x_1, x_2) + P_{LT_1} + f(x_1 + 1, x_2) + P_{EP_1} \text{ (by (7), resp. (8))}, \\
& f(x_1, x_2) + P_{EP_1} + f(x_1 + 1, x_2 + 1) + P_{EP_1} \geq f(x_1 + 1, x_2) + P_{EP_1} + f(x_1, x_2 + 1) + P_{EP_1} \text{ (by (8))}.
\end{aligned}$$

This implies that:

$$\begin{aligned}
& H_1 f(x_1, x_2) + H_1 f(x_1 + 1, x_2 + 1) \\
& \geq \min \left\{ f(x_1, x_2) + f(x_1 - 1, x_2 + 1), 2f(x_1, x_2) + P_{LT_1}, \right. \\
& \quad f(x_1, x_2) + f(x_1, x_2 + 1) + P_{EP_1}, f(x_1, x_2) + f(x_1 + 1, x_2 - 1) + 2P_{LT_1}, \\
& \quad \left. f(x_1, x_2) + P_{LT_1} + f(x_1 + 1, x_2) + P_{EP_1}, f(x_1 + 1, x_2) + f(x_1, x_2 + 1) + 2P_{EP_1} \right\} \\
& \geq \min \left\{ f(x_1, x_2), f(x_1 + 1, x_2 - 1) + P_{LT_1}, f(x_1 + 1, x_2) + P_{EP_1} \right\} \\
& \quad + \min \left\{ f(x_1 - 1, x_2 + 1), f(x_1, x_2) + P_{LT_1}, f(x_1, x_2 + 1) + P_{EP_1} \right\} \\
& = H_1 f(x_1 + 1, x_2) + H_1 f(x_1, x_2 + 1).
\end{aligned}$$

(ii) SuperC(1,2)

For $x_1 > 0, x_2 > 0$:

$$\begin{aligned}
& H_1 f(x_1 + 2, x_2) + H_1 f(x_1, x_2 + 1) \\
& = \min \left\{ f(x_1 + 1, x_2), f(x_1 + 2, x_2 - 1) + P_{LT_1}, f(x_1 + 2, x_2) + P_{EP_1} \right\} \\
& \quad + \min \left\{ f(x_1 - 1, x_2 + 1), f(x_1, x_2) + P_{LT_1}, f(x_1, x_2 + 1) + P_{EP_1} \right\} \\
& = \min \left\{ f(x_1 + 1, x_2) + f(x_1 - 1, x_2 + 1), f(x_1 + 1, x_2) + f(x_1, x_2) + P_{LT_1}, \right. \\
& \quad f(x_1 + 1, x_2) + f(x_1, x_2 + 1) + P_{EP_1}, f(x_1 + 2, x_2 - 1) + P_{LT_1} + f(x_1 - 1, x_2 + 1), \\
& \quad f(x_1 + 2, x_2 - 1) + P_{LT_1} + f(x_1, x_2) + P_{LT_1}, f(x_1 + 2, x_2 - 1) + P_{LT_1} + f(x_1, x_2 + 1) + P_{EP_1}, \\
& \quad f(x_1 + 2, x_2) + P_{EP_1} + f(x_1 - 1, x_2 + 1), f(x_1 + 2, x_2) + P_{EP_1} + f(x_1, x_2) + P_{LT_1}, \\
& \quad \left. f(x_1 + 2, x_2) + P_{EP_1} + f(x_1, x_2 + 1) + P_{EP_1} \right\}
\end{aligned}$$

It holds that:

$$\begin{aligned}
& f(x_1 + 1, x_2) + f(x_1 - 1, x_2 + 1) \geq f(x_1, x_2) + f(x_1, x_2 + 1) \text{ (by (9))}, \\
& f(x_1 + 2, x_2 - 1) + P_{LT_1} + f(x_1 - 1, x_2 + 1) \geq f(x_1 + 1, x_2 - 1) + f(x_1 + 1, x_2) - f(x_1, x_2) \\
& \qquad \qquad \qquad + P_{LT_1} + f(x_1 - 1, x_2 + 1) \\
& \qquad \qquad \qquad \geq f(x_1 + 1, x_2 - 1) + P_{LT_1} + f(x_1, x_2 + 1) \text{ (by twice (9))}, \\
& f(x_1 + 2, x_2 - 1) + P_{LT_1} + f(x_1, x_2) + P_{LT_1} \geq f(x_1 + 1, x_2 - 1) + P_{LT_1} + f(x_1 + 1, x_2) + P_{LT_1} \text{ (by (9))}, \\
& f(x_1 + 2, x_2 - 1) + P_{LT_1} + f(x_1, x_2 + 1) + P_{EP_1} \geq f(x_1 + 1, x_2 - 1) + f(x_1 + 1, x_2) - f(x_1, x_2) + P_{LT_1} \\
& \qquad \qquad \qquad + f(x_1, x_2 + 1) + P_{EP_1} \\
& \qquad \qquad \qquad \geq 2f(x_1 + 1, x_2) + P_{LT_1} + P_{EP_1} \text{ (by (9), resp. (10))}, \\
& f(x_1 + 2, x_2) + P_{EP_1} + f(x_1 - 1, x_2 + 1) \geq f(x_1 + 1, x_2) + f(x_1 + 1, x_2 + 1) - f(x_1, x_2 + 1) + P_{EP_1} \\
& \qquad \qquad \qquad + f(x_1 - 1, x_2 + 1) \\
& \qquad \qquad \qquad \geq f(x_1 + 1, x_2) + f(x_1, x_2 + 1) + P_{EP_1} \text{ (by (9), resp. (6))}, \\
& f(x_1 + 2, x_2) + P_{EP_1} + f(x_1, x_2) + P_{LT_1} \geq 2f(x_1 + 1, x_2) + P_{EP_1} + P_{LT_1} \text{ (by (6))}, \\
& f(x_1 + 2, x_2) + P_{EP_1} + f(x_1, x_2 + 1) + P_{EP_1} \geq f(x_1 + 1, x_2) + P_{EP_1} + f(x_1 + 1, x_2 + 1) + P_{EP_1} \text{ (by (9))}.
\end{aligned}$$

This implies that:

$$\begin{aligned}
& H_1 f(x_1 + 2, x_2) + H_1 f(x_1, x_2 + 1) \\
& \geq \min \left\{ f(x_1, x_2) + f(x_1, x_2 + 1), f(x_1 + 1, x_2) + f(x_1, x_2) + P_{LT_1}, \right. \\
& \quad f(x_1 + 1, x_2 - 1) + f(x_1, x_2 + 1) + P_{LT_1}, f(x_1 + 1, x_2 - 1) + f(x_1 + 1, x_2) + 2P_{LT_1}, \\
& \quad f(x_1 + 1, x_2) + f(x_1, x_2 + 1) + P_{EP_1}, 2f(x_1 + 1, x_2) + P_{LT_1} + P_{EP_1}, \\
& \quad \left. f(x_1 + 1, x_2) + f(x_1 + 1, x_2 + 1) + 2P_{EP_1} \right\} \\
& \geq \min \left\{ f(x_1, x_2), f(x_1 + 1, x_2 - 1) + P_{LT_1}, f(x_1 + 1, x_2) + P_{EP_1} \right\} \\
& \quad + \min \left\{ f(x_1, x_2 + 1), f(x_1 + 1, x_2) + P_{LT_1}, f(x_1 + 1, x_2 + 1) + P_{EP_1} \right\} \\
& = H_1 f(x_1 + 1, x_2) + H_1 f(x_1 + 1, x_2 + 1).
\end{aligned}$$

(iii) SuperC(2,1)

For $x_1 > 0, x_2 > 0$:

$$\begin{aligned}
& H_1 f(x_1, x_2 + 2) + H_1 f(x_1 + 1, x_2) \\
& = \min \left\{ f(x_1 - 1, x_2 + 2), f(x_1, x_2 + 1) + P_{LT_1}, f(x_1, x_2 + 2) + P_{EP_1} \right\} \\
& \quad + \min \left\{ f(x_1, x_2), f(x_1 + 1, x_2 - 1) + P_{LT_1}, f(x_1 + 1, x_2) + P_{EP_1} \right\} \\
& = \min \left\{ f(x_1 - 1, x_2 + 2) + f(x_1, x_2), f(x_1 - 1, x_2 + 2) + f(x_1 + 1, x_2 - 1) + P_{LT_1}, \right. \\
& \quad f(x_1 - 1, x_2 + 2) + f(x_1 + 1, x_2) + P_{EP_1}, f(x_1, x_2 + 1) + P_{LT_1} + f(x_1, x_2), \\
& \quad f(x_1, x_2 + 1) + P_{LT_1} + f(x_1 + 1, x_2 - 1) + P_{LT_1}, f(x_1, x_2 + 1) + P_{LT_1} + f(x_1 + 1, x_2) + P_{EP_1}, \\
& \quad f(x_1, x_2 + 2) + P_{EP_1} + f(x_1, x_2), f(x_1, x_2 + 2) + P_{EP_1} + f(x_1 + 1, x_2 - 1) + P_{LT_1}, \\
& \quad \left. f(x_1, x_2 + 2) + P_{EP_1} + f(x_1 + 1, x_2) + P_{EP_1} \right\}.
\end{aligned}$$

(as V_n is Supermod). So, for $x_2 > 0$, $\Delta w_{x_1}(u, x_1, x_2)$ is increasing in u :

$$\Delta w_{x_1}(2, x_1, x_2) \geq \Delta w_{x_1}(1, x_1, x_2) \geq \Delta w_{x_1}(0, x_1, x_2).$$

This implies that, for every $n \geq 0$, there exists a threshold for the inventory level x_1 , which can depend on x_2 , say $T_{n,1}^{di}(x_2)$, from which on it is optimal to fulfill demands directly from stock. Next there exists a threshold, say $T_{n,1}^{lt}(x_2)$, such that $T_{n,1}^{lt}(x_2) \leq T_{n,1}^{di}(x_2)$, from which on (until $T_{n,1}^{di}(x_2) - 1$) it is optimal to fulfill demands via a lateral transshipment, and on the interval $x_1 = 0$ up till $T_{n,1}^{lt}(x_2) - 1$ an emergency procedure is optimal. Hence, if f_{n+1} is the minimizing policy in (1), then f_{n+1} is a threshold policy. Note that the transition probability matrix of every stationary policy is unichain (since every state can access (S_1, S_2)) and aperiodic (since the transition probability from state (S_1, S_2) to itself is positive). Then, by Puterman [25, Theorem 8.5.4], the long run average costs under the stationary policy f_{n+1} converges to the minimal long run average costs as n tends to infinity. Since there are only finitely many stationary threshold policies, this implies that there exists an optimal stationary policy that is a threshold type policy.

For $x_2 = 0$, lateral transshipments ($u = 1$) are not possible, and we have, for each $n \geq 0$:

$$\Delta w_{x_1}(2, x_1, x_2) - \Delta w_{x_1}(0, x_1, x_2) = V_n(x_1 + 1, x_2) - V_n(x_1, x_2) - V_n(x_1, x_2) + V_n(x_1 - 1, x_2) \geq 0,$$

(as V_n is Conv(1)). Hence $\Delta w_{x_1}(2, x_1, 0) \geq \Delta w_{x_1}(0, x_1, 0)$, and so, for the special case $x_2 = 0$, there exists only one threshold. By the analogous reasoning as for $x_2 > 0$, it follows that there exists a $T_1^{di}(0)$ (which is equal to $T_1^{lt}(0)$). As it is only possible to deliver directly from stock if $x_1 \geq 1$, it follows that $T_1^{di}(0) \geq 1$.

By interchanging the numbering of the stockpoints, the analogous results for stockpoint 2 directly follow. \square

A.4 Proof of Theorem 3.5

Proof. Analogously to the proof of Theorem 3.4, consider a demand at stockpoint 1, and define:

$$\Delta w_{x_2}(u, x_1, x_2) := w(u, x_1, x_2 + 1) - w(u, x_1, x_2),$$

where $w(u, x_1, x_2)$ is as defined in (18). Then for each $n \geq 0$, and for $x_1 > 0$:

$$\Delta w_{x_2}(0, x_1, x_2) - \Delta w_{x_2}(1, x_1, x_2) = V_n(x_1 - 1, x_2 + 1) - V_n(x_1 - 1, x_2) - V_n(x_1, x_2) + V_n(x_1, x_2 - 1) \geq 0$$

(as, by Theorem 3.3, V_n is SuperC(2,1)), and:

$$\Delta w_{x_2}(2, x_1, x_2) - \Delta w_{x_2}(0, x_1, x_2) = V_n(x_1, x_2 + 1) - V_n(x_1, x_2) - V_n(x_1 - 1, x_2 + 1) + V_n(x_1 - 1, x_2) \geq 0$$

(as V_n is Supermod). Hence, for $x_1 > 0$:

$$\Delta w_{x_2}(2, x_1, x_2) \geq \Delta w_{x_2}(0, x_1, x_2) \geq \Delta w_{x_2}(1, x_1, x_2).$$

Analogously to the reasoning in the proof of Theorem 3.4, it now follows that, for n to infinity, there exist two thresholds $\hat{T}_2^{di}(x_1)$ and $\hat{T}_2^{lt}(x_1)$, where $\hat{T}_2^{di}(x_1) \leq \hat{T}_2^{lt}(x_1)$, such that from $\hat{T}_2^{lt}(x_1)$ lateral transshipments are optimal, from $\hat{T}_2^{di}(x_1)$ to $\hat{T}_2^{lt}(x_1) - 1$ direct delivering from stock is optimal, and from 0 to $\hat{T}_2^{di}(x_1) - 1$ emergency procedures are optimal.

For $x_1 = 0$, directly satisfying a demand from stock ($u = 0$) is not possible, and we have, for each $n \geq 0$:

$$\Delta w_{x_2}(2, x_1, x_2) - \Delta w_{x_2}(1, x_1, x_2) = V_n(x_1, x_2 + 1) - V_n(x_1, x_2) - V_n(x_1, x_2) + V_n(x_1, x_2 - 1) \geq 0,$$

(as V_n is Conv(2)). Hence $\Delta w_{x_2}(2, x_1, 0) \geq \Delta w_{x_2}(1, x_1, 0)$, and so, for the special case $x_1 = 0$, there exists only one threshold: $\hat{T}_2^{lt}(0)$ (which is equal to $\hat{T}_2^{di}(0)$). As it is only possible to apply a lateral transshipment if $x_2 \geq 1$, it follows that $\hat{T}_2^{lt}(0) \geq 1$.

By interchanging the numbering of the stockpoints, the analogous results for stockpoint 2 directly follow. \square

A.5 Proof of Theorem 3.6

Proof. We prove part 1a). Part 1b) then directly follows by interchanging the stockpoints, and 2) is a trivial consequence of 1a) and 1b).

For 1a), we prove that $a_1^*(1, x_2) = 0$ for all $x_2 \in \{0, 1, \dots, S_2\}$, then it follows by Theorem 3.4 that $T_1^{di}(x_2) = 1$ for all x_2 . It suffices to prove that, for all $n \geq 0$:

$$V_n(1, x_2) + P_{EP_1} \geq V_n(0, x_2), \quad \text{for } x_2 \in \{0, \dots, S_2\}, \quad (19)$$

$$V_n(1, x_2 - 1) + P_{LT_1} \geq V_n(0, x_2), \quad \text{for } x_2 \in \{1, \dots, S_2\}. \quad (20)$$

For $S_1 = 0$ trivially $T_1^{di}(x_2) = 1$ for all x_2 , and for $S_1 > 0, S_2 = 0$ we only have to prove (19).

We prove the inequalities by induction, using that, by Theorem 3.3, V_n satisfies (4)–(10). For $V_0 \equiv 0$ both inequalities trivially hold. We first prove (i) the induction step of (19), then (ii) that of (20), both for $S_1 > 0$. All given inequalities hold by the induction hypothesis, unless stated otherwise.

(i) Assume that (19) holds for a given n (induction hypothesis), and let $S_1 > 0$. We consider the operators H_1, H_2, G_1 and G_2 separately.

For $x_2 = 0$:

$$\begin{aligned} H_1 V_n(1, 0) + P_{EP_1} &= \min\{P_{EP_1} + V_n(0, 0), 2P_{EP_1} + V_n(1, 0)\} \\ &\geq \min\{P_{EP_1} + V_n(0, 0), P_{EP_1} + V_n(0, 0)\} \\ &= P_{EP_1} + V_n(0, 0) = H_1 V_n(0, 0); \end{aligned}$$

and for $x_2 \in \{1, 2, \dots, S_2\}$:

$$\begin{aligned} H_1 V_n(1, x_2) + P_{EP_1} &= \min\{P_{EP_1} + V_n(0, x_2), P_{EP_1} + P_{LT_1} + V_n(1, x_2 - 1), 2P_{EP_1} + V_n(1, x_2)\} \\ &\geq \min\{P_{LT_1} + V_n(0, x_2 - 1), P_{EP_1} + V_n(0, x_2)\} = H_1 V_n(0, x_2). \end{aligned}$$

For $x_2 = 0$:

$$\begin{aligned} H_2 V_n(1, 0) + P_{EP_1} &= \min\{P_{EP_1} + P_{LT_2} + V_n(0, 0), P_{EP_1} + P_{EP_2} + V_n(1, 0)\} \\ &\geq \min\{P_{EP_1} + P_{LT_2} - P_{EP_2} + H_2 V_n(0, 0), H_2 V_n(0, 0)\} \\ &= H_2 V_n(0, 0) + \min\{P_{EP_1} + P_{LT_2} - P_{EP_2}, 0\}; \end{aligned}$$

and for $x_2 \in \{1, 2, \dots, S_2\}$:

$$\begin{aligned} H_2 V_n(1, x_2) + P_{EP_1} &= \min\{P_{EP_1} + V_n(1, x_2 - 1), P_{EP_1} + P_{LT_2} + V_n(0, x_2), P_{EP_1} + P_{EP_2} + V_n(1, x_2)\} \\ &\geq \min\{V_n(0, x_2 - 1), P_{EP_1} + P_{LT_2} + V_n(0, x_2), P_{EP_2} + V_n(0, x_2)\} \\ &\geq H_2 V_n(0, x_2) + \min\{P_{EP_1} + P_{LT_2} - P_{EP_2}, 0\}, \end{aligned}$$

as $H_2 V_n(0, x_2) = \min\{V_n(0, x_2 - 1), P_{EP_2} + V_n(0, x_2)\}$.

For the operator G_1 we obtain:

$$\begin{aligned} G_1 V_n(1, x_2) + (S_1 - 1)P_{EP_1} &= (S_1 - 1)V_n(2, x_2) + V_n(1, x_2) + (S_1 - 1)P_{EP_1} \\ &= (S_1 - 1)[V_n(2, x_2) - V_n(1, x_2)] + S_1 V_n(1, x_2) + (S_1 - 1)P_{EP_1} \\ &\geq (S_1 - 1)[V_n(1, x_2) - V_n(0, x_2)] + S_1 V_n(1, x_2) + (S_1 - 1)P_{EP_1} \\ &\geq S_1 V_n(0, 1) = G_1 V_n(0, x_2), \end{aligned}$$

where the first inequality holds as V_n is Conv(1) (cf. Theorem 3.3).

For $x_2 \in \{0, 1, \dots, S_2 - 1\}$ we obtain:

$$\begin{aligned} G_2 V_n(1, x_2) + S_2 P_{EP_1} &= (S_2 - x_2)V_n(1, x_2 + 1) + x_2 V_n(1, x_2) + S_2 P_{EP_1} \\ &\geq (S_2 - x_2)V_n(0, x_2 + 1) + x_2 V_n(0, x_2) = G_2 V_n(0, x_2); \end{aligned}$$

and for $x_2 = S_2$ trivially:

$$G_2V_n(1, S_2) + S_2P_{EP_1} = S_2V_n(1, S_2) + S_2P_{EP_1} \geq S_2V_n(0, S_2) = G_2V_n(0, S_2).$$

Combining these give, for all x_2 (recall $\nu = \lambda_1 + \lambda_2 + \mu S_1 + \mu S_2$):

$$\begin{aligned} & \nu(V_{n+1}(1, x_2) + P_{EP_1}) \\ &= \lambda_1 H_1 V_n(1, x_2) + \lambda_2 H_2 V_n(1, x_2) + \mu G_1 V_n(1, x_2) + \mu G_2 V_n(1, x_2) + \nu P_{EP_1} \\ &= \lambda_1 [H_1 V_n(1, x_2) + P_{EP_1}] + \lambda_2 [H_2 V_n(1, x_2) + P_{EP_1}] + \mu [G_1 V_n(1, x_2) + (S_1 - 1)P_{EP_1}] \\ & \quad + \mu [G_2 V_n(1, x_2) + S_2 P_{EP_1}] + \mu P_{EP_1} \\ &\geq \lambda_1 H_1 V_n(0, x_2) + \lambda_2 [H_2 V_n(0, x_2) + \min\{P_{EP_1} + P_{LT_2} - P_{EP_2}, 0\}] + \mu G_1 V_n(0, x_2) + \mu G_2 V_n(0, x_2) \\ &\geq \nu V_{n+1}(0, x_2), \end{aligned} \tag{21}$$

where the last inequality holds by condition (12). This completes the induction step, and hence (19) holds for all $n \geq 0$.

(ii) Assume that (20) holds for a given n (induction hypothesis), and let $S_1, S_2 > 0$. We consider the operators H_1, H_2 and $G_1 + G_2$ separately:

For $x_2 \in \{2, \dots, S_2\}$:

$$\begin{aligned} & H_1 V_n(1, x_2 - 1) + P_{LT_1} \\ &= \min\{P_{LT_1} + V_n(0, x_2 - 1), 2P_{LT_1} + V_n(1, x_2 - 2), P_{LT_1} + P_{EP_1} + V_n(1, x_2 - 1)\} \\ &\geq \min\{P_{LT_1} + V_n(0, x_2 - 1), P_{EP_1} + V_n(0, x_2)\} = H_1 V_n(0, x_2); \end{aligned}$$

and for $x_2 = 1$:

$$\begin{aligned} & H_1 V_n(1, 0) + P_{LT_1} \\ &= \min\{P_{LT_1} + V_n(0, 0), P_{LT_1} + P_{EP_1} + V_n(1, 0)\} \\ &\geq \min\{P_{LT_1} + V_n(0, 0), P_{EP_1} + V_n(0, 1)\} = H_1 V_n(0, 1). \end{aligned}$$

For $x_2 \in \{2, \dots, S_2\}$:

$$\begin{aligned} & H_2 V_n(1, x_2 - 1) + P_{LT_1} \\ &= \min\{P_{LT_1} + V_n(1, x_2 - 2), P_{LT_1} + P_{LT_2} + V_n(0, x_2 - 1), P_{LT_1} + P_{EP_2} + V_n(1, x_2 - 1)\} \\ &\geq \min\{V_n(0, x_2 - 1), P_{EP_2} + V_n(0, x_2)\} = H_2 V_n(0, x_2); \end{aligned}$$

and for $x_2 = 1$:

$$\begin{aligned} & H_2 V_n(1, 0) + P_{LT_1} \\ &= \min\{P_{LT_1} + P_{LT_2} + V_n(0, 0), P_{LT_1} + P_{EP_2} + V_n(1, 0)\} \\ &\geq \min\{V_n(0, 0), P_{EP_2} + V_n(0, 1)\} = H_2 V_n(0, 1). \end{aligned}$$

For $x_2 \in \{1, \dots, S_2 - 1\}$:

$$\begin{aligned} & (G_1 + G_2)V_n(1, x_2 - 1) + (S_1 + S_2)P_{LT_1} \\ &= (S_1 - 1)V_n(2, x_2 - 1) + V_n(1, x_2 - 1) \\ & \quad + (S_2 - x_2 + 1)V_n(1, x_2) + (x_2 - 1)V_n(1, x_2 - 1) + (S_1 + S_2)P_{LT_1} \\ &= (S_1 - 1)[V_n(2, x_2 - 1) - V_n(1, x_2)] + (S_1 - 1)V_n(1, x_2) + V_n(1, x_2 - 1) \\ & \quad + (S_2 - x_2)[V_n(1, x_2) - V_n(0, x_2 + 1)] + V_n(1, x_2) + (S_2 - x_2)V_n(0, x_2 + 1) \\ & \quad + x_2[V_n(1, x_2 - 1) - V_n(0, x_2)] - V_n(1, x_2 - 1) + x_2 V_n(0, x_2) + (S_1 + S_2)P_{LT_1} \\ &\geq (S_1 - 1)[V_n(1, x_2 - 1) - V_n(0, x_2)] + S_1 V_n(1, x_2) \\ & \quad + (S_2 - x_2)[V_n(1, x_2) - V_n(0, x_2 + 1)] + (S_2 - x_2)V_n(0, x_2 + 1) \\ & \quad + x_2[V_n(1, x_2 - 1) - V_n(0, x_2)] + x_2 V_n(0, x_2) + (S_1 + S_2)P_{LT_1} \\ &\geq S_1 V_n(1, x_2) + (S_2 - x_2)V_n(0, x_2 + 1) + x_2 V_n(0, x_2) + P_{LT_1} \\ &= (G_1 + G_2)V_n(0, x_2) + P_{LT_1}, \end{aligned}$$

where the first inequality holds as V_n is SuperC(1,2) (cf. Theorem 3.3). For $x_2 = S_2$:

$$\begin{aligned}
& (G_1 + G_2)V_n(1, S_2 - 1) + (S_1 + S_2)P_{LT_1} \\
&= (S_1 - 1)V_n(2, S_2 - 1) + V_n(1, S_2 - 1) \\
&\quad + V_n(1, S_2) + (S_2 - 1)V_n(1, S_2 - 1) + (S_1 + S_2)P_{LT_1} \\
&= (S_1 - 1)[V_n(2, S_2 - 1) - V_n(1, S_2)] + S_1 V_n(1, S_2) \\
&\quad + S_2[V_n(1, S_2 - 1) - V_n(0, S_2)] + S_2 V_n(0, S_2) + (S_1 + S_2)P_{LT_1} \\
&\geq (S_1 - 1)[V_n(1, S_2 - 1) - V_n(0, S_2)] + S_1 V_n(1, S_2) \\
&\quad + S_2[V_n(1, S_2 - 1) - V_n(0, S_2)] + S_2 V_n(0, S_2) + (S_1 + S_2)P_{LT_1} \\
&\geq S_1 V_n(1, S_2) + S_2 V_n(0, S_2) + P_{LT_1} = (G_1 + G_2)V_n(0, S_2) + P_{LT_1},
\end{aligned}$$

where the first inequality again holds as V_n is SuperC(1,2).

Combining these gives, for all $x_2 \in \{1, \dots, S_2\}$:

$$\begin{aligned}
& \nu(V_{n+1}(1, x_2 - 1) + P_{LT_1}) \\
&= \lambda_1 H_1 V_n(1, x_2 - 1) + \lambda_2 H_2 V_n(1, x_2 - 1) + \mu(G_1 + G_2)V_n(1, x_2 - 1) + \nu P_{LT_1} \\
&= \lambda_1 [H_1 V_n(1, x_2 - 1) + P_{LT_1}] + \lambda_2 [H_2 V_n(1, x_2 - 1) + P_{LT_1}] + \mu[(G_1 + G_2)V_n(1, x_2 - 1) + (S_1 + S_2)P_{LT_1}] \\
&\geq \lambda_1 H_1 V_n(0, x_2) + \lambda_2 H_2 V_n(0, x_2) + \mu(G_1 + G_2)V_n(0, x_2) = \nu V_{n+1}(0, x_2),
\end{aligned}$$

which completes the induction step, and hence (20) holds for all $n \geq 0$. \square

A.6 Proof of Theorem 3.7

Proof. We again prove only part 1a), as again part 1b) directly follows by interchanging the stockpoints, and 2) is a trivial consequence of 1a) and 1b).

For 1a), analogously to the proof of Theorem 3.6, we prove that $a_1^*(0, 1) = 1$, then it follows by Theorem 3.5 that $\hat{T}_1^{lt}(0) = 1$. By induction, we prove that, for all $n \geq 0$:

$$V_n(0, 1) + P_{EP_1} \geq V_n(0, 0) + P_{LT_1}. \quad (22)$$

For $V_0 \equiv 0$ this trivially holds.

Assume that (22) holds for a given n (induction hypothesis), and we consider the operators H_1, H_2, G_1 and G_2 separately:

$$\begin{aligned}
H_1 V_n(0, 1) + P_{EP_1} &= \min\{P_{EP_1} + P_{LT_1} + V_n(0, 0), 2P_{EP_1} + V_n(0, 1)\} \\
&\geq P_{EP_1} + P_{LT_1} + V_n(0, 0) = H_1 V_n(0, 0) + P_{LT_1};
\end{aligned}$$

$$\begin{aligned}
H_2 V_n(0, 1) + P_{EP_1} &= \min\{P_{EP_1} + V_n(0, 0), P_{EP_1} + P_{EP_2} + V_n(0, 1)\} \\
&\geq \min\{P_{EP_1} - P_{EP_2} + P_{EP_2} + V_n(0, 0), P_{EP_2} + V_n(0, 0) + P_{LT_1}\} \\
&= H_2 V_n(0, 0) + \min\{P_{EP_1} - P_{EP_2}, P_{LT_1}\},
\end{aligned}$$

as $H_2 V_n(0, 0) = P_{EP_2} + V_n(0, 0)$;

$$\begin{aligned}
G_1 V_n(0, 1) + S_1 P_{EP_1} &= S_1[V_n(1, 1) - V_n(1, 0) + V_n(1, 0) + P_{EP_1}] \\
&\geq S_1[V_n(0, 1) - V_n(0, 0) + V_n(1, 0) + P_{EP_1}] \\
&\geq S_1[V_n(1, 0) + P_{LT_1}] = G_1 V_n(0, 0) + S_1 P_{LT_1},
\end{aligned}$$

where the first inequality holds as V_n is Supermod;

$$\begin{aligned}
G_2 V_n(0, 1) + (S_2 - 1)P_{EP_1} &= (S_2 - 1)[V_n(0, 2) - V_n(0, 1) + P_{EP_1}] + S_2 V_n(0, 1) \\
&\geq (S_2 - 1)[V_n(0, 1) - V_n(0, 0) + P_{EP_1}] + S_2 V_n(0, 1) \\
&= S_2 V_n(0, 1) + (S_2 - 1)P_{LT_1} = G_2 V_n(0, 0) + (S_2 - 1)P_{LT_1},
\end{aligned}$$

where the first inequality holds as V_n is Conv(2).

Combining these, using condition (14), gives, analogously to (21), the induction step, and hence (22) holds for all $n \geq 0$. \square

A.7 Proof of Lemma 4.1

Proof. We give the proofs for the operator \tilde{G}_1 . By interchanging the numbering of the locations, the results directly follow for the operator \tilde{G}_2 as well.

(i) It is straightforward to check that if f is Decr(1) (cf. (4)), then $\tilde{G}_1 f$ is Decr(1) as well, and if f is Decr(2) (cf. (5)), then $\tilde{G}_1 f$ is Decr(2) as well. Combining this proves that \tilde{G}_1 preserves Decr.

(ii) Assume that f is Conv(1) (cf. (6)), then we show that $\tilde{G}_1 f$ is Conv(1) as well. For $x_1 + 2 < S_1$ this is straightforward to check, for the case $x_1 + 2 = S_1$ we need Decr(1):

$$\begin{aligned} \tilde{G}_1(f(x_1, x_2) + f(x_1 + 2, x_2)) &= f(x_1 + 1, x_2) + f(x_1 + 2, x_2) \\ &\geq f(x_1 + 2, x_2) + f(x_1 + 2, x_2) = 2\tilde{G}_1 f(x_1 + 1, x_2). \end{aligned}$$

The preservation of Conv(2) (cf. (7)) is again straightforward to check, and hence \tilde{G}_1 preserves Conv.

(iii) It is straightforward to check that if f is Supermod (cf. (8)), then $\tilde{G}_1 f$ is Supermod as well, hence \tilde{G}_1 preserves Supermod.

(iv) It is straightforward to check that if f is SuperC(1,2) (cf. (9)), then $\tilde{G}_1 f$ is SuperC(1,2) as well, hence \tilde{G}_1 preserves SuperC(1,2). Assume that f is SuperC(2,1) (cf. (9)), then we show that $\tilde{G}_1 f$ is SuperC(2,1) as well. For $x_1 + 1 < S_1$ this is straightforward to check, for the case $x_1 + 1 = S_1$ we need Conv(2):

$$\begin{aligned} \tilde{G}_1(f(x_1, x_2 + 2) + f(x_1 + 1, x_2)) &= f(x_1 + 1, x_2 + 2) + f(x_1 + 1, x_2) \\ &\geq f(x_1 + 1, x_2 + 1) + f(x_1 + 1, x_2 + 1) = \tilde{G}_1(f(x_1, x_2 + 1) + f(x_1 + 1, x_2 + 1)). \end{aligned}$$

(v) By (11), this is a direct consequence of parts (iii) and (iv). \square