RARE EVENT ASYMPTOTICS FOR A RANDOM WALK IN THE QUARTER PLANE

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Abstract. This paper presents a novel technique to derive asymptotic expressions for rare event probabilities for random walks in the quarter plane. For concreteness, we study a tandem queue with Poisson arrivals, exponential service times and coupled processors. The service rate for one queue is only a fraction of the global service rate when the other queue is non empty; when one queue is empty, the other queue has full service rate. The bivariate generating function of the queue lengths gives rise to a functional equation. In order to derive asymptotic expressions for large queue lengths, we combine the kernel method for functional equations with boundary value problems and singularity analysis.

1. Introduction

Stationary distributions of two-dimensional one-step random walks in the quarter plane can be obtained by solving functional equations. Malyshev pioneered this general problem in the 1970's, and the theory has advanced since via its use in applications like lattice path counting and two-server queueing models. The idea to reduce the functional equation for the generating function to a standard Riemann-Hilbert boundary value problem stems from the work of Fayolle and Iasnogorodski [10] on two parallel $M/M/1$ queues with *coupled processors* (the service speed of a server depends on whether or not the other server is busy). Extensive treatments of the boundary value technique for functional equations can be found in Cohen and Boxma [6] and Fayolle, Iasnogorodski and Malyshev [11]. This technique concerns sophisticated complex analysis, Riemann surfaces and various boundary value problems.

This paper presents a novel technique to derive asymptotic estimates for the occurrence of rare events in random walks in the quarter plane. For concreteness, we shall do so by studying a tandem queue with Poisson arrivals, exponential service times and coupled processors. The presented technique can be applied to the general class of random walks covered in [11]. Denote by N_1 and N_2 the stationary number of customers in the first and second queue. The generating function $P(x, y) = \mathbb{E}(x^{N_1} y^{N_2})$ then satisfies the functional equation

$$h_1(x, y)P(x, y) = h_2(x, y)P(x, 0) + h_3(x, y)P(0, y) + h_4(x, y)P(0, 0), \qquad (1)$$

where the functions h_j are quadratic polynomials in x and y. Equation (1) cannot be solved directly for $P(x, y)$, because it contains other unknown functions $P(x, 0)$ and $P(0, y)$. This is the universal problem for random walks in the quarter plane.

The general approach is to consider the roots of the kernel $h_1(x, y)$ w.r.t. one of the variables x, y. Substituting such roots into (1) yields additional equations

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between the unknown functions $P(x,0)$ and $P(0,y)$ that are free of the term containing the full generating function $P(x,y)$. These additional equations in fact give rise to boundary value problems whose solutions lead to a specification of $P(x,0)$ and $P(0,y)$ and hence $P(x,y)$. For the tandem queue with coupled processors this was done in [21, 24]. The obtained formal solution, however, is too complicated to invert for the stationary distribution. We are particularly interested in the probabilities of large queue lengths (rare events), for which we develop a new asymptotic technique.

In order to find information on $\mathbb{P}(N_1 = n)$, for large n, we need to extract information from the generating function $P(x,1) = \sum_{n=0}^{\infty} \mathbb{P}(N_1 = n)x^n$. We shall employ the functional equation to determine the dominant (closest to the origin) singularities of the functions $P(x,0)$ and $P(x,1)$. Subsequently, by investigating $P(x,1)$ in the neighborhood of its dominant singularity, ξ say, we obtain exact asymptotic expressions for the tail of the probability distribution of N_1. While large deviations estimates yield results of the form

$$\lim_{n\to\infty} \frac{1}{n} \log \mathbb{P}(N_1 = n) = \xi^{-n},$$

we are also able to obtain the function $f(n)$ such that

$$\lim_{n\to\infty} \frac{\mathbb{P}(N_1 = n)}{f(n)} = 1. \tag{2}$$

The determination of f requires a full solution to $P(x,0)$. In [21, 24] solutions for $P(x,0)$ and $P(0,y)$ were derived that are valid only in specific parts of the complex planes. In this paper we provide complete solutions to $P(x,0)$ and $P(0,y)$, that are in fact the analytic continuations to the entire complex planes of the solutions in [21, 24].

The technique of investigating a function near its dominant singularity to obtain asymptotic expressions for its coefficients is known as *singularity analysis* and has a long history in areas of mathematics like analysis, combinatorics and number theory; for an elaborate exposition see Flajolet and Sedgewick [13]. In most cases the generating function is univariate and explicit, and extracting information on the coefficients boils down to the (asymptotic) evaluation of univariate contour integrals.

The extraction of asymptotics from multivariate generating functions has been strongly motivated by recursively defined combinatorial structures like trees, see e.g. [9, 12, 13], and specific random walks or queueing models [3, 14, 15, 18]. One of the central ideas in multivariate asymptotics is to exploit a functional equation to reduce multivariate problems to univariate contour integrals. In contrast to most functional equations that are subject to multivariate asymptotics (see [26] for an overview), our functional equation (1) does not allow for a closed form solution, which complicates considerably the application of singularity analysis. Our method can be considered as an extension of the technique of singularity analysis for bivariate generating functions.

There are two alternative methods: a method based on large deviations developed by Foley and McDonald [17], and the matrix-geometric method [23, 19, 28]. The further development of both techniques is a an active area of research. The matrix-geometric method aims at deriving the so-called *boundary condition*, under which the asymptotics show geometric behavior, which is to say that the function

f in (2) does not depend on n. This boundary condition plays a crucial role in the large deviations approach too, and is naturally the subject of much recent work [19, 20, 22, 23, 27]. Geometric decay requires the dominant singularity to be a pole, whereas it could be a singularity of a different nature like a branch point. Indeed, this is also the case for the tandem queue at hand. Foley and McDonald are able to obtain results for the non-geometric regimes, although for these regimes they need a highly involved case specific approach. For a modified Jackson network this is demonstrated in [16].

In the present paper, we make the following contributions:

- We provide in Propositions 3 and 4 exact solutions to $P(x, 0)$ and $P(0, y)$, in terms of meromorphic functions, that are valid in the entire complex x and y planes cut along some segments. The solutions follow from analytic continuations through the functional equation (1) and are generalizations of the partial solutions (valid in parts of the complex planes) obtained in [21, 24].
- We determine the domain of analyticity of the functions $P(x, 1)$ and $P(1, y)$. A crucial role is fulfilled by the resultant of the functions h_1 and h_2. The domains of analyticity lead to exact asymptotic expressions for $\mathbb{P}(N_1 = n)$ and $\mathbb{P}(N_2 = n)$.
- The parameter values determine the nature of the dominant singularities of $P(x, 1)$ and $P(1, y)$ that give rise to several different asymptotic regimes. Asymptotic estimates for the probabilities of large queue lengths are obtained using Laplace's method and Darboux's method. In Proposition 5 we identify four different regimes for queue 1, and Proposition 6 shows that there are three different regimes for queue 2.

Related work was done in [18] for two parallel $M/M/1$ queues with coupled processors, also leading to rare event probabilities. However, this latter model can be reduced to a Dirichlet problem (the boundary value problem has a boundary which is a circle, and the problem is solved by using the Poisson kernel; see [10, 18]). In the present paper, the boundary is a general smooth closed contour and we use a Riemann-Hilbert formulation, which allows us to directly extend the function outside the domain delineated by the boundary. In this respect, the problem considered in the present paper is much more general than the one considered in [18], and the same approach can be used for many models that fall into the class of random walks in the quarter plane.

The tandem queue with coupled servers, which we chose as our vehicle to present the asymptotic technique, is of independent interest. It arises as a natural model for bandwidth sharing of Internet capacity that is based on reservation procedures (see [7, 21, 24]). The two servers are coupled such that the server speed of server i is μ_i when the other server is busy, and μ_i^* when the other server is idle. This coupling became extremely popular in the last decade due to its relation to the Generalized Processor Sharing (GPS) discipline ($\mu_1^* = \mu_2^* = \mu_1 + \mu_2$), the prevalent discipline for bandwidth sharing in packet networks. See [1] for an overview on GPS. The different asymptotic regimes identified in this paper yield structural insights on the impact of GPS on rare events in a tandem queue.

We proceed as follows: Section 2 contains the model description and an extensive analysis of the zero-pairs of the kernel h_1 in Equation (1). In particular, various analytic continuations of these zero-pairs are constructed, which identify some of the singularities of the function $P(x,0)$ and $P(0,y)$. Further singularities are identified in Section 3 by considering the resultant of h_1 and h_2. In Section 4 we formulate $P(x,0)$ and $P(0,y)$ in terms of boundary value problems. The solutions to these boundary value problem yield solutions to $P(x,0)$ and $P(0,y)$ in terms of meromorphic functions, with a clear description of their singularities. In Section 5 this knowledge is used to obtain a complete characterization of the exact asymptotics for the stationary distributions of both queues.

2. Model description and preliminary properties

Consider a two-stage tandem queue, where jobs arrive at queue 1 according to a Poisson process with rate λ, demanding service at both queues before leaving the system. Each job requires an exponential amount of work with parameter ν_j at station j, $j = 1, 2$. The global service rate is set to one. The service rate for one queue is only a fraction (p for queue 1 and $1-p$ for queue 2) of the global service rate when the other queue is non empty; when one queue is empty, the other queue has full service rate. Therefore, when both queues are nonempty, the departure rates at queue 1 and 2 are $\nu_1 p$ and $\nu_2(1-p)$, respectively.

When one of the queues in empty, the departure rate of the nonempty queue j is temporarily increased to ν_j. With $N_j(t)$ the number of jobs at station j at time t, the two-dimensional process $\{(N_1(t), N_2(t)), t \geq 0\}$ is a Markov process, and upon uniformization, a random walk in the quarter plane.

The stability condition under which this Markov process has a unique stationary distribution is given by

$$\rho = \lambda/\nu_1 + \lambda/\nu_2 < 1. \tag{3}$$

This can be explained by the fact that, independent of p, the two stations together always work at capacity 1 (if there is work in the system), and that $\lambda/\nu_1 + \lambda/\nu_2$ equals the amount of work brought into the system per time unit. We henceforth assume that the ergodicity condition is satisfied.

Denote the joint stationary probabilities by

$$\mathbb{P}(N_1 = n, N_2 = k) = \lim_{t\to\infty} \mathbb{P}(N_1(t) = n, N_2(t) = k),$$

and let $P(x,y)$ represent the bivariate generating function

$$P(x,y) = \sum_{n=0}^{\infty}\sum_{k=0}^{\infty} \mathbb{P}(N_1 = n, N_2 = k)x^n y^k.$$

From the balance equations it follows (see [24]) that $P(x,y)$ satisfies the functional equation (1) with

$$\begin{aligned}
h_1(x,y) &= (\lambda + p\nu_1 + (1-p)\nu_2)xy - \lambda x^2 y - p\nu_1 y^2 - (1-p)\nu_2 x,\\
h_2(x,y) &= (1-p)\left(\nu_1 y(y-x) + \nu_2 x(y-1)\right),\\
h_3(x,y) &= -\frac{p}{1-p} h_2(x,y),\\
h_4(x,y) &= \nu_2 x(y-1) - h_2(x,y).
\end{aligned}$$

We have $P(0,0) = 1-\rho$.

2.1. **Zero-pairs of the kernel.** Set $\hat{r} = 1 + 1/r_1 + 1/r_2$ with $r_1 = \lambda/(p\nu_1)$ and $r_2 = \lambda/((1-p)\nu_2)$. With this notation, equation $h_1(x,y) = 0$ in y has two roots:

$$X_\pm(y) = \frac{1}{2y}\left((\hat{r}y - 1/r_2) \pm \sqrt{D_2(y)}\right)$$

where

$$D_2(y) = (\hat{r}y - 1/r_2)^2 - 4y^3/r_1.$$

The functions $X_\pm(y)$ are well defined for $y \in \mathbb{R}\setminus\{0\}$ as long as $D_2(y) \geq 0$. It is easily checked that $\lim_{y\to 0} X_+(y) = 0$ (the point 0 is a removable singularity for the function $X_+(y)$) and $\lim_{y\to 0^+} X_-(y) = -\infty$ (the point 0 is a singularity for the function $X_-(y)$). In addition, as shown in [21], the discriminant $D_2(y)$ has three roots in $\mathbb{R}$. These three roots are denoted by y_1, y_2 and y_3 and are such that $0 < y_1 < y_2 \leq 1 < y_3$. We have $D_2(y) > 0$ for $y \in (-\infty, y_1) \cup (y_2, y_3)$ and $D_2(y) < 0$ for $y \in (y_1, y_2) \cup (y_3, \infty)$.

Similarly, the equation $h_1(x,y) = 0$ in x has two roots:

$$Y_\pm(x) = \frac{r_1}{2}\left((\hat{r} - x)x \pm \sqrt{D_1(x)}\right),$$

where

$$D_1(x) = ((\hat{r} - x)x)^2 - 4x/(r_1 r_2).$$

The functions $Y_\pm(x)$ are well defined for $x \in \mathbb{R}$ as long as the discriminant $D_1(x) \geq 0$. As shown in [21], the discriminant $D_1(x)$ has four real roots $x_1 = 0 < x_2 \leq 1 < x_3 < x_4$. We have $D_1(x) > 0$ for $x \in (-\infty, x_1) \cup (x_2, x_3) \cup (x_4, \infty)$ and $D_1(x) < 0$ for $x \in (x_1, x_2) \cup (x_3, x_4)$.

In the next section, we investigate how to analytically continue the functions $Y_\pm(x)$ in $\mathbb{C} \setminus ([x_1, x_2] \cup [x_3, x_4])$ and $X_\pm(y)$ in $\mathbb{C} \setminus ([y_1, y_2] \cup [y_3, \infty))$.

2.2. **Analytic continuation.** In the following, we assume that for $z \in \mathbb{C}$, $\arg(z) \in (-\pi, \pi]$, and we take the determination of the square such that $\sqrt{x^2} = x$ if $x \geq 0$ and $\sqrt{-1} = i$. The couple $(X_+(y), (-\infty, y_1))$ defines a germ of analytic function. We first investigate how this germ can be analytically continued in the complex plane deprived of the segments $[y_1, y_2]$ and $[y_3, \infty)$.

Lemma 1. *The function $X^*(y)$ defined in $\mathbb{C} \setminus ([y_1, y_2] \cup [y_3, \infty))$ by*

$$X^*(y) = \begin{cases} X_+(y) & \text{when } y \in \{z : \Re(z) \leq y_2, \Im(D_2(z^+)) < 0\} \cup (-\infty, y_1), \\ X_-(y) & \text{otherwise,} \end{cases} \tag{4}$$

where $z^+ = \Re(z) + i|\Im(z)|$, is analytic.

Proof. Let $y = u + iv$ with $u, v \in \mathbb{R}$. We have $D_2(y) = \Re(D_2(y)) + i\Im(D_2(y))$ with

$$\begin{aligned} \Re(D_2(y)) &= \left(\hat{r}u - \frac{1}{r_2}\right)^2 - \hat{r}^2 v^2 - \frac{4}{r_1}(u^3 - 3uv^2), \\ \Im(D_2(y)) &= v\left(\frac{4}{r_1}v^2 - \left(\frac{12}{r_1}u^2 - 2\hat{r}^2 u + \frac{2}{r_2}\right)\right). \end{aligned}$$

The imaginary part vanishes for u and v satisfying the equation

$$\frac{4}{r_1}v^2 = \frac{12}{r_1}u^2 - 2\hat{r}^2 u + \frac{2}{r_2}. \tag{5}$$

For sufficiently large u the term in the right hand side of the above equation is positive. If we assume that this terms does not cancel for u describing the whole of $\mathbb{R}$, then we can define two curves in $\mathbb{C}$ along which the imaginary part of $D_2(y)$ vanishes, one curve entirely lies in the positive half-plane $\{y : \Im(y) > 0\}$ and the other one in the negative half-plane $\{y : \Im(y) < 0\}$.

Along one of these curves, the sign of the real part $\Re(D_2(y))$ is constant since the imaginary and the real parts cancel only for $y \in \mathbb{R}$ (namely for y equal to one of the roots y_1, y_2 and y_3). For the curve in the upper half-plane we have $v^2 \sim 3u^2$ for $|u| \to +\infty$. But in this case, we would have $\Re(D_2(y)) \sim 32u^3/r_1$, which contradicts the fact that $\Re(D_2(y))$ should keep the same sign as u describes the whole of $\mathbb{R}$. Hence, the polynomial in the right hand side of Equation (5) has roots in $\mathbb{R}$, which are positive since the value of this polynomial at point 0 is $2/r_2 > 0$. Let y_1^* and y_2^* denote these roots with $0 < y_1^* \leq y_2^*$.

Equation (5) defines two hyperbolic branches as depicted in Figure 1. The left branch intersects the real axis at point y_1^* and for y on this branch such that $\Im(y) \neq 0$, $\Re(D_2(y)) < 0$. By continuity of the real part, which is a polynomial in u and v, we have $\Re(D_2(y_1^*)) \leq 0$ and hence $y_1 \leq y_1^* \leq y_2$. The right branch intersects the real axis at point y_2^*. For y on this branch such that $\Im(y) \neq 0$, $\Re(D_2(y)) > 0$ and by continuity of the real part, we have $\Re(D_2(y_2^*)) \geq 0$, which implies that $y_2 \leq y_2^* \leq y_3$.

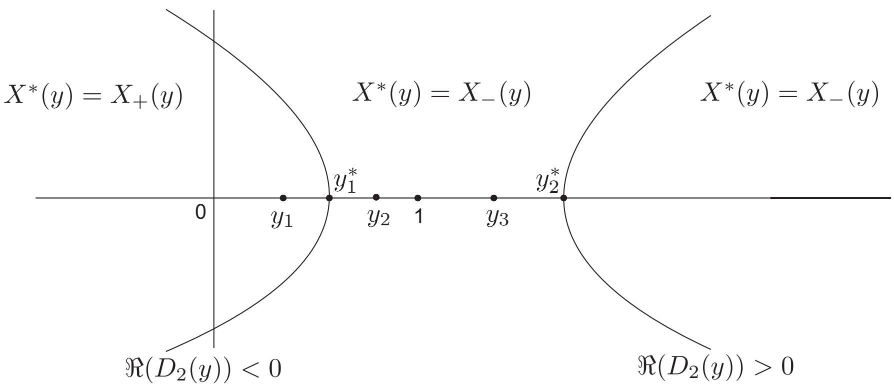

FIGURE 1. Branches on which $\Im(D_2(y)) = 0$.

The function $X_+(y)$ is analytic in the domain $\{y : \Re(y) \leq y_1^*, \Im(D_2(y^+)) < 0\} \cup (-\infty, y_1)$. The function $X_-(y)$ is analytic in the complementary domain of the closure of this set in $\mathbb{C} \setminus ([y_1, y_2] \cup [y_3, \infty))$. To show that the function $X^*(y)$ is analytic in the whole of $\mathbb{C}$ deprived of the segments $[y_1, y_2]$ and $[y_3, \infty)$, from Moreira's theorem, it is sufficient to show that this function is continuous on the branch $\{y : \Im(D_2(y)) = 0,\ \Re(D_2(y) \leq 0\}$ separating the two above domains. But this is straightforwardly checked from the choice of the determination of the square root. $\square$

By using exactly the same arguments as in the proof of Lemma 1, we can prove the following result.

Lemma 2. *The function $X_*(y)$ defined in $\mathbb{C} \setminus ([y_1, y_2] \cup [y_3, \infty))$ by*

$$X_*(y) = \begin{cases} X_-(y) & \text{when } y \in \{z : \Re(z) \leq y_2, \Im(D_2(z^+)) < 0\} \cup (-\infty, y_1), \\ X_+(y) & \text{otherwise}, \end{cases} \tag{6}$$

where $z^+ = \Re(z) + i|\Im(z)|$, *is analytic.*

We now turn to the functions $Y_\pm(x)$. First note that $Y_\pm(0) = 0$. As shown in [21], when x is close to the segment $[x_1, x_2]$, $Y_\pm(x)$ is close to a contour ∂D_y in the y-plane included in the half-plane $\{y : \Re(y) \geq 0\}$; in particular the point 0 lies in ∂D_y. In addition, when y is close to the segment $[y_1, y_2]$, $X(y)$ is in the x-plane close to a contour ∂D_x surrounding the point 0. The contours ∂D_x and ∂D_y delineate bounded open domains in the x-plane deprived of the segment $[x_1, x_2]$ and the y-plane deprived of the segment $[y_1, y_2]$ denoted by D_x and D_y, respectively. Since our ultimate goal is to exhibit a conformal mapping between these two domains and since $Y_\pm(-i\varepsilon) \sim \pm(\cos(\pi/4) + i\sin(\pi/4))\sqrt{\varepsilon/(r_1r_2)}$ for small $\varepsilon > 0$, we are led to pick up the function $Y_+(x)$ as a candidate for the desired conformal mapping because $Y_+(-i\varepsilon) \in D_y$ while $Y_-(-i\varepsilon) \notin D_y$ for sufficiently small $\varepsilon > 0$.

Lemma 3. *The function $Y^*(x)$ defined in $\mathbb{C} \setminus ([x_1, x_2] \cup [x_3, x_4])$ by*

$$Y^*(x) = \begin{cases} Y_+(x) & \text{when } x \in \{z : \Re(z) \leq x_2, \Im(D_1(z^+)) < 0\} \cup (-\infty, x_1), \\ Y_+(x) & \text{when } x \in \{z : \Re(z) \geq x_3, \Im(D_2(z^+)) > 0\} \cup (x_4, \infty), \\ Y_-(x) & \text{otherwise}, \end{cases} \tag{7}$$

where $z^+ = \Re(z) + i|\Im(z)|$, *is analytic.*

Proof. Let $x = u + iv$ with $u, v \in \mathbb{R}$. We have $D_1(x) = \Re(D_1(x)) + i\Im(D_1(x))$ with

$$\begin{aligned} \Re(D_1(x)) &= \left((\hat{r} - u)u + v^2\right)^2 - v^2(\hat{r} - 2u)^2 - \frac{4u}{r_1r_2}, \\ \Im(D_1(x)) &= 2v\left((\hat{r} - 2u)v^2 + u(\hat{r} - u)(\hat{r} - 2u) - \frac{2}{r_1r_2}\right). \end{aligned}$$

The imaginary part $\Im(D_1(x)) = 0$ if (u, v) satisfies

$$(2u - \hat{r})v^2 = u(2u - \hat{r})(u - \hat{r}) - \frac{2}{r_1r_2}. \tag{8}$$

Let $d_1(u)$ be the polynomial in the right hand side of the above equation. This polynomial is of degree 3 and has at least one real root (say, u_1). Since $\lim_{u\to+\infty} d_1(u) = +\infty$ and $d_1(\hat{r}) = -2/(r_1r_2) < 0$, $u_1 > \hat{r}$. The polynomial $d_1(u)$ can then be decomposed as $d_1(u) = (u - u_1)d_{11}(u)$. If the polynomial $d_{11}(u)$ had no real roots, then this polynomial would be positive in the whole of $\mathbb{R}$ since $d_1(u)$ is positive for large u. When $u < \hat{r}/2$, Equation (8) would have two roots, namely

$$v = \pm\sqrt{\frac{(u - u_1)d_{11}(u)}{2u - \hat{r}}}.$$

We would then obtain two curves, one entirely included in the half-plane $\{x : \Im(x) > 0\}$ and the other in the half-plane $\{x : \Im(x) < 0\}$. Along each of these curves, the sign of $\Re(D_1(x))$ should be constant (see the arguments in the proof of Lemma 1). But when $u \to -\infty$, $v^2 \sim u^2$ and then $\Re(D_1(x)) < 0$, and when $u \to \hat{r}/2$, $v^2 \sim -2/(r_1r_2(2u - \hat{r}))$ and $\Re(D_1(x)) > 0$, which is in contradiction with the fact that the sign of $\Re(D_1(x))$ should be constant along the curves $\Im(D_1(x)) = 0$. As a consequence, the polynomial $d_1(u)$ has three real roots. Let us denote these roots by x_1^*, x_2^* and x_3^* with $x_1^* \leq x_2^* \leq x_3^*$. Their product is equal to $1/(r_1r_2)$ and since one of them is positive, the two others have the same sign.

We already know that $x_3^* > \hat{r}$. If $x_1^* \geq \hat{r}/2$, then Equation (8) defines two curves for $u \leq \hat{r}/2$, one is included in the upper half plane and the other in the lower half plane, which is not possible for the same reasons as above. Hence, $x_1^* \leq \hat{r}/2$. This also implies that $x_2^* < \hat{r}/2$ since $d_1(\hat{r}/2) = -2/(r_1 r_2) < 0$. Hence, we have

$$0 \leq x_1^* \leq x_2^* < \hat{r}/2 < \hat{r} < x_3^*.$$

Let us consider the three curves defined by

$$v = \pm\sqrt{\frac{(u - x_1^*)(u - x_2^*)(u - x_3^*)}{2u - \hat{r}}} \quad \text{when } u \leq x_1^* \quad \text{or} \quad x_2^* \leq u < \hat{r}/2 \quad \text{or } u \geq x_3^*.$$

See Figure 2.

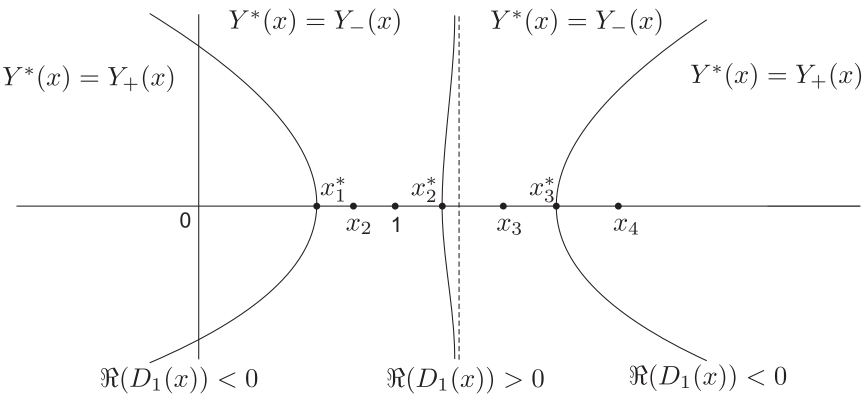

FIGURE 2. Branches on which $\Im(D_1(x)) = 0$.

For the curve defined for $u \leq x_1^*$ it is easily checked that $\Re(D_1(x)) < 0$ when $v \neq 0$ and by continuity we deduce that $\Re(D_1(x)) \leq 0$. This implies that $x_1 \leq x_1^* \leq x_2$. Similar arguments show that $x_3 \leq x_3^* \leq x_4$. For the curve defined for $x_2 \leq u < \hat{r}/2$, we have $\Re(D_1(x)) > 0$ when $v \neq 0$ and hence $\Re(D_1(x)) \geq 0$ all along the curve. This implies that $x_2 \leq x_2^* \leq x_3$. We finally have the ordering

$$x_1 \leq x_2^* \leq x_2 \leq x_2^* < \hat{r}/2 < x_3 \leq x_3^* \leq x_4.$$

Note that it is easily checked that $x_3 > \hat{r}/2$. Indeed, if we assume that $x_3 \leq \hat{r}/2 \leq x_3^* \leq x_4$, we would have $D_1(\hat{r}/2) \leq 0$ and then $D_1(x)$ would be non positive for all $x \geq \hat{r}/2$ since the term $(\hat{r} - x)x$ is maximum at point $\hat{r}/2$; this is clearly not possible.

By invoking the same arguments as in the proof of Lemma 1, it is easily checked that the function $Y^*(x)$ defined by Equation (7) is analytic in the complex plane deprived of the segments $[x_1, x_2]$ and $[x_3, x_4]$. $\square$

With similar arguments, we can prove the following result.

Lemma 4. *The function $Y^*(x)$ defined in $\mathbb{C} \setminus ([x_1, x_2] \cup [x_3, x_4])$ by*

$$Y_*(x) = \begin{cases} Y_-(x) & \text{when } x \in \{z : \Re(z) \leq x_2, \Im(D_1(z^+)) < 0\} \cup (-\infty, x_1), \\ Y_-(x) & \text{when } x \in \{z : \Re(z) \geq x_3, \Im(D_2(z^+)) > 0\} \cup (x_4, \infty), \\ Y_+(x) & \text{otherwise}, \end{cases} \tag{9}$$

where $z^+ = \Re(z) + i|\Im(z)|$, is analytic.

To conclude this section, let us examine the images of the contours ∂D_x and ∂D_y by the analytic functions Y^* and X^*, respectively. First note that for $x \in \mathbb{C} \setminus ([x_1, x_2] \cap [x_3, x_4])$, $X^*(Y^*(x)) = x$ and for $y \in \mathbb{C} \setminus ([y_1, y_2] \cap [y_3, \infty))$, $Y^*(X^*(y)) = y$. To prove the first equality, consider $x \in (-\infty, 0)$ sufficiently close to 0, then $Y^*(x) = Y_+(x) \sim \sqrt{-r_1 x / r_2}$ and $X^*(Y^*(x)) = X_+(Y^*(x)) \sim x$. It follows that the equality $X^*(Y^*(x)) = x$ holds for a neighborhood of 0 and since the function $X^*(Y^*(x))$ is analytic in $\mathbb{C} \setminus ([x_1, x_2] \cap [x_3, x_4])$ this equality holds for the whole of $\mathbb{C} \setminus ([x_1, x_2] \cap [x_3, x_4])$. Similar arguments can be invoked to prove the second equality.

Corollary 1. *We have $X^*(\partial D_y) \subset [x_1, x_2]$ and $Y^*(\partial D_x) \subset [y_1, y_2]$.*

Proof. Consider ∂D_y; the case of ∂D_x is completely symmetrical. Let us consider $y \in \partial D_y$. By construction, there exists $x \in [x_1, x_2]$, such that

$$y = Y_+(x + 0i), \quad \bar{y} = Y_+(x - 0i), \quad y = Y_-(x - 0i), \quad \bar{y} = Y_-(x + 0i).$$

Note that we use the notation $x+0i$ (resp. $x-0i$) to designate the limit of a sequence in the upper (resp. lower) half plane converging to $x \in \mathbb{R}$. From the definition of $Y^*(x)$, the determination of this function at point $x \pm 0i$ is either $Y_+(x \pm 0i)$ or $Y_-(x \pm 0i)$. It follows that $y = Y^*(x + \varepsilon 0i)$ where $\varepsilon = \pm 1$ depending on the determination of $Y^*(x)$. It follows that $X^*(y) = X^*(Y^*(x + \varepsilon 0i)) = x \in [x_1, x_2]$. Hence, $X^*(\partial D_y) \subset [x_1, x_2]$. □

2.3. **Conformal mappings.** We are now able to exhibit the conformal mappings which will play a crucial role in the derivation of the boundary functions $P(0, y)$ and $P(x, 0)$.

Proposition 1. *The function $X^*(y)$ is a conformal mapping from D_y onto D_x. The reciprocal function is $Y^*(x)$.*

Proof. As noted before, when y is in D_y and sufficiently close to 0, $X_+(y) \equiv X^*(y) \in D_x$. Since the set D_y is an open and simply connected domain and since $X^*(y)$ is an analytic function, $X^*(D_y) \cap D_x$ is a non null, open and simply connected domain included in D_x.

If D_x is not a subset of $X^*(D_y)$, let us consider the complementary set $X^*(D_y)^c \cap D_x \neq \emptyset$ in D_x. Let x be a point on the boundary between this set and $X^*(D_y) \cap D_x$. There exist a sequence (x_n) in $X^*(D_y) \cap D_x$ and a sequence (x'_n) in the interior of $X^*(D_y)^c \cap D_x$ both converging to x. Since (x_n) is in $X^*(D_y) \cap D_x$, there exists a sequence (y_n) in D_y such that $X^*(y_n) = x_n$. Moreover, as we have $X^*(Y^*(x)) = x$ for all x in the x-plane deprived of the segments $[x_1, x_2]$ and $[x_3, x_4]$, and $Y^*(X^*(y)) = y$ for all y in the y-plane deprived of the segments $[y_1, y_2]$ and $[y_3, \infty)$, the sequence (y_n) and $(Y^*(x_n))$ converge to the same point. But by definition the points $Y^*(x_n)$ lie outside of the domain D_y. It follows that these two sequences converge to a point on ∂D_y. By Corollary 1, this implies that $x \in [x_1, x_2]$, which is not possible. It follows that $D_x \subset X^*(D_y)$.

If the above inclusion is strict, we consider a point x on the boundary ∂D_x. There should exist a point y in D_y such that $X^*(y) = x$ but this is not possible since y should be in $[y_1, y_2]$ since $Y^*(\partial D_x) \subset [y_1, y_2]$. It follows that $X^*(D_y) = D_x$. In addition, the function $X^*(y)$ is one to one since $Y^*(X^*(y)) = y$. It follows that this function is a conformal mapping from D_y onto D_x and the reciprocal function is Y^*. □

The conformal mappings X^* and Y^* between the domains $D_x \setminus [x_1, x_2]$ and $D_y \setminus [y_1, y_2]$ are illustrated in Figure 3. While X^* maps $D_y \setminus [y_1, y_2]$ onto $D_x \setminus [x_1, x_2]$, the set $X_*(D_y \setminus [y_1, y_2])$ is an open domain surrounding D_x in the x-plane. Similarly, $Y_*(D_x \setminus [x_1, x_2])$ is an open set surrounding D_y in the y-plane.

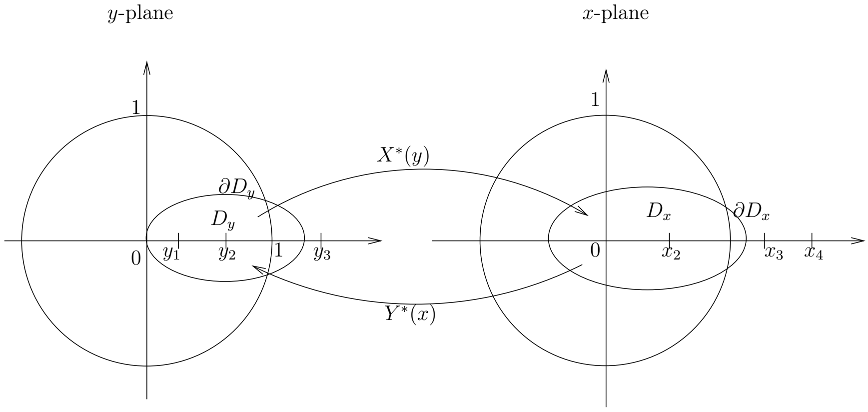

FIGURE 3. Fundamental domains D_y and D_x.

It is worth noting that $X^*(\xi) \to x \in \partial D_x$ from inside D_x when $\xi \to y \in [y_1, y_2]$. Similarly, $Y^*(\xi) \to y \in \partial D_y$ from inside D_y when $\xi \to x \in [x_1, x_2]$. We also have $X_*(\xi) \to x \in \partial D_x$ from outside D_x when $\xi \to y \in [y_1, y_2]$ and $Y_*(\xi) \to y \in \partial D_y$ from outside D_y when $\xi \to x \in [x_1, x_2]$.

3. Intersection points of the curves $h_1(x, y) = 0$ and $h_2(x, y) = 0$

When $h_1(x, y) = 0$, we see from Equation (1) that we can express $P(x, 0)$ (resp. $P(0, y)$) in function of $P(0, y)$ (resp. $P(x, 0)$) and $h_4(x, y)$ where the function $h_2(x, y)$ appears in the denominator. The common solutions of the equations $h_1(x, y) = 0$ and $h_2(x, y) = 0$ are then potential singularities for the function $P(x, 0)$ and $P(0, y)$.

3.1. **The common roots in variable y.** Let $y \in \mathbb{C} \setminus ([y_1, y_2] \cup [y_3, \infty))$ and $h_1(x, y) = 0$, $x = X_\pm(y)$. If in addition $h_2(x, y) = 0$, then y is a root of the resultant in x of the two polynomials $h_1(x, y)$ and $h_2(x, y)$ (see Appendix A); this resultant, denoted by $Q_x(y)$, is a polynomial of degree 5 in y, which has at most four distinct zeros in $\mathbb{C}$ (the point 0 is a double root).

One trivial root of the resultant is of course 0. Another trivial root is 1 since $h_1(1, 1) = 0$ and $h_2(1, 1) = 0$. As shown in Appendix A, the resultant $Q_x(y)$ can actually be decomposed as

$$Q_x(y) = c_x y^2 (y - 1) \mathcal{Q}_x(y),$$

where $\mathcal{Q}_x(y)$ is the quadratic polynomial

$$\mathcal{Q}_x(y) = \lambda \nu_1 y^2 + \nu_2(\nu_2 - \nu_1 + \lambda) y - \nu_2^2$$

and c_x is a constant.

When y describes the segment $[y_2, y_3]$, the curves $y \to x = X_\pm(y)$ describe the contour of a closed domain Ω_y in the (y, x)-plane as illustrated in Figure 4; the contour $\partial\Omega_y$ of Ω_y contains the point $(1, 1)$.

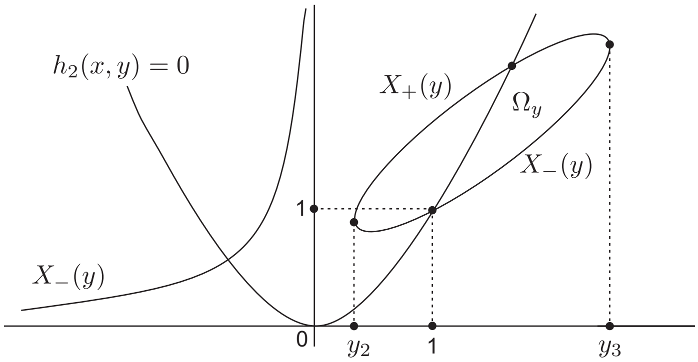

FIGURE 4. Intersection points of the functions $X_\pm(y)$ and the curve $h_2(x,y)=0$ when $r_1 \le 1$.

When $h_2(x,y)=0$,

$$x = \frac{\nu_1 y^2}{(\nu_1-\nu_2)y+\nu_2}. \tag{10}$$

As illustrated in Figure 4, when $r_1 < 1$, the hyperbolic branch defined by Equation (10) intersects the branch $x = X_-(y)$ at some point with a negative abscissa. The same observation is true when $r_1 \ge 1$. It follows that the resultant $Q_y(x)$ has four real roots and the quadratic polynomial $\mathcal{Q}_x(y)$ has two real roots, one is negative and the other is in $[y_2, y_3]$. The positive root is

$$y^* = \frac{\nu_2}{2\lambda\nu_1}\left(-(\nu_2-\nu_1+\lambda)+\sqrt{(\nu_2-\nu_1+\lambda)^2+4\lambda\nu_1}\right)$$

and the negative root is

$$y_* = \frac{\nu_2}{2\lambda\nu_1}\left(-(\nu_2-\nu_1+\lambda)-\sqrt{(\nu_2-\nu_1+\lambda)^2+4\lambda\nu_1}\right).$$

Note that the root y^* can be rewritten as

$$y^* = \frac{\rho_1}{2\rho_2}\left(\frac{1}{\rho_1}-\frac{1}{\rho_2}-1+\sqrt{\left(\frac{1}{\rho_1}-\frac{1}{\rho_2}-1\right)^2+\frac{4}{\rho_1}}\right).$$

It is worth noting that y^* does not depend on the probability p. From Appendix A, we know that $y^* \in (1, y_3]$.

3.2. **The common roots in variable x.** The resultant in x of the polynomials $h_1(x,y)$ and $h_2(x,y)$ is a polynomial of degree 5 with trivial roots 0 and 1 (0 is a double root). If $x \neq 0$ and (x,y) is an intersection point of the curves $h_1(x,y)=0$ and $h_2(x,y)=0$, then

$$y = \frac{\nu_2}{\lambda+\nu_2-\lambda x}. \tag{11}$$

For $x \in [x_2, x_3]$, the curves $y = Y_\pm(x)$ delineate a closed domain Ω_x such that its contour $\partial\Omega_x$ contains the point $(1,1)$. Note that if $r_1 < r_2$, then $Y_+(1)=1$ and if $r_1 > r_2$, then $Y_-(1)=1$.

The hyperbolic branch defined by Equation (11) intersects the branch $y = Y_-(x)$ or $y = Y_+(x)$ at a point with abscissa $x > x_4$. It follows that the resultant in y of the polynomials $h_1(x, y)$ and $h_2(x, y)$, denoted by $Q_y(x)$, can be decomposed as

$$Q_y(x) = c_y x^2(x-1)\mathcal{Q}_y(x),$$

where c_y is a constant and

$$\mathcal{Q}_y(x) = \lambda^2 x^2 - (\lambda + \nu_1 + \nu_2)\lambda x + \nu_1\nu_2.$$

The roots x^* and x_* are given by

$$x^* = \frac{\lambda + \nu_1 + \nu_2 - \sqrt{(\lambda + \nu_1 + \nu_2)^2 - 4\nu_1\nu_2}}{2\lambda}$$

and

$$x_* = \frac{\lambda + \nu_1 + \nu_2 + \sqrt{(\lambda + \nu_1 + \nu_2)^2 - 4\nu_1\nu_2}}{2\lambda},$$

and are such that $x^* \leq x_3 < x_4 \leq x_*$. In addition, we know that $x^* > 1$ and hence $x^* \in (1, x_3]$. The variable x^* can be written as

$$x^* = \frac{1}{2}\left(1 + \frac{1}{\rho_1} + \frac{1}{\rho_2} - \sqrt{\left(1 + \frac{1}{\rho_1} + \frac{1}{\rho_2}\right)^2 - \frac{4}{\rho_1\rho_2}}\right)$$

and does not depend on the probability p.

From the above observations, we deduce the following result.

Proposition 2. *The equation $Q_y(X^*(y)) = 0$ has a solution in $(-\infty, y_3]$, which is necessarily equal to $y^* \in (1, y_3]$, if and only if $x^* = X_-(y^*)$.*

Symmetrically, the equation $Q_x(Y^*(x)) = 0$ has a solution in $(-\infty, x_3]$, which is necessarily equal to $x^* \in (1, x_3]$, if and only if $y^* = Y_-(x^*)$.

It is worth noting that we can have $x^* = X^*(y^*)$ only if $1 = X^*(1)$, that is $r_1 \leq 1$. Similarly, we can have $y^* = Y^*(x^*)$ only if $1 = Y^*(1)$, that is $r_1 \geq r_2$.

4. Boundary value problems

We first determine the function $P(x, 0)$; the derivation of the function $P(0, y)$ is completely symmetrical.

Proposition 3. *The function $P(x, 0)$ is given by*

$$P(x,0) = \begin{cases} \dfrac{1}{2\pi i}\displaystyle\int_{\partial D_x} \frac{g_x(z)}{z-x}dz & \text{for} \quad x \in D_x, \\ g_x(x) + \dfrac{1}{2\pi i}\displaystyle\int_{C_x} \frac{g_x(z)}{z-x}dz & \text{for} \quad x \in \mathbb{C} \setminus D_x, \end{cases} \tag{12}$$

where C_x is a contour in D_x surrounding the slit $[x_1, x_2]$ and such that the function g_x given by

$$g_x(x) = (1-\rho)\frac{\nu_2 Y^*(x)(p\nu_1 Y^*(x) - \lambda x^2)}{(1-p)x\mathcal{Q}_x(Y^*(x))}$$

is analytic in the strip delineated by the contours C_x and ∂D_x. The function $P(x, 0)$ is a meromorphic function in $\mathbb{C} \setminus [x_3, x_4]$ with singularities at the solutions to the equation $\mathcal{Q}_x(Y^*(x)) = 0$ if they exist.

Proof. From the analysis carried out in Section 2, we know that for y in a neighborhood $V_y(0)$ of 0^+, $X^*(y)$ is close to 0 in $D_x(0,1)$ (the unit disk in the x-plane). For $y \in V_y(0)$, we deduce from Equation (1) that

$$h_2(X^*(y),y)P(X^*(y),0) + h_3(X^*(y),y)P(0,y) + h_4(X^*(y),y)P(0,0) = 0,$$

which implies that

$$P(X^*(y),0) = \frac{p}{1-p}P(0,y) - (1-\rho)\frac{h_4(X^*(y),y)}{h_2(X^*(y),y)}.$$

Note that $h_2(X^*(y),y) = 0$ if and only if $Q_y(X^*(y)) = 0$, which has only real solutions (see Section 3). From Proposition 2, this equation has a solution in $(-\infty, y_3]$ if and only if $x^* = X^*(y^*)$, which is then the unique solution and which is in $(1, y_3]$. If $\alpha = Y^*(x_2) \leq 1$, the domain D_y is included in the unit disk $D_y(0,1)$ and in that case the function $h_4(X^*(y),y)/h_2(X^*(y),y)$ has no singularities in D_y. If $\alpha > 1$, then $r_1 > r_2$. In this case, x^* is not equal to $X^*(y^*)$ and the function $h_4(X^*(y),y)/h_2(X^*(y),y)$ has no singularities in D_y. Hence, by using the same arguments as in [21], we deduce that the function $P(x,0)$ can be analytically continued to the domain D_x. (We use the fact that the function $P(x,0)$ can be expanded in a power series of x at point 0 with positive coefficients and $P(0,y_2) < \infty$, which implies that $P(x,0)$ is analytic in the disk with center 0 and radius $X^*(y_2)$ containing D_x.)

Now, if we use the function $X_*(y)$, we obtain a meromorphic function in a domain surrounding from outside the domain D_x. If we take y in a sufficiently small neighborhood of $[y_1, y_2]$ we can analytically define $P(x,0)$ in an outer neighborhood of D_x.

Consider $x_0 \in \partial D_x$. Then there exists $y_0 \in [y_1, y_2]$ such that $X^*(y) \to x_0$ from inside when $y \to y_0$. In that case, $X_*(y) \to \bar{x}_0$ from outside. Let us define the interior (resp. exterior) limit $P_i(x,0)$ (resp. $P_e(x,0)$) of the function $P(x,0)$ with respect to the contour ∂D_y by

$$P_i(x_0,0) = \lim_{x \to x_0, x \in D_x} P(x,0) \ \left(\text{resp. } P_e(x_0,0) = \lim_{x \to x_0, x \in \mathbb{C}\setminus D_x} P(x,0)\right).$$

We then deduce from the above observation that for $x \in \partial D_y$ and $y = Y^*(x)$

$$P_i(x,0) = \frac{pP(0,y)}{1-p} - (1-\rho)\frac{h_4(x,y)}{h_2(x,y)}, \quad P_e(x,0) = \frac{pP(0,y)}{1-p} - (1-\rho)\frac{h_4(\bar{x},y)}{h_2(\bar{x},y)},$$

since $P(.,0)$, h_2 and h_4 have real coefficients. Hence, we arrive at the fact that for $x \in \partial D_x$ and $y = Y^*(x)$

$$P_i(x,0) - P_e(x,0) = -2i(1-\rho)\Im\left(\frac{h_4(x,y)}{h_2(x,y)}\right).$$

Note that for $x \in \partial D_x$, we have $x\bar{x} = y/r_1 = Y^*(x)/r_1$ since x and $\bar{x}$ are the two solutions to Equation (1) in x. In addition, from Appendix A, we know that the resultant $Q_x(y)$ can be written as

$$Q_x(y) = p_x(x,y)h_1(x,y) + q_x(x,y)h_2(x,y),$$

where $p_x(x,y)$ and $q_x(x,y)$ are polynomials in x and y. For $y = Y^*(x)$, we have $h_1(x,y) = 0$ and then

$$Q_x(y) = q_x(x,y)h_2(x,y).$$

Simple computations show that

$$\frac{h_4(x,y)}{h_2(x,y)} = -1 + \frac{\nu_2 x(y-1)}{h_2(x,y)}$$

and

$$q_x(x,y) = \lambda y b_1(y) x - (\lambda(1-p)\nu_1 y^3 + a_1(y)b_1(y)),$$

where

$$\begin{aligned} a_1(y) &= (\lambda + p\nu_1 + (1-p)\nu_2)y - (1-p)\nu_2, \\ b_1(y) &= (1-p)((\nu_2-\nu_1)y - \nu_2). \end{aligned}$$

Hence, for $x \in \partial D_y$ and $y = Y^*(x)$, we have

$$\Im\left(\frac{h_4(x,y)}{h_2(x,y)}\right) = \Im\left(\frac{\nu_2 x(y-1)(\lambda y b_1(y)x - (\lambda(1-p)\nu_1 y^3 + a_1(y)b_1(y)))}{-\nu_1(1-p)^2 y^2 (y-1)\mathcal{Q}_x(y)}\right).$$

By using the fact that $\lambda y x^2 - a_1(y)x = -p\nu_1 y^2$, we have

$$\Im\left(\frac{h_4(x,y)}{h_2(x,y)}\right) = \Im\left(\frac{\nu_2(p\nu_1 b_1(y) + \lambda(1-p)\nu_1 y x)}{\nu_1(1-p)^2\mathcal{Q}_x(y)}\right)$$

and then,

$$\Im\left(\frac{h_4(x,y)}{h_2(x,y)}\right) = \frac{\nu_2\lambda y \Im(x)}{(1-p)\mathcal{Q}_x(y)} = \frac{\nu_2\lambda y(r_1x^2 - y)}{2ir_1x(1-p)\mathcal{Q}_x(y)}$$

We finally arrive at the classical Riemann-Hilbert problem: for $x \in \partial D_x$,

$$P_i(x,0) - P_e(x,0) = (1-\rho)\frac{\nu_2 Y^*(x)(p\nu_1 Y^*(x) - \lambda x^2)}{x(1-p)\mathcal{Q}_x(Y^*(x))} = g_x(x).$$

The solution to this Riemann-Hilbert problem is given by

$$P(x,0) = \frac{1}{2\pi i}\int_{\partial D_x}\frac{g_x(z)}{z-x}dz \quad \text{for} \quad x \notin \partial D_x.$$

The above formula defines an analytic function in D_x. For $x \in \mathbb{C}\setminus D_x$, let us pick up a closed contour C_x in D_x surrounding the slit $[x_1, x_2]$ and so that the function g_x is analytic in the strip delineated by the contours ∂D_x and C_x. Then, we have

$$\frac{1}{2\pi i}\int_{\partial D_x}\frac{g_x(z)}{z-x}dz = g_x(x) + \frac{1}{2\pi i}\int_{C_x}\frac{g_x(z)}{z-x}dz.$$

The function in the right hand side of the above equation defines a meromorphic function in $\mathbb{C}\setminus[x_3, x_4]$. □

We can replace the integrals appearing in Equation (12) with integrals along the segment $[y_1, y_2]$. We then obtain elliptic integrals. Since these integrals do not appear as simple combinations of Jacobi elliptic functions, we do not investigate further the connection between the function $P(x,0)$ and elliptic functions. Finally, it is worth noting that the radius of convergence of the function $P(x,0)$ is equal to either x_3 or else x^* if $y^* = Y^*(x^*)$.

By adapting the above proof to the function $P(0,y)$, we obtain the following result.

Proposition 4. *The function $P(0,y)$ is given by*

$$P(0,y) = \begin{cases} \dfrac{1}{2\pi i}\displaystyle\int_{\partial D_y} \frac{g_y(z)}{z-y}dz & \text{for} \quad y \in D_y, \\ g_y(y) + \dfrac{1}{2\pi i}\displaystyle\int_{C_y} \frac{g_y(z)}{z-y}dz & \text{for} \quad y \in \mathbb{C} \setminus D_y, \end{cases}$$

where C_y is a closed contour in D_y surrounding the slit $[y_1, y_2]$ such that the function g_y given by

$$g_y(y) = (1-\rho)\frac{\lambda(p\nu_1 y^2 - (1-p)\nu_2 X^*(y))}{py\mathcal{Q}_y(X^*(y))}$$

is analytic in the strip delineated by the contours C_y and ∂D_y. The function $P(0,y)$ is a meromorphic function in $\mathbb{C} \setminus [y_3, y_4]$ with singularities at the solutions to the equation $\mathcal{Q}_y(X^*(y)) = 0$ if they exist.

Proof. Denote by $P_i(0,y)$ and $P_e(0,y)$ the interior and exterior limits of the function $P(0,y)$ with respect to the contour ∂D_y. We have for $y \in \partial D_y$ and $x = X^*(y)$

$$P_i(0,y) - P_e(0,y) = 2i(1-\rho)\frac{1-p}{p}\Im\left(\frac{h_4(x,y)}{h_2(x,y)}\right).$$

We have $Q_y(x) = q_y(x,y)h_2(x,y)$ for $x = x^*(y)$ with

$$\begin{aligned} q_y(x,y) = (1-p)\nu_1[&-yp\nu_1\left(p(\nu_2-\nu_1)x + \alpha_1(x)\right) \\ &+ p\alpha_1(x)(\nu_2-\nu_1)x + \alpha_1(x)^2 - p\nu_1\nu_2 x]. \end{aligned}$$

Then

$$\Im\left(\frac{h_4(x,y)}{h_2(x,y)}\right) = \frac{\nu_2 x}{Q_y(x)}\Im\left((y-1)q_y(x,y)\right) = \frac{\lambda(p\nu_1 y^2 - (1-p)\nu_2 x)}{2i(1-p)y\mathcal{Q}_y(x)},$$

which implies that

$$\begin{aligned} P_i(0,y) - P_e(0,y) &= (1-\rho)\frac{\lambda(p\nu_1 y^2 - (1-p)\nu_2 x)}{py\mathcal{Q}_y(x)} \\ &= (1-\rho)\frac{\lambda(p\nu_1 y^2 - (1-p)\nu_2 X^*(y))}{py\mathcal{Q}_y(X^*(y))} = g_y(y). \end{aligned}$$

Note that 0 is a removable singularity of the function $g_y(y)$ since $X^*(y) \sim -r_2 y^2/r_1$ when $y \to 0$. □

5. Asymptotic analysis

We derive in this section the tail of the distribution of the numbers of customers in the first and the second queue. For this purpose, we consider the generating functions $P(x,1)$ and $P(1,y)$ which satisfy

$$P(x,1) = \sum_{n=0}^{\infty} \mathbb{P}(N_1 = n)x^n \quad \text{and} \quad P(1,y) = \sum_{n=0}^{\infty} \mathbb{P}(N_2 = n)y^n,$$

where N_1 and N_2 are the numbers of customers in the first and the second queue, respectively. From Equation (1), we clearly have

$$P(x,1) = \nu_1\frac{(1-p)P(x,0) - pP(0,1) - (1-p)(1-\rho)}{\lambda x - p\nu_1}$$

and

$$P(1,y) = \frac{(\nu_1 y + \nu_2)((1-p)P(1,0) - pP(0,y) - (1-p)(1-\rho)) + \nu_2(1-\rho)}{(1-p)\nu_2 - p\nu_1 y}.$$

Note that the normalizing condition $P(1,1) = 1$ implies that

$$(1-p)P(1,0) - pP(0,1) = (1-p)(1-\rho) + \rho_1 - p. \tag{13}$$

Lemma 5. *If $r_2 \leq 1$, then*

$$(1-p)P\left(r_1^{-1},0\right) - pP(0,1) - (1-p)(1-\rho) = 0, \tag{14}$$

which implies that the point $1/r_1$ is a removable singularity for the function $P(x,1)$. If $r_2 > 1$ (and then $r_1 \leq 1$ by the stability condition (3)), we have

$$(1-p)P\left(r_1^{-1},0\right) - pP(0,1) - (1-p)(1-\rho) < 0 \tag{15}$$

and the point $1/r_1$ is a singularity for the function $P(x,1)$.

Proof. We know that $P(x,0)$ is a meromorphic function in the disk with center 0 and radius x_3, with a unique potential singularity at point x^*. Equation (1) implies for $x \neq x^*$ when x^* is a singularity for $P(x,0)$

$$h_2(x,Y^*(x))P(x,0) + h_3(x,Y^*(x))P(0,Y^*(x)) + h_4(x,Y^*(x)) = 0. \tag{16}$$

When $r_2 \leq 1$, we have $Y^*(1/r_1) = 1$ and the above equation implies Equation (14). When $r_2 > 1$ (and hence $r_1 \leq 1$), we have $Y^*(1/r_1) = 1/r_2 < 1$, and Equation (16) implies

$$(1-p)P(1/r_1,0) - pP(0,1/r_2) - (1-p)(1-\rho) = \\ (1-\rho)\frac{\frac{\nu_2}{r_1}\left(1 - \frac{1}{r_2}\right)}{\frac{\nu_1}{r_2}\left(\frac{1}{r_2} - \frac{1}{r_1}\right) + \frac{\nu_2}{r_1}\left(\frac{1}{r_2} - 1\right)} < 0$$

Since $P(0,1/r_2) \leq P(0,1)$, Inequality (15) follows. □

Similar arguments yield the following result for the function $P(1,y)$; the proof is omitted.

Lemma 6. *We have*

$$(1-p)P(1,0) - pP(0,r_1/r_2) - (1-p)(1-\rho) + p(1-\rho) = 0. \tag{17}$$

and the point r_1/r_2 is a removable singularity for the function $P(1,y)$.

By using the two above lemmas, we are now able to determine the tails of the probability distributions of the random variables N_1 and N_2.

Proposition 5. *The quantities $\mathbb{P}(N_1 = n)$ are when $n \to \infty$ as follows:*

Case I: *If $y^* = Y^*(x^*)$, which can occur only if $r_1 \geq r_2$, then*

$$\mathbb{P}(N_1 = n) \sim \kappa_1^{(1)} \left(\frac{1}{x^*}\right)^n. \tag{18}$$

Case II: *If $y^* \neq Y^*(x^*)$, we distinguish two subcases:*

Case II.1: *If $r_2 > 1$ (and then $r_1 \leq 1$),*

$$\mathbb{P}(N_1 = n) \sim \kappa_2^{(1)} (r_1)^n. \tag{19}$$

Case II.2: *If $r_2 \leq 1$, $1/r_1$ is a removable singularity for $P(x,1)$ and we have*

$$\mathbb{P}(N_1 = n) \sim \kappa_3^{(1)} \frac{1}{n\sqrt{n}} \left(\frac{1}{x_3}\right)^n. \tag{20}$$

Case III: *If $x^* = x_3$ and $y^* = Y^*(x_3)$,*

$$\mathbb{P}(N_1 = n) \sim \kappa_4^{(1)} \frac{1}{\sqrt{n}} \left(\frac{1}{x^*}\right)^n. \tag{21}$$

Here,

$$\begin{aligned}
\kappa_1^{(1)} &= (1-\rho)\frac{\nu_1\nu_2((1-p)\nu_2 x^* - p\nu_1(y^*)^2)}{(\lambda x^* - p\nu_1)x^* y^* \mathcal{Q}_x'(y^*)}, \\
\kappa_2^{(1)} &= \frac{1}{p}\left(-(1-p)P\left(r_1^{-1}, 0\right) + pP(0,1) + (1-p)(1-\rho)\right), \\
\kappa_3^{(1)} &= \frac{(1-\rho)\nu_1}{4\sqrt{\pi}(p\nu_1 - \lambda x_3)} \frac{\lambda^2(1-p)x_3^2 + 2p\nu_2\lambda x_3 - p\nu_2(p\lambda+\nu_1)}{\mathcal{Q}_y(x_3)\mathcal{Q}_y^*(x_3)}\sqrt{x_3}\tau_x, \\
\kappa_4^{(1)} &= \frac{(1-\rho)\nu_1}{2\sqrt{\pi}(p\nu_1 - \lambda x_3)} \frac{\lambda^2(1-p)x_3^2 + 2p\nu_2\lambda x_3 - p\nu_2(p\lambda+\nu_1)}{\sqrt{x_3}\mathcal{Q}_y'(x_3)\mathcal{Q}_y^*(x_3)}\tau_x,
\end{aligned}$$

with $\tau_x = \sqrt{(x_3 - x_1)(x_3 - x_2)(x_4 - x_3)}$ and

$$\mathcal{Q}_y^*(x) = \left(x - \frac{p\nu_1 y^*}{\lambda x^*}\right)\left(x - \frac{p\nu_1 y_*}{\lambda x_*}\right). \tag{22}$$

Proof. Note first that we always have $1/r_1 \leq x_3$ since

$$D_1(1/r_1) = (1 - 1/r_2)^2/r_1^2 \geq 0.$$

In case I, note that if $r_2 \leq 1$, $1/r_1$ is a removable singularity for the function $P(x,1)$. If $r_2 > 1$, then $x^* < 1/r_1 \leq x_3$ since

$$Q_y(1/r_1) = \nu_1\lambda(1/r_2 - p - (1-p)/r_1) < 0.$$

This implies that x^* is the singularity with the smallest module. The residue of the function $P(x,0)$ at point x^* is equal to

$$(1-\rho)\frac{\nu_2 y^*(p\nu_1 y^* - \lambda(x^*)^2)}{(1-p)x^*\mathcal{Q}_x'(y^*)\left.\frac{\partial Y^*}{\partial x}\right|_{x=x^*}}$$

Since $h_1(x, Y^*(x)) = 0$, we deduce that

$$\left.\frac{\partial Y^*}{\partial x}\right|_{x=x^*} = -\frac{\frac{\partial h_1}{\partial x}(x^*,y^*)}{\frac{\partial h_1}{\partial y}(x^*,y^*)} = \frac{(y^*)^2(p\nu_1 y^* - \lambda(x^*)^2)}{x^*(p\nu_1(y^*)^2 - (1-p)\nu_2 x^*)}.$$

A direct application of Darboux's method then yields Equation (18).

In case II.1, the point $r_1 \leq 1$ is the pole with the smallest module for the function $P(x,1)$ and Darboux's method yields Equation (19).

In case II.2, the function $P(x,1)$ has no singularities in the disk $D(0,x_3)$ with center 0 and radius x_3. In that case, the function $P(x,0)$ can be represented as follows: for $|x| < x_3$,

$$P(x,0) = \frac{1}{2i\pi}\int_{C(x_3)} \frac{g_x(z)}{z-x}dz,$$

where $C(x_3)$ is the circle with center 0 and radius x_3. By using Equation (14), we have

$$P(x,1) = \nu_1(1-p)\frac{P(x,0)-P(1/r_1,0)}{\lambda x - p\nu_1} = \frac{1}{2i\pi}\int_{C(x_3)}\frac{h_x(z)}{z-x}dz,$$

where

$$h_x(z) = \nu_1(1-p)\frac{g_x(z)}{\lambda z - p\nu_1}.$$

As shown in Section 3, the point x_* may be a pole of the function h_x. Let $\mathrm{Res}(h_x;x_*)$ denote the residue of the function h_x at point x_*. By deforming the integration contour so as to encompass the segment $[x_3,x_4]$ and since $|h_x(z)| < K_x/|z|$ for some constant $K_x > 0$ when $|z| \to \infty$, we deduce that

$$P(x,1) = \frac{1}{2i\pi}\int_{x_3}^{x_4}\frac{h_x(z+0i)-h_x(z-0i)}{z-x}dz + \frac{\mathrm{Res}(h_x;x_*)}{x-x_*}$$

and then

$$\begin{aligned} P(x,1) &= \frac{-1}{\pi}\int_{x_3}^{x_4}\frac{(1-\rho)\nu_1\nu_2}{\xi(\lambda\xi-p\nu_1)(\xi-x)}\Im\left(\frac{Y^*(\xi)(\lambda\xi^2-p\nu_1Y^*(\xi))}{\mathcal{Q}_x(Y^*(\xi))}\right)d\xi \\ &\quad + \frac{\mathrm{Res}(h_x;x_*)}{x-x_*}. \end{aligned}$$

We have

$$\Im\left(\frac{Y^*(\xi)(\lambda\xi^2-p\nu_1Y^*(\xi))}{\mathcal{Q}_x(Y^*(\xi))}\right) = \frac{\Im\left(Y^*(\xi)(\lambda\xi^2-p\nu_1Y^*(\xi))\mathcal{Q}_x(\overline{Y^*(\xi)})\right)}{\mathcal{Q}_x(Y^*(\xi))\mathcal{Q}_x(\overline{Y^*(\xi)})}.$$

When $\xi \in [x_3,x_4]$, we have

$$\overline{Y^*}(\xi) = Y_*(\xi)$$

and tedious computations show that $\mathcal{Q}_x(Y^*(\xi))\mathcal{Q}_x(Y_*(\xi))$ is a quadratic polynomial in ξ. We specifically have

$$\begin{aligned} &\mathcal{Q}_x(Y^*(\xi))\mathcal{Q}_x\left(Y_*(\xi)\right) = \\ &\qquad (\lambda\nu_1)^2(Y^*(\xi)-y^*)(Y^*(\xi)-y_*)\left(Y_*(\xi)-y^*\right)\left(Y_*(\xi)-y_*\right) \end{aligned}$$

By definition, we know that the above quantity vanishes for x equal to x^* or x_*. More precisely, in case II.2, we have $Y_*(x^*) = y^*$. In addition, $Y_*(x_*)$ or $Y^*(x_*)$ is equal to y_*. Finally, we note that if x is such that $h_1(x,y)=0$ then $p\nu_1 y/(\lambda x)$ is also such that $h_1(x,y)=0$. This implies that the four roots of the polynomial $\mathcal{Q}_x(Y^*(\xi))\mathcal{Q}_x(Y_*(\xi))$ are x_*, x^*, $p\nu_1y^*/(\lambda x^*)$ and $p\nu_1 y_*/(\lambda x_*)$. Hence,

$$\begin{aligned} \mathcal{Q}_x(Y^*(\xi))\mathcal{Q}_x\left(Y_*(\xi)\right) &= -\frac{\lambda^3\nu_2^2}{p^2\nu_1}(\xi-x_*)(\xi-x^*)\left(\xi-\frac{p\nu_1y^*}{\lambda x^*}\right)\left(\xi-\frac{p\nu_1y_*}{\lambda x_*}\right) \\ &= -\frac{\lambda\nu_2^2}{p^2\nu_1}\mathcal{Q}_y(\xi)\mathcal{Q}_y^*(\xi), \end{aligned}$$

where the polynomial $\mathcal{Q}_y^*(x)$ is defined by (22).

Moreover, we have

$$\begin{aligned} &\Im\left(Y^*(\xi)(p\nu_1Y^*(\xi)-\lambda\xi^2)\mathcal{Q}_x(\overline{Y^*(\xi)})\right) = \\ &\qquad \frac{\lambda}{2p\nu_1}\left(-\frac{\lambda^2\nu_2(1-p)}{p}\xi^3+\nu_2^2(p\lambda+\nu_1)\xi-2\nu_2^2\lambda\xi^2\right)\sqrt{-D_1(\xi)}. \end{aligned}$$

It follows that
$$P(x,1) = \frac{1}{\pi}\int_{x_3}^{x_4} \frac{H_x(\xi)}{\xi - x} d\xi + \frac{\mathrm{Res}(h_x; x_*)}{x - x_*},$$
where
$$H_x(\xi) = \frac{(1-\rho)\nu_1}{p\nu_1 - \lambda\xi} \frac{\lambda^2(1-p)\xi^2 + 2p\lambda\nu_2\xi - p\nu_2(p\lambda+\nu_1)}{2\mathcal{Q}_y(\xi)\mathcal{Q}_y^*(\xi)}\sqrt{-D_1(\xi)}$$
and then
$$\mathbb{P}(N_1 = n) = \frac{1}{\pi}\int_{x_3}^{x_4} \frac{H_x(\xi)}{\xi} e^{-n\log\xi} d\xi - \frac{\mathrm{Res}(h_x; x_*)}{(x_*)^{n+1}}. \tag{23}$$
In the neighborhood of x_3, we have
$$-\log\xi = -\log x_3 - \frac{1}{x_3}(\xi - x_3) + o(\xi - x_3)$$
and
$$\frac{H_x(\xi)}{\pi\xi} = k_1^{(1)}\sqrt{\xi - x_3} + o(\sqrt{\xi - x_3}),$$
where
$$k_1^{(1)} = \frac{(1-\rho)\nu_1}{2\pi(p\nu_1 - \lambda x_3)} \frac{\lambda^2(1-p)x_3^2 + 2p\nu_2\lambda x_3 - p\nu_2(p\lambda+\nu_1)}{x_3\mathcal{Q}_y(x_3)\mathcal{Q}_y^*(x_3)}\tau_x.$$
A direct application of Laplace's method [2, 8] then yields
$$\mathbb{P}(N_1 = n) \sim k_1^{(1)}\Gamma(3/2)\frac{1}{n^{3/2}}\left(\frac{1}{x_3}\right)^{n-\frac{3}{2}}$$
when $n \to \infty$. Since $\Gamma(3/2) = \sqrt{\pi}/2$, Equation (20) follows.

In case III, we have for ξ in the neighborhood of x_3
$$\mathcal{Q}_y(\xi) = \mathcal{Q}_y'(x_3)(\xi - x_3) + o(\xi - x_3)$$
and then
$$\frac{H_x(\xi)}{2\pi\xi} = k_2^{(1)}(\xi - x_3)^{-1/2} + o((\xi - x_3)^{-1/2}),$$
where
$$k_2^{(1)} = \frac{(1-\rho)\nu_1}{2\pi(p\nu_1 - \lambda x_3)} \frac{\lambda^2(1-p)x_3^2 + 2p\nu_2\lambda x_3 - p\nu_2(p\lambda+\nu_1)}{x_3\mathcal{Q}_y'(x_3)\mathcal{Q}_y^*(x_3)}\tau_x.$$
Laplace's method then yields
$$\mathbb{P}(N_1 = n) \sim k_2^{(1)}\Gamma(1/2)\frac{1}{n^{1/2}}\left(\frac{1}{x_3}\right)^{n-\frac{1}{2}}$$
and by using the fact that $\Gamma(1/2) = \sqrt{\pi}$, Equation (21) follows. $\square$

Remark. *(*Priority case) When we set $p = 0$ we give full priority to queue 2 and the functional equation greatly simplifies due to $h_3(x,y) = 0$. Then, for $\zeta(x) = \nu_2/(\lambda + \nu_2 - \lambda x)$, we see that $h_1(x,\zeta(x)) = 0$ and hence
$$\begin{aligned} P(x,0) &= -\frac{h_4(x,\zeta(x))P(0,0)}{h_2(x,\zeta(x))} = \frac{(\nu_1\nu_2 - \lambda\nu_1 x)(1-\rho)}{\mathcal{Q}_y(x)} \\ &= \frac{(\nu_1\nu_2 - \lambda\nu_1 x)(1-\rho)}{\lambda^2(x - x_*)(x - x^*)} = \frac{c_1}{x - x_*} + \frac{c_2}{x - x^*}, \end{aligned}$$

with
$$c_1 = \frac{(\nu_1\nu_2 - \lambda\nu_1 x_*)(1-\rho)}{\lambda^2(x_* - x^*)}, \quad c_2 = \frac{(\nu_1\nu_2 - \lambda\nu_1 x^*)(1-\rho)}{\lambda^2(x_* - x^*)}.$$
This gives
$$P(x,1) = \frac{\nu_1}{\lambda x}\left[\frac{c_1}{x-x_*} + \frac{c_2}{x-x^*} - (1-\rho)\right]$$
and
$$\mathbb{P}(N_1 = n) \sim \frac{\nu_1^2\lambda x^* - \nu_1^2\nu_2}{\lambda^3(x^* - x_*)(x^*)^2}(1-\rho)\left(\frac{1}{x^*}\right)^n.$$
Note that this agrees with regime I in Proposition 5 if
$$\frac{\nu_1(\lambda x^* - \nu_2)}{\lambda^2(x^* - x_*)x^*} = \frac{\nu_2^2}{\nu_2^2 + \lambda\nu_1(y_*)^2},$$
which can indeed be shown to be true.

For the second queue, we first note by using Lemma 6 that the point r_1/r_2 is always a removable singularity for the function $P(1,y)$.

Proposition 6. *The quantities $\mathbb{P}(N_2 = n)$ are when $n \to \infty$ as follows:*

Case I: *If $x^* = X^*(y^*)$, which can occur only when $r_1 \leq 1$, then*
$$\mathbb{P}(N_2 = n) \sim \kappa_1^{(2)}\left(\frac{1}{y^*}\right)^n. \tag{24}$$

Case II: *If $x^* \neq X^*(y^*)$, then*
$$\mathbb{P}(N_2 = n) \sim \kappa_2^{(2)}\frac{1}{n\sqrt{n}}\left(\frac{1}{y_3}\right)^n. \tag{25}$$

Case III: *If $x^* = X^*(y^*)$ and $y^* = y_3$,*
$$\mathbb{P}(N_2 = n) \sim \kappa_3^{(2)}\frac{1}{\sqrt{n}}\left(\frac{1}{y^*}\right)^n. \tag{26}$$

Here,
$$\kappa_1^{(2)} = \frac{(1-\rho)\lambda(\nu_1 y^* + \nu_2)(p\nu_1 y^* - \lambda(x^*)^2)}{((1-p)\nu_2 - p\nu_1 y^*)x^*\mathcal{Q}_y'(x^*)},$$

$$\kappa_2^{(2)} = \frac{(1-\rho)(\nu_2 + \nu_1 y_3)\left(\lambda p(p\nu_2 + (1-p)\nu_1)y_3^2 + 2\lambda p(1-p)\nu_2 y_3 - (1-p)\nu_2^2\right)}{2\sqrt{\pi}p^2(p\nu_1 y_3 - (1-p)\nu_2)\mathcal{Q}_x(y_3)\mathcal{Q}_x^*(y_3)}\sqrt{y_3}\tau_y,$$

$$\kappa_3^{(2)} = \frac{(1-\rho)(\nu_2 + \nu_1 y_3)\left(\lambda p(p\nu_2 + (1-p)\nu_1)y_3^2 + 2\lambda p(1-p)\nu_2 y_3 - (1-p)\nu_2^2\right)}{\sqrt{\pi}\sqrt{y_3}p^2(p\nu_1 y_3 - (1-p)\nu_2)\mathcal{Q}_x'(y_3)\mathcal{Q}_x^*(y_3)}\tau_y,$$
with $\tau_y = \sqrt{p\nu_1(y_3 - y_1)(y_3 - y_2)/\lambda}$ and
$$\mathcal{Q}_x^*(y) = \left(y - \frac{(1-p)\nu_2 x^*}{p\nu_1 y^*}\right)\left(y - \frac{(1-p)\nu_2 x_*}{p\nu_1 y_*}\right). \tag{27}$$

Proof. In case I, y^* is the pole with the smallest module for the function $P(1,y)$ and a direct application of Darboux's method yields

$$\mathbb{P}(N_2 = n) \sim \frac{(1-\rho)\lambda}{(1-p)\nu_2 - p\nu_1 y^*} \frac{(1-p)\nu_2 x^* - p\nu_1 (y^*)^2}{(y^*)^2 \mathcal{Q}'_y(x^*) \left.\frac{\partial X^*}{\partial y}\right|_{y=y^*}} \left(\frac{1}{y^*}\right)^n$$

and Equation (24) follows.

In case II, the function $P(1,y)$ is analytic in the disk with center 0 and radius y_3 and we have

$$P(0,y) = \frac{1}{2i\pi} \int_{C(y_3)} \frac{g_y(z)}{z-y} dz,$$

where $C(y_3)$ is the circle with center 0 and radius y_3. By using Equation (17), we have

$$P(1,y) = 1 - \rho + \frac{(\nu_1 y + \nu_2) p (P(0, r_1/r_2) - P(0,y))}{(1-p)\nu_2 - p\nu_1 y} = 1 - \rho + \frac{1}{2i\pi} \int_{C(y_3)} \frac{h_y(z)}{z-x} dz,$$

where

$$h_y(z) = \frac{p(\nu_2 + \nu_1 z) g_y(z)}{p\nu_1 z - (1-p)\nu_2}.$$

By deforming the integration contour along the segment $[y_3, \infty)$ and since the function $h_y(z)$ is such that $|h_y(z)| < K_y/|z|$ for some constant $K_y > 0$ when $|z| \to \infty$, we deduce that

$$P(1,y) = (1-\rho) + \frac{1}{2i\pi} \int_{y_3}^{\infty} \frac{h_y(z+0i) - h_y(z-0i)}{z-y} dz$$

and then

$$P(1,y) = (1-\rho) + \frac{-1}{\pi} \int_{y_3}^{\infty} \frac{(1-\rho)\lambda(\nu_2 + \nu_1 z)}{z(p\nu_1 z - (1-p)\nu_2)} \Im\left(\frac{(p\nu_1 y^2 - (1-p)\nu_2 X^*(y))}{\mathcal{Q}_y(X^*(y))}\right) dz.$$

There holds

$$\Im\left(\frac{(p\nu_1 y^2 - (1-p)\nu_2 X^*(y))}{\mathcal{Q}_y(X^*(y))}\right) = \frac{\Im\left((p\nu_1 y^2 - (1-p)\nu_2 X^*(y))\mathcal{Q}_y(\overline{X^*(y)})\right)}{\mathcal{Q}_y(X^*(y))\mathcal{Q}_y(\overline{X^*(y)})}.$$

When $z \in [y_3, \infty)$, we have

$$\overline{X^*}(z) = X_*(z).$$

It is easily checked that the function $z \to z^2 \mathcal{Q}_y(X^*(z))\mathcal{Q}_y(X_*(z))$ is a quadratic polynomial in z. By definition, we know that this polynomial vanishes for y equal to y^* or y_*. More precisely, in case II, we have $X_*(y^*) = x^*$. In addition $X_*(y_*)$ or $X^*(y_*)$ is equal to x_*. If y is such that $h_1(x,y) = 0$ then $(1-p)\nu_2 x/(p\nu_1 y)$ is also such that $h_1(x,y) = 0$. This implies that the four roots of the polynomial $z^2 \mathcal{Q}_y(X^*(z))\mathcal{Q}_y(X_*(z))$ are y_*, y^*, $(1-p)\nu_2 x^*/(p\nu_1 y^*)$ and $(1-p)\nu_2 x_*/(p\nu_1 y_*)$. Hence,

$$z^2 \mathcal{Q}_y(X^*(z))\mathcal{Q}_y(X_*(z)) = \lambda^2 p^2 \nu_1^2 (z - y_*)(z - y^*) \left(z - \frac{(1-p)\nu_2 x^*}{p\nu_1 y^*}\right) \left(z - \frac{(1-p)\nu_2 x_*}{p\nu_1 y_*}\right)$$

and then

$$z^2 \mathcal{Q}_y(X^*(z))\mathcal{Q}_y(X_*(z)) = \lambda\nu_1 p^2 \mathcal{Q}_x(z)\mathcal{Q}^*_x(z),$$

where the polynomial $\mathcal{Q}^*_x(z)$ is defined by Equation (22).

Moreover, we have in the neighborhood of y_3

$$\Im\left((p\nu_1 y^2-(1-p)\nu_2 X^*(y))\mathcal{Q}_y(\overline{X^*(y)})\right) = \\ -\frac{\nu_1}{2y}\left(\lambda p(p\nu_2+(1-p)\nu_1)y^2+2\lambda p(1-p)\nu_2 y-(1-p)\nu_2^2\right)\sqrt{-D_2(y)}.$$

It follows that

$$P(1,y) = (1-\rho)+\frac{1}{\pi}\int_{y_3}^{\infty}\frac{H_y(z)}{z-x}dz,$$

where the function $H_y(z)$ is defined for z in the neighborhood of y_3 by

$$H_y(z) = \frac{(1-\rho)(\nu_2+\nu_1 z)}{2p^2(p\nu_1 z-(1-p)\nu_2)\mathcal{Q}_x(z)\mathcal{Q}_x^*(z)} \\ \times\left(\lambda p(p\nu_2+(1-p)\nu_1)z^2+2\lambda p(1-p)\nu_2 z-(1-p)\nu_2^2\right)\sqrt{-D_2(z)}.$$

Then, for $n\geq 1$,

$$\mathbb{P}(N_2=n) = \frac{1}{\pi}\int_{y_3}^{\infty}\frac{H_y(z)}{z}e^{-n\log z}dz. \tag{28}$$

In the neighborhood of y_3, we have

$$-\log z = -\log y_3 - \frac{1}{y_3}(z-y_3)+o(z-y_3)$$

and

$$\frac{H_y(z)}{\pi z} = k_1^{(2)}\sqrt{z-y_3}+o(\sqrt{z-y_3}),$$

where

$$k_1^{(2)} = \frac{(1-\rho)(\nu_2+\nu_1 y_3)\left(\lambda p(p\nu_2+(1-p)\nu_1)y_3^2+2\lambda p(1-p)\nu_2 y_3-(1-p)\nu_2^2\right)}{2\pi y_3 p^2(p\nu_1 y_3-(1-p)\nu_2)\mathcal{Q}_x(y_3)\mathcal{Q}_x^*(y_3)} \\ \times\sqrt{4p\nu_1(y_3-y_1)(y_3-y_2)/\lambda}.$$

A direct application of Laplace's method [8, Paragraph IV.2] then yields

$$\mathbb{P}(N_2=n)\sim k_1^{(2)}\Gamma(3/2)\frac{1}{n^{3/2}}\left(\frac{1}{y_3}\right)^{n-\frac{3}{2}}$$

when $n\to\infty$. Since $\Gamma(3/2)=\sqrt{\pi}/2$, Equation (25) follows.

In case III, we have for z in the neighborhood of y_3

$$\mathcal{Q}_x(z) = \mathcal{Q}_x'(y_3)(z-y_3)+o((z-y_3))$$

and then

$$\frac{H_y(z)}{2\pi z} = k_2^{(2)}(z-y_3)^{-1/2}+o((z-y_3)^{-1/2}),$$

where

$$k_2^{(2)} = \frac{(1-\rho)(\nu_2+\nu_1 y_3)\left(\lambda p(p\nu_2+(1-p)\nu_1)y_3^2+2\lambda p(1-p)\nu_2 y_3-(1-p)\nu_2^2\right)}{2\pi y_3 p^2(p\nu_1 y_3-(1-p)\nu_2)\mathcal{Q}_x'(y_3)\mathcal{Q}_x^*(y_3)} \\ \times\sqrt{4p\nu_1(y_3-y_1)(y_3-y_2)/\lambda}.$$

Laplace's method then yields

$$\mathbb{P}(N_2=n)\sim k_2^{(2)}\Gamma(1/2)\frac{1}{n^{1/2}}\left(\frac{1}{y_3}\right)^{n-\frac{1}{2}}$$

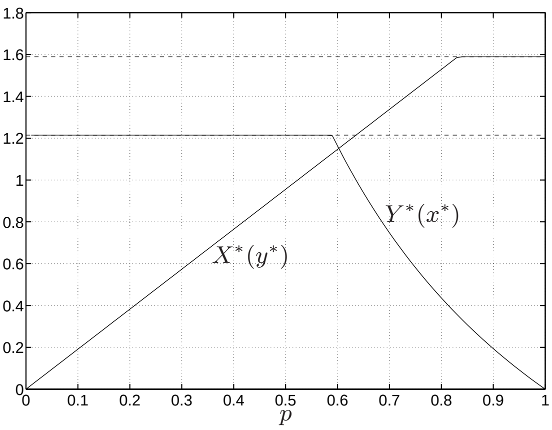

FIGURE 5. Comparisons between x^*, $X^*(y^*)$ and y^*, $Y^*(x^*)$ when p varies. Here, $\lambda = 1.5$, $\rho_1 = .4$, $\rho_2 = .3$ and $x^* = 1.5890$, $y^* = 1.2146$.

and by using the fact that $\Gamma(1/2) = \sqrt{\pi}$, Equation (26) follows. □

5.1. **Numerical examples.** We shall now compare the asymptotic estimates in Propositions 5 and 6 against results obtained by numerical calculations. Truncating the state space by bounding one of the queue lengths leads to a Markov process on an infinite strip, better known as a Quasi-Birth-Death (QBD) process. For these processes, fast numerical algorithms are available (see [25]). All numerical results presented were obtained by imposing an upper bound on the second queue of 500.

For a first scenario we take $\lambda = 1.5$, $\rho_1 = .4$ and $\rho_2 = .3$. Figure 5 compares $X^*(y^*)$ with x^* and $Y^*(x^*)$ with y^*, when p varies. For example, we see that for $p < .6$, $Y^*(x^*) = y^*$. For $p = .5$, we have regime (18) for queue 1 and regime (25) for queue 2. Results for this case are presented in Table 1. Note that (18) converges fast to the true (numerical) value. The convergence of the branch point asymptotics (25) seems slower, in particular the convergence of the last column in Table 1 to the value $\kappa_2^{(2)} = 20.7454$. In order to demonstrate that $\kappa_2^{(2)}$ is indeed the leading constant, we compare (25) against the integral representation (28) (omitting the residue term); see Table 2. Indeed, this confirms the correctness of $\kappa_2^{(2)} = 20.7454$.

Results for $p = .65$ are presented in Table 3 in which case we have regime (20) for queue 1 and regime (25) for queue 2. Note again the slow convergence to the asymptotic constants $\kappa_3^{(1)}$ and $\kappa_2^{(2)}$.

Table 4 illustrates some results for $\lambda = 1.5$, $\rho_1 = .2$, $\rho_2 = .4$ and $p = .4$, in which case we have regimes (19) and (24).

APPENDIX A. THE RESULTANT OF THE POLYNOMIALS h_1 AND h_2

Generally speaking, when we have two polynomials in two variables, say,

$$\begin{aligned} f_1(x,y) &= a_0(y) + a_1(y)x + \cdots + a_n(y)x^n, \\ f_2(x,y) &= b_0(y) + b_1(y)x + \cdots + b_m(y)x^m, \end{aligned}$$

n	$\mathbb{P}(N_1 = n)$	$\kappa_1^{(1)}(x^*)^{-n}$	$\mathbb{P}(N_2 = n)$	$n^{-3/2}(y_3)^{-n}$	$\frac{\mathbb{P}(N_2=n)}{n^{-3/2}(y_3)^{-n}}$
5	2.8301e-002	2.4891e-002	1.0567e-002	5.1414e-003	2.0553e+000
10	2.5852e-003	2.4569e-003	3.6384e-004	1.0449e-004	3.4821e+000
15	2.4842e-004	2.4252e-004	1.5032e-005	3.2693e-006	4.5978e+000
20	2.4237e-005	2.3938e-005	6.7391e-007	1.2206e-007	5.5210e+000
50	2.2151e-011	2.2140e-011	1.0132e-014	1.1140e-015	9.0958e+000
100	1.9438e-021	1.9438e-021	1.8829e-027	1.5511e-028	1.2139e+001
200	1.4983e-041	1.4983e-041	1.2762e-052	8.5067e-054	1.5002e+001
300	1.1549e-061	1.1549e-061	1.1804e-077	7.1825e-079	1.6434e+001

TABLE 1. Illustration of (18) and (25) for $\lambda = 1.5$, $\rho_1 = .4$, $\rho_2 = .3$, $p = .5$. In this case $x^* = 1.5890$, $X^*(y^*) = 0.9555$, $y^* = Y^*(x^*) = 1.2146$. We find that $\kappa_2^{(2)} = 20.7454$.

n	(28)	$n^{-3/2}(y_3)^{-n}$	$\frac{\mathbb{P}(N_2=n)}{n^{-3/2}(y_3)^{-n}}$
10^2	1.8301e-27	1.5509e-28	1.1801e+1
10^3	4.8227e-252	2.5453e-253	1.8947e+1
10^4	2.3446e-2486	1.1415e-2487	2.0540e+1
10^5	2.4607e-24816	1.1873e-24817	2.0725e+1
10^6	1.1550e-248102	5.5682e-248104	2.0743e+1
10^7	1.8797e-2480952	9.0611e-2480954	2.0745e+1

TABLE 2. Comparison of (28) and (25) for $\lambda = 1.5$, $\rho_1 = .4$, $\rho_2 = .3$, $p = .5$ and $\kappa_2^{(2)} = 20.7454$.

n	$\mathbb{P}(N_1 = n)$	$n^{-3/2}(x_3)^{-n}$	$\frac{\mathbb{P}(N_1=n)}{n^{-3/2}(x_3)^{-n}}$	$\mathbb{P}(N_2 = n)$	$n^{-3/2}(y_3)^{-n}$	$\frac{\mathbb{P}(N_2=n)}{n^{-3/2}(y_3)^{-n}}$
5	2.0854e-002	7.4520e-003	2.7985e+000	2.6154e-002	2.7103e-002	9.6499e-001
10	1.2811e-003	2.1951e-004	5.8359e+000	4.1746e-003	2.9037e-003	1.4377e+000
15	8.6268e-005	9.9552e-006	8.6656e+000	8.3828e-004	4.7896e-004	1.7502e+000
20	6.0730e-006	5.3873e-007	1.1273e+001	1.8669e-004	9.4268e-005	1.9804e+000
50	1.0651e-012	4.5586e-014	2.3364e+001	4.9780e-008	1.8464e-008	2.6961e+000
100	9.3290e-024	2.5976e-025	3.5914e+001	1.3411e-013	4.2613e-014	3.1472e+000
200	1.1821e-045	2.3856e-047	4.9552e+001	2.2323e-024	6.4202e-025	3.4770e+000
300	1.9248e-067	3.3732e-069	5.7061e+001	5.3829e-035	1.4892e-035	3.6146e+000

TABLE 3. Illustration of (20) and (25) for $\lambda = 1.5$, $\rho_1 = .4$, $\rho_2 = .3$, $p = .65$. In this case $x^* = 1.5890$, $X^*(y^*) = 1.2421$, $y^* = 1.2146$ and $Y^*(x^*) = 0.9392$. We find that $\kappa_3^{(1)} = 81.6727$ and $\kappa_2^{(2)} = 3.7799$.

the resultant of the polynomials f_1 and f_2 with respect to x is the determinant $\mathrm{Res}_x(f_1, f_2)$ of the matrix

$$\begin{pmatrix} a_n & \cdots & a_0 & 0 & \cdots & \cdots \\ 0 & a_n & \cdots & a_0 & 0 & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & 0 & a_n & \cdots & a_0 \\ b_m & \cdots & b_0 & 0 & \cdots & \cdots \\ 0 & b_m & \cdots & b_0 & 0 & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & 0 & b_m & \cdots & b_0 \end{pmatrix} \begin{array}{l} \left.\vphantom{\begin{array}{c}a\\a\\a\\a\end{array}}\right\} m \text{ rows} \\ \left.\vphantom{\begin{array}{c}a\\a\\a\\a\end{array}}\right\} n \text{ rows} \end{array}$$

n	$\mathbb{P}(N_1 = n)$	$\kappa_2^{(1)}(r_1)^n$	rel. error	$\mathbb{P}(N_2 = n)$	$\kappa_1^{(2)}(y^*)^{-n}$
5	2.5017e-003	2.1008e-003	1.1909	3.7599e-002	3.7227e-002
10	1.0008e-005	8.6452e-006	1.1577	2.8912e-002	2.8894e-002
15	4.0423e-008	3.5577e-008	1.1362	2.2428e-002	2.2427e-002
20	1.6403e-010	1.4641e-010	1.1204	1.7407e-002	1.7407e-002
50	7.6060e-025	7.1109e-025	1.0696	3.8058e-003	3.8058e-003
100	1.0262e-048	9.9052e-049	1.0360	3.0199e-004	3.0199e-004
200	1.9451e-096	1.9219e-096	1.0120	1.9014e-006	1.9014e-006
300	3.7427e-144	3.7292e-144	1.0036	1.1972e-008	1.1972e-008

TABLE 4. Illustration of (19) and (24) for $\lambda = 1$, $\rho_1 = .1$, $\rho_2 = .85$, $p = .3$. In this case $x^* = X^*(y^*) = 1.0581$, $y^* = 1.0520$ and $Y^*(x^*) = 0.2761$.

which is a polynomial in y. The polynomials f_1 and f_2 have a non trivial root (x_0, y_0) in common if and only if the resultant with respect to x is 0 at y_0. This leads to the resolution of a polynomial equation. Note that by adding to the $(m+n)$th column, the ith column multiplied by x^{m+n-i} for $0 \leq i < n+m$, $\mathrm{Res}_x(f_1, f_2)$ is equal to the determinant of the matrix

$$\begin{pmatrix} a_n & \cdots & a_0 & 0 & \cdots & x^{m-1}f_1 \\ 0 & a_n & \cdots & a_0 & 0 & x^{m-2}f_1 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & 0 & a_n & \cdots & f_1 \\ b_m & \cdots & b_0 & 0 & \cdots & x^{n-1}f_2 \\ 0 & b_m & \cdots & b_0 & 0 & x^{n-2}f_2 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & 0 & b_m & \cdots & f_2 \end{pmatrix},$$

which can written as $p(x,y)f_1(x,y) + q(x,y)f_2(x,y)$, where p and q are polynomials in variables x and y.

A.1. **Resultant in x.** In the case of the polynomials $h_1(x,y)$ and $h_2(x,y)$, the resultant in x, denoted by $Q_x(y)$, is the determinant of the matrix

$$\begin{pmatrix} -\lambda y & a_1(y) & -p\nu_1 y^2 \\ b_1(y) & (1-p)\nu_1 y^2 & 0 \\ 0 & b_1(y) & (1-p)\nu_1 y^2 \end{pmatrix},$$

where $a_1(y) = (\lambda + p\nu_1 + (1-p)\nu_2)y - (1-p)\nu_2$ and $b_1(y) = (1-p)((\nu_2 - \nu_1)y - \nu_2)$. Straightforward computations show that

$$Q_x(y) = -\nu_1(1-p)^2 y^2 (y-1)\mathcal{Q}_x(y),$$

where

$$\mathcal{Q}_x(y) = \lambda\nu_1 y^2 + \nu_2(\nu_2 - \nu_1 + \lambda)y - \nu_2^2.$$

It is easily checked that the quadratic polynomial $\mathcal{Q}_x(y)$ has two roots with opposite sign, as stated in Section 3. The positive root is

$$y^* = \frac{\nu_2}{2\lambda\nu_1}\left(-(\nu_2 - \nu_1 + \lambda) + \sqrt{(\nu_2 - \nu_1 + \lambda)^2 + 4\lambda\nu_1}\right)$$

and the negative root is

$$y_* = \frac{\nu_2}{2\lambda\nu_1}\left(-(\nu_2-\nu_1+\lambda) - \sqrt{(\nu_2-\nu_1+\lambda)^2+4\lambda\nu_1}\right).$$

In addition, the value of this polynomial at point 1 is equal to $\lambda(\nu_1+\nu_2)-\nu_1\nu_2 = \nu_1\nu_2(\rho_1+\rho_2-1)<0$, which implies that $y^*>1$.

A.2. **Resultant in y.** The resultant in y of the polynomials $h_1(x,y)$ and $h_2(x,y)$ is denoted by $Q_y(x)$ and is equal to the determinant of the matrix

$$\begin{pmatrix} -p\nu_1 & \alpha_1(x) & -(1-p)\nu_2 x & 0 \\ 0 & -p\nu_1 & \alpha_1(x) & -(1-p)\nu_2 x \\ (1-p)\nu_1 & (1-p)(\nu_2-\nu_1)x & -\nu_2(1-p)x & 0 \\ 0 & (1-p)\nu_1 & (1-p)(\nu_2-\nu_1)x & -\nu_2(1-p)x \end{pmatrix},$$

where $\alpha_1(x) = x(\lambda+p\nu_1+(1-p)\nu_2-\lambda x)$. Straightforward computations show that

$$Q_y(x) = -\nu_2\nu_1(1-p)^2x^2(x-1)\left(\lambda^2x^2-(\lambda+\nu_1+\nu_2)\lambda x+\nu_1\nu_2\right).$$

The quadratic polynomial in the right hand side of the above equation has two positive roots equal to

$$x^* = \frac{\lambda+\nu_1+\nu_2-\sqrt{(\lambda+\nu_1+\nu_2)^2-4\nu_1\nu_2}}{2\lambda}$$

and

$$x_* = \frac{\lambda+\nu_1+\nu_2+\sqrt{(\lambda+\nu_1+\nu_2)^2-4\nu_1\nu_2}}{2\lambda}.$$

If we set $\mathcal{Q}_y(x) = \lambda^2x^2-(\lambda+\nu_1+\nu_2)\lambda x+\nu_1\nu_2$, x_* and x^* are the two roots of this polynomial with $x^*<x_*$ and since $\mathcal{Q}_y(1) = \nu_1\nu_2(1-\rho_1-\rho_2)>0$, $x^*>1$.

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