FIRST PASSAGE OF TIME-REVERSIBLE SPECTRALLY-NEGATIVE MARKOV ADDITIVE PROCESSES

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Abstract. We study the first passage process of a spectrally-negative Markov additive process (MAP). The focus is on the background Markov chain at the times of the first passage. This process is a Markov chain itself with a transition rate matrix Λ. Assuming time-reversibility we show that all the eigenvalues of Λ are real with algebraic and geometric multiplicities being the same, which allows us to identify the Jordan normal form of Λ. Furthermore, this fact simplifies the analysis of fluctuations of a MAP. We provide an illustrative example and show that our findings greatly reduce the computational efforts required to obtain Λ in the time-reversible case.

Key words: Markov additive processes; time-reversibility; first passage process; Jordan normal form; queueing

1. Introduction

Continuous-time Markov additive processes (MAPs) have proven an important modelling tool in communications networking [19, Ch. 6-7] as well as finance [3, 13]. This has led to a vast body of literature; for an overview see for instance [2, Ch. XI]. A MAP is essentially a Lévy process whose Laplace exponent depends on the state of a (finite-state) Markovian background process. It is a non-trivial generalization of the standard Lévy process, with many analogous properties and characteristics, as well as new mathematical objects associated to it, posing new challenges. It should be remarked that the original definition of MAP [6] does not require the background process to live on a finite (or even countable) state space. In the case of infinite state space the structure of MAP can be very complicated, see [2, Ch. XI]. It is hence usual to assume that the state space of the background process is finite [2, Ch. XI], in which case the MAP is sometimes called a Markov-modulated Lévy process.

Just as for standard Lévy processes, the class of MAPs with one-sided jumps is of high importance. On the one hand this class is rich, as it covers for instance Markov-modulated one-sided compound Poisson processes with drift, Markov-modulated Brownian motions, as well as 'Markov fluids' [2, Section XI.1b], but on the other hand it allows for fairly explicit results. In this paper we consider *spectrally-negative* MAPs, that is, processes which are only allowed to have negative jumps. We denote the MAP by $(X(\cdot), J(\cdot))$, with $X(\cdot)$ being the value of the MAP and $J(\cdot)$ the state of the Markovian background process. The focus of the present paper is on the Markov chain $J(\tau_x)$ associated with the first passage process $\tau_x \coloneqq \inf\{t \geq 0 : X(t) > x\}$. Here it is noted that the (possibly defective) transition rate matrix Λ of the Markov chain $J(\tau_x)$ plays a crucial role in the fluctuation theory for one-sided MAPs, see e.g. [8, 18].

The problem of identifying Λ received a lot of attention, see [20, 21, 1, 8] for modulated compound Poisson, linear drift, Brownian motion and a general spectrally one-sided Lévy process respectively. There are two (related) approaches to characterize Λ available in the literature. Firstly, it is known that Λ solves a specific matrix integral equation, see [8]. Secondly, a spectral method can be used to identify Λ. The advantage of the second approach is that one gets an explicit expression. This approach, however, requires to assume that the eigenvalues of Λ are distinct, unless one considers a Markov-modulated linear drift model in which case both

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approaches result in the same identity. Recently, it was shown in [7] how to circumvent the assumption of distinct eigenvalues, but then one has to resort to considerably heavier mathematical machinery, such as analytic matrix function theory.

In this paper we show that if a MAP is time-reversible, then the eigenvalues of Λ are *real with algebraic and geometric multiplicities being the same.* In other words, Λ is similar to a real diagonal matrix D, that is, $\Lambda = PDP^{-1}$ where P is an invertible matrix. We provide a short and simple proof of this result, and construct the matrices D and P *under no additional assumptions*, see Section 2. It should be noted that this property greatly simplifies the analysis of Markov-modulated storage systems: there is no need to assume that the eigenvalues are distinct or to resort to analytic matrix theory. We will demonstrate this point in Section 3, where we consider a simple example of a queue fed by a spectrally-negative MAP. Moreover, our findings considerably reduce, in this time-reversible case, the computational efforts required to obtain numerical output, one of the reasons being that computations involve only real numbers. Further computational aspects, particularly for systems in which the driving MAP is a superposition of multiple time-reversible MAPs, are discussed in Section 4.

The rest of this introduction is devoted to developing a set of prerequisites.

1.1. **Spectrally-negative MAP.** Before formally defining the class of MAPs, we first introduce some notation. Throughout this paper we use bold symbols to denote (column) vectors. For example, $\mathbf{1}$ and $\mathbf{0}$ denote vectors of 1-s and 0-s respectively, whereas $\boldsymbol{e}_i$ stands for a vector of 0-s but with the i-th element being 1. Moreover, $\boldsymbol{a} < \boldsymbol{b}$ means that $a_i < b_i$ for all indices i.

A MAP is a bivariate Markov process $(X(\cdot), J(\cdot)) \equiv (X(t), J(t))_t$ defined as follows. Let $J(\cdot)$ be an irreducible continuous-time Markov chain with finite state space $E := \{1, \ldots, N\}$ and $N \times N$ transition rate matrix $Q = (q_{ij})$. For each state i of $J(\cdot)$ let $X_i(\cdot)$ be a Lévy process with Laplace exponent $\phi_i(\alpha) := \log(\mathbb{E}e^{\alpha X_i(1)})$. Letting T_n and T_{n+1} be two successive transition epochs of $J(\cdot)$, and given that $J(\cdot)$ jumps from state i to state j at T_n, we define the additive process $X(\cdot)$ in the time interval $[T_n, T_{n+1})$ through

$$X(t) := X(T_n-) + U_{ij}^n + [X_j(t) - X_j(T_n)],$$

where (U_{ij}^n) is a sequence of independent and identically distributed random variables with Laplace-Stieltjes transform

$$\tilde{G}_{ij}(\alpha) := \mathbb{E}e^{\alpha U_{ij}^1}, \quad \text{where } U_{ii}^1 \equiv 0,$$

describing the jumps at transition epochs. To make the MAP spectrally-negative, it is required that $U_{ij}^1 \leq 0$ (for all $i, j \in \{1, \ldots, N\}$) and that $X_i(\cdot)$ is allowed to have only negative jumps (for all $i \in \{1, \ldots, N\}$).

We partition the index set E into two disjoint sets E^+ and $E^\downarrow$ letting $N^+ = |E^+|$ and $N^\downarrow = |E^\downarrow|$. It is assumed that the processes $X_i(\cdot)$ with index in $E^\downarrow$ are those and only those Lévy processes which are *downward subordinators*, i.e., stochastic processes with non-increasing paths a.s. As will turn out, these downward subordinators play a special role in our analysis. We use $\boldsymbol{v}^+$ and $\boldsymbol{v}^\downarrow$ to denote the restrictions of a vector $\boldsymbol{v}$ to the indices from E^+ and $E^\downarrow$ respectively. Finally, in order to exclude trivialities it is assumed that E^+ is non-empty.

A central object, which can be considered as the multi-dimensional analog of a Laplace exponent, defining the law of a MAP, is the associated *matrix exponent* $F(\alpha)$, given by

$$F(\alpha) := Q \circ \tilde{G}(\alpha) + \mathrm{diag}(\phi_1(\alpha), \ldots, \phi_N(\alpha)),$$

where $\tilde{G}(\alpha) := (\tilde{G}_{ij}(\alpha))$ and for matrices A and B of the same dimensions $A \circ B := (a_{ij}b_{ij})$. The matrix exponent is finite for all $\alpha \geq 0$, and in addition

$$\mathbb{E}_i[e^{\alpha X(t)} 1_{\{J(t)=j\}}] = (e^{F(\alpha)t})_{ij},$$

cf. [2, Prop. XI.2.2], where $\mathbb{E}_i[\cdot]$ denotes expectation given that $J(0) = i$. Let $k(\alpha)$ be the eigenvalue of $F(\alpha)$ with maximal real part, which is, by Perron-Frobenius theory [2, Thm. I.6.5], well-defined, simple and real.

An important quantity associated to a MAP is the *asymptotic drift* $\lim_{t\to\infty} X(t)/t$, which does not depend on the initial state i of $J(t)$ and is given by

$$\lim_{t\to\infty} \frac{X(t)}{t} = k'(0) = \sum_i \pi_i \left(\phi_i'(0) + \sum_{j\neq i} q_{ij} \tilde{G}_{ij}'(0) \right),$$

where $\boldsymbol{\pi}$ is a unique stationary distribution of $J(\cdot)$ [2, Cor. XI.2.7].

In this paper we assume that the MAP $(X(\cdot), J(\cdot))$ under consideration is *time-reversible*, which is equivalent to saying that the Markov chain $J(\cdot)$ is time-reversible (that is, $\pi_i q_{ij} = \pi_j q_{ji}$) and U_{ij}^1 has the same law as U_{ji}^1 for all $i, j \in \{1, \dots, N\}$, see [2, Section XI.2e]. Yet another equivalent definition of time-reversibility, which we will use in the present paper, is, with $\Delta_{\boldsymbol{x}}$ being a diagonal matrix with the vector $\boldsymbol{x}$ on its diagonal,

$$\Delta_{\boldsymbol{\pi}} F(\alpha) = (\Delta_{\boldsymbol{\pi}} F(\alpha))^{\mathrm{T}}. \tag{1}$$

1.2. **First Passage Process.** Define the first passage time over level $x > 0$ for the process $X(\cdot)$ as

$$\tau_x := \inf\{t \geq 0 : X(t) > x\}.$$

Note that due to the absence of positive jumps the time-changed process $J(\tau_x)$ is again a Markov chain but taking values in the set E^+ only (see also [18]). Denote the corresponding $N^+ \times N^+$ dimensional transition rate matrix by Λ, so that

$$\mathbb{P}(J(\tau_x) = j, \tau_x < \infty | J(\tau_0) = i) = (e^{\Lambda x})_{ij}, \text{ where } i, j \in E^+. \tag{2}$$

Another matrix of interest is the $N \times N^+$ matrix Π defined as follows

$$\Pi_{ij} := \mathbb{P}(J(\tau_0) = j, \tau_0 < \infty | J(0) = i), \text{ where } i \in E, j \in E^+. \tag{3}$$

In the following we have to distinguish between two cases:

- if $k'(0) \geq 0$, then Λ is a non-defective transition rate matrix: $\Lambda \mathbf{1}^+ = \mathbf{0}^+$;
- if $k'(0) < 0$, then Λ is a defective transition rate matrix: $\Lambda \mathbf{1}^+ \leq \mathbf{0}^+$, with at least one strict inequality.

This follows from [2, Prop. XI.2.10], which states that in the case of a non-negative asymptotic drift $\limsup_{t\to\infty} X(t) = +\infty$ $\mathbb{P}_i$-a.s. for all i, and thus $\mathbb{P}_i(\tau_x < \infty) = 1$. In the case of a negative asymptotic drift $\lim_{t\to\infty} X(t) = -\infty$ $\mathbb{P}_i$-a.s. for all i, and thus $\mathbb{P}_i(\tau_x < \infty) < 1$. This also means that $\Pi \mathbf{1}^+ = \mathbf{1}$ if $k'(0) \geq 0$, and $\Pi \mathbf{1}^+ < \mathbf{1}$ if $k'(0) < 0$.

2. Main Result

In this section we prove that under the time-reversibility assumption the transition rate matrix Λ is *similar* to some real diagonal matrix D, in the sense that $\Lambda = PDP^{-1}$ for some invertible matrix P. Moreover, we provide a procedure to construct the matrices D and P.

Let $\alpha_1, \dots, \alpha_k$ and $m_1, \dots, m_k$ be the zeros of $\det(F(\alpha))$ in $(0, \infty)$ and their multiplicities. For all $i = 1, \dots, k$ let p_i denote the dimension of the null space of $F(\alpha_i)$ (geometric multiplicity of the null-eigenvalue) and $\boldsymbol{v}_i^1, \dots, \boldsymbol{v}_i^{p_i}$ be some basis of this null space. It is well known, see e.g. [11, Lemma 2.4], that $m_i \geq p_i$. In the special case when $m_i = p_i$ the zero α_i is called *semi-simple.*

Let Υ_i be a $p_i \times p_i$ diagonal matrix with α_i on the diagonal and $V_i := [\boldsymbol{v}_i^1, \dots, \boldsymbol{v}_i^{p_i}]$. Define

$$\Upsilon := \mathrm{diag}(\Upsilon_1, \dots, \Upsilon_k) \text{ and } V := [V_1, \dots, V_k], \qquad \text{if } k'(0) < 0$$

$$\Upsilon := \mathrm{diag}(0, \Upsilon_1, \dots, \Upsilon_k) \text{ and } V := [\mathbf{1}, V_1, \dots, V_k], \qquad \text{if } k'(0) \geq 0 \tag{4}$$

and let V^+ denote the matrix V restricted to the rows corresponding to E^+. It is the difficult part of the proof of our main result, Thm. 1, to show that V is composed of N^+ columns, which implies that Υ and V^+ are square $N^+ \times N^+$-dimensional matrices.

Theorem 1. *Let $(X(\cdot), J(\cdot))$ be a time-reversible spectrally-negative MAP. Then Υ and V^+ are $N^+ \times N^+$-dimensional matrices, V^+ is invertible, and*

$$\Lambda = -V^+\Upsilon(V^+)^{-1} \text{ and } \Pi = V(V^+)^{-1}.$$

We start the proof of Thm. 1 by establishing a lemma, which can be considered as a weak analog of this theorem.

Lemma 2. *If $\alpha > 0$ and $\boldsymbol{v}$ are such that $F(\alpha)\boldsymbol{v} = \boldsymbol{0}$ then*

$$\Lambda\boldsymbol{v}^+ = -\alpha\boldsymbol{v}^+ \text{ and } \boldsymbol{v} = \Pi\boldsymbol{v}^+. \tag{5}$$

Proof. By choosing $\lambda(\alpha) = 0$ and $\boldsymbol{h}(\alpha) = \boldsymbol{v}$ in [4, Lemma 2.1], we obtain that for any distribution of $J(0)$

$$M(t) := e^{\alpha X(t)} v_{J(t)}$$

is a martingale. Apply the optional sampling theorem to see that, for any $t > 0$ and any $x \geq 0$,

$$v_i = \mathbb{E}_i[e^{\alpha X(\tau_x \wedge t)} v_{J(\tau_x \wedge t)}].$$

Note that $M(t)$ is bounded in absolute value on $[0, \tau_x]$, due to the facts that $\alpha > 0$ and $X(t) \leq x$ on $[0, \tau_x]$. It moreover always holds that either $\mathbb{P}_i(\tau_x = \infty) = 0$ or $\lim_{t\to\infty} X(t) = -\infty$ a.s. (where the former case corresponds to $k'(0) \geq 0$, and the latter case to $k'(0) < 0$), so by applying 'dominated convergence' we have

$$v_i = \mathbb{E}_i[1_{\{\tau_x < \infty\}} e^{\alpha x} v_{J(\tau_x)}] = e^{\alpha x} \sum_{j \in E^+} \mathbb{P}_i(J(\tau_x) = j, \tau_x < \infty) v_j.$$

Choosing $x = 0$ we obtain

$$\boldsymbol{v} = \Pi\boldsymbol{v}^+,$$

see also (3). On the other hand, considering $i \in E^+$ we get

$$\boldsymbol{v}^+ = e^{\alpha x} e^{\Lambda x} \boldsymbol{v}^+,$$

see also (2). The first equality appearing in (5) is obtained by differentiating the above equality with respect to x and setting $x = 0$. □

Recall that if $k'(0) \geq 0$ then $\mathbb{P}_i(\tau_x < \infty) = 1$, so $\Lambda\boldsymbol{1}^+ = \boldsymbol{0}^+$ and $\Pi\boldsymbol{1}^+ = \boldsymbol{1}$. Using Lemma 2 one can see now that $\Lambda V^+ = -V^+\Upsilon$ and $V = \Pi V^+$. Note that the columns of the matrix V^+ are vectors from the bases of the eigenspaces of the matrix Λ, so they are linearly independent. But then the matrix V^+ (and so also V) cannot have more than N^+ columns. Thus, in order to prove Thm. 1, it remains to show that

$$V \text{ is composed of at least } N^+ \text{ column vectors.} \tag{6}$$

We devote the rest of this section to proving this claim.

Lemma 3. *The eigenvalues of $F(\alpha), \alpha \geq 0$ are real with algebraic and geometric multiplicities being the same.*

Proof. Recall that time-reversibility of $(X(\cdot), J(\cdot))$ implies that the matrix $\Delta_{\boldsymbol{\pi}} F(\alpha)$, $\alpha \geq 0$ is real symmetric, see (1), and hence the same applies to $\Delta_{\boldsymbol{\pi}}^{1/2} F(\alpha) \Delta_{\boldsymbol{\pi}}^{-1/2}$, $\alpha \geq 0$. It is well known that a real symmetric matrix has real eigenvalues with algebraic and geometric multiplicities coinciding, see [10, Thms. 2.5.4 and 4.1.3]. The claim is now immediate in view of [10, Thm. 1.4.8]. □

Remark 2.1. There are three types of multiplicities mentioned in this section: the algebraic and geometric multiplicities of eigenvalues of $F(\alpha)$ for some fixed α, and the multiplicities of zeros of $\det(F(\alpha))$. Hence to every zero $\alpha_i > 0$ of $\det(F(\alpha))$, having a multiplicity that we denoted by m_i, we can associate a null-eigenvalue of the matrix $F(\alpha_i)$, which has the same algebraic and geometric multiplicities according to Lemma 3; we recall that this multiplicity is denoted by p_i.

Let $g_i(\alpha)$ be the i-th largest eigenvalue of $F(\alpha), \alpha \geq 0$ (so that $g_1(\alpha) = k(\alpha)$, the Perron-Frobenius eigenvalue defined earlier). Then $g_i : [0, \infty) \mapsto \mathbb{R}$ is a continuous function. The next lemma presents some properties of the functions $g_i(\cdot)$.

Lemma 4. *It holds that*

- $g_1(0) = 0$ *and* $g_i(0) < 0$ *for* $i = 2, \ldots, N$,
- $g_i(\alpha) \to \infty$ *as* $\alpha \to \infty$ *for* $i = 1, \ldots, N^+$.

Proof. The first statement follows immediately by noting that $F(0) = Q$ is an irreducible transition rate matrix; see also [2, Cor. I.4.9]. Consider the second statement. It is well known that $\lim_{\alpha\to\infty} \phi_i(\alpha) = \infty$ if $X_i(\cdot)$ is not a downward subordinator. Let $f(\alpha)$ be the N^+-th largest number out of $\phi_i(\alpha)$ for $i \in \{1, \ldots, N\}$, so $\lim_{\alpha\to\infty} f(\alpha) = \infty$ and $F(\alpha)/f(\alpha)$ goes to a diagonal matrix with at least N^+ positive (possibly infinite) elements. Take an arbitrary sequence $\alpha_n \to \infty$ and apply Lemma 7 of the Appendix to $F(\alpha_n)/f(\alpha_n)$ to obtain the result. □

Proof of Theorem 1. Recall that we are left to prove that the matrix V is composed of at least N^+ columns, see (6). Lemma 4 shows that the functions $g_2(\alpha), \ldots, g_{N^+}(\alpha)$, and in addition $g_1(\alpha)$ provided that $k'(0) < 0$, hit 0 in the interval $(0, \infty)$ at least once (recall that $k(\alpha) = g_1(\alpha)$). If these functions hit 0 for distinct α, then the claim is immediate, see the definition of matrix V given by (4). Suppose now that n of these functions hit 0 for some $\alpha = \alpha^*$. Then the algebraic multiplicity of the null-eigenvector of $F(\alpha^*)$ is n. But the algebraic and geometric multiplicities of all the eigenvalues of $F(\alpha), \alpha > 0$ are the same according to Lemma 3, so the null space of $F(\alpha^*)$ is of dimension n, and the claim follows. □

We conclude with the following immediate corollary.

Corollary 5. $\det(F(\alpha))$ *has* $N^+ - 1_{\{k'(0)\geq 0\}}$ *zeros in* $\{\alpha \in \mathbb{C} : \mathrm{Re}(\alpha) > 0\}$ *(counting multiplicities), all of which are real and semi-simple.*

Proof. It is only required to show that $\det(F(\alpha))$ has no zeros in $\{\alpha \in \mathbb{C} : \mathrm{Re}(\alpha) > 0\}$, which are not real. Suppose there is such a zero, and denote it by α_0. It is easy to check that the arguments in the proof of Lemma 2 still hold for α_0. Hence $-\alpha_0$ is another eigenvalue of Λ, which is impossible. □

The above result partly holds without the assumption of time-reversibility. It has been shown in [12] (for the case when $k'(0)$ is non-zero and finite) that the total number of zeros (taken according to their multiplicities) of $\det(F(\alpha))$ in $\{\alpha \in \mathbb{C} : \mathrm{Re}(\alpha) > 0\}$ is $N^+ - 1_{\{k'(0)\geq 0\}}$. It is, however, not true in general that the zeros are real or semi-simple.

3. A Simple Example of a Markov-modulated Queue

In this section we consider a queue fed by a spectrally-negative MAP. Our goal is to illustrate on a simple example that the time-reversibility assumption can considerably simplify the analysis of fluctuations of a MAP.

We recall that for a given MAP $(X(\cdot), J(\cdot))$ the *workload process* $(W(\cdot), J(\cdot))$ is defined through

$$W(t) := X(t) + \max\left\{-\inf_{s\in[0,t]} X(s), 0\right\}.$$

It is well known that this process has a unique stationary distribution if the asymptotic drift of $(X(\cdot), J(\cdot))$ is negative (i.e., the stability condition $k'(0) < 0$ holds), which we assume in the sequel. Let a pair of random variables (W, J) have the stationary distribution of $(W(\cdot), J(\cdot))$, and denote the all-time maximum attained by $X(\cdot)$ through $\overline{X} := \sup_{t\geq 0} X(t)$. It is an immediate consequence of [2, Prop. XI.2.11] and time-reversibility that

$$(W|J = i) \text{ and } \big(\overline{X}|J(0) = i\big) \text{ have the same distribution.}$$

But $\overline{X}$ is the life-time of $J(\tau_x)$, thus it is of phase type [2, Section III.4], where Λ is the *phase generator*, the ith row of Π is the *initial distribution*, and $-\Lambda\mathbf{1}^+$ is the *exit vector*. Hence

$$\boldsymbol{p}(x) = -\Pi e^{\Lambda x}\Lambda\mathbf{1}^+, \tag{7}$$

where $p_i(x)$ is a density of $(W|J = i)$ at $x > 0$. In addition, $(W|J = i)$ can have mass at 0. It is easy to see from normalization that the vector of masses is given by $\mathbf{1} - \Pi\mathbf{1}^+$, so that there is no mass if $i \in E^+$.

It should be noted that a similar result holds true for a general (not time-reversible MAP). The advantage of time-reversibility is that we can compute the matrices Λ and Π without any additional assumptions regarding simplicity of eigenvalues as in [20, 1, 14], neither we use objects from analytic matrix theory as was done in [7]. Moreover, we can simplify (7) and show that the density $\boldsymbol{p}(x)$ is of a particularly simple form. Using Theorem 1 we write

$$\boldsymbol{p}(x) = Ve^{-\Upsilon x}\Upsilon(V^+)^{-1}\mathbf{1}^+ = \sum_{i=1}^{N^+} e^{-\alpha_i x}a_i\boldsymbol{v}_i,$$

where $\alpha_i = \Upsilon_{ii} > 0$ and $\boldsymbol{v}_i$ is the ith column of V, hence $F(\alpha_i)\boldsymbol{v}_i = \mathbf{0}$. The elements a_i are given by $\boldsymbol{a} = \Upsilon(V^+)^{-1}\mathbf{1}^+$.

We now compare our findings with those in [14], where the stationary distribution of Markov-modulated Brownian motion (MMBM) is given. Our result is in line with the expressions in [14]; one can use identity (1) to see this. It was shown in [14] that a_i-s satisfy a system of N^+ linear equations, but it was *not* shown that these equations are linearly independent, and hence it was not clear whether they uniquely determine the unknowns a_i. Note that our result provides an explicit expression for these unknowns. It is valid for any time-reversible spectrally-negative MAP. In fact, the result in [14] is valid for any MMBM which has the property that all α_i-s are distinct.

We finally note that a time-reversible MMBM also appears in [15], where authors succeed in identifying the stationary distribution of this process reflected at two barriers. This is however done under a rather restrictive assumption, namely all the pairs (μ_i, σ_i^2) are assumed to be proportional to (μ, σ^2), which e.g. implies that all drifts μ_i have the same sign.

4. Computational Aspects

In this final section we consider the problem of finding the zeros of $\det(F(\alpha))$, which are then used to construct matrices Υ and V and hence the transition rate matrix Λ of $J(\tau_x)$. The assumption of time-reversibility greatly reduces the computational efforts required to find the zeros. Firstly, we can restrict ourselves to the domain of reals, whereas in general the right half of the complex plane is to be considered. Secondly, it turns out (as proved in Section 4.1) that the functions $g_i(\alpha)/\alpha, \alpha > 0$ are strictly increasing. Hence a simple root finding procedure can be employed to find the zeros of functions $g_i(\alpha)$, which are exactly the zeros of $\det(F(\alpha))$. Finally, this idea can be extended further for systems in which the driving MAP is a superposition of multiple time-reversible MAPs. It is shown in Section 4.2 that one can reduce the computational burden for such systems by several orders of magnitude.

It is important to note that the zeros of $\det(F(\alpha)), \alpha > 0$ are bounded by $\max_i\{\psi_i(2q_i) : i \in E^+\}$, where $q_i := -Q_{ii}$ and $\psi_i(\cdot)$ is the right-inverse of $\phi_i(\cdot)$, see [17] for a definition. The above claim is true, because for larger α the matrix $F(\alpha)$ is diagonally dominant and thus non-singular (use basic properties of the functions $\phi_i(\cdot)$, or see [12]).

4.1. **Monotonicity of Eigenvalue Functions.** It is immediate from Lemma 4 and Corollary 5 that the functions $g_2(\alpha), \ldots, g_{N^+}(\alpha)$, and in addition $g_1(\alpha)$ provided that $k'(0) < 0$, hit 0 exactly once on the interval $(0, \infty)$. Moreover, these are the only functions $g_i(\alpha)$ hitting 0 for some $\alpha \in (0, \infty)$.

Let $h_i(\alpha) := g_i(\alpha)/\alpha$ for $\alpha > 0$ and define d_j to be the deterministic drift of the Lévy process $X_j(\cdot)$ if this process has paths of bounded variation, and ∞ otherwise.

Lemma 6. *The functions $h_i(\alpha)$ are strictly increasing with*

(i) $\lim_{\alpha\downarrow 0} h_1(\alpha) = k'(0)$ *and* $\lim_{\alpha\downarrow 0} h_i(\alpha) = -\infty$ *for* $i > 1$,

(ii) $\lim_{\alpha\to\infty} h_i(\alpha) = c_i$, *where c_i is the i-th largest number among the d_i-s.*

Proof. Fix a $c \in \mathbb{R}$, and define the time-reversible matrix exponent $\tilde{F}(\alpha) := F(\alpha) - c\alpha$. Trivially $\tilde{g}_i(\alpha) = g_i(\alpha) - c\alpha$ and $\tilde{h}_i(\alpha) := h_i(\alpha) - c$. But functions $\tilde{g}_i(\alpha)$, and hence also $\tilde{h}_i(\alpha)$, hit 0 in the interval $(0, \infty)$ at most once. This shows that $h_i(\alpha)$ are strictly increasing, because c is arbitrary.

Claim (i) now follows immediately from Lemma 4. Finally, note that $\tilde{N}^+$ (in self-evident notation) is decreasing in c. More precisely, $\tilde{N}^+$ decreases when $c = d_j$ for some j, because then $X_j(t) - ct$ becomes a downward subordinator. This means that one of the functions $h_i(\alpha)$ approaches d_j but does not hit it, which proves the second claim. □

We note that this monotonicity result makes it possible to use a simple root finding procedure to obtain the zeros of $\det(F(\alpha))$.

4.2. **Aggregates of Multiple MAPs.** We now consider the situation in which the MAP $X(\cdot)$ consists of the superposition of multiple independent MAPs $X^{(1)}(\cdot), \ldots, X^{(M)}(\cdot)$ [16]. Then $F(\alpha)$ can be written as $F^{(1)}(\alpha) \oplus \ldots \oplus F^{(M)}(\alpha)$, with $\oplus$ denoting the Kronecker sum [5], and $F^{(1)}(\alpha), \ldots, F^{(M)}(\alpha)$ being matrix exponents. If $X^{(1)}(\cdot), \ldots, X^{(M)}(\cdot)$ are spectrally-negative and time-reversible, clearly $X(\cdot)$ inherits these properties. Following the procedure outlined above, one can identify V by equating the eigenvalue functions $h_i(\cdot)$ of $\bar{F}(\alpha)$ to 0. If N_m is the dimension of matrix exponent $F^{(m)}(\alpha)$, this would require solving eigensystems of dimension $\prod_{m=1}^{M} N_m$ (as this is the dimension of $F(\alpha)$).

One can, however, find V in a considerably more efficient manner, following the approach presented in [22, Section 4]. There it is explained how to convert the $\sum_{m=1}^{M} N_m$ eigenvalue functions

$$h_i^{(m)}(\alpha),\ \ i = 1, \ldots, N_m,\ \ m = 1, \ldots, M$$

of the (low-dimensional) MAPs $X^{(1)}(\cdot), \ldots, X^{(M)}(\cdot)$ into the $\prod_{m=1}^{M} N_m$ eigenvalue functions of the (high-dimensional) MAP $X(\cdot)$. It essentially entails that the bulk of the computations can be performed at the level of individual MAPs $X^{(1)}(\cdot), \ldots, X^{(M)}(\cdot)$. The corresponding eigenvector (solving $\bar{F}(\alpha)\boldsymbol{v} = \mathbf{0}$) is then the Kronecker product of the eigenvectors associated to the lower dimensional MAPs, cf. [22, Eqn. (60)]. This procedure may lead to reducing the computational burden by several orders of magnitude. The function $h_1^{(m)}(\alpha) = k^{(m)}(\alpha)/\alpha$ is often referred to as the *effective bandwidth function* [9].

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Appendix A

Lemma 7. *If a sequence of $N \times N$ matrices A_n goes to a diagonal matrix A with elements in $[-\infty, \infty]$, then the N eigenvalues of A_n go to the different (with regard to index) diagonal elements of A.*

Proof. It trivially follows from Gershgorin's theorem [10, Thm. 6.1.1] that for any $\delta > 0$ there exists n_0 such that the eigenvalues of A_n for $n > n_0$ belong to the discs $D_i := \{z \in \mathbb{C} : |z - (A_n)_{ii}| < \delta\}$. Moreover, if a union of k of these discs is disjoint from all the remaining $N-k$ discs then there are exactly k eigenvalues of A_n in this union. Clearly, for sufficiently small δ and large enough n_0 the discs D_i and D_j are disjoint if $(A_n)_{ii}$ and $(A_n)_{jj}$ do not have the same limit. This concludes the proof. $\square$

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