Algebraic polynomials and moments of stochastic integrals

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Abstract: We propose a new proof of the following variation of the Burkholder-Davis-Gundy inequality. Let $b(s)$, $s \in [0,t]$, be a progressively measurable process, $b \in \mathcal{L}_2[0,T]$, $t \leq T$. Then for every $n \geq 2$ there exists constants $C_1 > 0$, $C_2 > 0$ such that

$$C_1\,\mathbb{E}\Big(\int_0^t b^2(s)\,ds\Big)^n \quad\leq\quad \mathbb{E}\Big(\int_0^t b(s)\,dW(s)\Big)^{2n} \quad\leq\quad C_2\,\mathbb{E}\Big(\int_0^t b^2(s)\,ds\Big)^n .$$

Our proof is based on using qualitative properties of roots of algebraic polynomials from certain general classes.

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1. Introduction

Connections between special algebraic polynomials and stochastic integrals have a long history (see Wiener [1938], Itô [1951]), and received considerable attention in stochastic analysis (Ikeda and Watanabe [1989], Carlen and Krée [1991], Borodin and Salminen [2002]). Fruitful applications of special polynomials have been found in the theory of Markov processes (Kendall [1959], Karlin and Mc-

Gregor [1957]), financial mathematics (Schoutens [2000]), statistics (Diaconis and Zabell [1991]). The book Schoutens [2000] contains an extensive overview of this field of stochastic analysis and its applications.

In this paper, we study a different type of applications of polynomials to stochastic integration. We show that not only properties of special systems of orthogonal polynomials can be used in stochastic analysis, but in fact that elementary properties of many general classes of polynomials lead to fruitful applications in stochastics.

2. The main result

We propose an algebraic proof for the following classic variation of the Burkholder-Davis-Gundy inequality.

Theorem 1. *Let $b(s)$, $s \in [0,t]$, be a progressively measurable process, $b \in \mathcal{L}_2[0,T]$, $t \le T$. Then for every $n \ge 2$ there exists constants $C_1 > 0$, $C_2 > 0$ such that*

$$C_1\,\mathbb{E}\biggl(\int_0^t b^2(s)\,ds\biggr)^n \;\le\; \mathbb{E}\biggl(\int_0^t b(s)\,dW(s)\biggr)^{2n} \;\le\; C_2\,\mathbb{E}\biggl(\int_0^t b^2(s)\,ds\biggr)^n . \tag{1}$$

The constants C_1 and C_2 depend on the choice of n, but not on b.

The specific feature of our approach is that we make a heavy use of properties of algebraic polynomials.

Proof. In the proof below we can assume that b is bounded, since the general case follows by the usual truncation argument.

We denote for brevity

$$\int_0^t b^2(s)\,ds \;=\; \int b^2\,ds\,, \qquad \int_0^t b(s)\,dW(s) \;=\; \int b\,dW\,.$$

Let us write for $n \ge 1$

$$\begin{aligned}\rho_{2n}(t) &= H_{2n}\Big(\int b^2\,ds\,,\int b\,dW\Big)\\ &= \sum_{0\le k\le n}(-1)^k a_k\Big(\int b\,dW\Big)^{2n-2k}\Big(\int b^2\,ds\Big)^k\,, \qquad (2)\end{aligned}$$

where we denote

$$a_k = \frac{1}{2^k k!(2n-2k)!}\,.$$

Taking in (2) the expectation of both sides and noting that $\mathbb{E}\,\rho_{2n} = 0$ (see Borodin and Salminen [2002] or Ikeda and Watanabe [1989]), we get

$$\sum_{0\le k\le n}(-1)^k a_k\mathbb{E}\Big\{\Big(\int b\,dW\Big)^{2n-2k}\Big(\int b^2\,ds\Big)^k\Big\} = 0\,. \qquad (3)$$

Lemma 2. *Let $b(s)$, $s\in[0,t]$, be progressively measurable process, $b\in\mathcal{L}_2[0,T]$, $t\le T$. Then for all $k\ge 1$*

$$\begin{aligned}\mathbb{E}\Big\{\Big(\int b\,dW\Big)^{2n-2k}\Big(\int b^2\,ds\Big)^k\Big\} &\le \qquad (4)\\ &\le \mathbb{E}^{\frac{n-k}{n}}\Big(\int b\,dW\Big)^{2n}\mathbb{E}^{\frac{k}{n}}\Big(\int b^2\,ds\Big)^n.\end{aligned}$$

Proof. (Lemma 2) This lemma follows by applying the Jensen inequality for $\int b\,dW$ and $\int b^2\,ds$ with powers $p = n/(n-k)$ and $q = n/k$ respectively. □

***Part* I**. Consider first the case of even n, and let $n = 2m$ in (1). Since $a_k \ge 0$ for all k, and also

$$\mathbb{E}\Big\{\Big(\int b\,dW\Big)^{2n-2k}\Big(\int b^2\,ds\Big)^k\Big\} \ge 0$$

for all k, after throwing out from (3) all the summands with even k, except for $k=0$ and $k=n$, we get

$$a_0\,\mathbb{E}\Big(\int b\,dW\Big)^{2n} \quad - \sum_{0\le 2l+1\le n} a_{2l+1}\mathbb{E}\Big\{\Big(\int b\,dW\Big)^{2n-2k(l)}\Big(\int b^2\,ds\Big)^{k(l)}\Big\}$$
$$+\quad a_n\,\mathbb{E}\Big(\int b^2\,ds\Big)^{n} \quad\le\quad 0\,, \tag{5}$$

where for integer $l \ge 0$ we denoted $k(l) = 2l + 1$.

Applying Lemma 2 to (5), we get

$$a_0\,\mathbb{E}\Big(\int b\,dW\Big)^{2n} \quad - \sum_{0\le 2l+1\le n} a_{2l+1}\mathbb{E}^{\frac{n-k(l)}{n}}\Big(\int b\,dW\Big)^{2n}\mathbb{E}^{\frac{k(l)}{n}}\Big(\int b^2\,ds\Big)^{n}$$
$$+\quad a_n\,\mathbb{E}\Big(\int b^2\,ds\Big)^{n} \quad\le\quad 0\,. \tag{6}$$

Divide both parts of (6) by $\mathbb{E}\big(\int b^2\,ds\big)^n$ and put

$$z \;:=\; \frac{\mathbb{E}^{1/n}\big(\int b\,dW\big)^{2n}}{\mathbb{E}^{1/n}\big(\int b^2\,ds\big)^{n}}\;, \tag{7}$$

then we obtain

$$a_0\,z^n \;-\sum_{0\le 2l+1\le n} a_{2l+1}z^{n-k(l)} \;+\; a_n \;\le\; 0\,,$$

or equivalently

$$a_0\,z^n \;+\; a_n \;\le \sum_{0\le 2l+1\le n} a_{2l+1}z^{n-k(l)}\,. \tag{8}$$

Lemma 3. *Consider real polynomials*

$$P_1(z) \;=\; \sum_{k=0}^{2m} b_k z^{2k} \quad \text{and} \quad P_2(z) \;=\; z\sum_{i=0}^{2m_1} c_i z^{2i}\,, \tag{9}$$

where $m_1 < m$ is integer and nonnegative, $b_k \ge 0$ for all k, $b_0 > 0$, $b_m > 0$, $c_i \ge 0$ for all i. Then there exists $d_1 > 0$, $d_2 > 0$ such that only for $z \in [d_1, d_2]$ one can have $P_1(z) \le P_2(z)$, but for $z \notin [d_1, d_2]$ one has $P_1(z) > P_2(z)$.

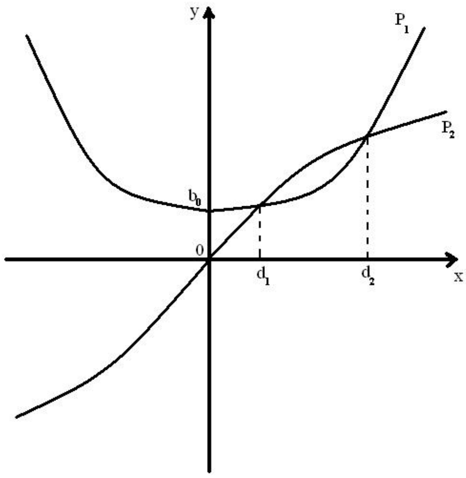

FIG 1. *Illustration for Lemma 3.*

Proof. (Lemma 3) Note first that $P_1(z)$ is symmetric, $P_1(z) \geq b_0$ for all $z \in \mathbb{R}$, and $P_1(z) \sim b_m z^{2m}$ as $z \to \infty$. Furthermore, $P_2(-z) = -P_2(z)$, and for $z \geq 0$ one has $P_2(z) \geq 0$, $P_2(0) = 0$, and $deg\, P_1(z) < deg\, P_2(z)$.

This implies that for $z < 0$ one has $P_1(z) > 0 > P_2(z)$. At $z = 0$ it holds that $P_1(0) = b_0 > 0 = P_2(z)$. This shows that all possible solutions of the inequality $P_1(z) \leq P_2(z)$ are positive, i.e. bounded from below by a positive number d_1.

Since $P_1(z)/P_2(z) \to \infty$ as $z \to \infty$, it follows that for sufficiently large $z \geq z_0$ always $P_1(z) > P_2(z)$. Therefore, all possible solutions of the inequality $P_1(z) \leq P_2(z)$ lies in some interval $[d_1, d_2]$ with $d_1 > 0$ and $d_2 > 0$.

□

Let us now put in (8) $P_1(z) = a_0\, z^n \,+\, a_n$, $P_2(z) = \sum_{0 \leq 2l+1 \leq n} a_{2l+1} z^{n-k(l)}$. By Lemma 3, there exists positive constants C_1, C_2 such that $0 < C_1 \leq z \leq C_2$, i.e. $C_1^n \leq z^n \leq C_2^n$, and this proves (1) for the case of $n = 2m$.

***Part* II**. Consider now the case of odd n, and let $n = 2m + 1$ in (1). Throwing away from (3) all the summands with even k, except for $k = 0$, we get

$$a_0\, \mathbb{E}\bigg(\int b\, dW \bigg)^{2n} - \sum_{0 \leq 2l+1 \leq n} a_{2l+1} \mathbb{E}\bigg\{ \bigg(\int b\, dW \bigg)^{2n-2k(l)} \bigg(\int b^2\, ds \bigg)^{k(l)} \bigg\} \;\leq\; 0\,,$$

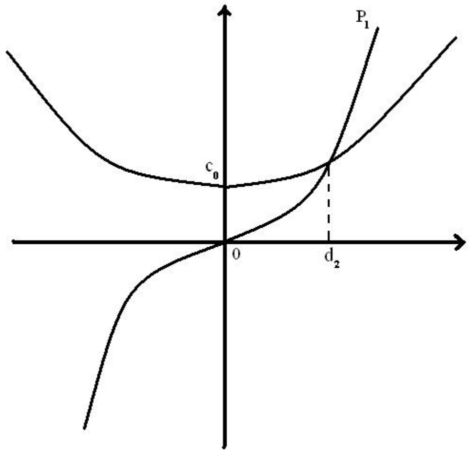

FIG 2. *Illustration for Lemma 4.*

$$ \tag{10} $$

and analogously to (8) we derive

$$a_0\, z^n \;\le \sum_{0\le 2l+1\le n} a_{2l+1} z^{n-k(l)}\,, \tag{11}$$

where z is defined by (7).

Lemma 4. *Consider real polynomials*

$$P_1(z) \;=\; b_0 z^{2m+1} \quad \textit{and} \quad P_2(z) \;=\; z\sum_{i=0}^{m_1} c_i z^{2i}\,, \tag{12}$$

where $m_1 < m$ is integer and nonnegative, $b_0 > 0$, $c_i \ge 0$ for all i, $c_0 > 0$, $c_{m_1} > 0$. Then there exists $d_2 > 0$ such that only for $z \in [-\infty, d_2]$ one can have $P_1(z) \le P_2(z)$, but for $z > d_2$ one always has $P_1(z) > P_2(z)$.

Proof. (Lemma 4) The proof is analogous to the one of Lemma 3. □

After applying Lemma 4 to $P_1(z) = a_0\, z^n$ and $P_2(z) = \sum_{0\le 2l+1\le n} a_{2l+1} z^{n-k(l)}$ in (11), we obtain from (11) that $z \le d_2$ for some positive d_2. Since n is odd, this implies $z^n \le d_2^n$ and the upper bound in (1) follows.

It remains only to prove the lower bound in (1) for $n = 2m + 1$. In this case, we leave in (3) only the summands with even k and $k = n$, thus getting

$$\sum_{0 \le 2k < n} a_{2k}\, \mathbb{E}\Big\{ \Big(\int b\, dW \Big)^{2n-4k} \Big(\int b^2\, ds \Big)^{2k} \Big\} - a_n\, \mathbb{E}\Big\{ \Big(\int b^2\, ds \Big)^{n} \Big\} \ \ge\ 0\,. \tag{13}$$

Analogously to our previous derivations, this implies the inequality

$$\sum_{0 \le 2k < n} a_{2k}\, z^{n-2k} \ -\ a_n \ \ge\ 0\,, \quad \text{i.e.}$$

$$P(z) \ :=\ \sum_{0 \le 2k < n} a_{2k}\, z^{n-2k} \ \ge\ a_n\,, \tag{14}$$

where z is again as in (7). Since $P(z)$ is a polynomial of the form $\sum_{i=1}^{m} b_i z^{2i+1}$, it easily follows that (14) is equivalent to $z \ge C_1$ for some constant $C_1 = C_1(n) > 0$. Therefore, $z^n \ge C_1^n$, and the lower bound in (1) is proved for $n = 2m+1$. $\square$

The idea of proving the Burkholder-Davis-Gundy inequality via the use of Hermitian polynomials have been already used by different authors (see, for example, Ikeda and Watanabe [1989] or Lecture Notes "Stochastic Calculus" by Andrei Borodin). However, the previous proofs used properties of polynomials in a different way and worked only for $n \le 4$. Our proof is valid for general n.

In the above proof we have used only some elementary and entirely qualitative facts about certain general types of polynomials, together with such a crude technique as simple throwing out every second term in the starting martingale identity. Nontheless, we were able to prove a rather general Burkholder-Davis-Gundy theorem. This shows that our approach can lead to substantially stronger results in estimation of stochastic integrals.

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